NON-HERMITIAN QUANTUM MECHANICS: HOW WE CAME WITH THE HATANO-NELSON MODEL

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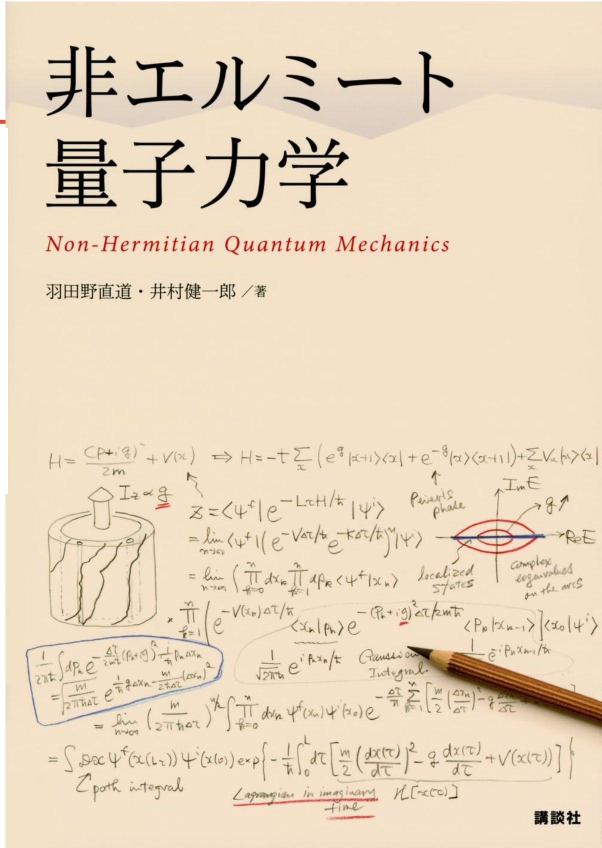


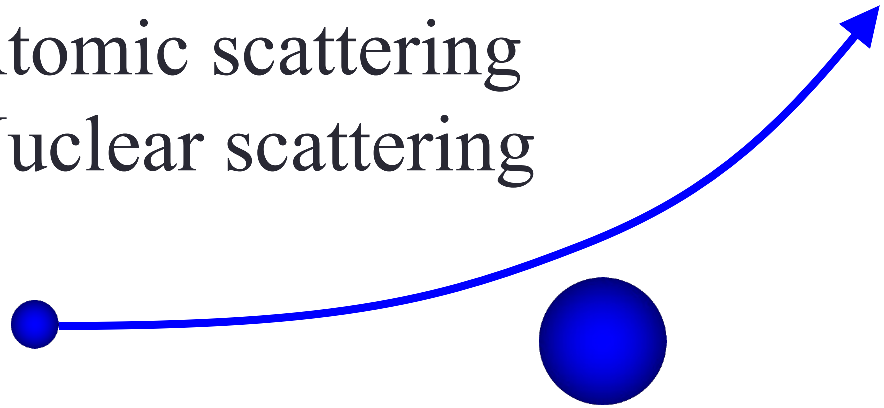
Non-Hermitian Quantum Mechanics

- Open quantum systems: a part of a Hermitian world (dates back to early era of QM)
- PT symmetric systems: “The world is non-Hermitian!” (C.M. Bender, 1998)
- Non-Hermitian systems transformed from other systems: including the Hatano-Nelson model (1996)

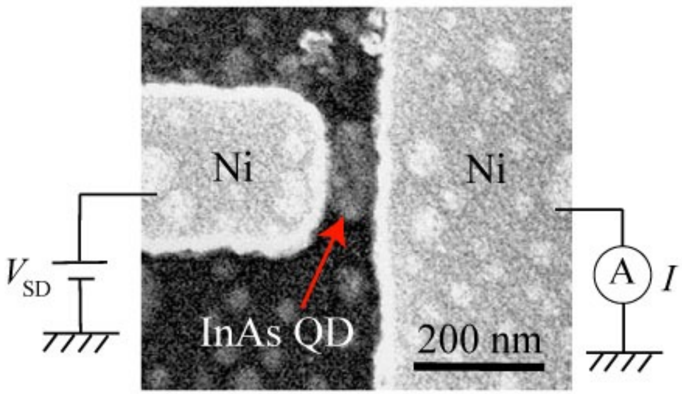
N. Hatano and D.R. Nelson, PRL 77 (1996) 570

Open Quantum Systems

Atomic scattering
Nuclear scattering



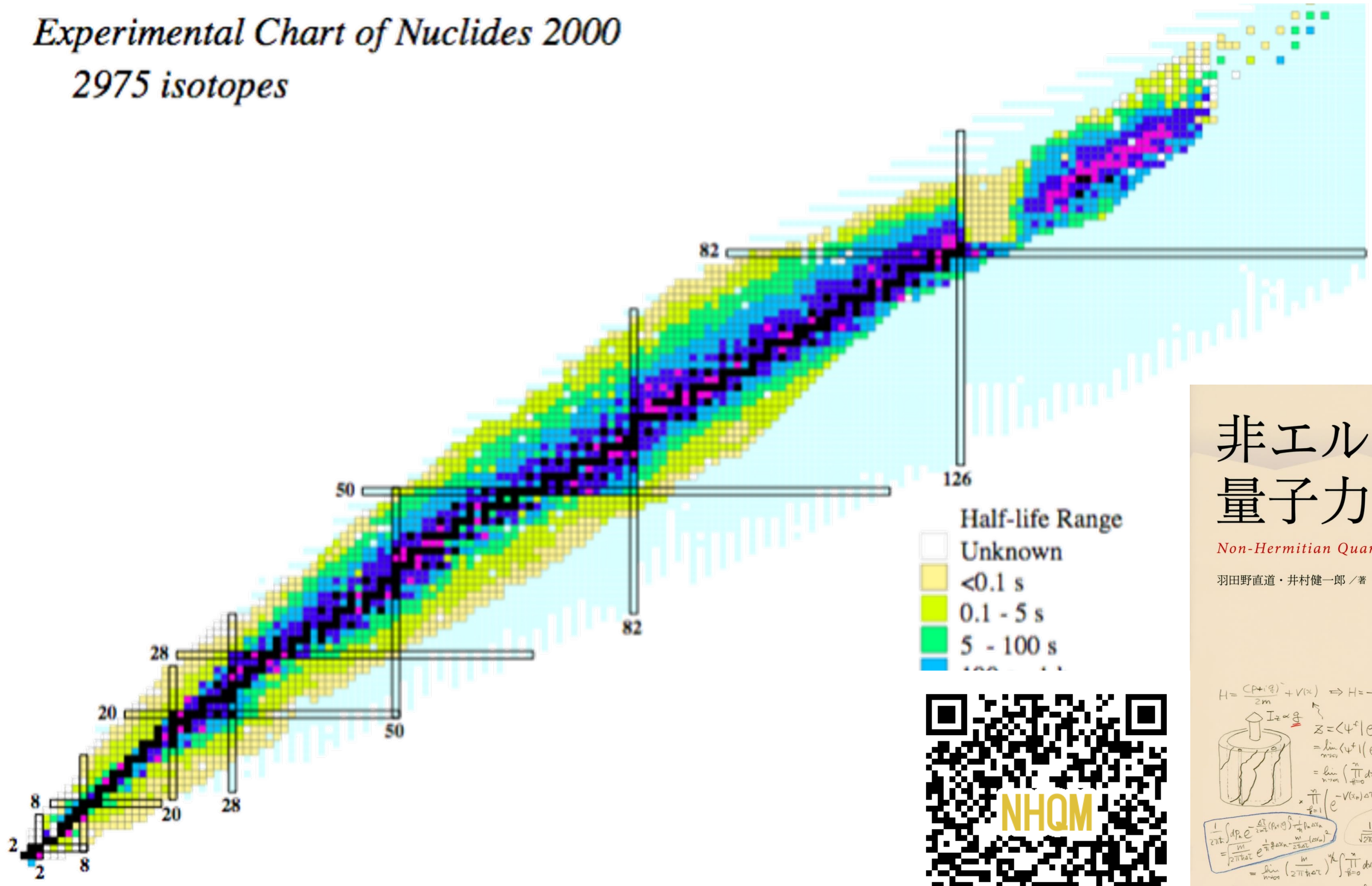
Electronic conduction
in mesoscopic systems

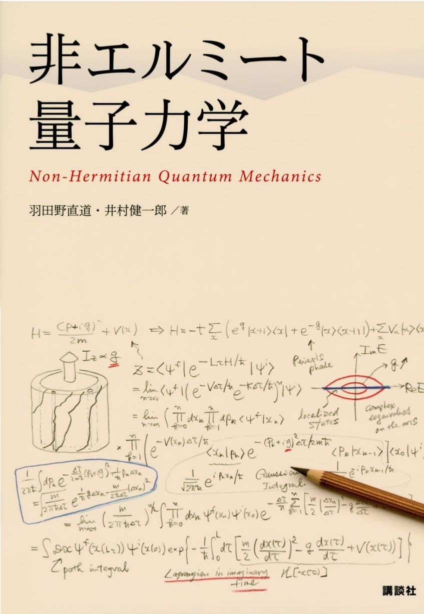


Once the particle leaves the central system, it never comes back coherently.

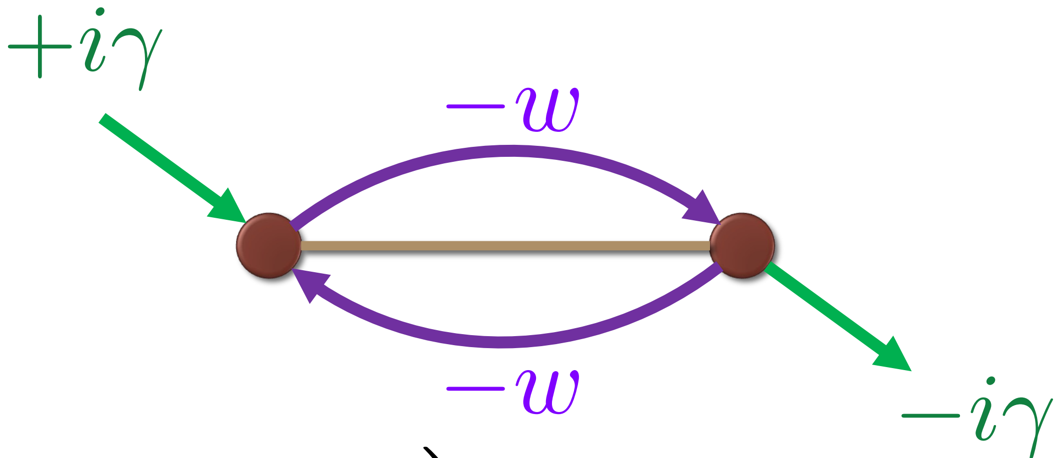
Resonant states in open systems

Unstable nuclides = resonant states





PT-Symmetric Systems

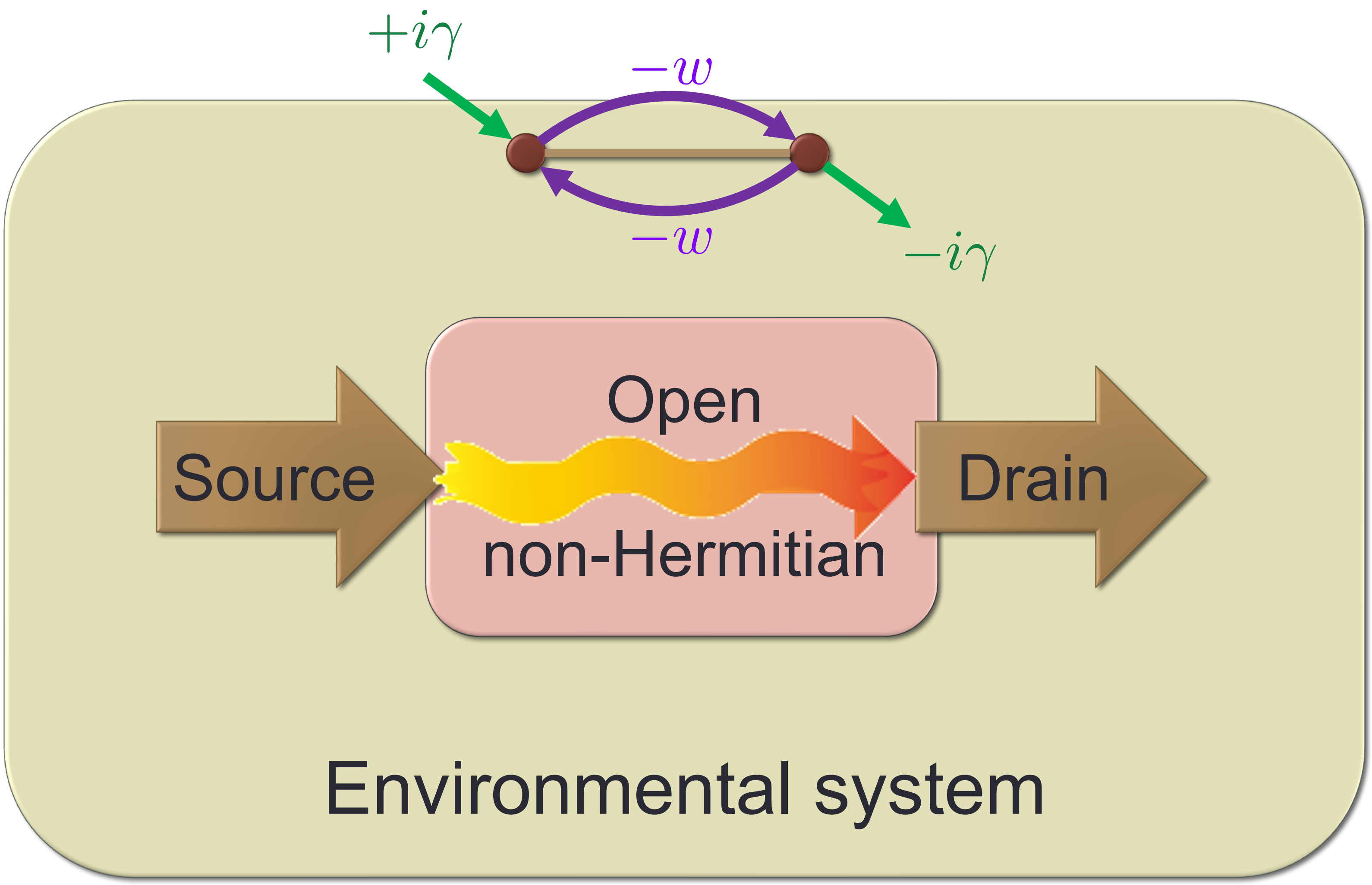


$$H = \begin{pmatrix} +i\gamma & -w \\ -w & -i\gamma \end{pmatrix} \qquad E_{\pm} = \pm\sqrt{w^2 - \gamma^2}$$

$|w| > |\gamma|$: *PT* unbroken phase
(Eigenvecs are also *PT*-symmetric)

$|w| < |\gamma|$: *PT* broken phacs
(Eigenvecs break *PT*-symmetry)

Quantum Transport (Open systems)



Closed non-Hermitian systems

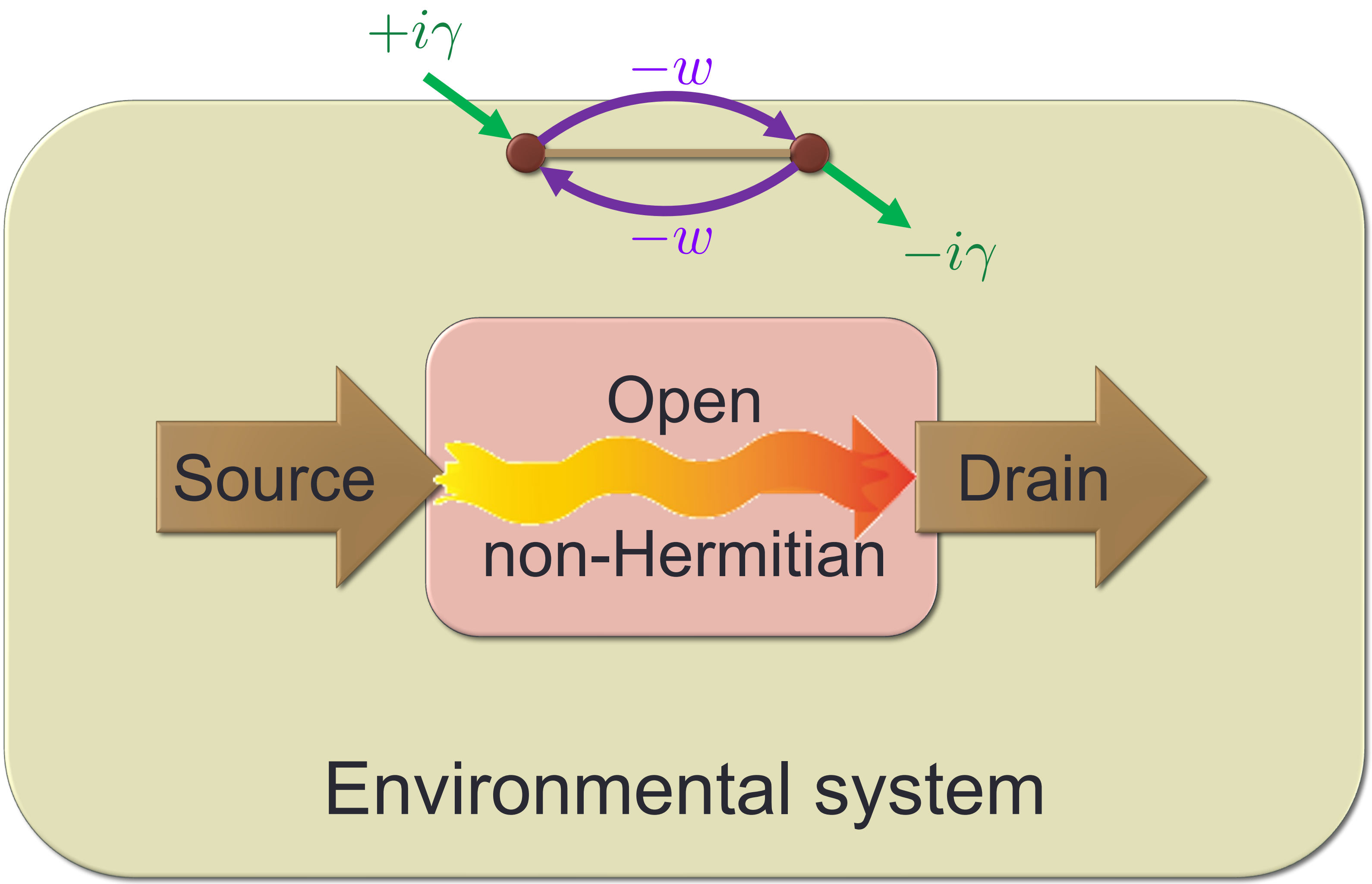
C.M. Bender “For guaranteeing the reality of the energy, the Hermiticity is too mathematical!”

Closed

non-Hermitian

The universe may be in the *PT* unbroken phase of a *PT*-symmetric non-Hermitian system??

Experiments in open systems



“Non-Hermitian Quantum Mechanics”

VOLUME 77, NUMBER 3 PHYSICAL REVIEW LETTERS 15 JULY 1996

PRL'96

Localization Transitions in Non-Hermitian Quantum Mechanics

Naomichi Hatano* and David R. Nelson
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138
(Received 15 March 1996)

We study the localization transitions which arise in both one and two dimensions when quantum mechanical particles described by a random Schrödinger equation are subjected to a constant imaginary vector potential. A path-integral formulation relates the transition to flux lines depinned from columnar defects by a transverse magnetic field in superconductors. The theory predicts that, close to the depinning transition, the transverse Meissner effect is accompanied by stretched exponential relaxation of the field into the bulk and a diverging penetration depth. [S0031-9007(96)00677-1]

Used the words “Non-Hermitian Quantum Mechanics” in the title for the first time (perhaps)

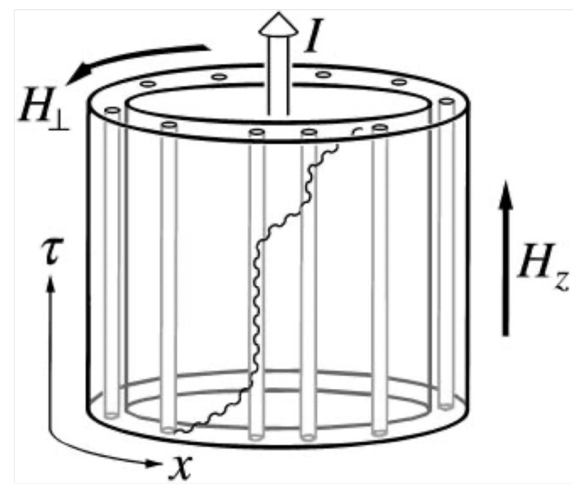
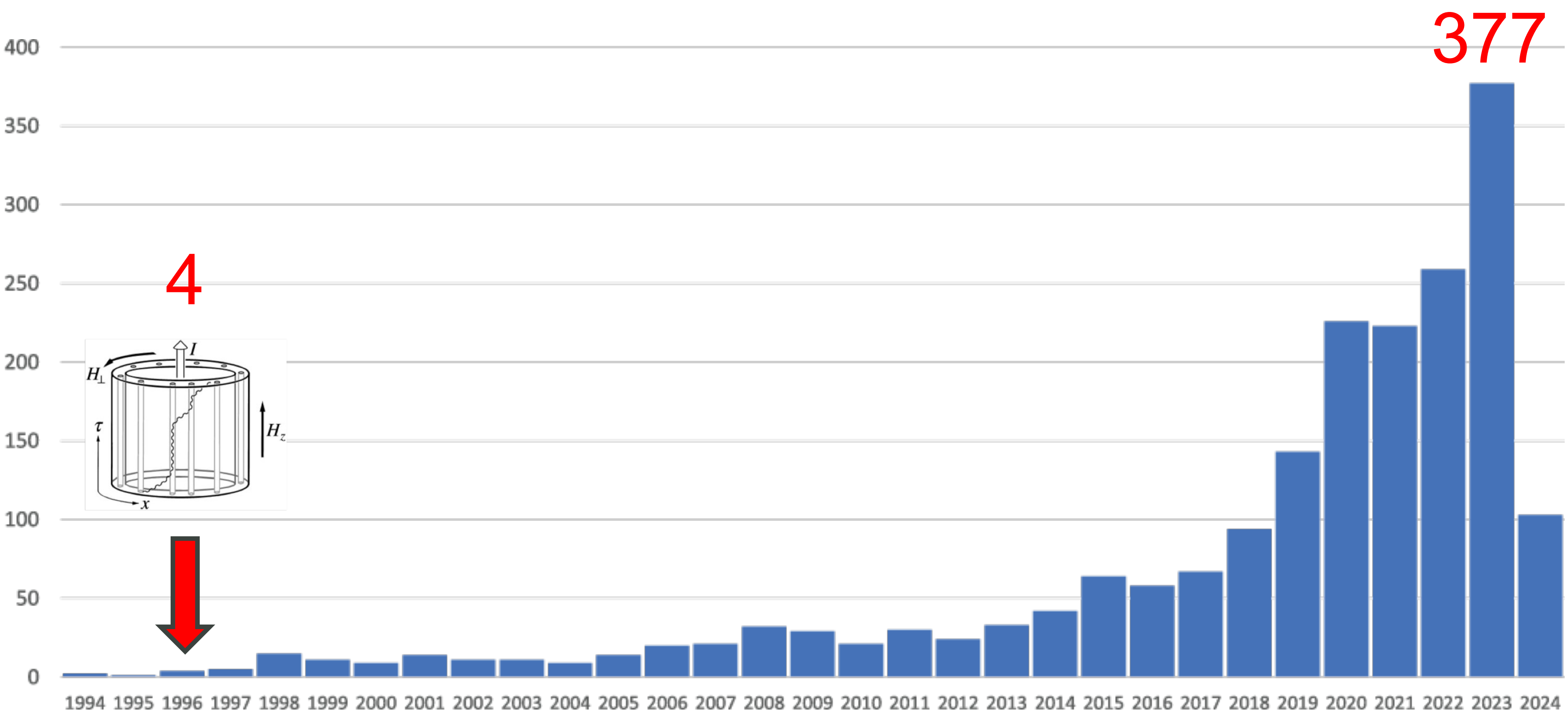


FIG. 1. One flux line (wavy curve) induced by the field $\boldsymbol{H}_z$ and interacting with columnar pins in a cylindrical superconducting shell with radial thickness smaller than the penetration depth of the defect-free material. The field $\boldsymbol{H}_\perp$ is generated by the current $\boldsymbol{I}$ threading the ring.

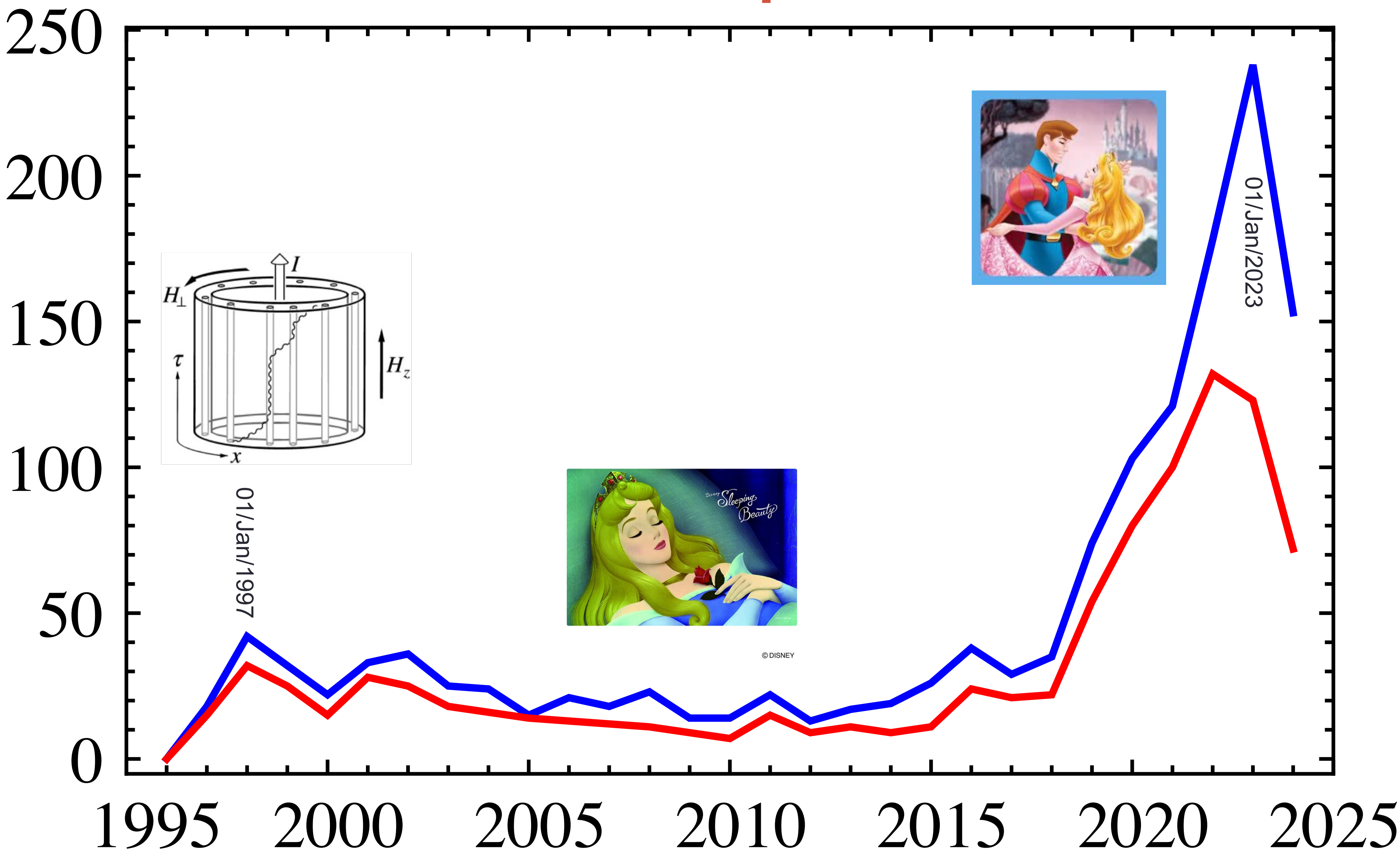
1313 citations（Google Scholar as of March 18., 2024）

Non-Hermitian Papers

\# of arXiv preprints having “non-Hermitian” in the titles

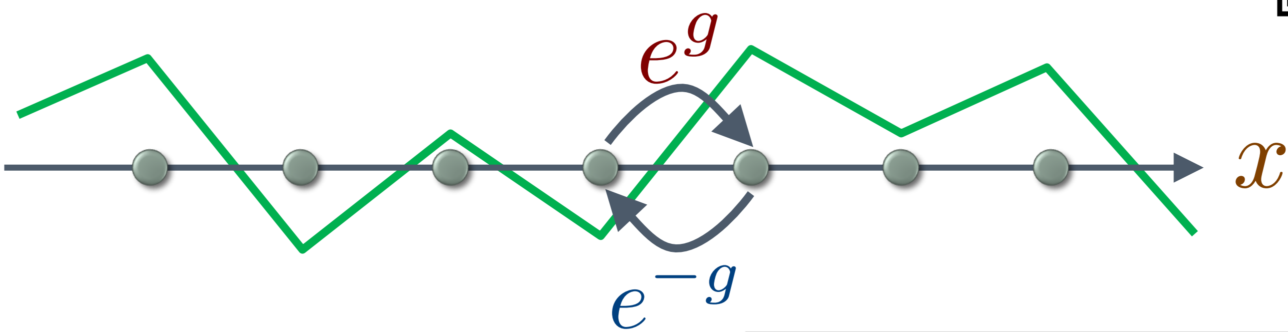


Citation of Our Paper

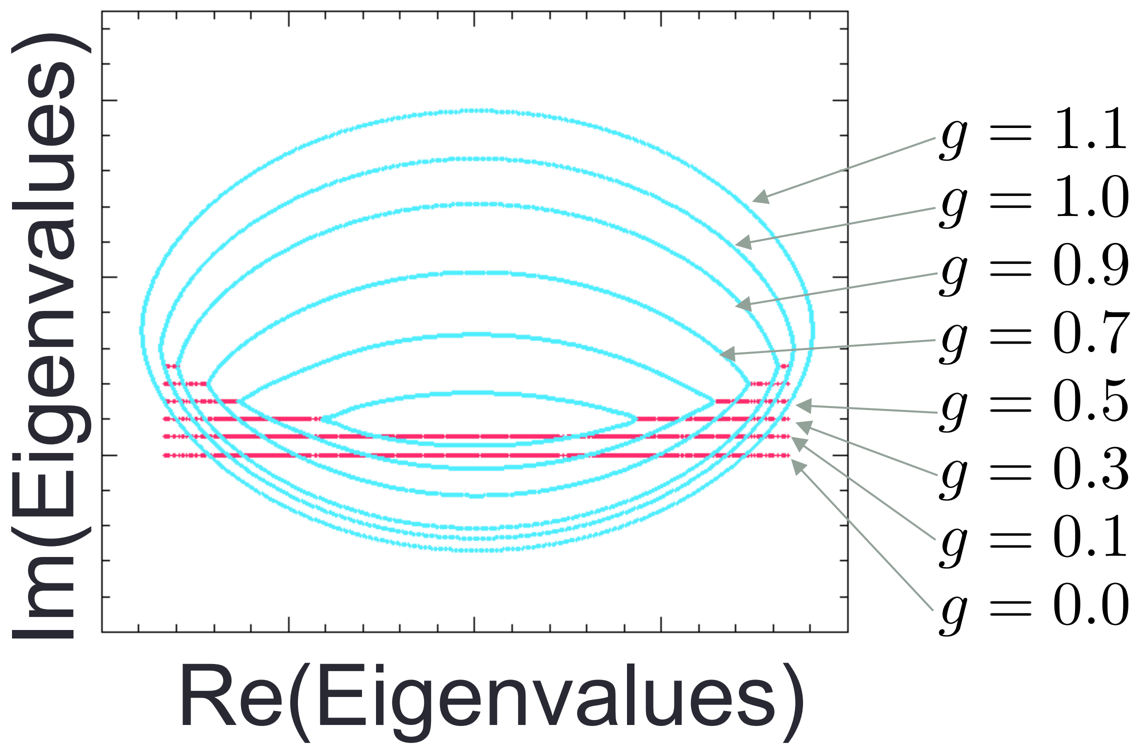


Hatano-Nelson model

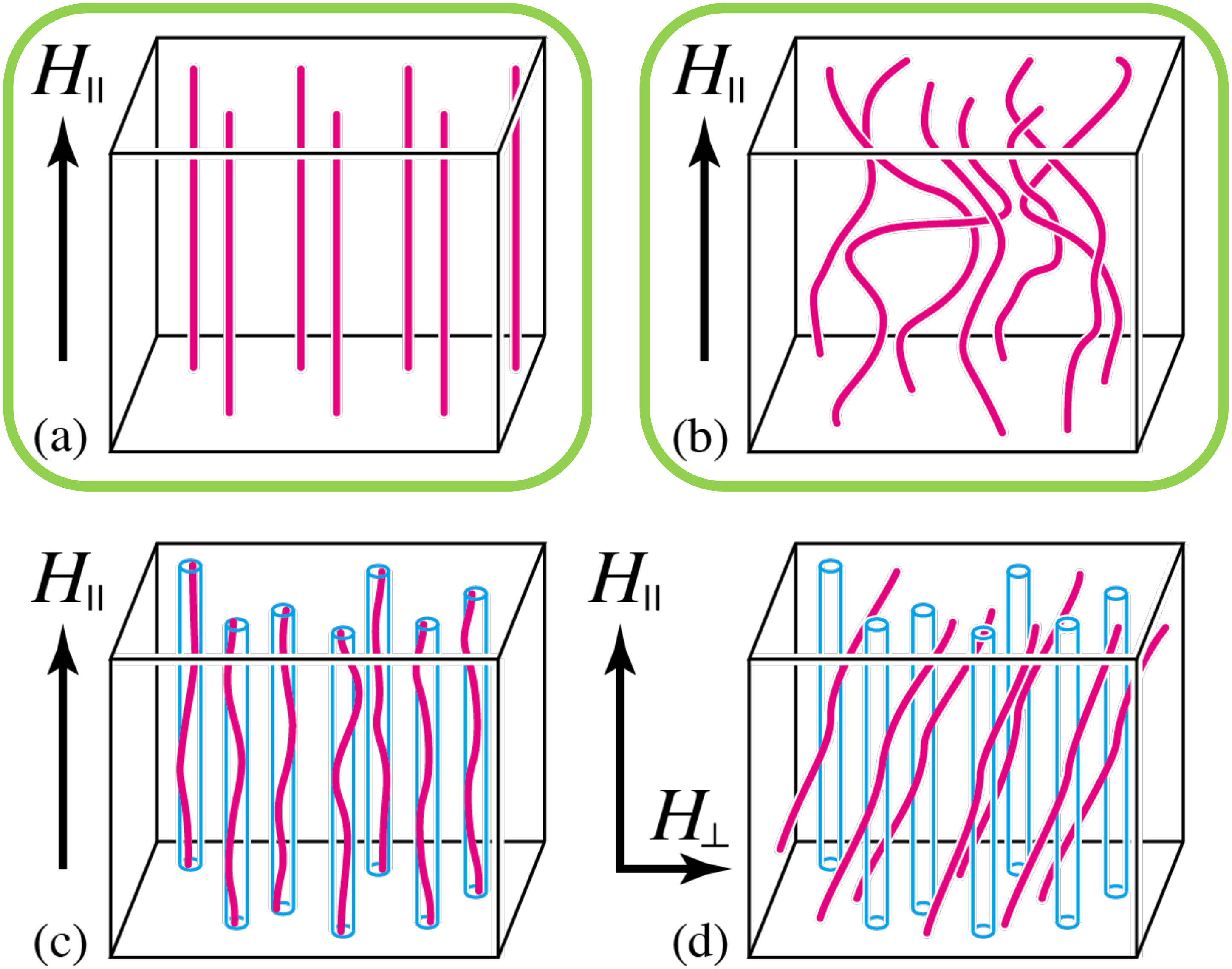
N. Hatano and D.R. Nelson, PRL 77 (1996) 570



Competition between the random potential and the non-Hermiticity

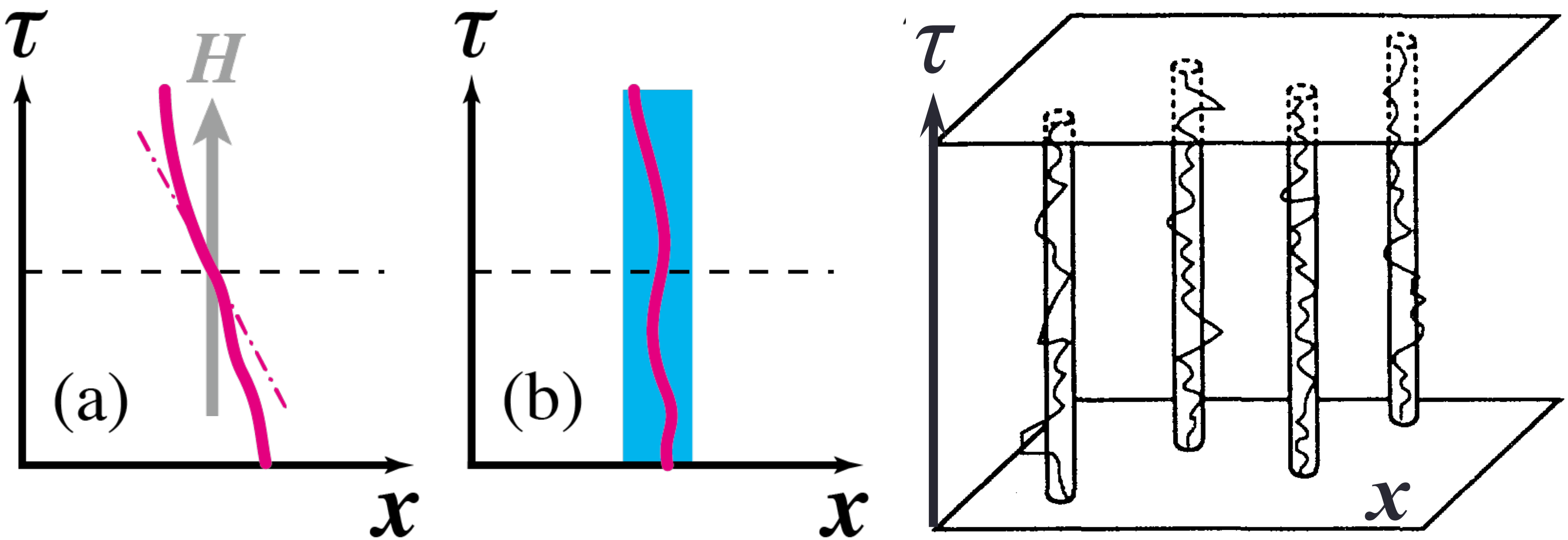


Flux lines in superconductors



Classical model for flux lines

D.R. Nelson and V.M. Vinokur, PRB 48, 13060 (1993)



$$\left(\frac{\mathrm{d}\mathbf{x_i}(\tau)}{\mathrm{d}\tau}\right)^2 \qquad V_1(\mathbf{x_i}(\tau)) \qquad V_2(\mathbf{x}_i(\tau) - \mathbf{x}_j(\tau))$$

Random potential

Interaction potential

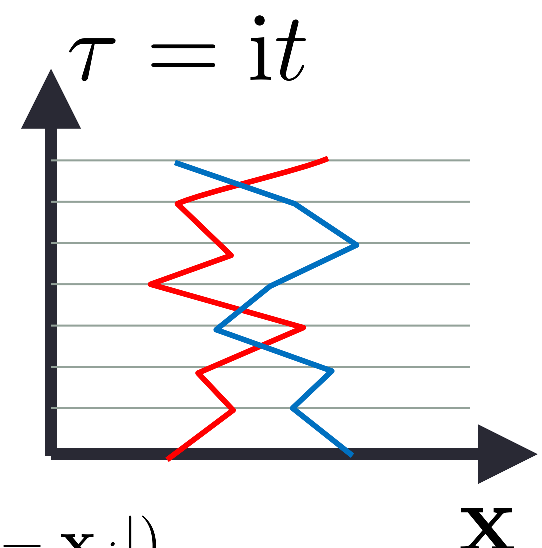
Classical model for flux lines

D.R. Nelson and V.M. Vinokur, PRB 48, 13060 (1993)

$$Z = \prod_i \int \mathcal{D}\mathbf{x}_i(\tau) \exp\left[-\beta \int \mathrm{d}\tau E[\{\mathbf{x}_i(\tau)\}]\right]$$

$$E[\{\mathbf{x}_i(\tau)\}]] = \sum_i \left[\frac{m}{2}\left(\frac{\mathrm{d}\mathbf{x}_i}{\mathrm{d}\tau}\right)^2 + V_1(\mathbf{x}_i)\right] + \sum_{i<j} V_2(|\mathbf{x}_i - \mathbf{x}_j|)$$

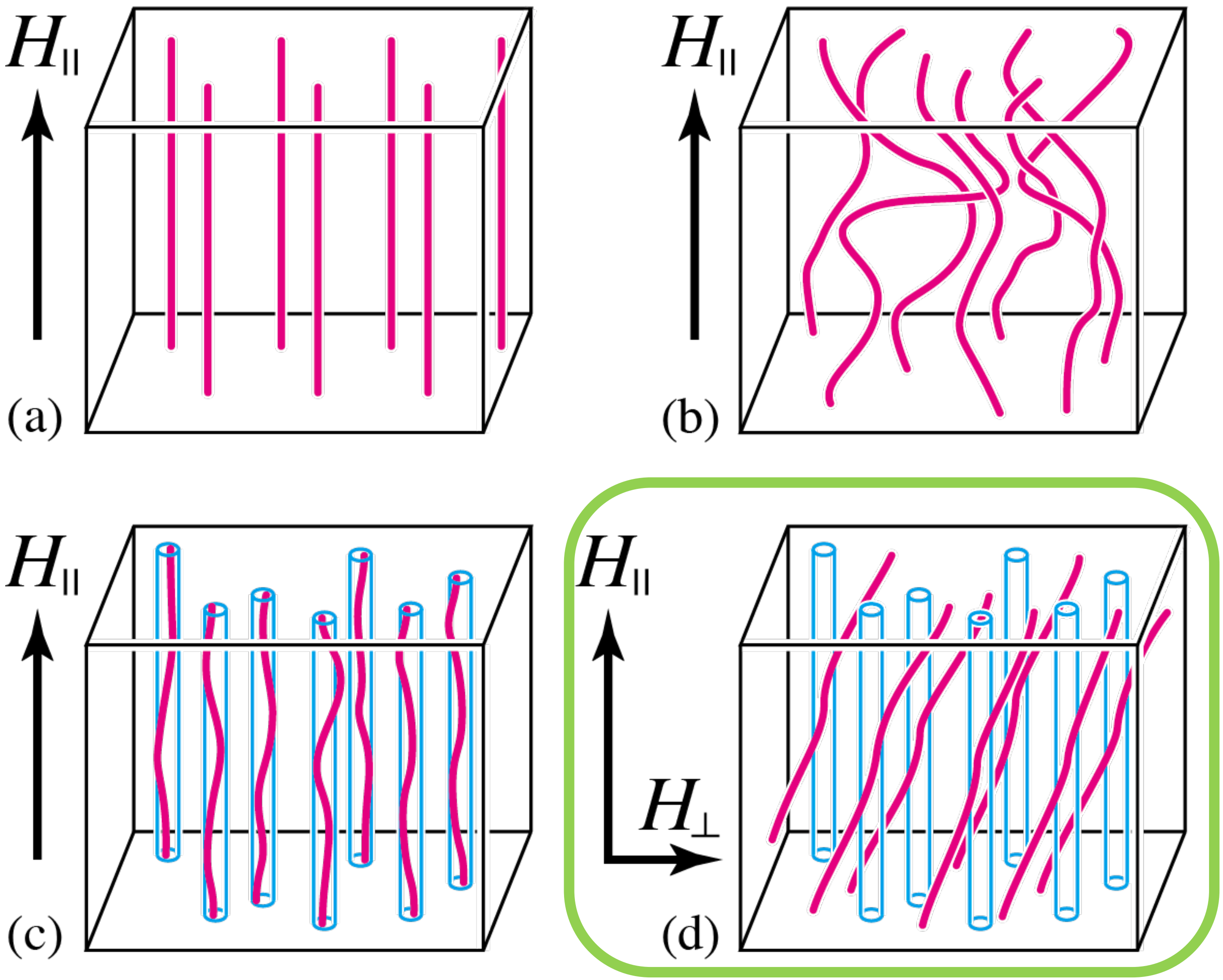
Path-integral mapping



$$Z = \mathrm{Tr}\, \mathrm{e}^{-\mathrm{i}Ht/\hbar}$$

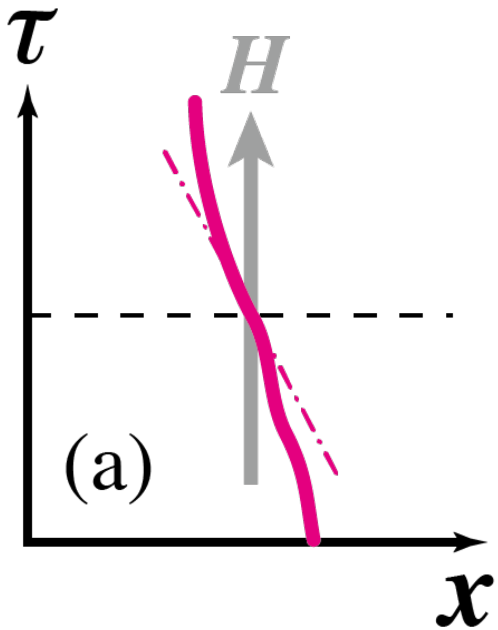
$$H = \sum_i \left[\frac{{\mathbf{p}_i}^2}{2m} + V_1(\mathbf{x}_i)\right] + \sum_{i<j} V_2(|\mathbf{x}_i - \mathbf{x}_j|)$$

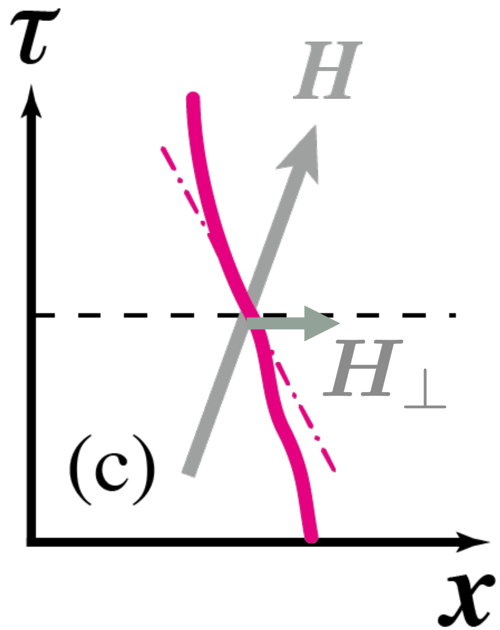
Flux lines in superconductors



Classical model for tilted flux lines

D.R. Nelson, private communication





$$\left(\frac{\mathrm{d}\mathbf{x_i}(\tau)}{\mathrm{d}\tau}\right)^2$$

$$\left(\frac{\mathrm{d}\mathbf{x}_i(\tau)}{\mathrm{d}\tau} - \mathbf{v}_0\right)^2$$

$$\mathbf{v}_0 \propto \boldsymbol{H}_\perp$$

Effective model for tilted flux lines

D.R. Nelson, private communication

$$E[\{\mathbf{x}_i(\tau)\}] = \sum_i \left[\frac{m}{2} \left(\frac{\mathrm{d}\mathbf{x}_i}{\mathrm{d}\tau} - \mathbf{v}_0 \right)^2 + V_1(\mathbf{x}_i) \right] + \sum_{i<j} V_2(|\mathbf{x}_i - \mathbf{x}_j|)$$

Random potential

Interaction potential

Path-integral mapping

$$\mathbf{g} \propto \mathbf{v}_0 \propto \boldsymbol{H}_\perp$$

$$H = \sum_i \left[\frac{(\mathbf{p}_i - \mathrm{i}\mathbf{g})^2}{2m} + V_1(\mathbf{x}_i) \right] + \sum_{i<j} V_2(|\mathbf{x}_i - \mathbf{x}_j|)$$

See Appendix of N. Hatano, Physica A 254 (1998) 317

Effective model for tilted flux lines

D.R. Nelson, private communication

$$H = \sum_i \left[\frac{(\mathbf{p}_i - \mathbf{ig})^2}{2m} + V_1(\mathbf{x}_i) \right] + \sum_{i<j} V_2(|\mathbf{x}_i - \mathbf{x}_j|)$$

Lattice version

$$H = -\sum_x \left[(t+w) b^\dagger_{x+1} b_x + (t-w) b^\dagger_x b_{x+1} \right] + \sum_x V_x b^\dagger_x b_x + \sum_x U_x b^\dagger_x b_x b^\dagger_{x+1} b_{x+1}$$

Random potential

Interaction potential

My contributions

1. Giving up the interaction!
→ Simplest nontrivial model
→ Numerical diagonalization

$$H = -\sum_x \left[(t+w) b^\dagger_{x+1} b_x + (t-w) b^\dagger_x b_{x+1} \right] + \sum_x V_x b^\dagger_x b_x + \sum_x U_x b^\dagger_x b_x b^\dagger_{x+1} b_{x+1}$$

Random potential

Interaction potential

Can find all eigenvalues of *L*= 1000 in a few minuites (then).

My contributions

2. Imaginary vector potential
→ Exponential coefficients
→ Anderson localization

Imaginary vector potentential

$$H = -\sum_x \left[e^{+g} b_{x+1}^\dagger b_x + e^{-g} b_x^\dagger b_{x+1} \right]$$

$$H = -\sum_x \left[(t+w) b_{x+1}^\dagger b_x + (t-w) b_x^\dagger b_{x+1} \right]$$

$$+ \sum_x V_x b_x^\dagger b_x$$

Random potential

Eigenvalue change tells us the localization of eigenvectors.

Imaginary vector potential

vector potential

$$H = \frac{(\mathbf{p} - e\mathbf{A})^2}{2m}$$

Lattice version

Peierls phase

$$H = -\sum_x \left[\mathrm{e}^{-\mathrm{i}eA} b^\dagger_{x+1} b_x + \mathrm{e}^{\mathrm{i}eA} b^\dagger_x b_{x+1} \right]$$

imaginary vector potential

$$H = \frac{(\mathbf{p} - \mathrm{i}\mathbf{g})^2}{2m}$$

Lattice version

Complex Peierls phase

$$H = -\sum_x \left[\mathrm{e}^{+g} b^\dagger_{x+1} b_x + \mathrm{e}^{-g} b^\dagger_x b_{x+1} \right]$$

Gauge transform. vanishes *A*

$$H_A = -\sum_x \left[\mathrm{e}^{-\mathrm{i}eA} b_{x+1}^\dagger b_x + \mathrm{e}^{\mathrm{i}eA} b_x^\dagger b_{x+1} \right] + \sum_x V_x b_x^\dagger b_x$$

$$A = 0: \; H_0 \psi_0 = E_0 \psi_0$$

⬇

$$A \neq 0: \; H_A \psi_A = E_A \psi_A$$

$$\psi_A = \psi_0 e^{ieAx}; \; E_A = E_0$$

Imag. gauge transform. vanishes *g*?

$$H_g = -\sum_i \left[e^{+g} b^\dagger_{x+1} b_x + e^{-g} b^\dagger_x b_{x+1} \right] + \sum_x V_x b^\dagger_x b_x$$

$$g = 0: \ H_0 \psi_0 = E_0 \psi_0$$

$$\Downarrow$$

Really??

$$g \neq 0: \ H_g \psi_g = E_g \psi_g$$

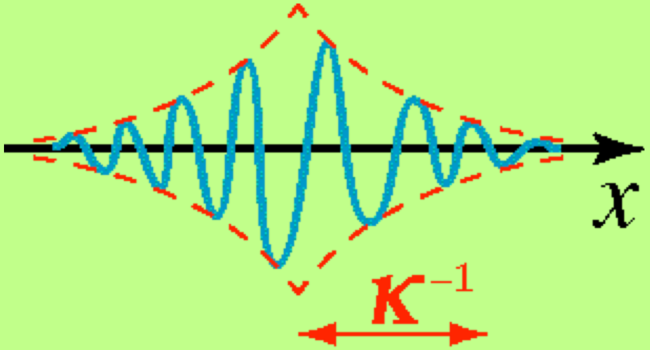
$$\psi_g = \psi_0 e^{gx}; \ E_g = E_0$$

Anderson localization (1D)

All states are localized in a random potential

$$\psi_0 \sim e^{-\kappa|x|}$$

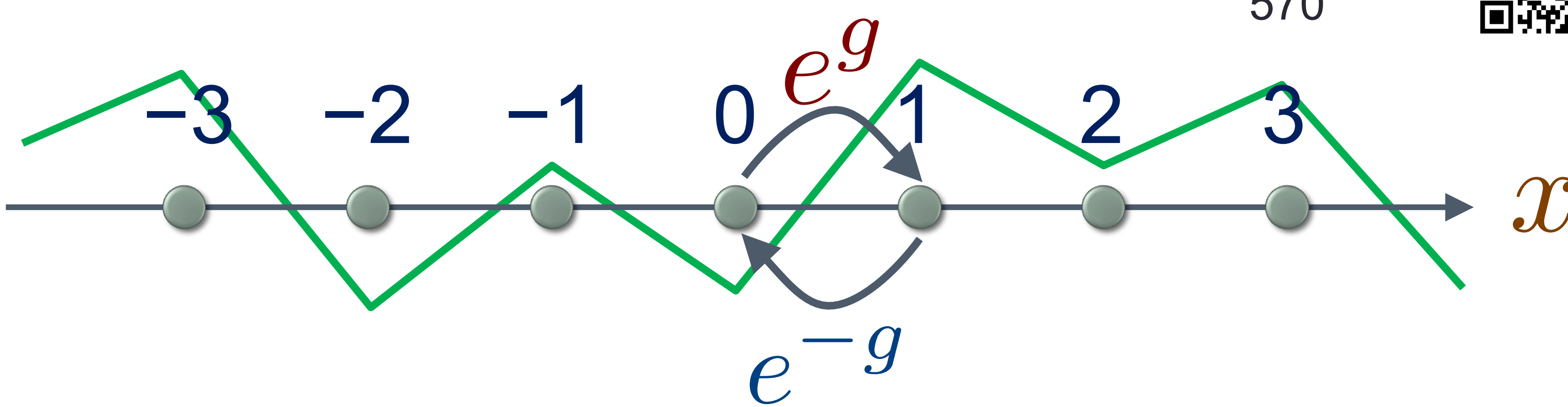
Inv. loc. length



V(*x*): (real) random scalar potential

Hatano-Nelson moel

N. Hatano and D.R. Nelson, PRL 77 (1996) 570



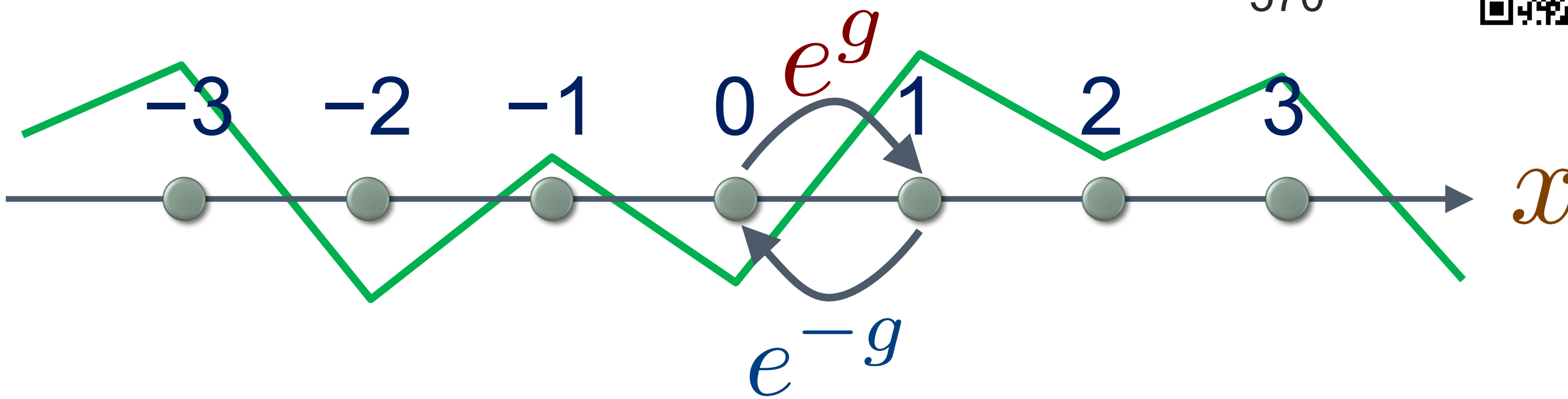
Asymmetric hop.

$$H_g = -\frac{t}{2}\sum_x \left(e^{g} c_{x+1}^{\dagger} c_x + e^{-g} c_x^{\dagger} c_{x+1} \right) + \sum_x V_x c_x^{\dagger} c_x$$

Real random potential

Hatano-Nelson model

N. Hatano and D.R. Nelson, PRL 77 (1996) 570



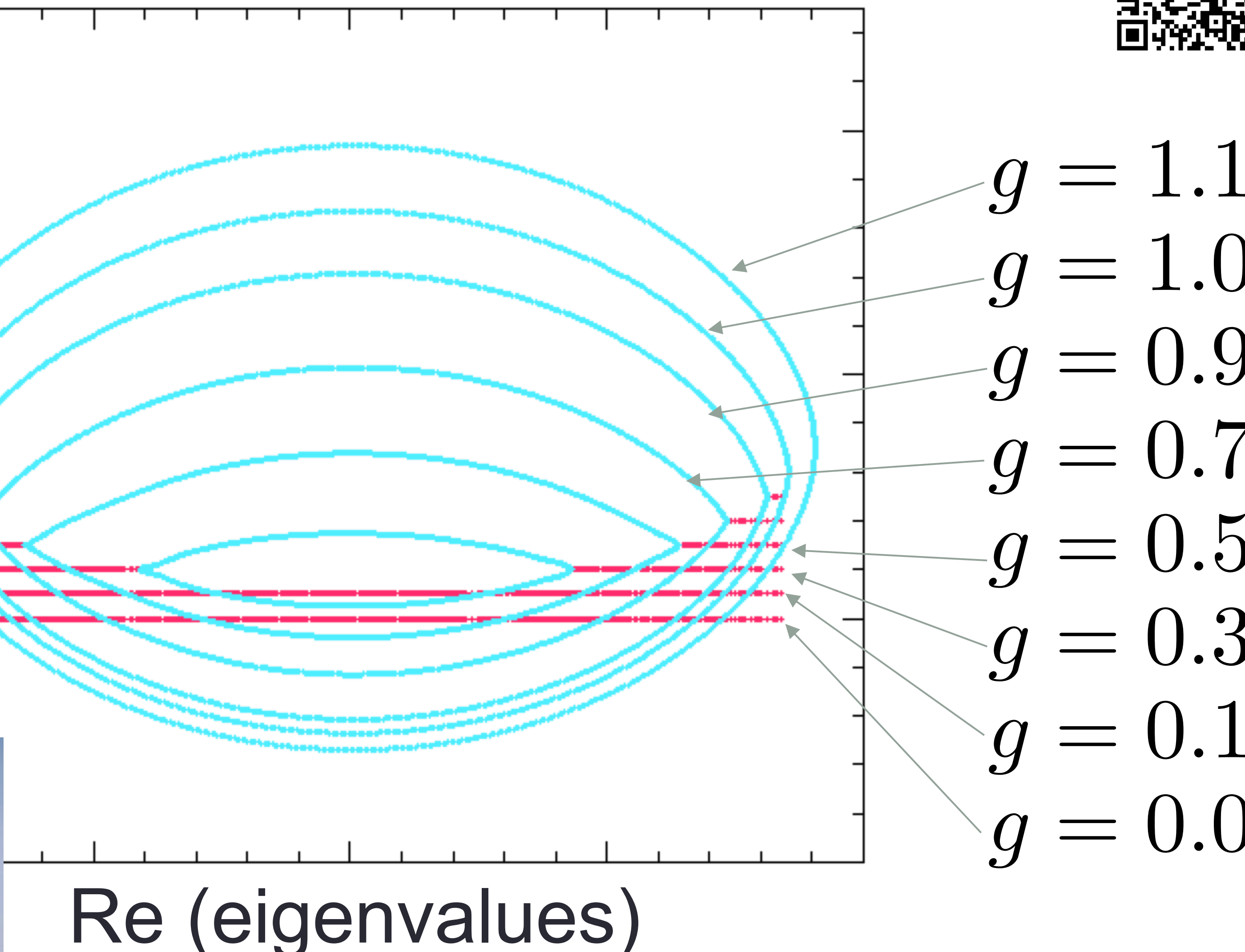
Randomness κ
$\rightarrow$ Localizing
Asymmetric hop. g
$\rightarrow$ Delocalizing

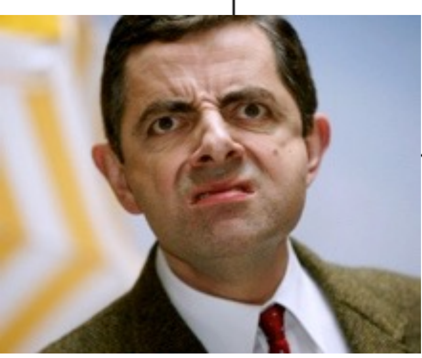
Hatano-Nelson model

N. Hatano and D.R. Nelson, PRL 77 (1996) 570

$$
H = \begin{pmatrix}
\ddots & \ddots & & & & e^{g} \\
\ddots & V_{-1} & e^{-g} & & & \\
 & e^{g} & V_{0} & e^{-g} & & \\
 & & e^{g} & V_{1} & e^{-g} & \\
 & & & e^{g} & V_{2} & \ddots \\
e^{-g} & & & & \ddots & \ddots
\end{pmatrix}
$$

Complex eigenv. transition



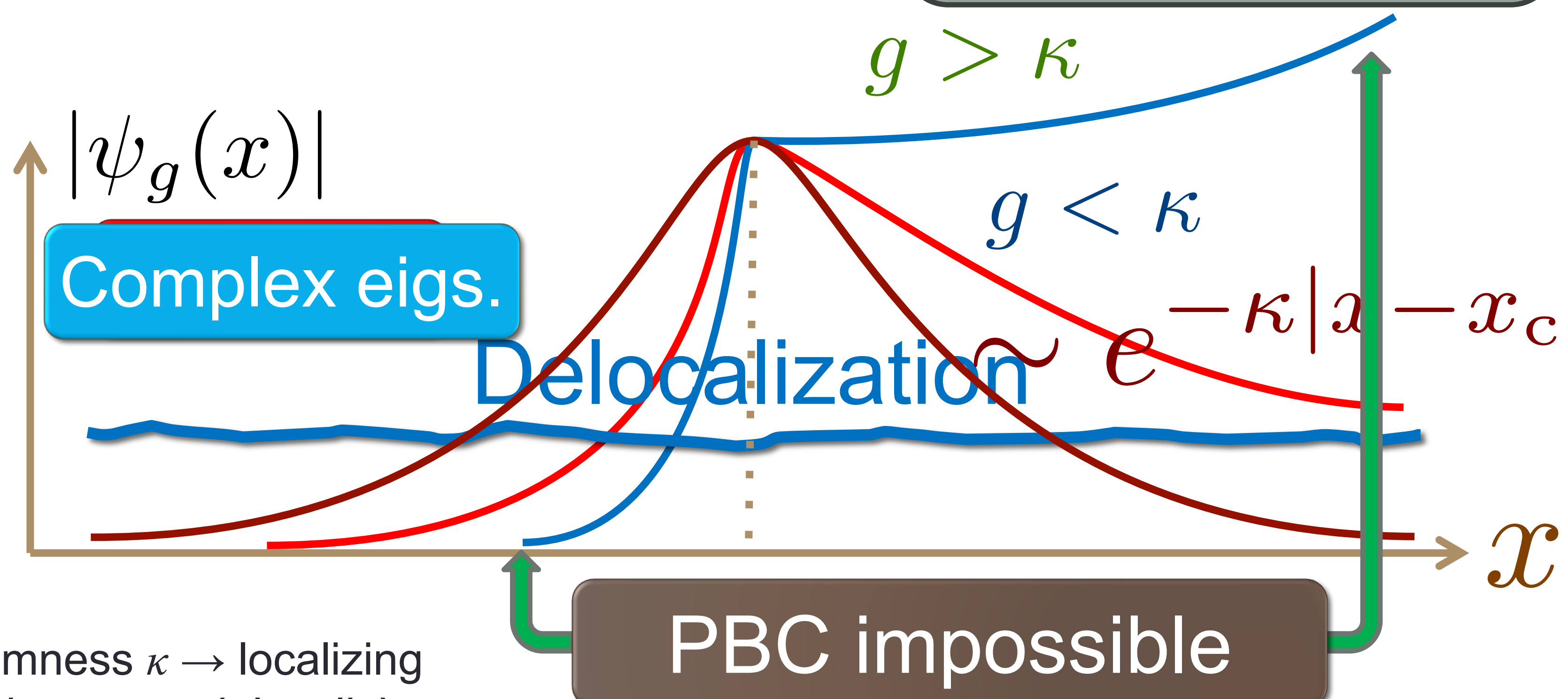


$V_x \in [-1, 1]$ 1000 sites Periodic B.C.

Imag. gauge transformation

$$\psi_0(x) \longrightarrow \psi_g(x) \quad \cdots c$$

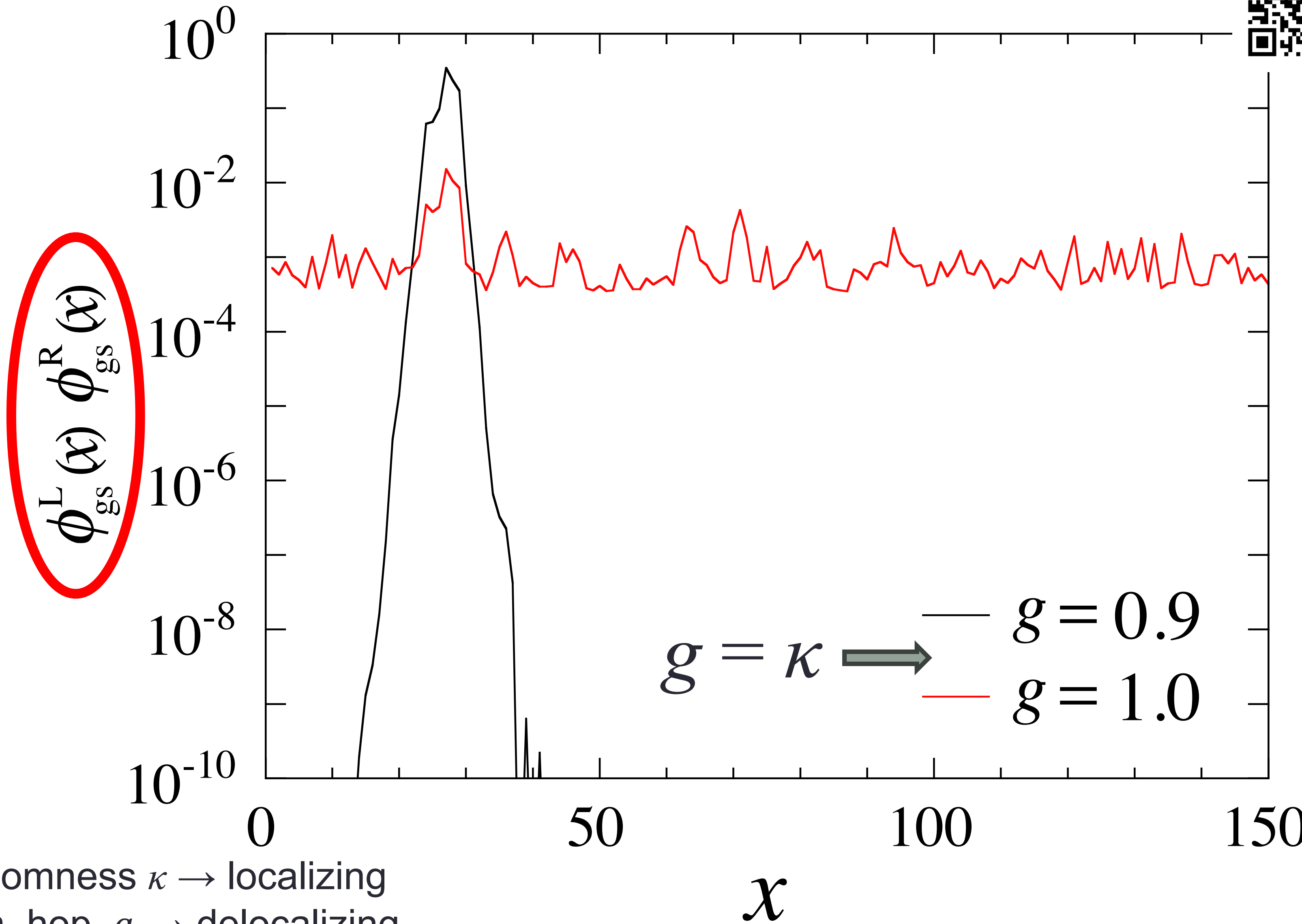
Non-Hermitian Skin Effect



Randomness $\kappa \rightarrow$ localizing
Asym. hop. $g \rightarrow$ delocalizing

Non-Hermitian delocalization

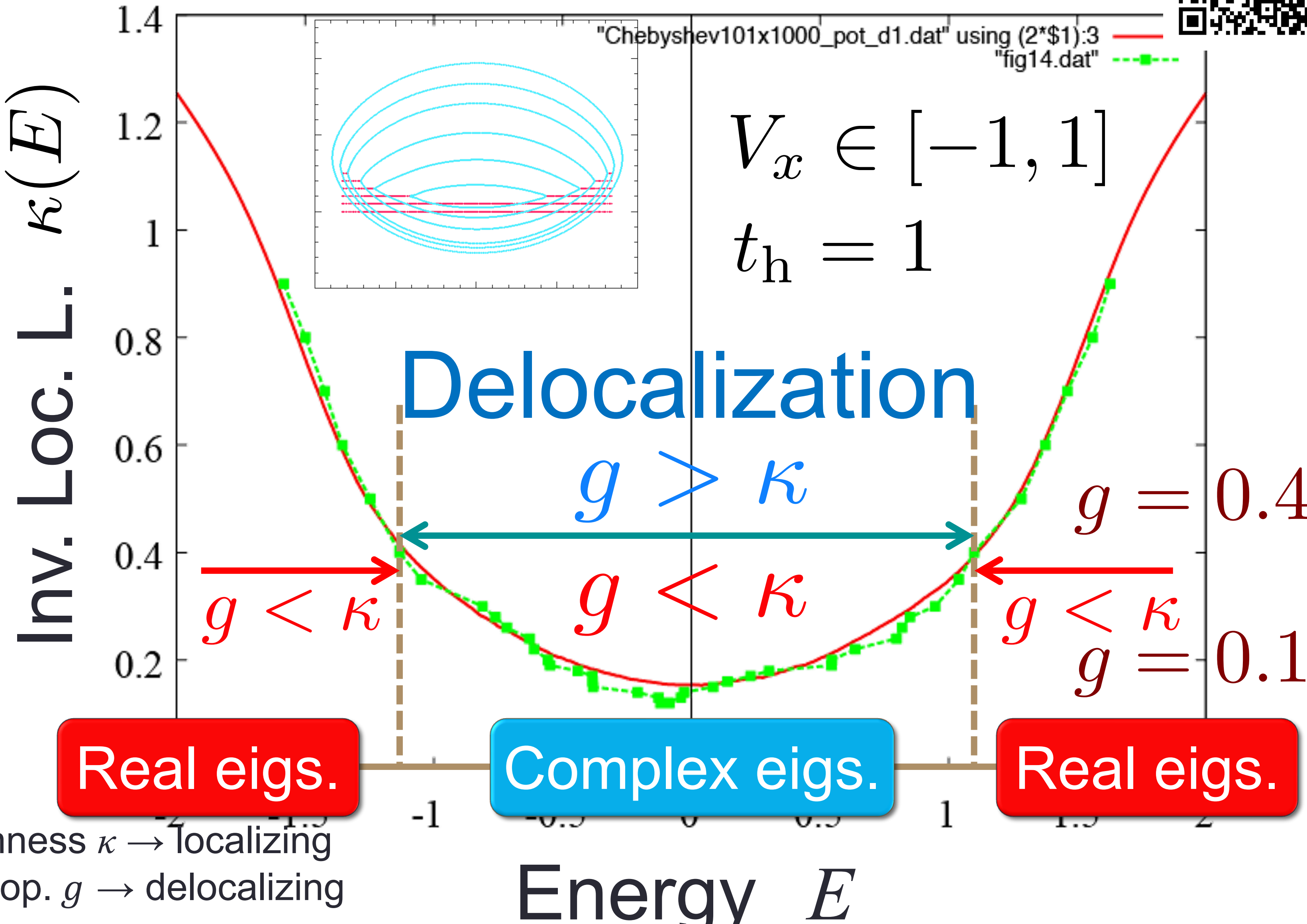
PRL 96



Randomness $\kappa \rightarrow$ localizing

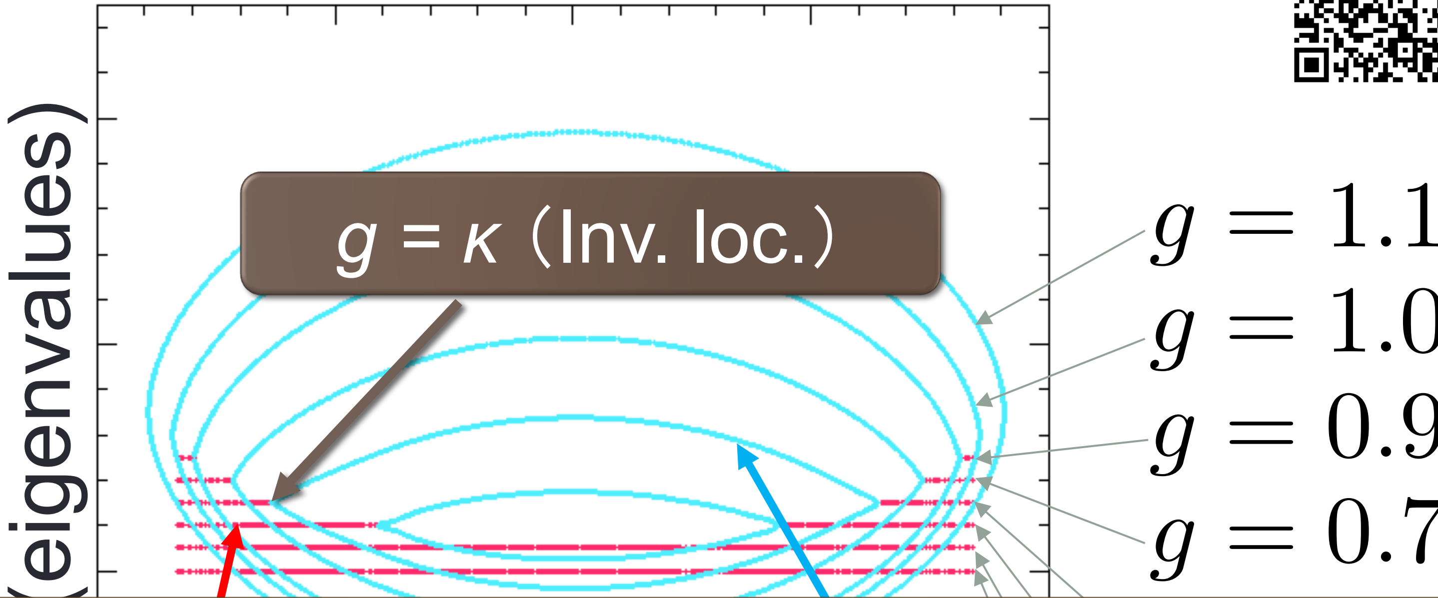
Asym. hop. $g \rightarrow$ delocalizing

Non-Hermitian delocalization



Randomness $\kappa \to$ localizing
Asym. hop. $g \to$ delocalizing

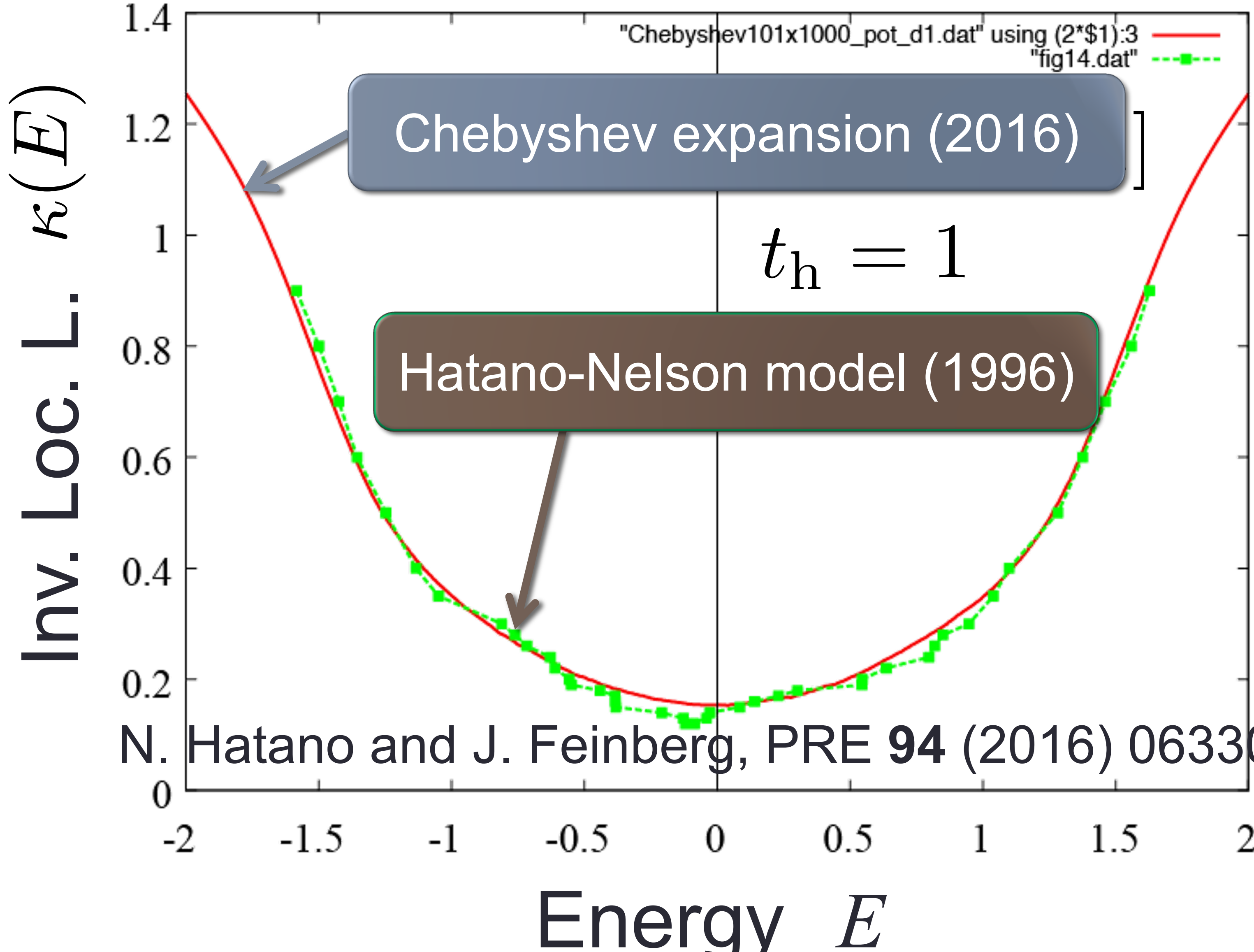
Non-Hermitian delocalization



Complex eig. transition ⇔ Non-Hermitian deloc. transtion

Eigenvalues tell about eigenvectors

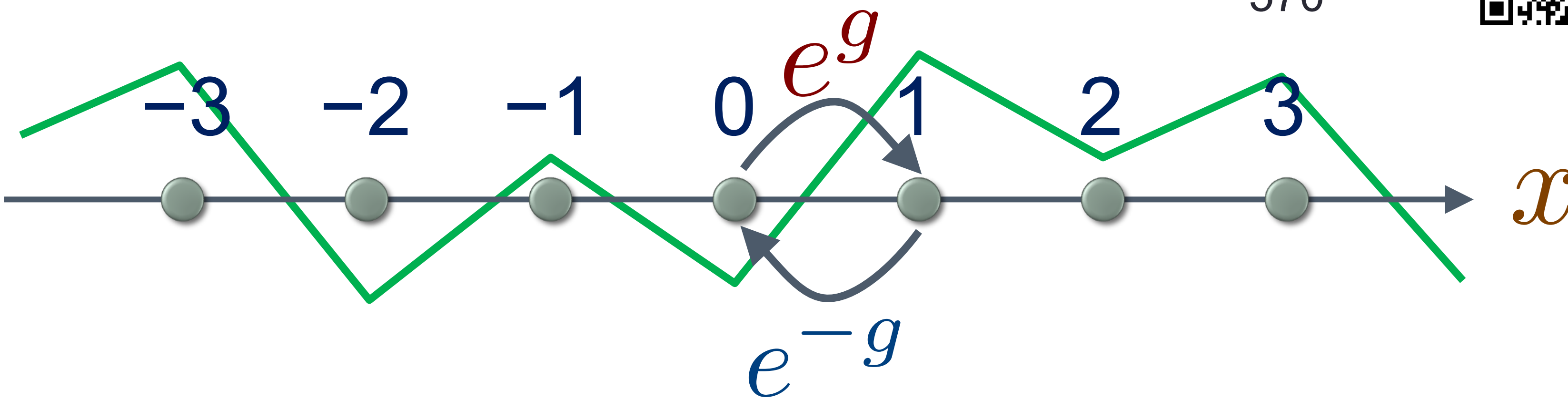
Energy Dep. of Inv. Loc. Length



N. Hatano and J. Feinberg, PRE **94** (2016) 063305

Hatano-Nelson model

N. Hatano and D.R. Nelson, PRL 77 (1996) 570



Randomness κ

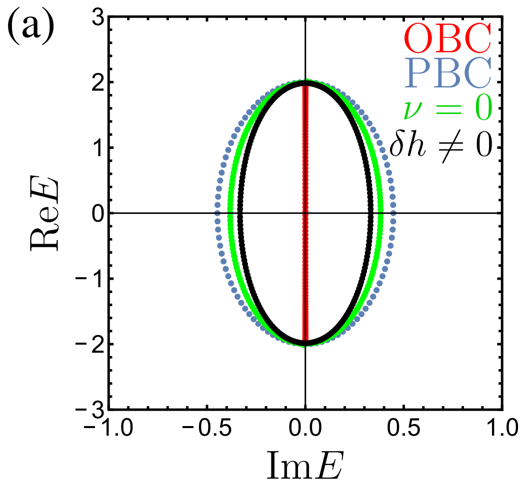
$\rightarrow$ Localizing

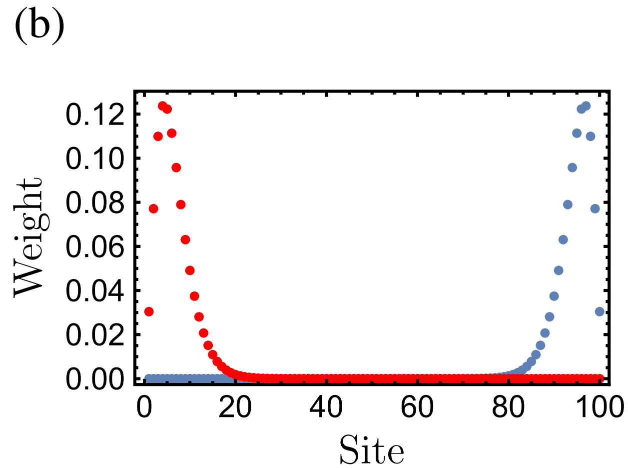
Asymmetric hop. g

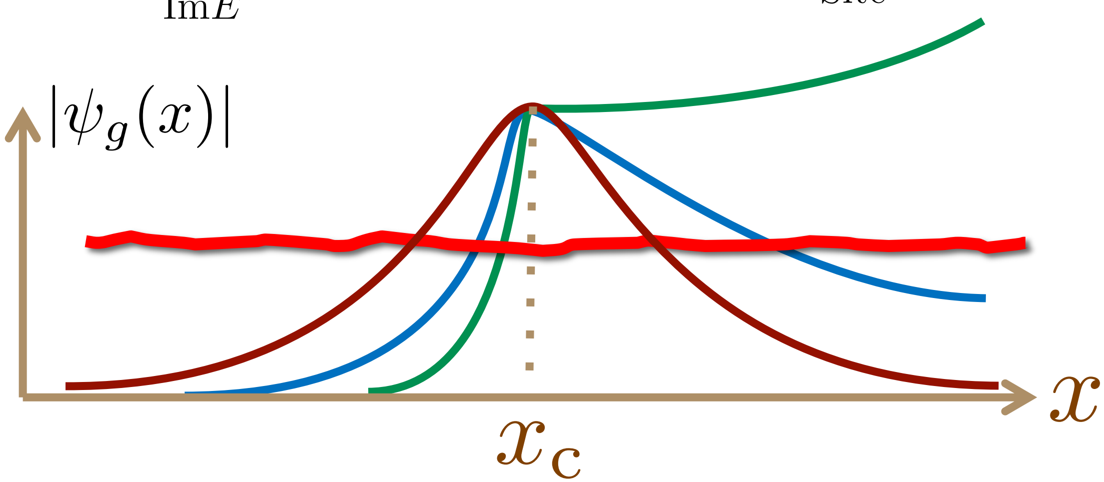
$\rightarrow$ Delocalizing

Non-Hermitian Skin Effect

S. Yao, Z. Wang, PRL **121** (2018) 086803
N. Okuma, K. Kawabata, K. Shiozaki, M. Sato, PRL **124** (2020) 086801







Topology of Point Gap

Z. Gong, Y. Ashida, K. Kawabata, K. Takasan *et al.*, PRX **8** (2018) 031079
K. Kawabata, K. Shiozaki, M. Ueda, M. Sato, PRX **9** (2019) 041015

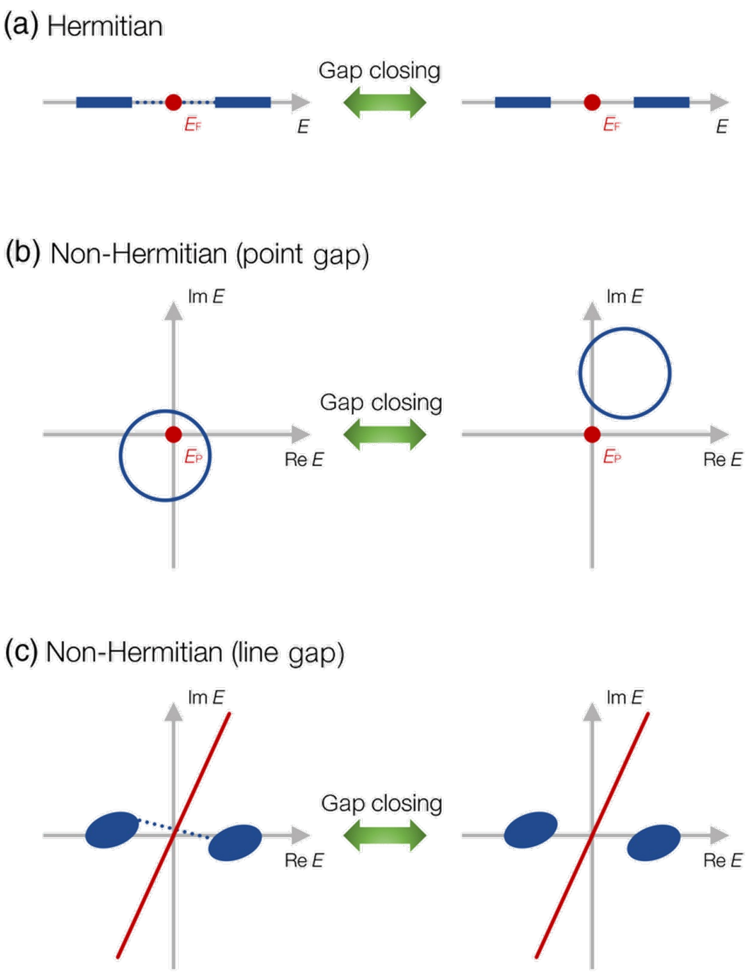
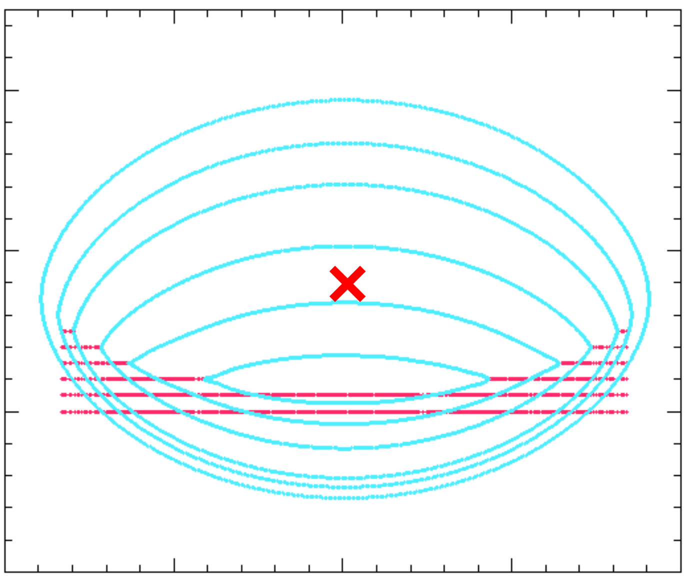
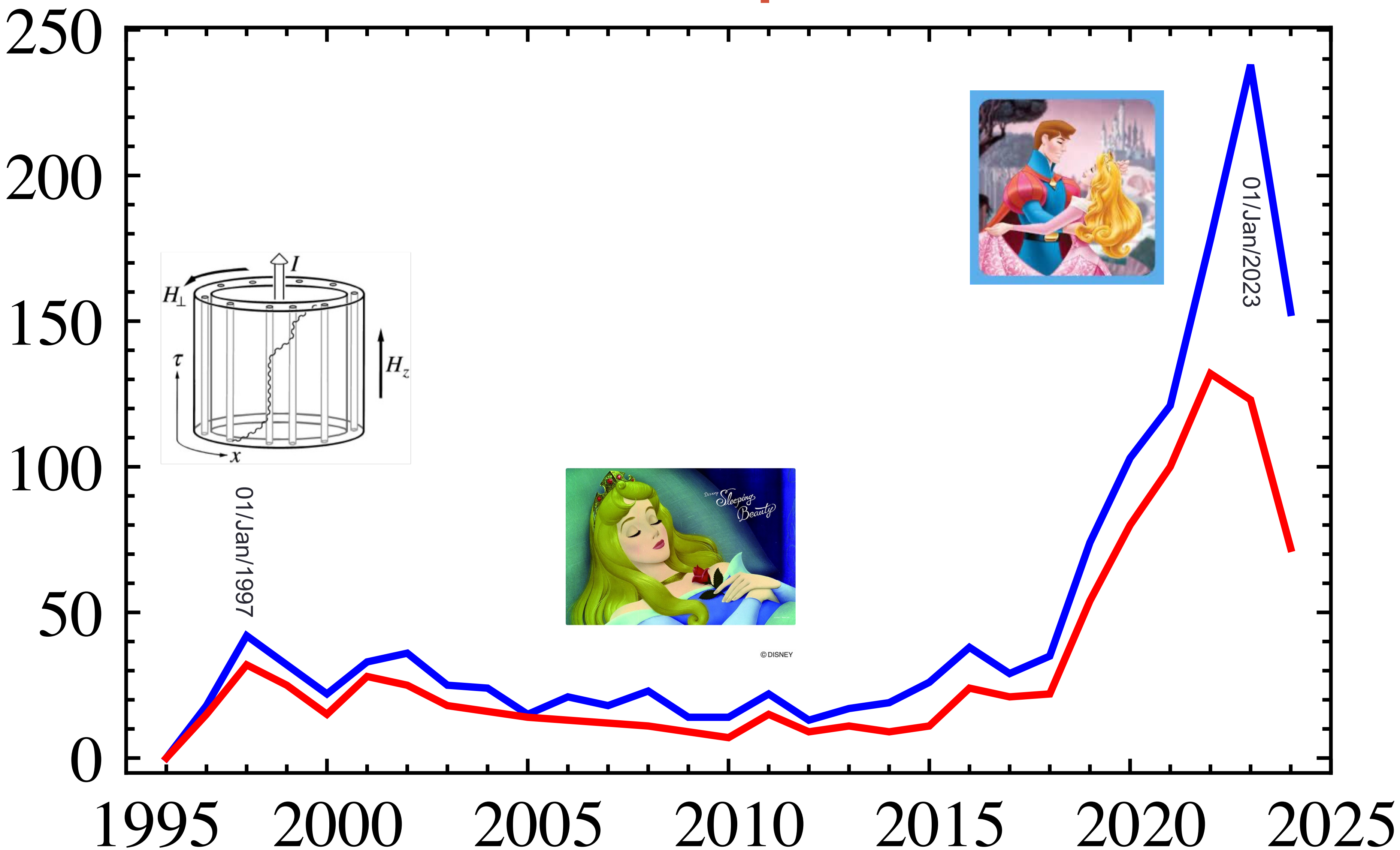


TABLE III. Topological classification table for non-Hermitian systems in the complex AZ symmetry class. Non-Hermitian topological phases are classified according to the AZ symmetry class, the spatial dimension d, and the definition of complex-energy point (P) or line (L) gaps. The subscript of L specifies the line gap for the real or imaginary part of the complex spectrum.

AZ class	Gap	Classifying space	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
A	P	$\mathcal{C}_1$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
	L	$\mathcal{C}_0$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	P	$\mathcal{C}_0$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
	L_r	$\mathcal{C}_1$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
	L_i	$\mathcal{C}_0 \times \mathcal{C}_0$	$\mathbb{Z} \oplus \mathbb{Z}$	0	$\mathbb{Z} \oplus \mathbb{Z}$	0	$\mathbb{Z} \oplus \mathbb{Z}$	0	$\mathbb{Z} \oplus \mathbb{Z}$	0



Citation of Our Paper



“Non-Hermitian Quantum Mechanics”

1313 citations（Google Scholar as of March 18., 2024）

ducting shell with radial thickness smaller than the penetration depth of the defect-free material. The field $\boldsymbol{H}_{\perp}$ is generated by the current $\boldsymbol{I}$ threading the ring.