# MANY BODY THERMALIZATION

## close to integrability

**Sergej FLACH**

Center for Theoretical Physics of Complex Systems
Institute for Basic Science
Daejeon South Korea

- measuring thermalization = measuring time scales
- ergodization time scales and Lyapunov time scales
- thermalization slowing down approaching integrability
- two 'universality classes' of slowing down
- results for classical and quantum systems

Reads:

PRE 95 060202 (2017)
PRL 122 054102 (2019)
PRE 100 032217 (2019)
PRE 104 014218 (2021)
Chaos 32 063113 (2022)
PRL 128 134102 (2022)
PRE 108 L062301 (2023)
PRR 6 L012064 (2024)
Chaos 34 033107 (2024)
arXiv:2405.00786

PCS Center for Theoretical Physics of Complex Systems

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# Dynamics Days Asia Pacific 13 / YKIS2024

Yukawa Institute for Theoretical Physics, Kyoto University, Japan

July 1st - 5th, 2024

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PRE 108 L062301 (2023)
PRR 6 L012064 (2024)
Chaos 34 033107 (2024)
arXiv:2405.00786

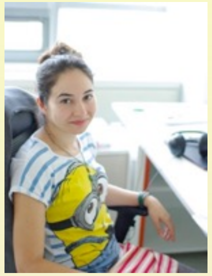
# People in chronological project order

David Campbell

Carlo Danieli

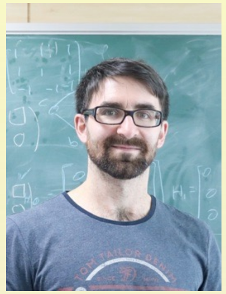
Mithun Thudiyangal

Yagmur Kati



Alexander Cherny

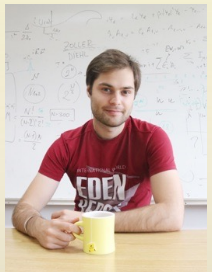
Thomas Engl



Mikhail Fistul

Boris Altshuler

Merab Malishava



Weihua Zhang

Barbara Dietz

Emil Yuzbashyan

Gabriel Lando

Aniket Patra

Budhaditya Bhattacharjee

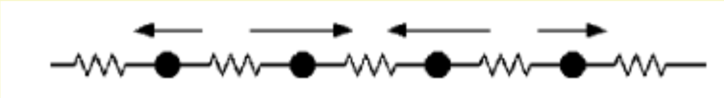
Alexei Andreanov

Xiaodong Zhang

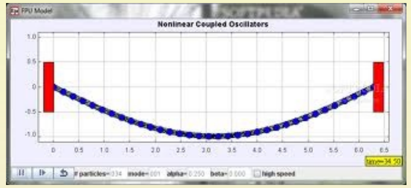
# The FPUT Problem

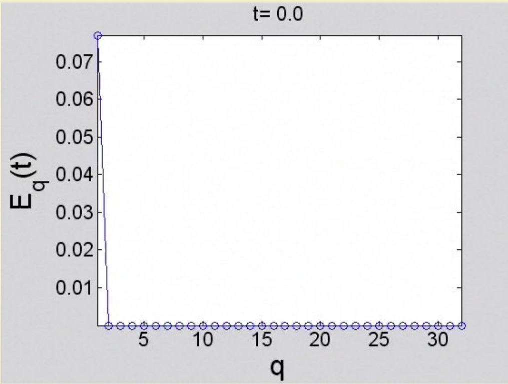
**1955**

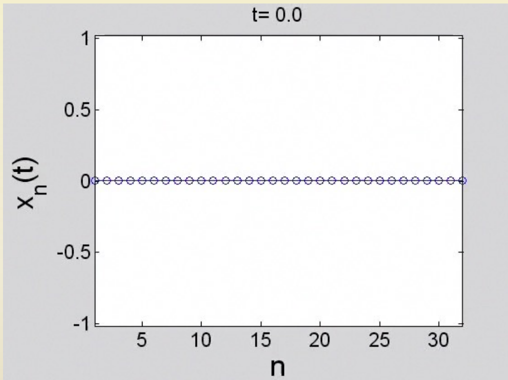
## Fermi, Pasta, Ulam, Tsingou



$$\ddot{x}_n = (x_{n+1} - 2x_n + x_{n-1}) + \alpha[(x_{n+1} - x_n)^2 - (x_n - x_{n-1})^2]$$






**FPUT problem:**
**excited mode q=1**
**did not observe equipartition**
**energy stays localized in few modes**
**recurrences after more integrations**
**thresholds in energy, system size etc**

**two time scales**
**$T_1$: formation of exponentially localized packets**
**$T_2$: gradual destruction and equipartition**

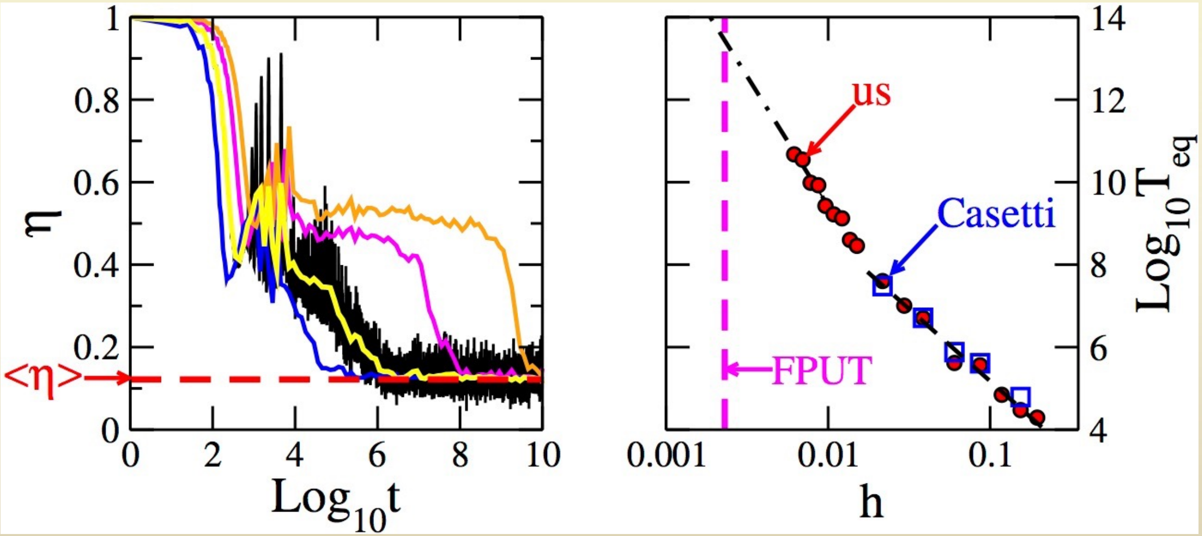
# FPUT Problem

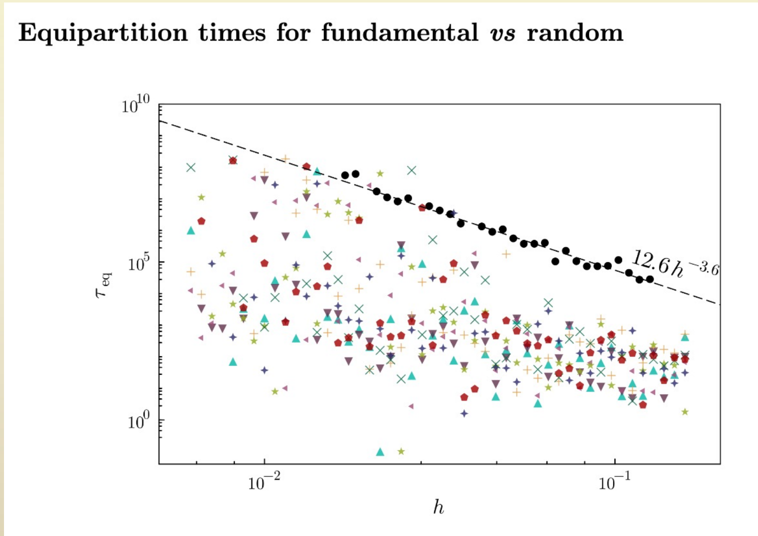
PRE 95 060202(R) 2017
+ in preparation

**How to measure $T_2$ ?**

**→ Use normalized entropy η = [0,1]**

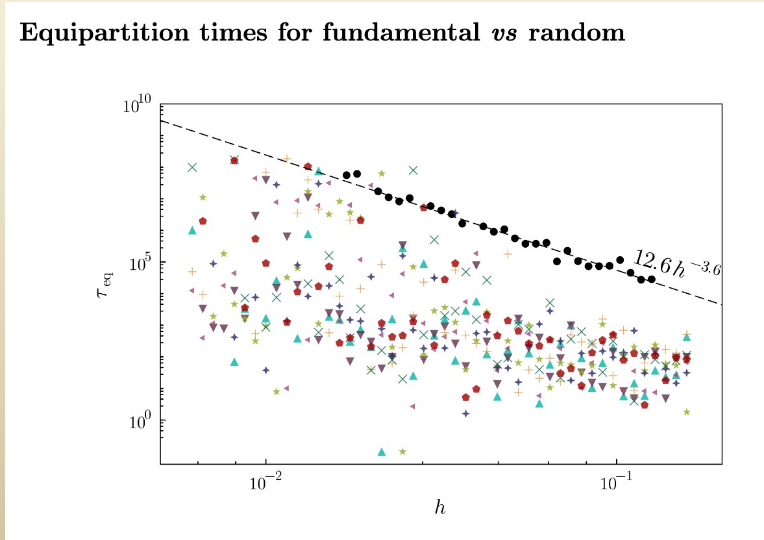
**→ $T_2$ anomalous large due to specific initial condition choice**



## History and some lessons

- FPUT paradox is probably NOT primarily about integrability
- FPUT is about specific initial conditions with ‘bottlenecks’
- FPUT is NOT about thermalization
- FPUT is about pre-thermalization
- The time scale $T_2$ is way larger than thermalization times of typical states
- Now we have a neat way to measure thermalization times of typical states!



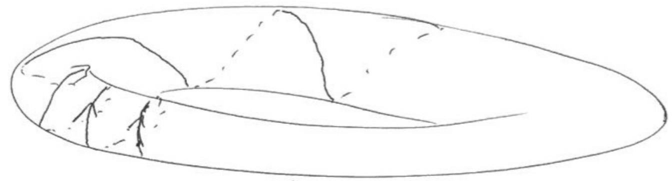
# Integrable System: measure Ø!

$\nu = N$, canonical transformation:

$\vec{p}, \vec{q} \Rightarrow \vec{J}, \vec{\Theta}$

$\Rightarrow H(\vec{J})$, $\dot{\vec{J}} = 0$, $\dot{\vec{\Theta}} = H_{,\vec{J}} = \vec{\omega}$, $\dot{\vec{\omega}} = 0$

$\Rightarrow$ trajectory on N-dimensional torus, embedded in 2N-dimensional phase space



$\vec{\omega}$ components incommensurate: torus densely covered (measure 1)

$\vec{\omega}$ components commensurate (measure Ø) trajectory periodic, closes after finite time

## Goals

**Thermalization dynamics slowing down of many body system in proximity to integrable limit:**

- **Use unique action-angle coordinates**
- **Identify different classes of nonintegrable perturbations networks**
- **Quantify thermalization process**
- **Identify novel dynamical regimes**
- **Analyze classical systems**
- **Extend to quantum systems**

## Measuring Thermalization : measuring time scales !

- **Measure dynamics of observables**
- **Measure and compare their time averages with ensemble averages**
- **Extract ergodization time scales $T_E$**

- **Measure Lyapunov spectra**
- **Invert to obtain Lyapunov times $T_L$**

## Approaching integrable limits

- Time scales will diverge (length scales perhaps as well)
- How will they diverge? How many will diverge? Which ones will diverge?
- Are there different universality classes?
- Can we observe and compute critical exponents?
- Are there further universal quantities?

## Nutshell summary

**Choose unique action-angle coordinates at the integrable limit**

**Analyze network spanned by the nonintegrable perturbation**

***We found two different classes of weak nonintegrability***

***LRN: one diverging time scale controls all thermalization dynamics***

***SRN: one diverging time scale and one diverging (length) scale control the thermalization dynamics***

***SRN: dramatical slowing down of thermalization***

## Nutshell summary

**Choose unique action-angle coordinates at the integrable limit**

**Analyze network spanned by the nonintegrable perturbation**

***We found two different classes of weak nonintegrability***

***LRN: ordered systems***
***weak nonlinearity (weak two-body interaction)***
***all to all nonintegrable interaction between***
***conserved quantities***

***SRN: weak finite range lattice coupling/hopping***
***short range nonintegrable interaction between***
***conserved quantities***
***and also***
***disordered systems and weak nonlinearity (weak two-body int.)***

## ERGODICITY NR 1:
## Finite time average distribution evolution of action observables

**Is the system ergodic?**

**Do infinite time averages equal ensemble averages?**

**We don’t have infinite time at our disposal!**

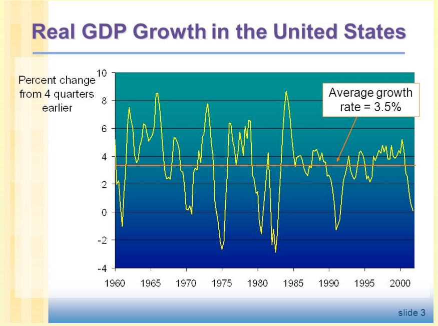
**Finite time averages (FTA)**

**FTA distributions must tend to delta functions for infinite times**

**Convergence for large averaging times**

**How large is large?**

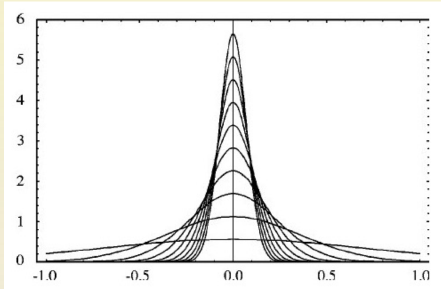
# ERGODICITY NR 1:
# Finite time average distribution evolution of action observables




## Observables f:

- the actions J of the corresponding integrable limit
- or some simple monotonous functions of them

**Choose the right observables!**

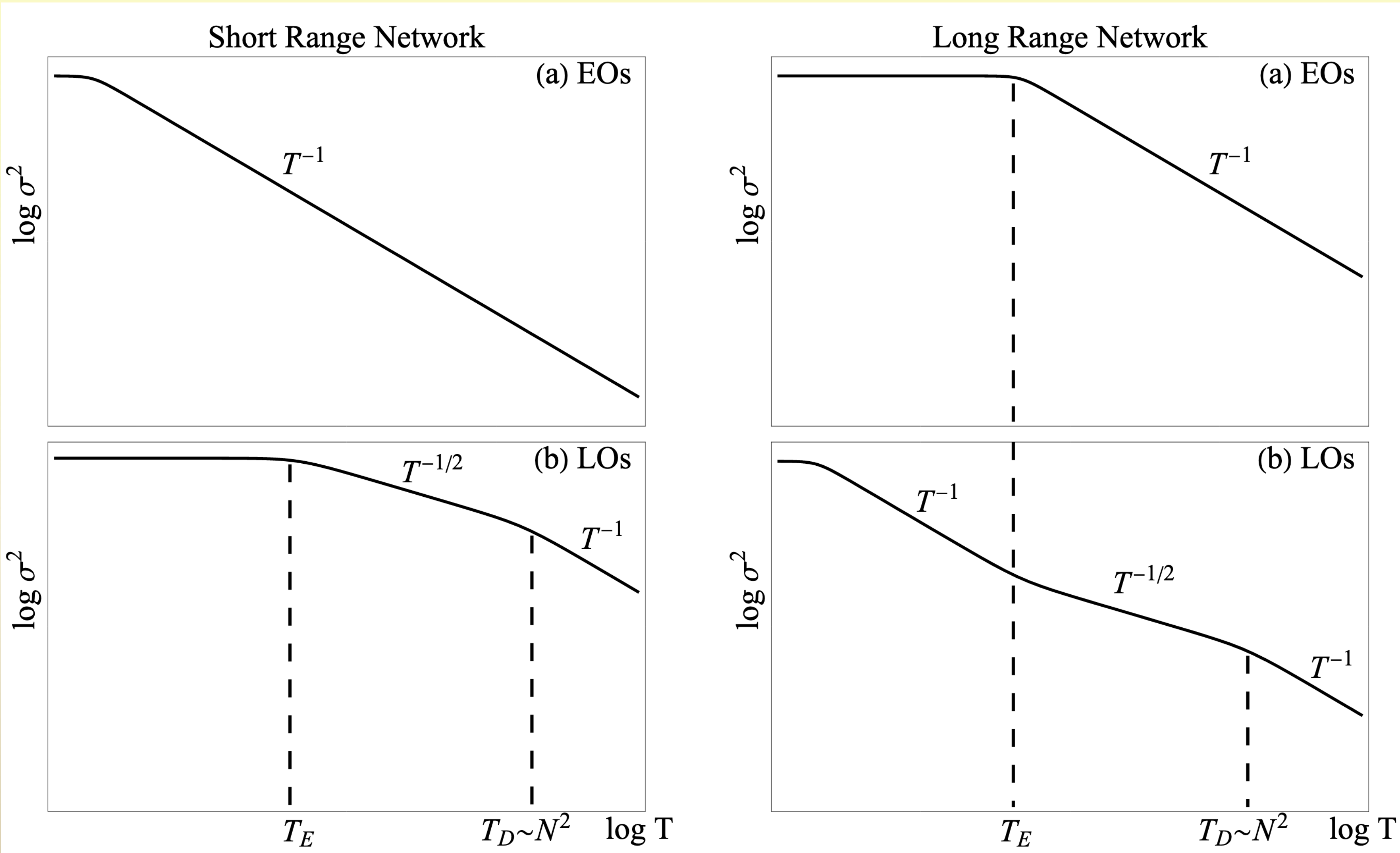



## Measuring:

- f(t) = J(t)
- finite time averages (FTA) for finite averaging time T
- distributions of finite time averages
- width of distribution of FTA as function of T

# ERGODICITY NR 1:
# Finite time average distribution evolution of action observables

## a quick and incomplete schematic overview



**ERGODICITY NR 2:**
**Poincare sectioning time intervals from evolution of action observables**

**compute statistics of fluctuations instead of correlation functions**

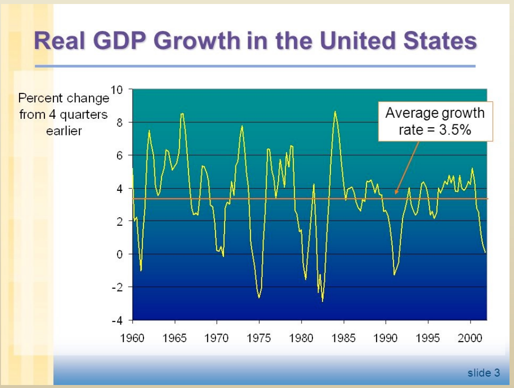
**obtain <f> - defines a generalized Poincare equilibrium manifold f=<f>**

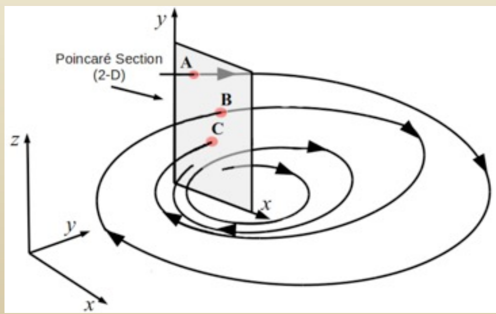
**if system is ergodic, trajectory will pierce infinitely many times**

**measure excursion times $\tau$ between piercings**

**compute probability distribution function P($\tau$) and its moments**

**compute amplitude distributions, obtain lifetimes of excitations**



# ERGODICITY: Finite time average distribution evolution of action observables

RAPID COMMUNICATIONS

PHYSICAL REVIEW E **95**, 060202(R) (2017)

**Intermittent many-body dynamics at equilibrium**

C. Danieli,[1,2] D. K. Campbell,[3] and S. Flach[2,1]

PHYSICAL REVIEW LETTERS **122**, 054102 (2019)

**Dynamical Glass and Ergodization Times in Classical Josephson Junction Chains**

Thudiyangal Mithun,[1] Carlo Danieli,[1] Yagmur Kati,[1,2] and Sergej Flach[1]

PHYSICAL REVIEW E **100**, 032217 (2019)

**Dynamical glass in weakly nonintegrable Klein-Gordon chains**

Carlo Danieli,[1] Thudiyangal Mithun,[1] Yagmur Kati,[1,2] David K. Campbell,[3] and Sergej Flach[1,4]

PHYSICAL REVIEW E **104**, 014218 (2021)

**Fragile many-body ergodicity from action diffusion**

Thudiyangal Mithun,[1,2] Carlo Danieli,[3,2] M. V. Fistul,[2,4,5] B. L. Altshuler,[6,2] and Sergej Flach[2,7]

Chaos ARTICLE scitation.org/journal/cha

**Thermalization dynamics of macroscopic weakly nonintegrable maps**

Cite as: Chaos 32, 063113 (2022); doi: 10.1063/5.0092032
Submitted: 20 March 2022 · Accepted: 16 May 2022 ·
Published Online: 3 June 2022

View Online Export Citation CrossMark

Merab Malishava[1,2,a)] and Sergej Flach[1,2,a)]

**Nutshell:**

**approaching the integrable limit for**

**LRN: $T_E / T_{L1}$ constant**

**SRN: $T_E / T_{L1}$ diverges**

# ERGODICITY: Finite time average distribution evolution of action observables

## Open Issues

Numerical implementation very demanding

Ambiguity in observable choice

Can be fooled by law of large numbers

Even integrable systems are ergodic, but not mixing!

Lack of aesthetic satisfaction

## The Lyapunov Spectrum

**Number of Lyapunov exponents = phase space dimension**

**Lyapunov exponents come in $\pm\lambda$ pairs**

**Per each integral of motion: two zero Lyapunov exponents**

| TITLE | CITED BY | YEAR |
|---|---|---|
| Lyapunov characteristic exponents for smooth dynamical systems and for Hamiltonian systems; a method for computing all of them. Part 1: Theory<br>G Benettin, L Galgani, A Giorgilli, JM Strelcyn<br>Meccanica 15 (1), 9-20 | 2377 | 1980 |

## The Lyapunov Spectrum

PRL 128 134102 (2022)
Chaos 32 063113 (2022)
PRE 108 L062301 (2023)
PRR 6 L012064 (2024)

- **initial thermal state: use proper Gibbs distributions**
- **run the trajectory**
- **add small perturbation, linearize and obtain the tangent map**
- **run 2N trajectories in the tangent map TM**
- **TM1: obtain LLE**
- **TM2: project $\perp$ to TM1 and obtain 2nd largest LE**
- **… and so on**

$N$ DoF (sites), $2N$ sorted LEs: $\Lambda_i > \Lambda_j$ with $\{i < j\} \in 1, ..., 2N$

total norm conserved: $\Lambda_N = \Lambda_{N+1} = 0$

unitary evolution: $\Lambda_{i<N} = -\Lambda_{2N-i+1}$

LLE : $\Lambda_1$

Irreducible normalized LEs: $\bar{\Lambda}(\rho) = \Lambda_i/\Lambda_1$, $\rho = i/N$

# The Lyapunov Spectrum

PRL 128 134102 (2022)
Chaos 32 063113 (2022)
PRE 108 L062301 (2023)
PRR 6 L012064 (2024)

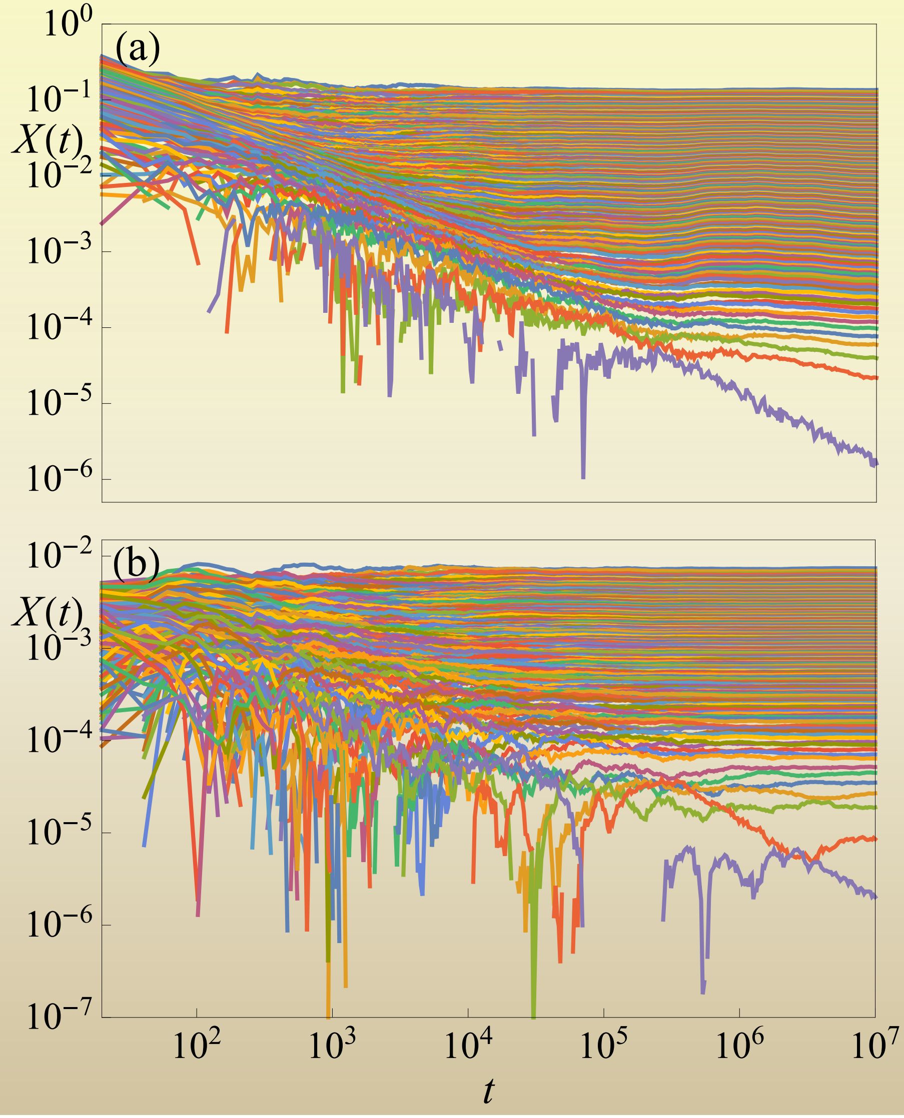
## Fitting the rescaled LS

$$\bar{\lambda}(\rho) = (1 - \rho^{\alpha})\mathrm{e}^{-\beta\rho}$$

# The Lyapunov Spectrum

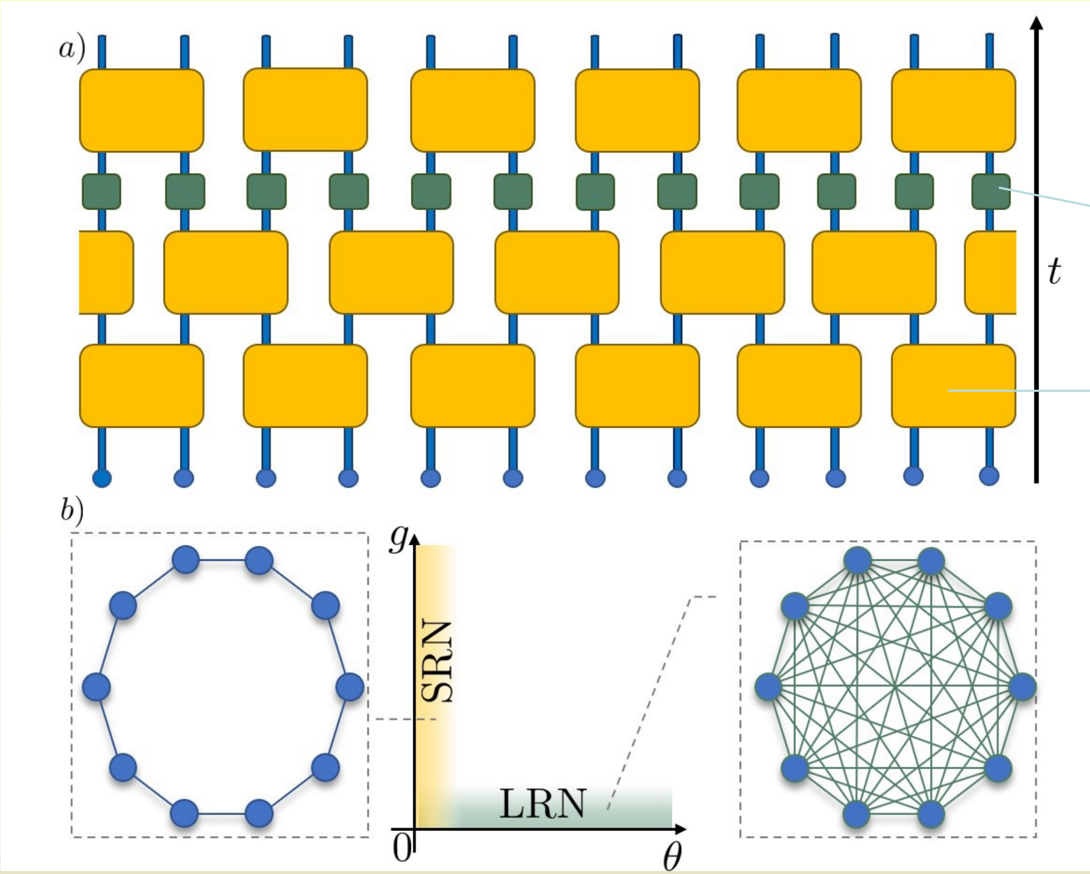
PRL 128 134102 (2022)
Chaos 32 063113 (2022)
PRE 108 L062301 (2023)
PRR 6 L012064 (2024)

**raw data:**



# Unitary Circuits for Thermalization

**Merab Malishava, SF**
**PRL 128 134102 (2022)**
**Chaos 32 063113 (2022)**




$$\hat{G}_n = e^{ig|\psi_n|^2}|n\rangle\langle n|$$

$$\hat{C}_{n,n+1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\vec{\Psi}(t+1) = \widehat{U}\vec{\Psi}(t)$$

$$g \to 0 : \ \mathrm{LRN}$$

$$\theta \to 0 : \ \mathrm{SRN}$$

- **Fast numerical evolution due to parallelization**
- **No time discretization roundoff errors (except for roundoff errors)**
- **Versatile, highly efficient unitary map toolbox for long time evolution**

# Unitary Circuits for Thermalization

$$\vec{\Psi}(t) = \{\psi_n^A(t), \psi_n^B(t)\}_{n=1}^{N/2}$$

$$\alpha_n^A(t) \equiv \cos^2\theta\psi_n^A(t) - \cos\theta\sin\theta\psi_{n-1}^B(t) + \sin^2\theta\psi_{n-1}^A(t) + \cos\theta\sin\theta\psi_n^B(t)$$

$$\alpha_n^B(t) \equiv \sin^2\theta\psi_{n+1}^B(t) - \cos\theta\sin\theta\psi_n^A(t) + \cos^2\theta\psi_n^B(t) + \cos\theta\sin\theta\psi_{n+1}^A(t) \ .$$

$$\psi_n^A(t+1) = e^{ig|\alpha_n^A|^2}\left[\cos^2\theta\psi_n^A(t) - \cos\theta\sin\theta\psi_{n-1}^B(t) + \sin^2\theta\psi_{n-1}^A(t) + \cos\theta\sin\theta\psi_n^B(t)\right]$$

$$\psi_n^B(t+1) = e^{ig|\alpha_n^B|^2}\left[\sin^2\theta\psi_{n+1}^B(t) - \cos\theta\sin\theta\psi_n^A(t) + \cos^2\theta\psi_n^B(t) + \cos\theta\sin\theta\psi_{n+1}^A(t)\right]$$

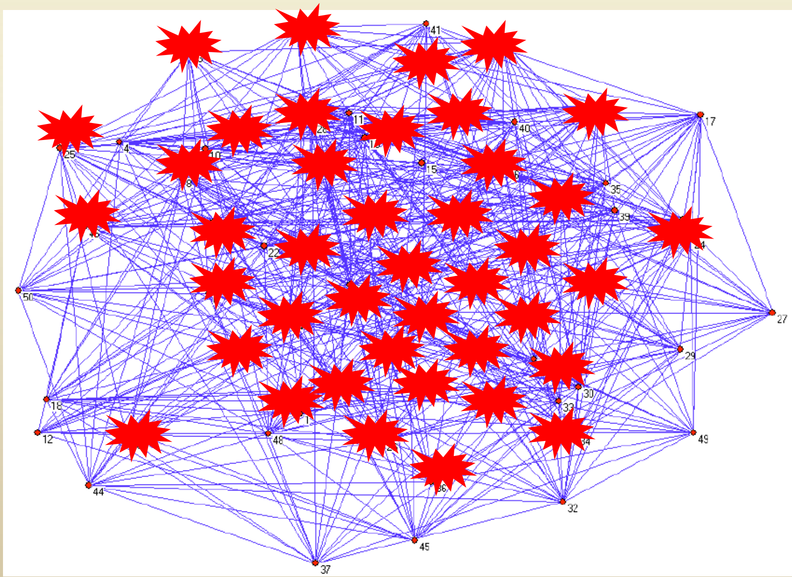
## Long Range Network (small g)

$$\left(\psi_n^A(t), \psi_n^B(t)\right)^T = e^{-i(\omega_k t - kn)} \left(\psi_k^A, \psi_k^B\right)^T \qquad \vec{\Psi}(t) = \sum_k c_k^r(t) \vec{\Psi}_k^r$$

$$\omega(k) = \pm \arccos\left(\cos^2\theta + \sin^2\theta \cos k\right)$$

$$c_k^r(t+1) = e^{i\omega_k} c_k^r(t) + \frac{ig}{N} \sum_{\substack{r_1,r_2,r_3 \\ k_1,k_2,k_3}} e^{i(\omega_{k_1}^{r_1} + \omega_{k_2}^{r_2} - \omega_{k_3}^{r_3})} I_{k,k_1,k_2,k_3}^{r,r_1,r_2,r_3} c_{k_1}^{r_1}(t) c_{k_2}^{r_2}(t) \left(c_{k_3}^{r_3}(t)\right)^*$$

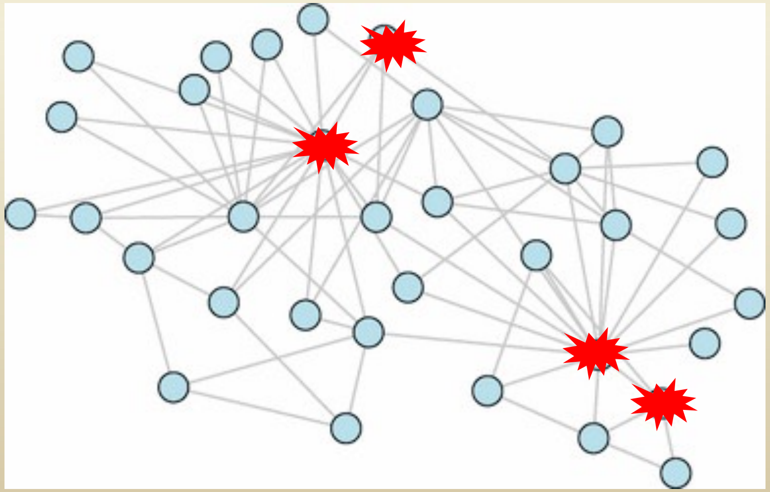
$$I_{k,k_1,k_2,k_3}^{r,r_1,r_2,r_3} = \delta_{k_1+k_2-k_3-k,0} \sum_p \psi_{k_1}^{r_1,p} \psi_{k_2}^{r_2,p} (\psi_{k_3}^{r_3,p})^* (\psi_k^{r,p})^*.$$



## Short Range Network (small ϴ)

$$\psi_n^A(t+1) = e^{ig|\alpha_n^A|^2}\left[\psi_n^A(t) - \theta(\psi_{n-1}^B(t) - \psi_n^B(t))\right]$$

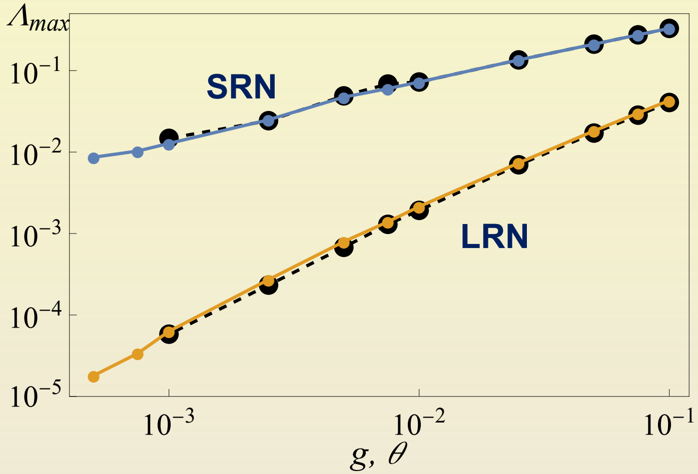
$$\psi_n^B(t+1) = e^{ig|\alpha_n^B|^2}\left[\psi_n^B(t) + \theta(\psi_{n+1}^A(t) - \psi_n^A(t))\right].$$



# Unitary Circuits for Thermalization

**Merab Malishava, SF**
**PRL 128 134102 (2022)**
**Chaos 32 063113 (2022)**

## processed data: LLE

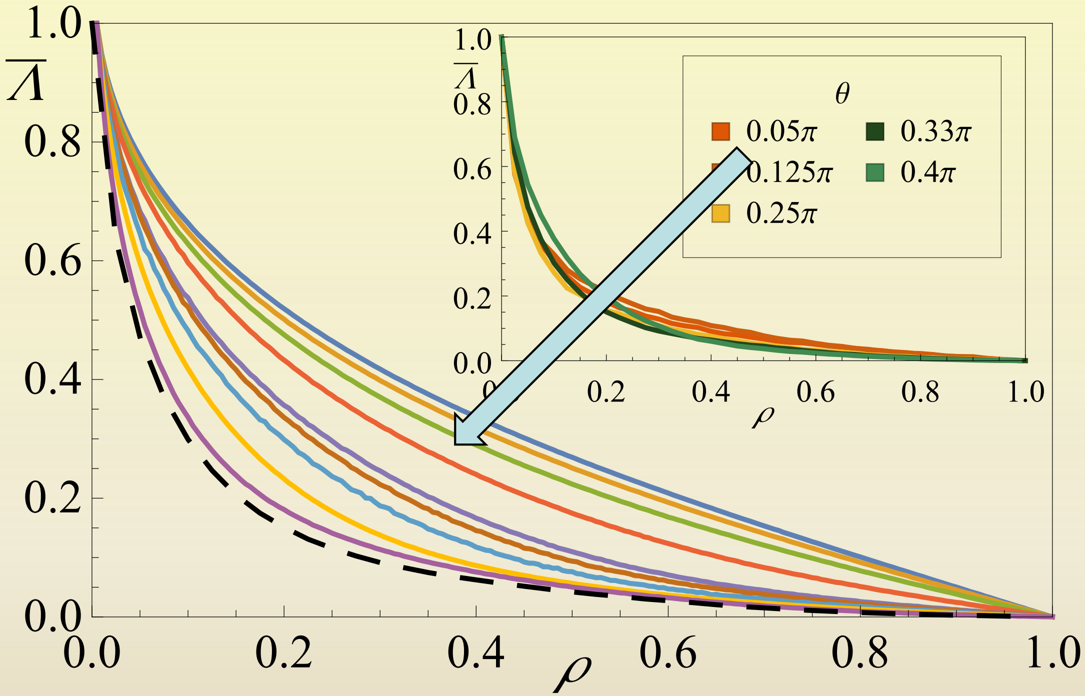



integrable limits

# Unitary Circuits for Thermalization

Merab Malishava, SF
PRL 128 134102 (2022)
Chaos 32 063113 (2022)

## LRN, one diverging scale




$\bar{\Lambda}(\rho)$: analytic function in the LRN limit

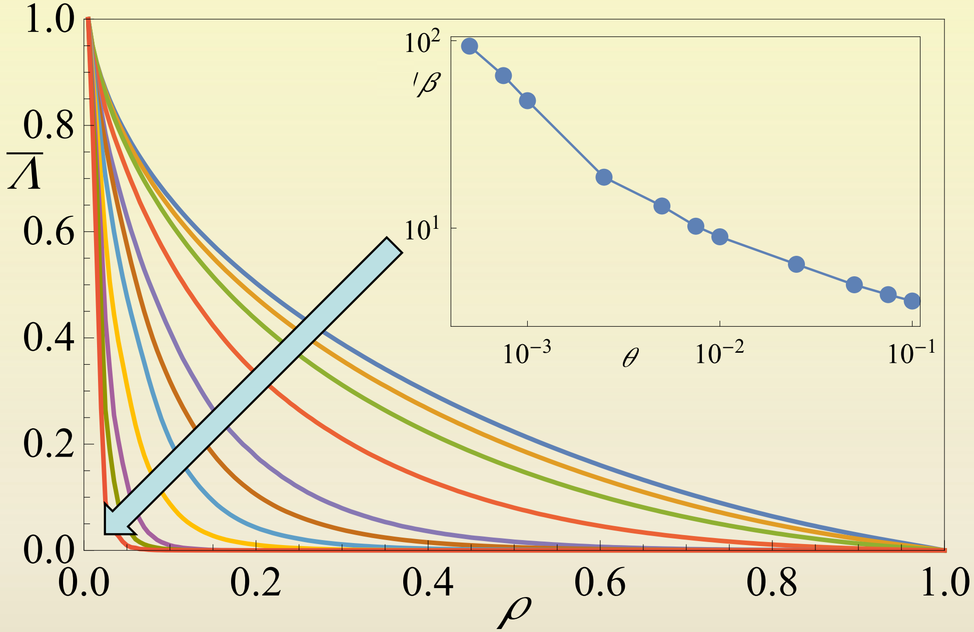
$\bar{\lambda}(\rho) = (1 - \rho^{\alpha})\mathrm{e}^{-\beta\rho}$

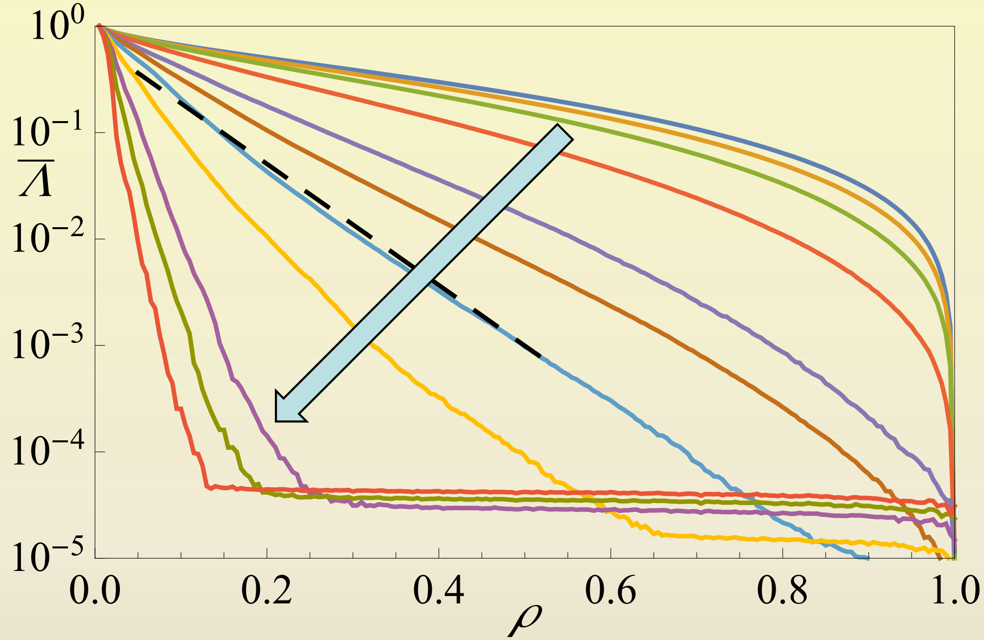
$\alpha$, $\beta$ and $\kappa$ stay constant nonzero for $g \to 0$

# Unitary Circuits for Thermalization

**Merab Malishava, SF**
**PRL 128 134102 (2022)**
**Chaos 32 063113 (2022)**

## SRN, two diverging scales




$\bar{\Lambda}(\rho)$: non-analytic function in the SRN limit

$$\bar{\lambda}(\rho) = (1 - \rho^{\alpha})e^{-\beta\rho}$$

$\beta$ diverges and $\kappa$ vanishes for $\theta \to 0$

# Josephson junction networks in d=1,2,3

Gabriel Lando, SF
PRE 108 L062301 (2023)

$$H = \sum \frac{p_n^2}{2} + E_J[1 - \cos(q_n - q_{n-1})]$$

**Number of sites (volume) : N**

**Energy density: h = H/N**

**SRN: $E_J$ / h → 0**

**LRN: h / $E_J$ → 0**

## Long Range Network

Gabriel Lando, SF
PRE 108 L062301 (2023)

Josephson junction network, energy density $h$: $h/E_J \ll 1$

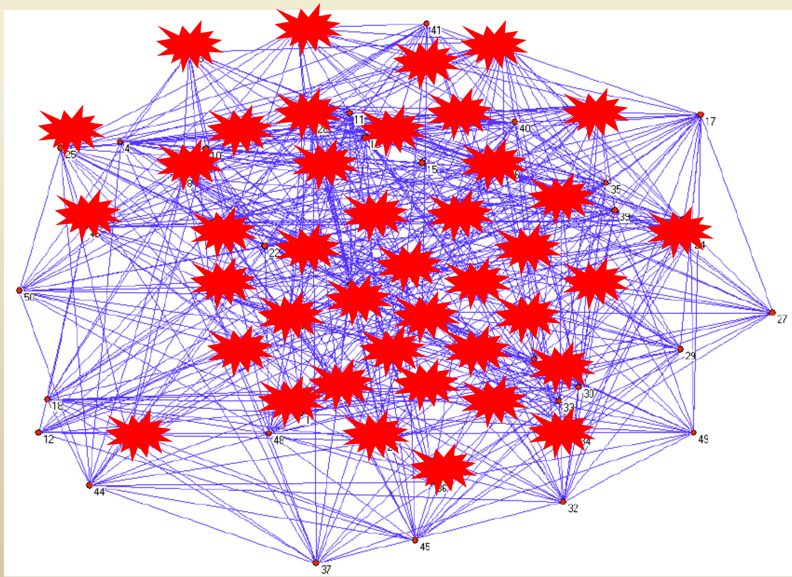
$$H = \sum \frac{p_n^2}{2} + E_J[1 - \cos(q_n - q_{n-1})]$$

$$H_0 = \sum \frac{p_n^2}{2} + \frac{E_J}{2}(q_n - q_{n-1})^2 \; : \text{ harmonic chain}$$

$$H_1 = -\frac{E_J}{4}\sum (q_n - q_{n-1})^4 \; : \text{ quartic anharmonicity}$$

$$L = 3, \; R \sim N^2$$

$\longrightarrow$ Long Range Network



**long range network**:

$$Q_q = \sqrt{2J_q}\sin\Theta_q \; , \; P_q = \omega_q\sqrt{2J_q}\cos\Theta_q$$

$$\dot{J}_q = -E_J \sum_{q_1,q_2,q_3} \omega_{q_1}\omega_{q_2}\omega_{q_3} A_{q,q_1,q_2,q_3}\sqrt{J_q J_{q_1} J_{q_2} J_{q_3}} cos\Theta_q \sin\Theta_{q_1} \sin\Theta_{q_2} \sin\Theta_{q_3}$$

Gabriel Lando, SF
PRE 108 L062301 (2023)

## Long Range Network

→ **exists due to weak nonlinear all-to-all interactions between normal modes**

→ **nonintegrable nonlinearity local in real space, normal modes extended**

**long range network:**

$$Q_q = \sqrt{2J_q}\sin\Theta_q \ , \ P_q = \omega_q\sqrt{2J_q}\cos\Theta_q$$

$$\dot{J}_q = -E_J \sum_{q_1,q_2,q_3} \omega_{q_1}\omega_{q_2}\omega_{q_3} A_{q,q_1,q_2,q_3} \sqrt{J_q J_{q_1} J_{q_2} J_{q_3}} cos\Theta_q \sin\Theta_{q_1} \sin\Theta_{q_2} \sin\Theta_{q_3}$$

## Short Range Network

Gabriel Lando, SF
PRE 108 L062301 (2023)

Josephson junction network, $h/E_J \gg 1$

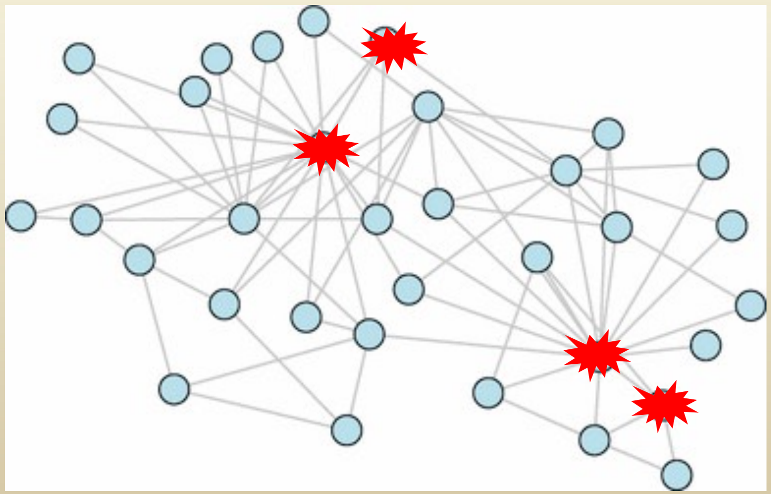
$$H = \sum \frac{p_n^2}{2} + E_J[1 - \cos(q_n - q_{n-1})]$$

$H_0 = \sum \frac{p_n^2}{2}$ : free rotors

$H_1 = E_J \sum [1 - \cos(q_n - q_{n-1})]$ : nearest neighbour coupling

$L = 1,\ R = 2$

$\longrightarrow$ Short Range Network



**short range network**:

$q_n = \Theta_n\ ,\ p_n = J_n$

$\dot{J}_n = -E_J\,(\sin(\Theta_n - \Theta_{n-1}) + \sin(\Theta_n - \Theta_{n+1}))$

## Short Range Network

Gabriel Lando, SF
PRE 108 L062301 (2023)

→ exists due to underlying lattice model structure

→ nonintegrable lattice coupling is local, rotor actions are local

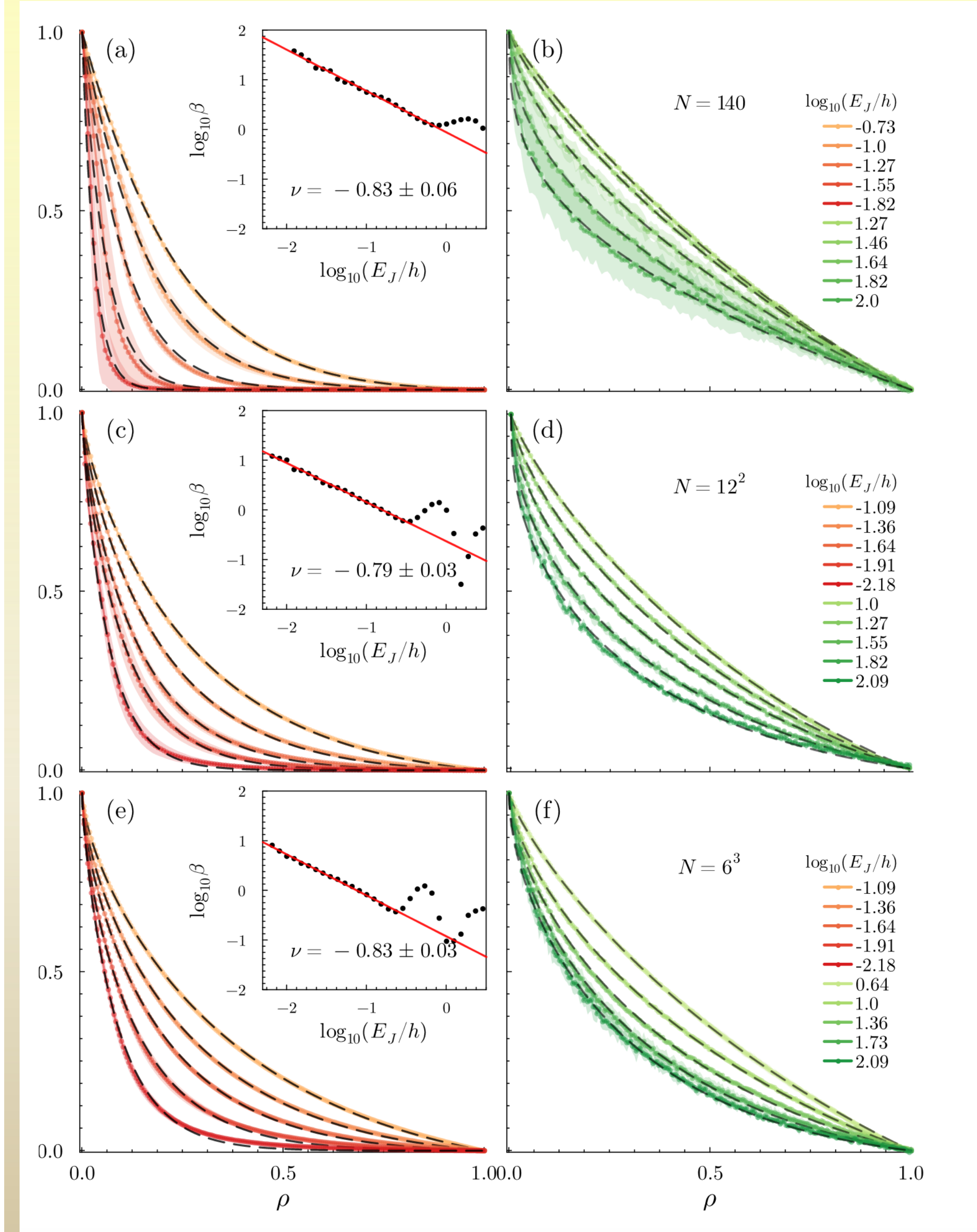
**short range network:**

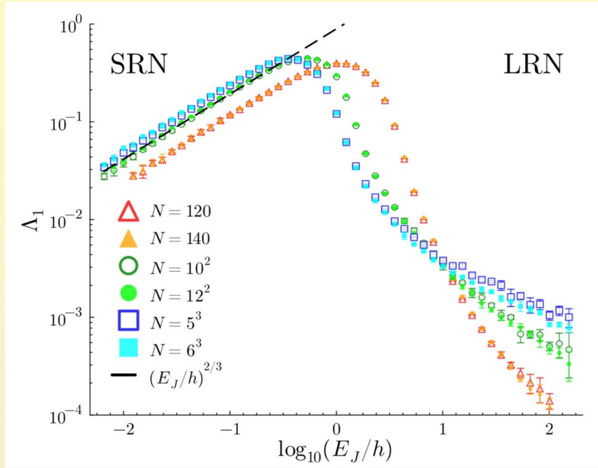
$$q_n = \Theta_n \ , \ p_n = J_n$$

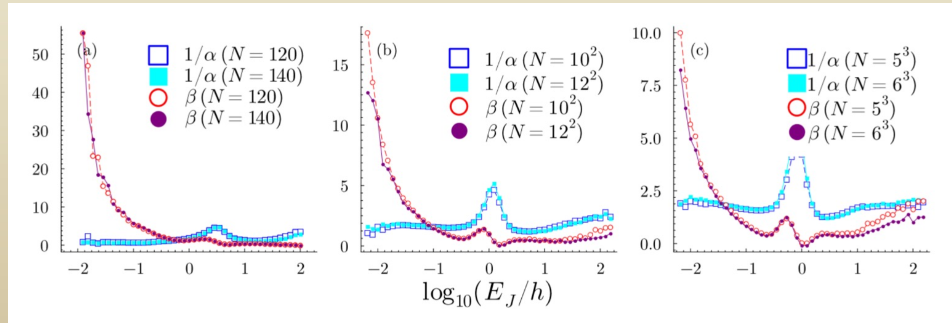
$$\dot{J}_n = -E_J \left(\sin(\Theta_n - \Theta_{n-1}) + \sin(\Theta_n - \Theta_{n+1})\right)$$

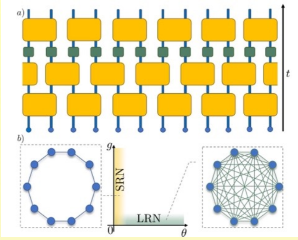
# Rescaled Lyapunov spectrum for d=1,2,3

**Gabriel Lando, SF**
**PRE 108 L062301 (2023)**

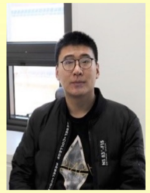




# Unitary Circuits for Thermalization

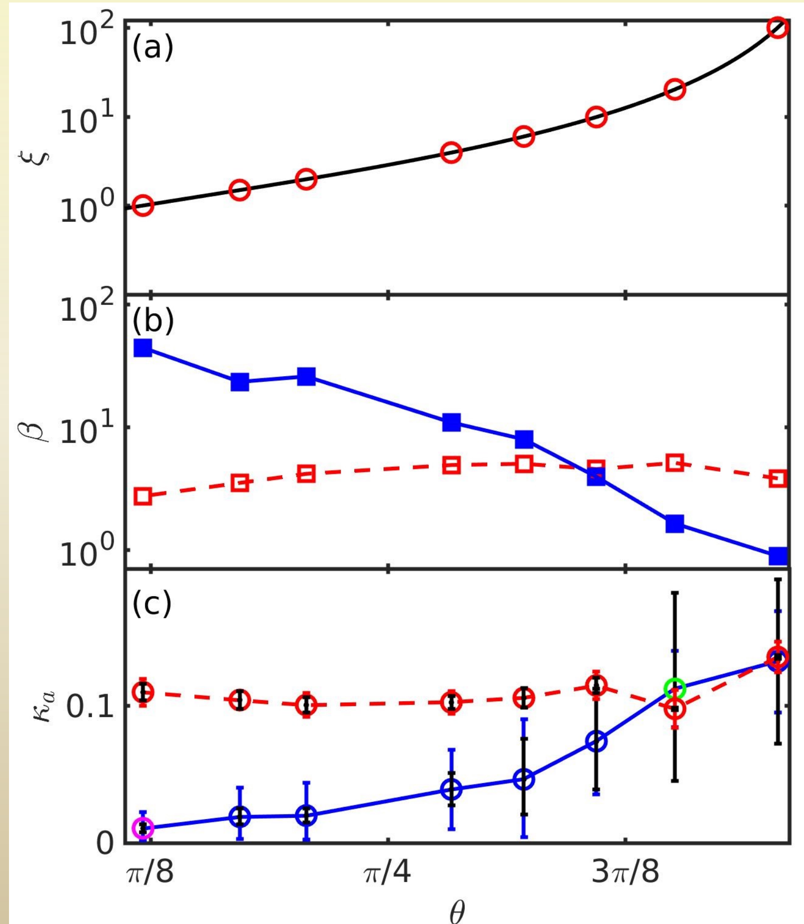


**Weihua Zhang**
**Gabriel Lando**
**Barbara Dietz**
**SF**
**PRR 6 L012064 (2024)**

## Disorder and Anderson Localization:

$$\hat{G}_n = e^{i\phi_n} e^{ig|\psi_n|^2 |n\rangle\langle n|} \;,\; \phi_n \in [0, 2\pi]$$

$$g = 0 \;:\; \xi^{-1} = |\ln(|\sin\Theta|)|$$



# going quantum


**Budhaditya Bhattacharjee**
**Alexei Andreanov, SF**
**arXiv:2405.00786**

$$H = -\sum_{i=1}^{N} \left( J\sigma_{i+1}^z \sigma_i^z + g\sigma_i^z + h\sigma_i^x \right)$$

**h → 0 : SRN, conserved quantity** $\sigma_i^z$

**g → 0: LRN, conserved quantity** $\prod_{i=1}^{N} \sigma_i^x$

**numerics: ED with 8 spins and PBC, infinite temperature**

## going quantum

**Budhaditya Bhattacharjee**
**Alexei Andreanov, SF**
**arXiv:2405.00786**

$$H = -\sum_{i=1}^{N} \left( J\sigma_{i+1}^{z}\sigma_{i}^{z} + g\sigma_{i}^{z} + h\sigma_{i}^{x} \right)$$

### Ergodization time scale from ETH:

$$f_{\mathcal{O}}(t) = \langle O(t) \rangle - \bar{\mathcal{O}}$$
$$= \sum_{m,n-\{m',n'\}} c_n c_m^{*} e^{i(E_n - E_m)t} \langle m|\mathcal{O}|n\rangle - \sum_{n} |c_n|^2 \langle n|\mathcal{O}|n\rangle$$

### Measure times $\mathcal{T}$ of spacings of zeros

## going quantum

$$H = -\sum_{i=1}^{N} \left( J\sigma_{i+1}^z \sigma_i^z + g\sigma_i^z + h\sigma_i^x \right)$$


**Budhaditya Bhattacharjee**
**Alexei Andreanov, SF**
**arXiv:2405.00786**

## Largest Lyapunov exponent from operator growth (Krylov Complexity)

$$\mathcal{A}_n = [H, \mathcal{O}_{n-1}] - b_{n-1}\mathcal{O}_{n-2}$$

$$\mathcal{O}(t) = e^{iHt}\mathcal{O}_0 e^{-iHt} = \sum_{n=0}^{\mathcal{K}} i^n \phi_n(t)\mathcal{O}_n$$

$$K(t) = \sum_n n|\phi_n(t)|^2 \sim \mathrm{e}^{2\lambda t} = \mathrm{e}^{2t/T_\lambda}$$

# going quantum

$$H = -\sum_{i=1}^{N} \left( J\sigma_{i+1}^z \sigma_i^z + g\sigma_i^z + h\sigma_i^x \right)$$


**Budhaditya Bhattacharjee**
**Alexei Andreanov, SF**
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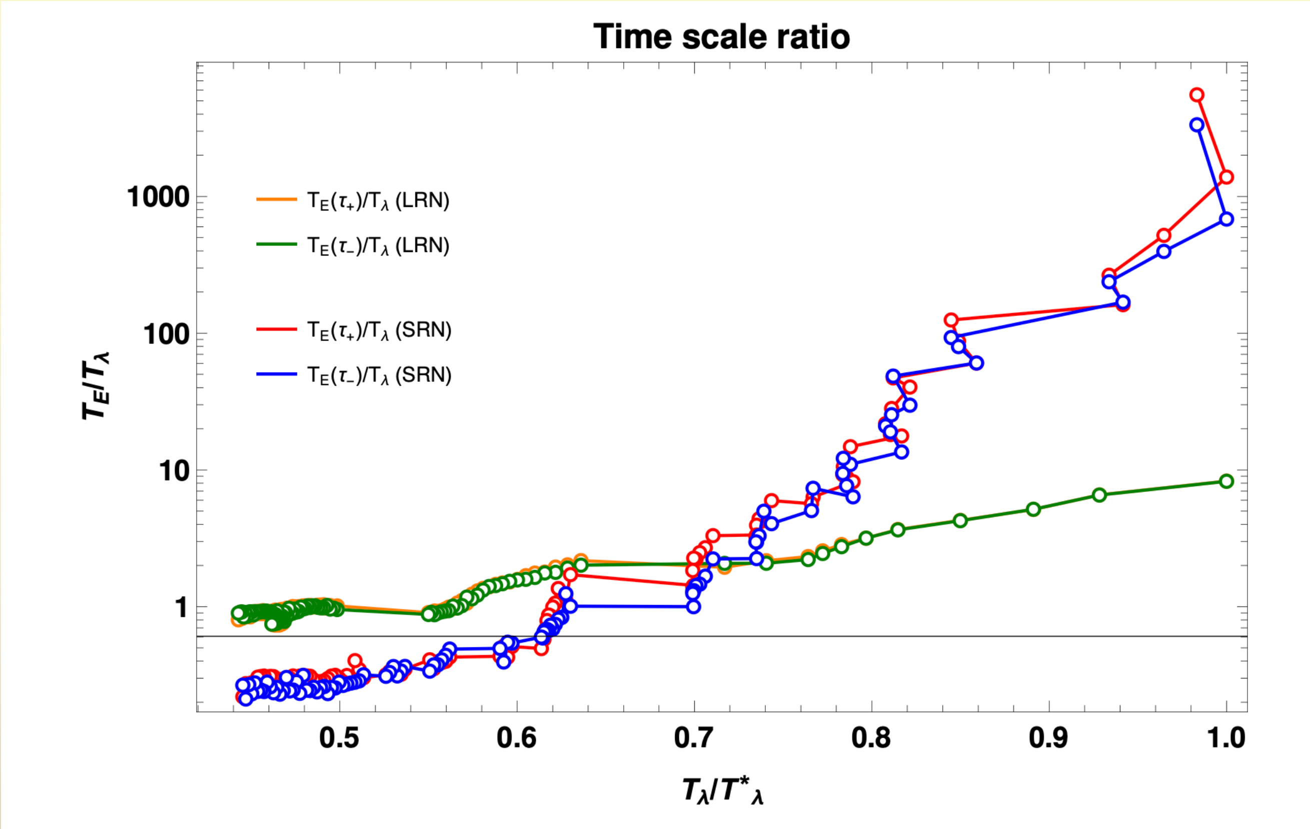

## Take Home Messages

Reads:

PRE 95 060202 (2017)
PRL 122 054102 (2019)
PRE 100 032217 (2019)
PRE 104 014218 (2021)
Chaos 32 063113 (2022)
PRL 128 134102 (2022)
PRE 108 L062301 (2023)
PRR 6 L012064 (2024)
Chaos 34 033107 (2024)
arXiv:2405.00786

**We observe two distinct universality classes**

**Classifier: nonintegrable perturbation network type**

**Long range - one LLE controls all thermalization time scales**
**- renormalized Lyapunov spectrum: analytic function**

**Short range - one LLE and one (length) scale control the thermalization**
**- renormalized Lyapunov spectrum: non-analytic function**

**Disorder and Anderson localization induce transition from LRN to SRN**

**Explains finite time average observations**

**Works in any lattice dimension, both classical and quantum**