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THE INSTITUTE FOR SOLID STATE PHYSICS
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Symmetry classification of non-Hermitian random matrices and open quantum systems

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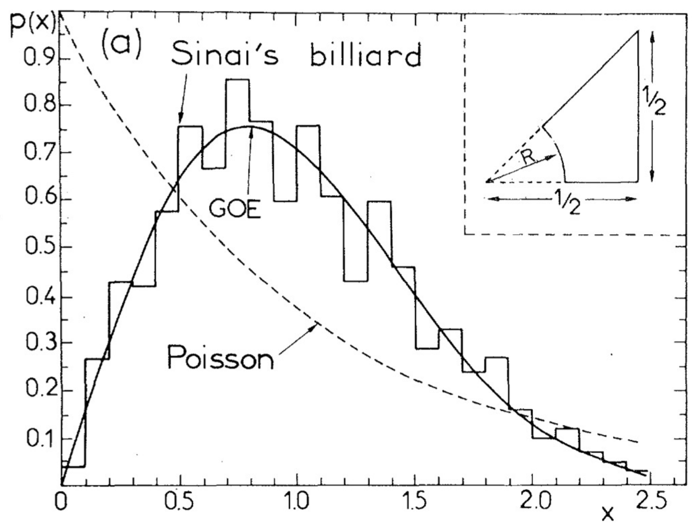
Random matrix theory

☆ **Random matrix theory has various applications in physics.**

- **Quantum chaos**



Bohigas *et al.*, PRL **52**, 1 (1984)

— Integrable: **Poisson statistics**

Berry & Tabor, Proc. R. Soc. A **356**, 375 (1977)

— Nonintegrable: **random-matrix statistics**

→ quantum chaos & thermalization

- **Electronic transport phenomena** Beenakker, RMP **69**, 731 (1997)

- **Quantum dynamics** Fisher *et al.*, Ann. Rev. Condens. Matter Phys. **14**, 335 (2023)

Altland-Zirnbauer symmetry

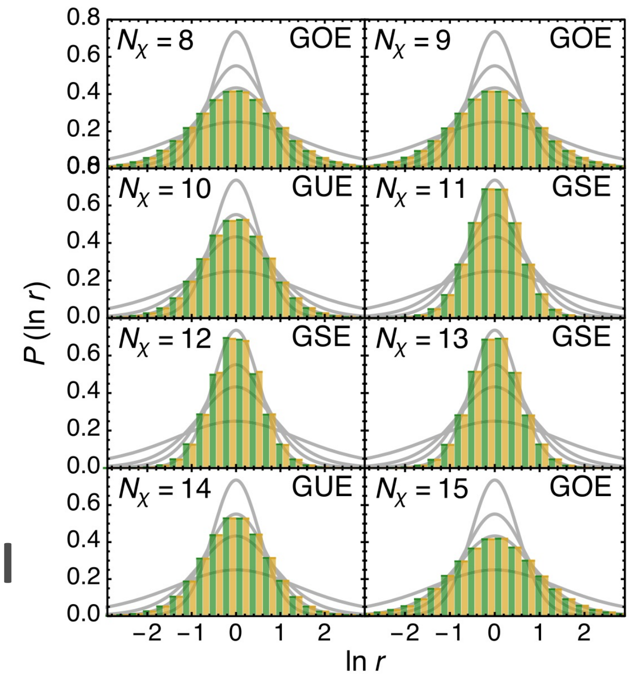
☆ **Random matrices are classified by the tenfold AZ symmetry.**

time reversal $\mathcal{T} H^* \mathcal{T}^{-1} = H$

particle hole $\mathcal{C} H^* \mathcal{C}^{-1} = -H$

chiral $\mathcal{S} H \mathcal{S}^{-1} = -H$

Altland & Zirnbauer, PRB **55**, 1142 (1997)



e.g., Tenfold symmetry classification of the SYK model

symmetry-enriched behavior of quantum chaos

You *et al.*, PRB **95**, 115150 (2017)
Cotler *et al.*, JHEP **2017**, 118

☆ **AZ symmetry is also relevant to the physics of free fermions.**

- Anderson localization and transition
- Topological insulators and superconductors

Periodic table for TIs and TSCs

General and comprehensive theoretical framework of TIs and TSCs:

Periodic table based on spatial dimension and symmetry

AZ Symmetry				Dimension							
Class	TRS	PHS	CS	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$					
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+1	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	+1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$						
DIII	-1	+1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	-1	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$					
CII	-1	-1	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	+1	-1	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Quantum Hall insulator

Kitaev/Majorana chain

Quantum spin Hall insulator

Schnyder, Ryu, Furusaki & Ludwig, PRB **78**, 195125 (2008)

Kitaev, AIP Conf. Proc. **1134**, 22 (2009)

Non-Hermitian physics

Despite the enormous success, the existing framework of condensed matter physics is **confined to Hermitian systems at equilibrium**.

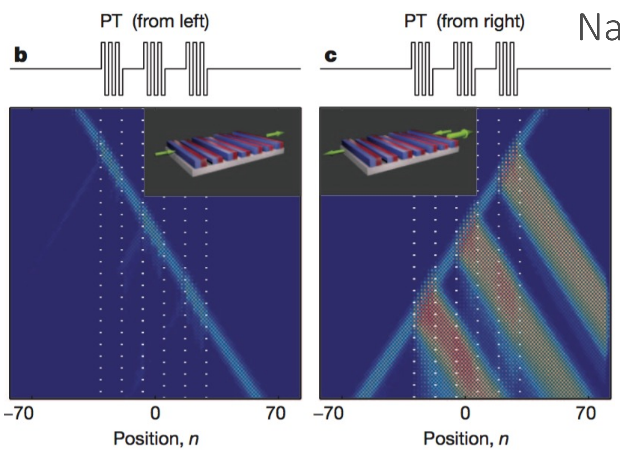
→ **Richer properties appear in non-Hermitian systems!**

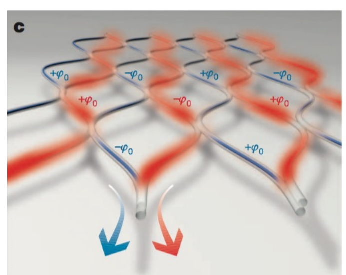
☆ Non-Hermiticity arises from **dissipation**, i.e., exchanges of energy or particles with an environment.

- **Photonic lattices with gain/loss**

Unidirectional light transport

Regensburger *et al.*, Nature **488**, 167 (2012)

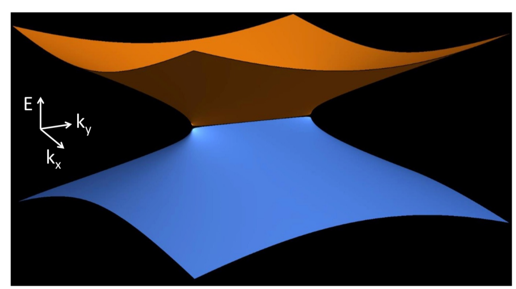


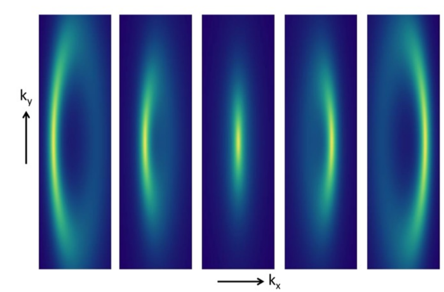


- **Finite-lifetime quasiparticles**

Bulk Fermi arc due to non-Hermitian self-energy

Kozii & Fu, arXiv: 1708.05841



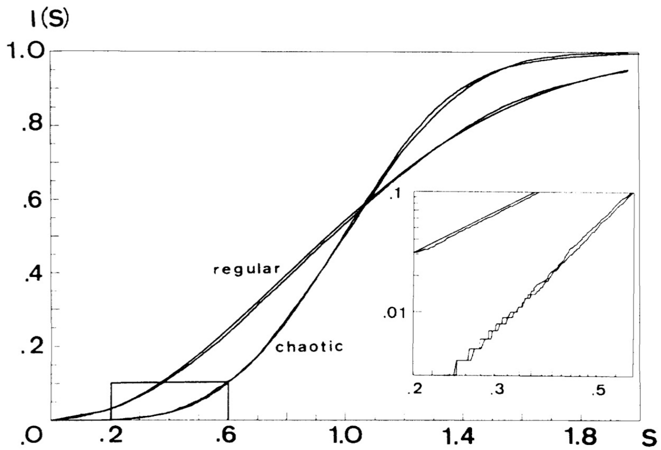


Dissipative quantum chaos

Characterization of quantum chaos is confined to closed quantum systems.

→ **Can we characterize chaos in open quantum systems?**

☆ Dissipative quantum chaos is captured by non-Hermitian random matrices!



Grobe, Haake & Sommers, PRL **61**, 1899 (1988)

Level-spacing distribution of periodically kicked tops with damping:

Integrable: complex (2D) Poisson

Chaotic: non-Hermitian random matrix (Ginibre ensembles)

Ginibre, J. Math. Phys. **6**, 440 (1965)

— Different types of open quantum systems (many-body, Lindbladians, …)

Hamazaki, **Kawabata** *et al.*, PRL **123**, 090603 (2019); Akemann *et al.*, PRL **123**, 254101 (2019)

— Different quantitative measures of dissipative quantum chaos

Sá *et al.*, PRX **10**, 021019 (2020); Li *et al.*, PRL **127**, 170602 (2021); Cipolloni & Kudler-Flam, PRL **130**, 010401 (2023)

Motivation

Non-Hermitian random matrices are relevant to the physics of open systems, including **dissipative quantum chaos**.

However, the **role of symmetry in non-Hermitian random matrices** has yet to be understood clearly.

How can we classify non-Hermitian random matrices with symmetry?

Results

We develop the **symmetry classification of non-Hermitian random matrices**.

We show that non-Hermiticity changes the nature of symmetry and leads to the **38-fold symmetry classification**.

Using symmetry, we classify the **universal spectral statistics of non-Hermitian random matrices** in the spectral bulk, around the real and imaginary axes, and around the spectral origin.

We also find **symmetry-enriched dissipative chaos** in various open quantum systems.

Outline

1. Introduction

2. 38-fold symmetry classification

3. Spectral statistics in the bulk

4. Spectral statistics around the real axis

5. Spectral statistics around the origin

6. Singular-value statistics

Symmetry classification of non-Hermitian systems

<u>**Kawabata**</u>, Higashikawa, Gong, Ashida & Ueda, Nat. Commun. **10**, 297 (2019)

<u>**Kawabata**</u>, Shiozaki, Ueda & Sato, PRX **9**, 041015 (2019)

10-fold symmetry class

☆ **3-fold symmetry class by Wigner & Dyson**

Wigner (1959)
Dyson, J. Math. Phys. **3**, 1199 (1962)

time reversal $$\underbrace{\mathcal{T}}_{\pm 1} H^* \mathcal{T}^{-1} = H$$

anti-unitary
(with complex conjugation)

☆ **10-fold symmetry class by Altland & Zirnbauer**

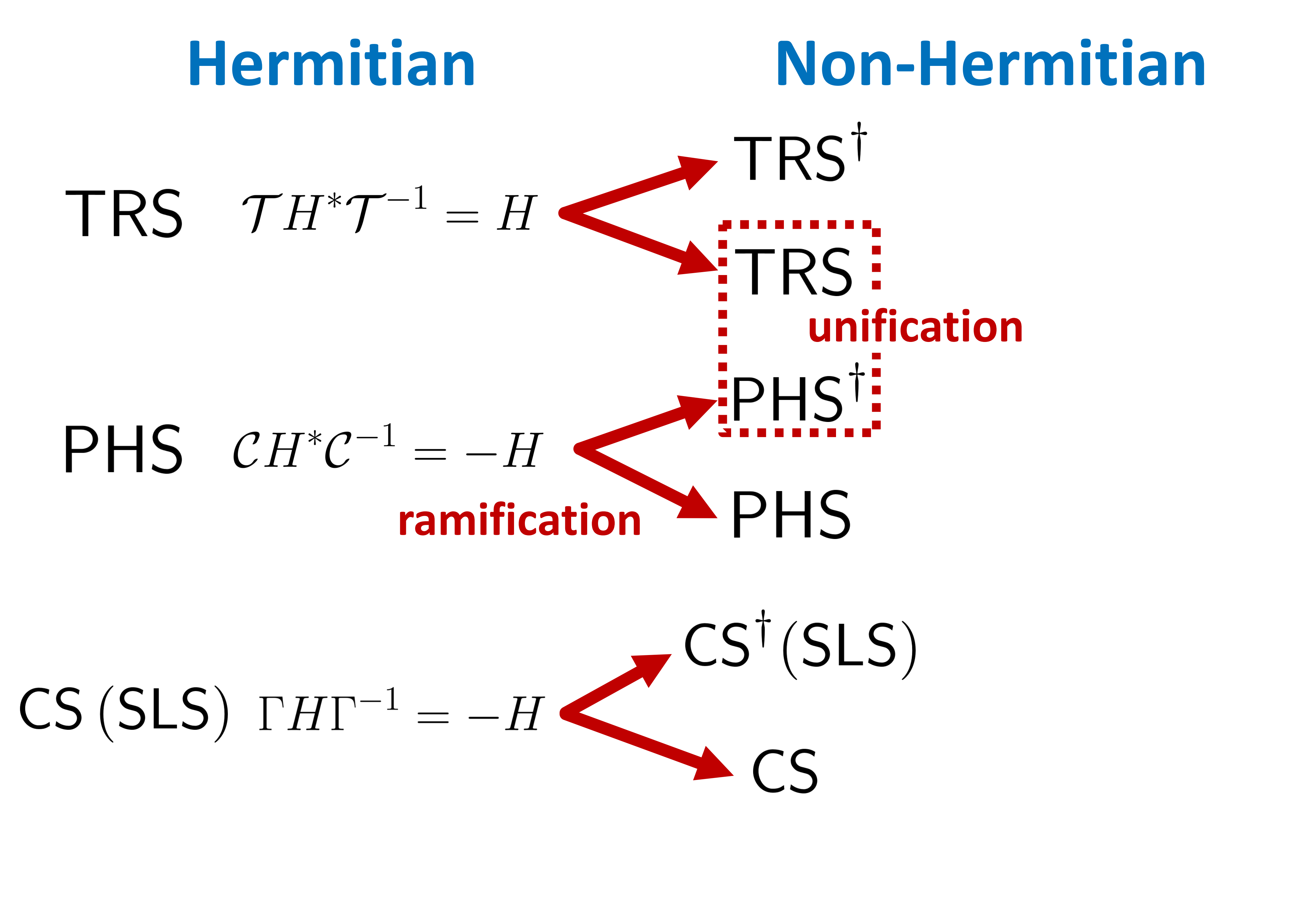
Altland & Zirnbauer, PRB **55**, 1142 (1997)

particle hole $$\mathcal{C} H^* \mathcal{C}^{-1} = -H$$ anti-unitary

chiral (sublattice) $$\Gamma H \Gamma^{-1} = -H$$ unitary

- **Universality**

Random matrix theory, Anderson transitions, topological phases, ……


Hermitian
Non-Hermitian
TRS $\mathcal{T}H^{*}\mathcal{T}^{-1} = H$
TRS†
TRS
unification
PHS†
PHS $\mathcal{C}H^{*}\mathcal{C}^{-1} = -H$
ramification
PHS
CS† (SLS)
CS (SLS) $\Gamma H \Gamma^{-1} = -H$
CS

Symmetry ramification (1)

☆ Symmetry **ramifies (bifurcates)** in non-Hermitian systems.

<u>**Kawabata**</u>, Shiozaki, Ueda & Sato,
PRX **9**, 041015 (2019).

Hermitian TRS | **Non-Hermitian TRS**

time reversal

$$\mathcal{T} H^* \mathcal{T}^{-1} = H \quad \longrightarrow \quad \mathcal{T} H^* \mathcal{T}^{-1} = H$$

$$\| \;\; H^* = H^T \qquad\qquad \nparallel \;\; H^* \neq H^T$$

$$\mathcal{T} H^T \mathcal{T}^{-1} = H \quad \longrightarrow \quad \mathcal{T} H^T \mathcal{T}^{-1} = H$$

reciprocity

Two types of symmetry appear due to the **distinction of complex conjugation and transpose operation** in non-Hermitian systems!

Symmetry ramification (2)

☆ Symmetry **ramifies (bifurcates)** in non-Hermitian systems.

Kawabata, Shiozaki, Ueda & Sato,
PRX **9**, 041015 (2019).

Hermitian CS=SLS

$$\Gamma H \Gamma^{-1} = -H$$

$$\| \quad H = H^\dagger$$

$$\Gamma H^\dagger \Gamma^{-1} = -H$$

Non-Hermitian CS/SLS

sublattice

$$\Gamma H \Gamma^{-1} = -H$$

$$\not\parallel \quad H \neq H^\dagger$$

$$\Gamma H^\dagger \Gamma^{-1} = -H$$

chiral

Two types of symmetry appear due to the **distinction of complex conjugation and transpose operation** in non-Hermitian systems!

Hermitian

TRS $\mathcal{T} H^* \mathcal{T}^{-1} = H$

PHS $\mathcal{C} H^* \mathcal{C}^{-1} = -H$

CS (SLS) $\Gamma H \Gamma^{-1} = -H$

ramification

Non-Hermitian

TRS† $\mathcal{T} H^T \mathcal{T}^{-1} = H$

TRS $\mathcal{T} H^* \mathcal{T}^{-1} = H$

unification

PHS† $\mathcal{C} H^* \mathcal{C}^{-1} = -H$

PHS $\mathcal{C} H^T \mathcal{C}^{-1} = -H$

CS† (SLS) $\Gamma H \Gamma^{-1} = -H$

CS $\Gamma H^\dagger \Gamma^{-1} = -H$

Symmetry unification (1)

☆ Antiunitary symmetries distinct in Hermitian systems are **unified** in non-Hermitian systems.

Kawabata *et al.*, Nat. Commun. **10**, 297 (2019)

$$\mathcal{T} H^* \mathcal{T}^{-1} = H$$

$$\mathcal{C} H^* \mathcal{C}^{-1} = -H$$

Two antiunitary symmetries are distinct for **Hermitian** H

→ If we allow **non-Hermitian** H, they are equivalent!

$$\mathcal{T} H^* \mathcal{T}^{-1} = H \longleftrightarrow \mathcal{T} \left[\mathrm{i} H \right]^* \mathcal{T}^{-1} = \underline{\underline{-}} \left[\mathrm{i} H \right]$$

one-to-one mapping
(wavefunctions are invariant)

Symmetry unification (2)

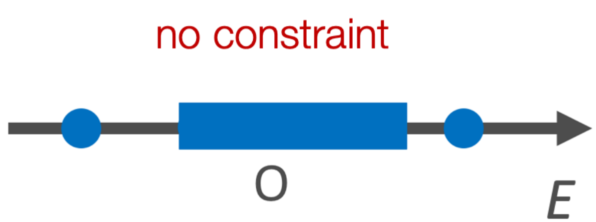
☆ Antiunitary symmetries distinct in Hermitian systems are **unified** in non-Hermitian systems.

Kawabata *et al.*, Nat. Commun. **10**, 297 (2019)

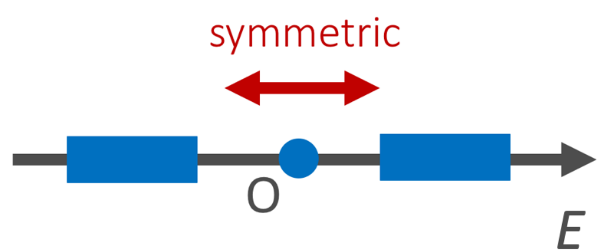
Symmetry	Hermitian	Non-Hermitian
$\mathcal{T}$	no constraints	$E \in \mathbb{R}$ or $(E,\ E^*)$
$\mathcal{C}$	$E = 0$ or $(E,\ -E)$	$E \in \mathrm{i}\mathbb{R}$ or $(E,\ -E^*)$

(Hermitian)

$\mathcal{T}$



$\mathcal{C}$



Symmetry unification (2)

☆ Antiunitary symmetries distinct in Hermitian systems are **unified** in non-Hermitian systems.

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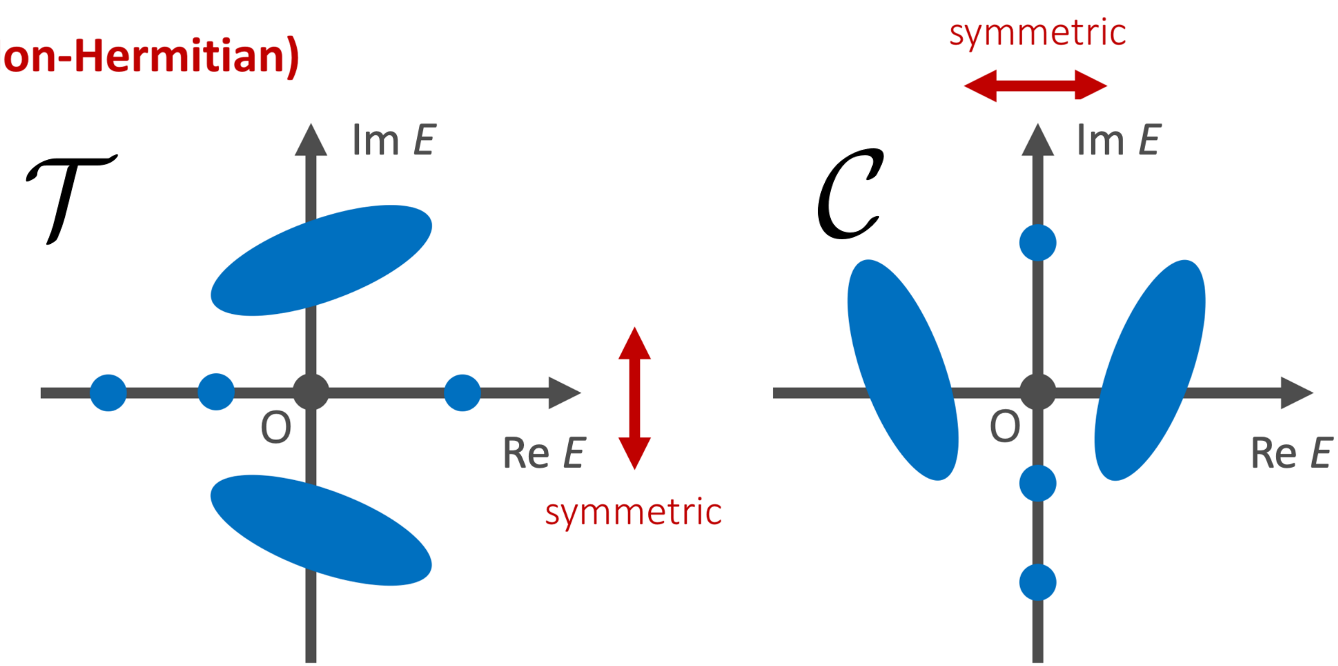
(Non-Hermitian)



38-fold symmetry class

- Hermitian case: **10 classes** (AZ symmetry class)

  time reversal, particle hole, and chiral (=sublattice)

Kawabata, Shiozaki, Ueda & Sato, PRX **9**, 041015 (2019)
cf. Bernard & LeClair, arXiv:cond-mat/0110649

- Non-Hermitian case: **38 classes**

  10-fold non-Hermitian AZ symmetry class

$$\mathcal{T} H^* \mathcal{T}^{-1} = H, \quad \mathcal{C} H^T \mathcal{C}^{-1} = -H, \quad \Gamma H^\dagger \Gamma^{-1} = -H$$

  10-fold non-Hermitian AZ† symmetry class

$$\mathcal{T} H^T \mathcal{T}^{-1} = H, \quad \mathcal{C} H^* \mathcal{C}^{-1} = -H, \quad \Gamma H^\dagger \Gamma^{-1} = -H$$

(**Hermitian conjugate** of the AZ class)

  22-fold non-Hermitian AZ symmetry class with **sublattice symmetry**

$$\mathcal{S} H \mathcal{S}^{-1} = -H$$ (**NOT** equivalent to chiral symmetry)

$$10 + 10 + 22 \underbrace{-\,4}_{\text{unification}} = 38 \text{ symmetry classes}$$

38-fold symmetry class

New symmetry classes lead to new physics:

- **Non-Hermitian random matrix theory and dissipative quantum chaos** [this talk!]

- **Non-Hermitian Anderson transitions**

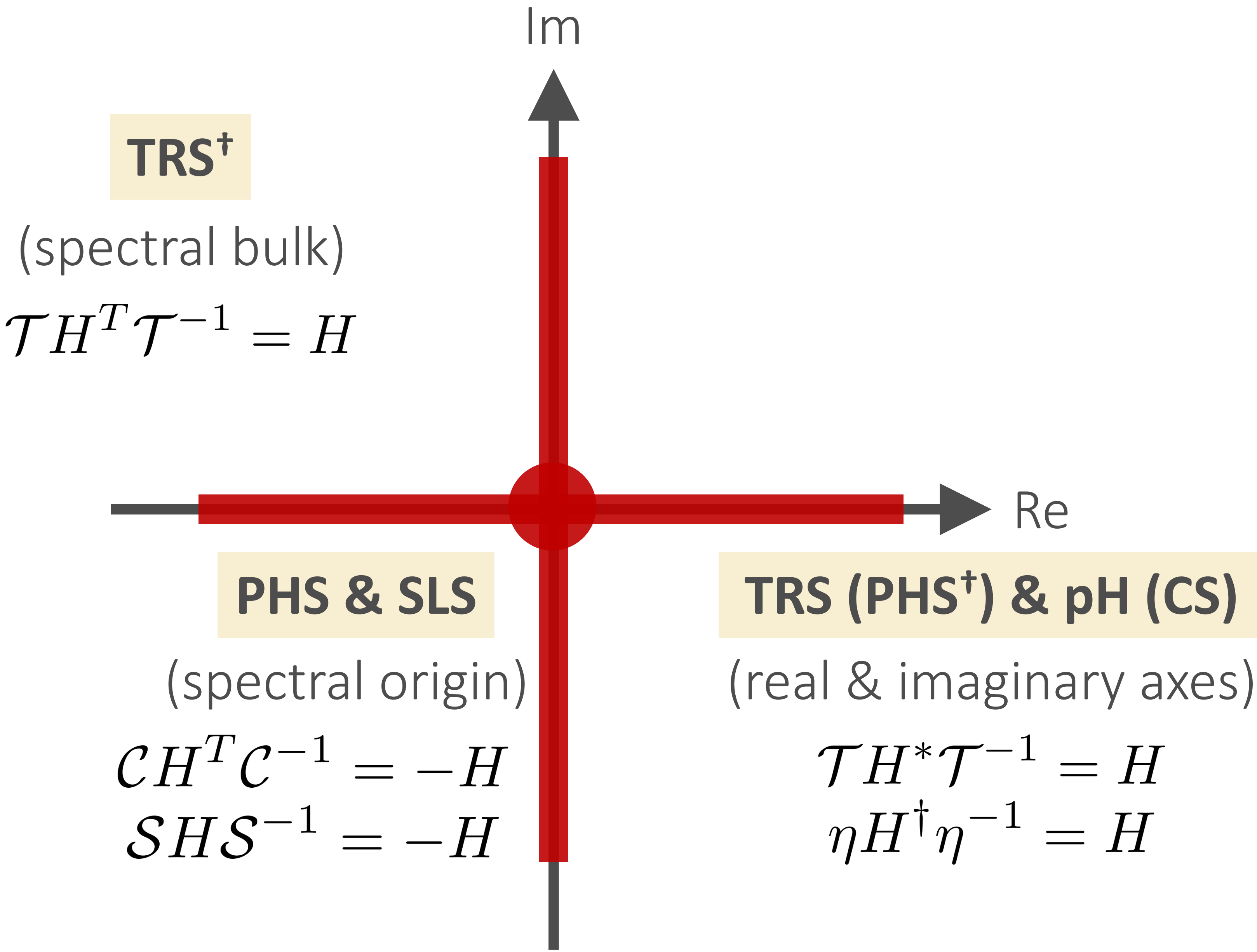
  Kawabata & Ryu, PRL **126**, 166801 (2020)
  Luo, Xiao, **Kawabata**, Ohtsuki & Shindou, PRR **4**, L022035 (2022)

- **Non-Hermitian topological phases**

  Kawabata, Shiozaki, Ueda & Sato, PRX **9**, 041015 (2019)
  Kawabata, Bessho & Sato, PRL **123**, 066405 (2019)
  Okuma, **Kawabata**, Shiozaki & Sato, PRL **124**, 086801 (2020)

10 + 10 + 22 – 4 = 38 symmetry classes

unification


Im
Re
TRS†
(spectral bulk)
$\mathcal{T} H^T \mathcal{T}^{-1} = H$
PHS & SLS
(spectral origin)
$\mathcal{C} H^T \mathcal{C}^{-1} = -H$
$\mathcal{S} H \mathcal{S}^{-1} = -H$
TRS (PHS†) & pH (CS)
(real & imaginary axes)
$\mathcal{T} H^* \mathcal{T}^{-1} = H$
$\eta H^\dagger \eta^{-1} = H$

Level statistics in the spectral bulk of non-Hermitian systems

Hamazaki, **Kawabata**, Kura & Ueda, PRR **2**, 023286 (2020)

Threefold way

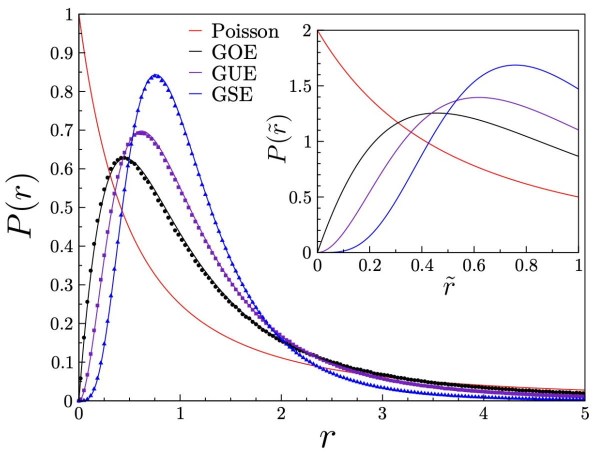
☆ Threefold universality classes of Hermitian random matrices

Gaussian unitary ensemble (GUE; class A): no symmetry

Gaussian orthogonal ensemble (GOE; class AI): TRS with +1

Gaussian symplectic ensemble (GSE; class AII): TRS with -1

Wigner (1959)
Dyson, J. Math. Phys. **3**, 1199 (1962)



Atas *et al.*, PRL **110**, 084101 (2013)

TRS changes the bulk spectral correlations.

$$\beta = 1, 2, 4$$

☆ The threefold way is also fundamental in condensed matter physics

Ginibre ensembles

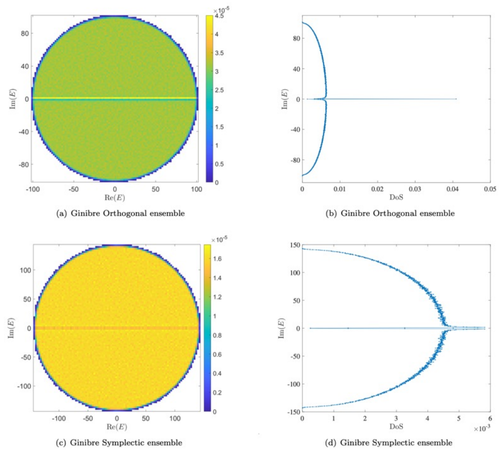
☆ Ginibre ensembles: a non-Hermitian extension of the threefold way

Ginibre unitary ensemble (GinUE; class A): no symmetry

Ginibre, J. Math. Phys. **6**, 440 (1965)

Ginibre orthogonal ensemble (GinOE; class AI): TRS with +1

Ginibre symplectic ensemble (GinSE; class AII): TRS with -1



Luo *et al.*, PRR **4**, L022035 (2022)

TRS changes the spectral correlations around the real axis, but **does NOT change the spectral correlations in the bulk**.

Universal cubic eigenvalue repulsion in contrast to the Hermitian case (β=1,2,4)

Grobe & Haake, PRL **62**, 2893 (1989)

☆ Can we have threefold universal spectral correlations also in non-Hermitian random matrices?

Threefold way in non-Hermitian RMT

☆ Two types of time-reversal symmetry

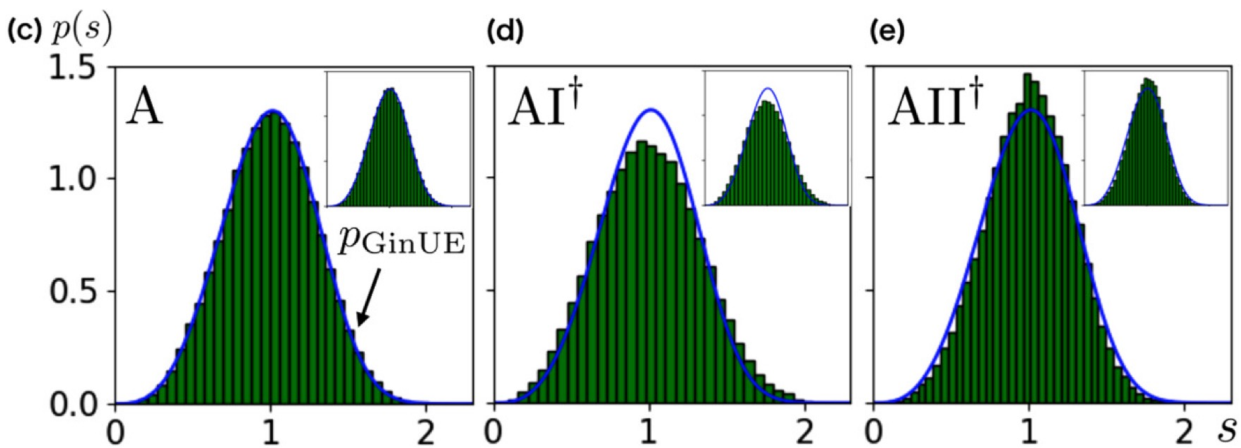
TRS: $\mathcal{T} H^* \mathcal{T}^{-1} = H$ (Ginibre's threefold way)

TRS†: $\mathcal{T} H^T \mathcal{T}^{-1} = H$

☆ We show that TRS† leads to the threefold level statistics in the bulk!

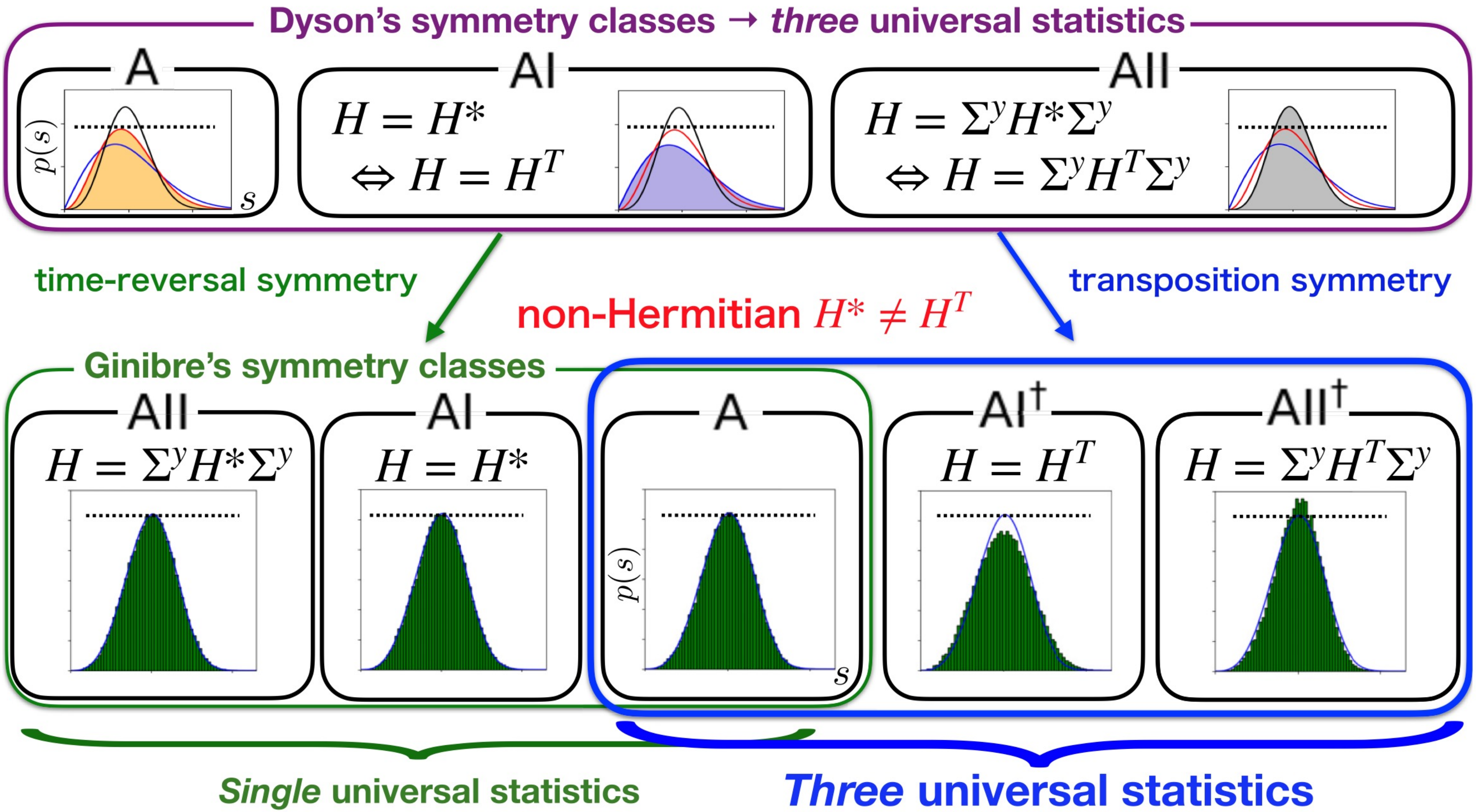
Level spacing for complex eigenvalues: $s_\alpha := \min_\beta |E_\beta - E_\alpha|$

The other symmetries are irrelevant to the bulk level statistics



(numerical results for 2000 × 2000 non-Hermitian random matrices in the Gaussian ensembles)

Threefold way in non-Hermitian RMT



Hamazaki, **Kawabata**, Kura & Ueda, PRR **2**, 023286 (2020)

Wigner surmise

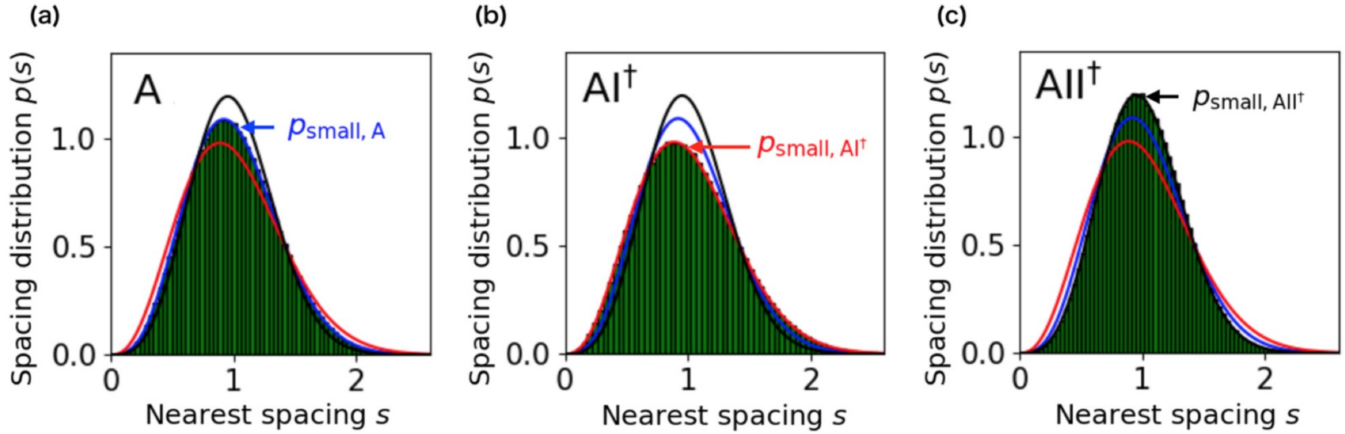
☆ We analytically derive the threefold level-spacing distributions for 2 × 2 or 4 × 4 non-Hermitian random matrices (like Wigner surmise).

$$p_{\mathrm{small}}(s) = \frac{(C_f s)^3}{\mathcal{N}_f} \underbrace{K_{\frac{f-2}{2}}\left[(C_f s)^2\right]}_{\text{modified Bessel function}} = \begin{cases} 2C_3^4 s^3 e^{-C_3^2 s^2} & (\text{class A}; f=3) \\ 2C_2^4 s^3 K_0\left(C_2^2 s^2\right) & (\text{class AI}^\dagger; f=2) \\ 2C_5^4/3 \cdot s^3 \left(1 + C_5^2 s^2\right) e^{-C_5^2 s^2} & (\text{class AII}^\dagger; f=5) \end{cases}$$

The level repulsion is universally cubic: $p_{\mathrm{small}}(s) \propto s^3 \quad (s \ll 1)$

(with a logarithmic correction for class AI†)

Qualitatively similar behavior to the large-*N* results (not quantitative, though)

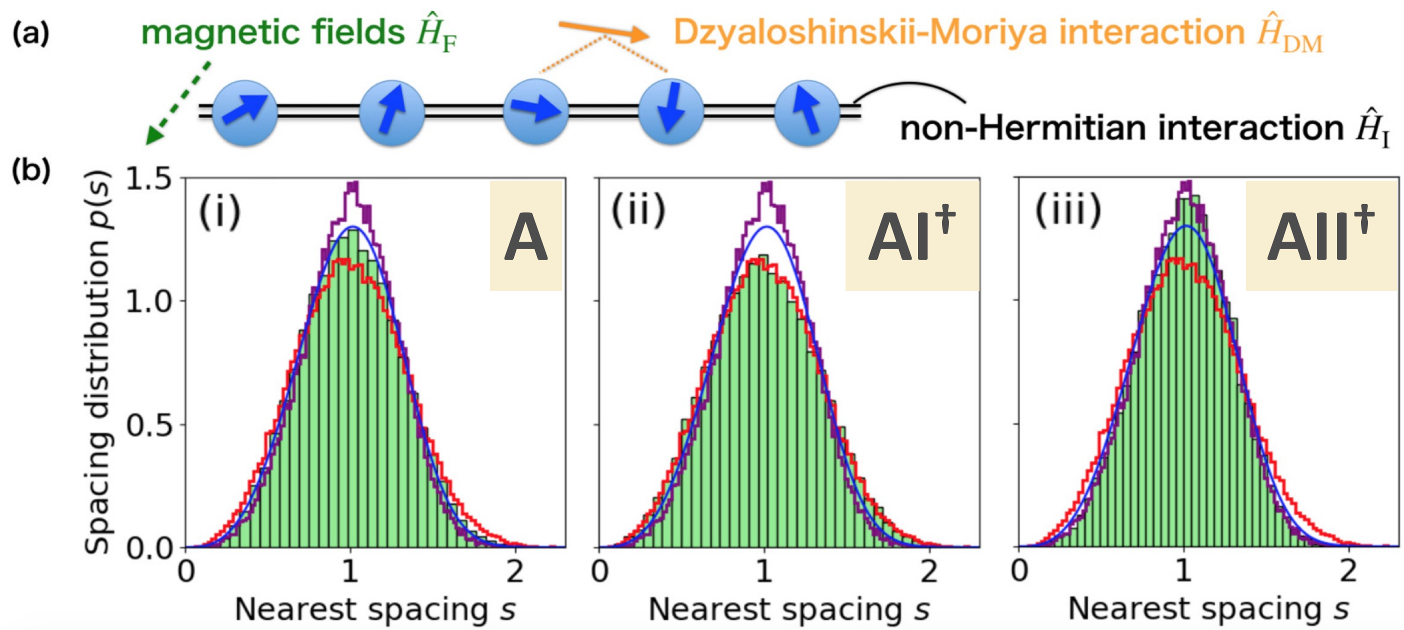


Dissipative quantum chaos

☆ **Non-Hermitian spin chain**

Hamazaki, **Kawabata**, Kura & Ueda, PRR **2**, 023286 (2020)

$$H = -\sum_{j=1}^{L-1} (1 + \mathrm{i}J\epsilon_j)\, \sigma_j^z \sigma_{j+1}^z - h \sum_{j=1}^{L} \left(-2.1\sigma_j^x + \sigma_j^z\right) + \sum_{j=1}^{L-1} \vec{D} \cdot (\vec{\sigma}_j \times \vec{\sigma}_{j+1})$$



Random-matrix behavior appears despite the sparsity of the matrices.
Signature of dissipative quantum chaos!

Level statistics of real eigenvalues in non-Hermitian systems

Xiao, **Kawabata**, Luo, Ohtsuki & Shindou, PRR **4**, 043196 (2022)

Time-reversal symmetry

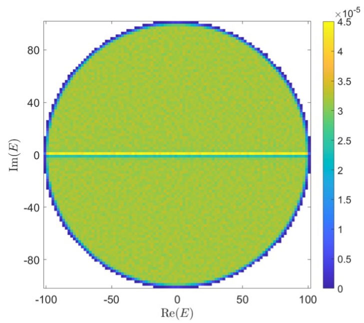
☆ Ginibre ensembles: a non-Hermitian extension of the threefold way

Ginibre, J. Math. Phys. **6**, 440 (1965)

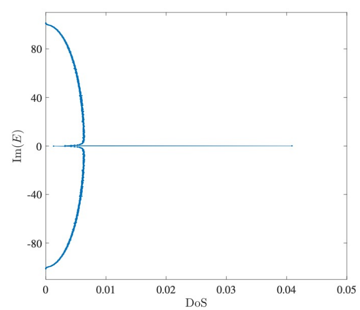
Ginibre unitary ensemble (GinUE; class A): no symmetry

Ginibre orthogonal ensemble (GinOE; class AI): TRS with +1

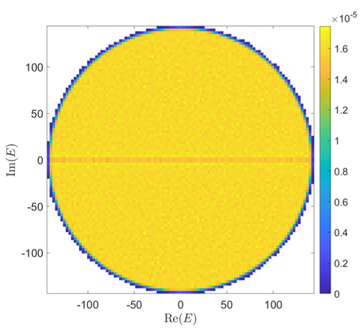
Ginibre symplectic ensemble (GinSE; class AII): TRS with -1



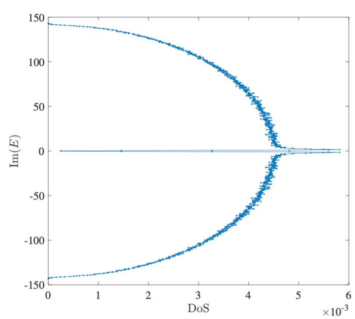
(a) Ginibre Orthogonal ensemble



(b) Ginibre Orthogonal ensemble



(c) Ginibre Symplectic ensemble



(d) Ginibre Symplectic ensemble

Luo *et al.*, PRR **4**, L022035 (2022)

- Density of states decays toward the real axis

  (GinOE) $\rho \propto |\mathrm{Im}\, E|$

  (GinSE) $\rho \propto |\mathrm{Im}\, E|^2$

- Real eigenvalues

  (GinOE) Present $(\bar{N}_{\mathrm{real}} \propto \sqrt{N})$

  (GinSE) Absent

Pseudo-Hermicity (chiral symmetry)

Can other symmetries change the level statistics around the real axis?

→ **Pseudo-Hermiticity (pH) is also relevant!**

Mostafazadeh, J. Math. Phys. **43**, 205 (2002)

$$\underline{\eta} H^{\dagger} \eta^{-1} = H$$

(unitary & Hermitian)

Symmetry unique to non-Hermitian systems

Leading to various nonequilibrium phenomena

cf. PT-symmetry breaking — Bender & Boettcher, PRL **80**, 5243 (1998)

Equivalent to chiral symmetry $\eta \, (\mathrm{i}H)^{\dagger} \, \eta^{-1} = - \, (\mathrm{i}H)$

☆ TRS and pH are only possible symmetries that can change the spectral statistics around the real axis.

Tenfold way: AZ† classification

☆ Tenfold AZ† classification

TRS: $\mathcal{T} H^* \mathcal{T}^{-1} = H$

pH: $\eta H^\dagger \eta^{-1} = H$

Combined symmetry gives TRS†: $\mathcal{T} H^T \mathcal{T}^{-1} = H$

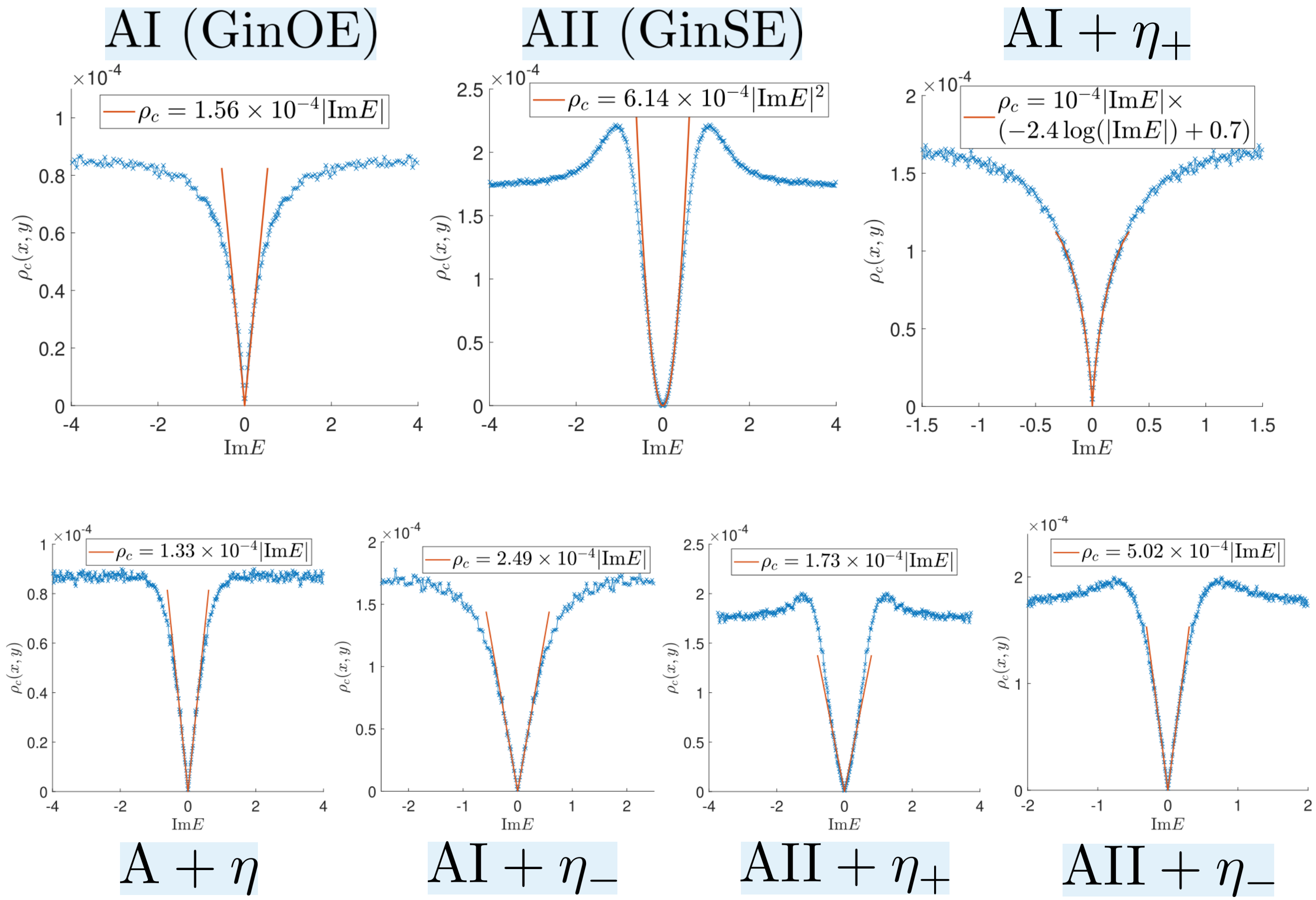
Tenfold universal spectral statistics on and around the real axis

symmetry class	symmetry class (equiv)	TRS (PHS†)	TRS†	pH (CS)	soft gap	$\delta(y)$	$\langle r \rangle$	χ
A	A							
A + η	AIII			$\surd$	$\lvert y\rvert$	$\surd$	0.4194(4)	0.83
AI	D†	+1			$\lvert y\rvert$	$\surd$	0.4858(3)	0.59
AII	C†	-1			$\lvert y\rvert^2$			
AI†	AI†		+1					
AII†	AII†		-1					
AI + η_+	BDI†	+1	+1	$\surd$	$-\lvert y\rvert \log(\lvert y\rvert)$	$\surd$	0.4451(4)	0.73
AI + η_-	DIII†	+1	-1	$\surd$	$\lvert y\rvert$	$\surd$	0.4943(4)	0.58
AII + η_+	CII†	-1	-1	$\surd$	$\lvert y\rvert$	$\surd$	0.3708(7)	1.11
AII + η_-	CI†	-1	+1	$\surd$	$\lvert y\rvert$			

Xiao, **Kawabata**, Luo, Ohtsuki & Shindou, PRR **4**, 043196 (2022)

Density of states around the real axis

☆ The density of states decays differently toward the real axis.



$$\rho(E) \propto \begin{cases} |\mathrm{Im}\, E| & (\mathrm{A}+\eta, \mathrm{AI}, \mathrm{AI}+\eta_-, \mathrm{AII}+\eta_\pm) \\ -|\mathrm{Im}\, E| \log |\mathrm{Im}\, E| & (\mathrm{AI}+\eta_+) \\ |\mathrm{Im}\, E|^2 & (\mathrm{AII}) \end{cases}$$

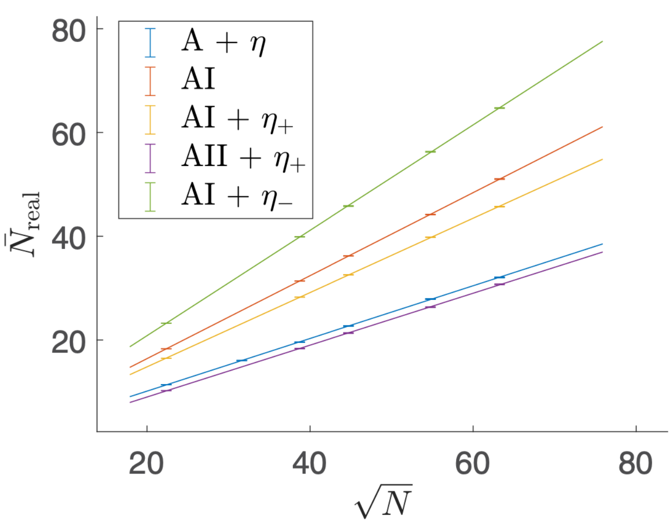
Level statistics of real eigenvalues

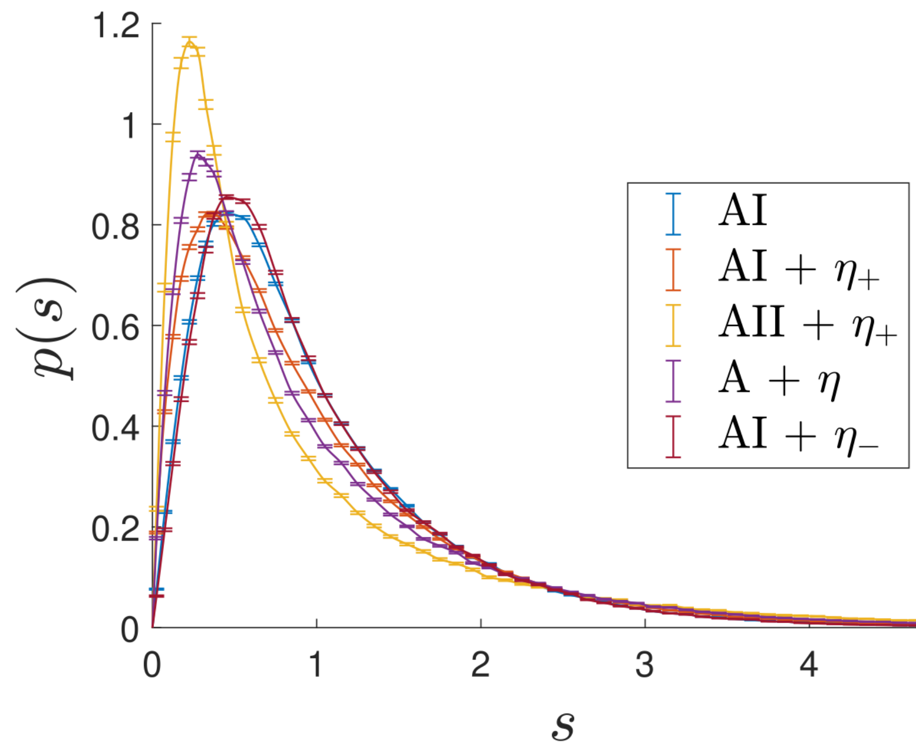
Five symmetry classes accompany a subextensive number of real eigenvalues

$(\bar{N}_{\mathrm{real}} \propto \sqrt{N})$

Five universal level statistics of real eigenvalues!

Not identical to any level statistics of Hermitian random matrices





Dissipative free fermions

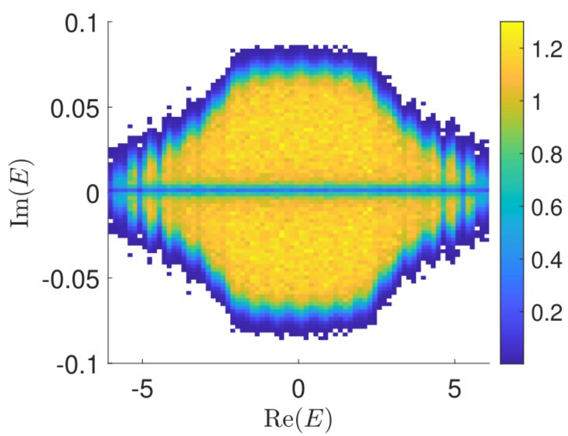
☆ Level statistics of real eigenvalues capture dissipative quantum chaos!

e.g., non-Hermitian 3D Anderson model

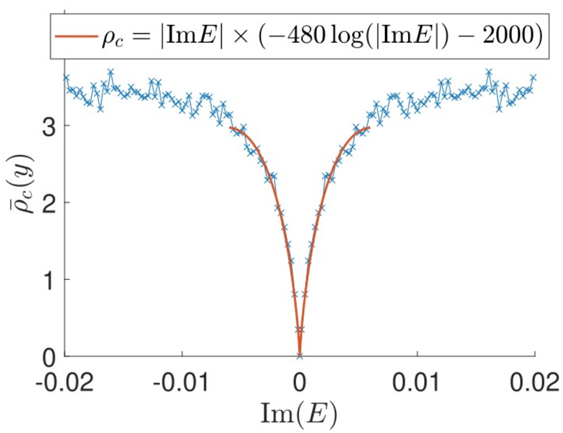
$$H = \sum_i \left(c_i^\dagger \left(\varepsilon_i \sigma_0 + \varepsilon_i' \sigma_z \right) c_i + \mathrm{i} \omega_i c_i^\dagger \sigma_y c_i \right) + t \sum_{\langle i,j \rangle} c_i^\dagger \sigma_0 c_j$$

$$\text{class AI} + \eta_+ : (\text{TRS})\; H^* = H, \quad (\text{pH})\; \sigma_z H^\dagger \sigma_z = H$$

Random-matrix behavior appears even in this physical model!

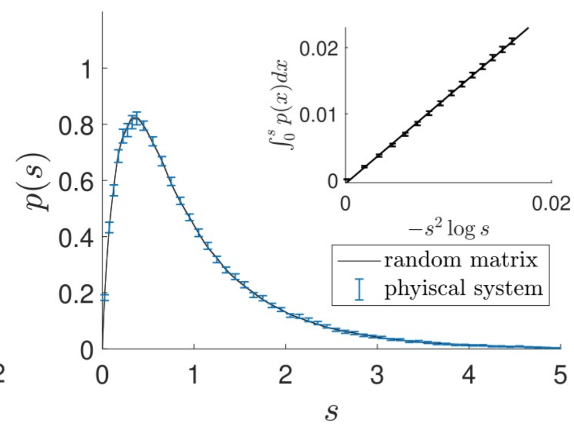


(a) density $\rho_c(x, y)$ of complex eigenvalues ($W_1 = 3, W_2 = 1$)



(b) $\bar{\rho}_c(y)$ ($W_1 = 3, W_2 = 1$)

DOS around the real axis



(c) $p(s)$ ($W_1 = 3, W_2 = 1$, metal phase)

level-spacing distributions of real eigenvalues

Universal hard-edge statistics of non-Hermitian systems

Xiao, Shindou & **Kawabata**, PRR **6**, 023303 (2024)

Particle-hole and sublattice symmetries

[Level statistics in the spectral bulk]

TRS†: $\mathcal{T} H^{T} \mathcal{T}^{-1} = H$

[Level statistics around the real and imaginary axes]

TRS: $\mathcal{T} H^{*} \mathcal{T}^{-1} = H$

pH: $\eta H^{\dagger} \eta^{-1} = H$

The two remaining symmetries:

particle-hole symmetry (PHS): $\mathcal{C} H^{T} \mathcal{C}^{-1} = -H$

(e.g., dissipative superconductors)

sublattice symmetry (SLS): $\mathcal{S} H \mathcal{S}^{-1} = -H$

(e.g., QCD with nonzero chemical potential)

Akemann & Wettig, PRL **92**, 102002 (2004)
Osborn, PRL **93**, 222001 (2004)

These symmetries are respected only for zero eigenvalue.

How do they change the level statistics around the spectral origin?

Tenfold way: AZ_0 classification

☆ Tenfold AZ_0 classification

cf. Splittorff & Verbaarschot, Nucl. Phys. B **683**, 467 (2004)
cf. Akemann *et al.*, PRE **80**, 065201(R) (2009)
cf. García-García *et al.*, PRX **12**, 0121040 (2022)

PHS: $\mathcal{C} H^T \mathcal{C}^{-1} = -H$

SLS: $\mathcal{S} H \mathcal{S}^{-1} = -H$

Combined symmetry gives TRS†: $\mathcal{T} H^T \mathcal{T}^{-1} = H$

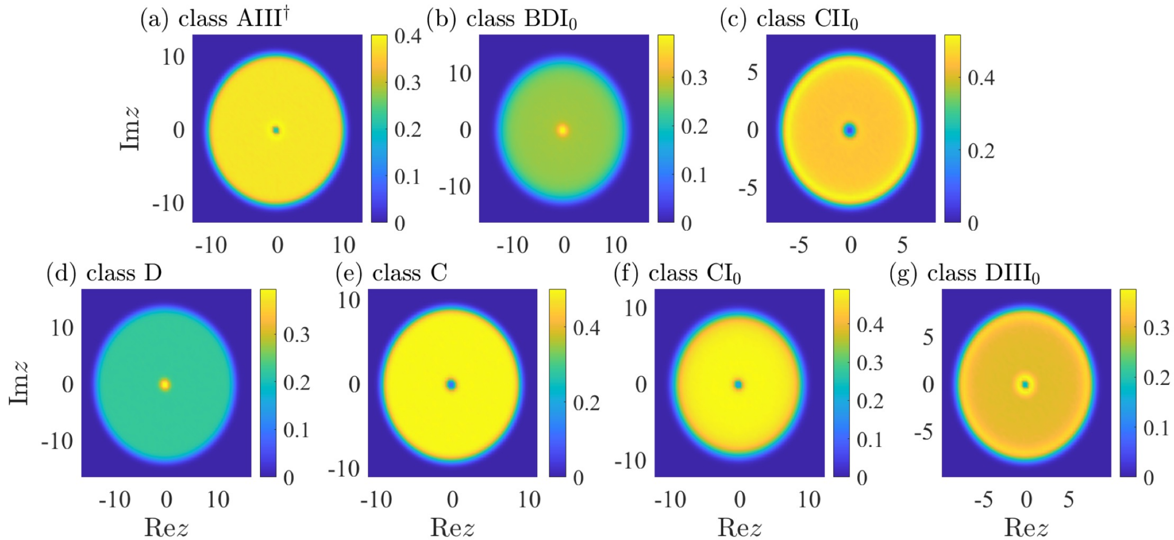
Tenfold universal spectral statistics on and around the spectral origin

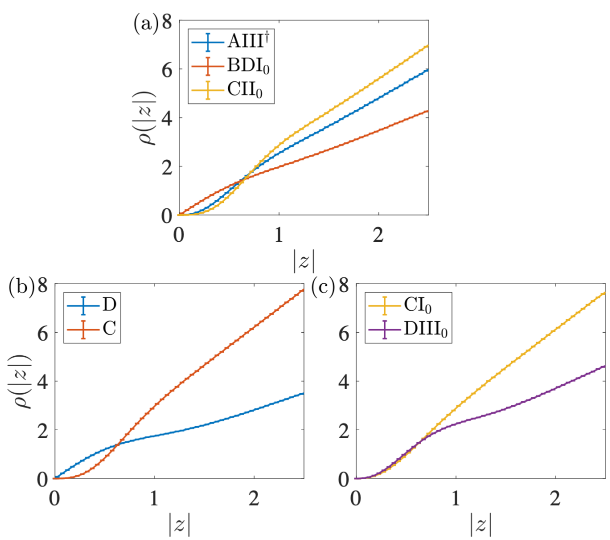
Class	Equivalent class	TRS†	PHS	SLS	$\rho(\lvert z\rvert)$ and $p_r(\lvert z\rvert)$	$\langle r\rangle$	$\langle\cos\theta\rangle$
A	-	0	0	0	-	-	-
AI†	-	+1	0	0	-	-	-
AII†	-	−1	0	0	-	-	-
AIII†	A + $\mathcal{S}$	0	0	1	$-\lvert z\rvert^3 \ln\lvert z\rvert$	0.6357(5)	0.5391(7)
BDI_0	D + $\mathcal{S}_+$, AI† + $\mathcal{S}_+$	+1	+1	1	$\lvert z\rvert$	0.5778(6)	0.5681(7)
CII_0	C + $\mathcal{S}_+$, AII† + $\mathcal{S}_+$	−1	−1	1	$\lvert z\rvert^3$	0.6623(5)	0.5147(7)
D	-	0	+1	0	$\lvert z\rvert$	0.5411(6)	0.5524(7)
C	-	0	−1	0	$\lvert z\rvert^3$	0.6746(5)	0.5343(7)
CI_0	C + $\mathcal{S}_-$, AI† + $\mathcal{S}_-$	+1	−1	1	$-\lvert z\rvert^3 \ln\lvert z\rvert$	0.6708(5)	0.5589(7)
DIII_0	D + $\mathcal{S}_-$, AII† + $\mathcal{S}_-$	−1	+1	1	$-\lvert z\rvert^3 \ln\lvert z\rvert$	0.5950(6)	0.5252(7)

Xiao, Shindou & Kawabata, PRR **6**, 023303 (2024)

Density of states

☆ **The density of states decays differently toward the spectral origin.**





Density of states for $|z| \ll 1$

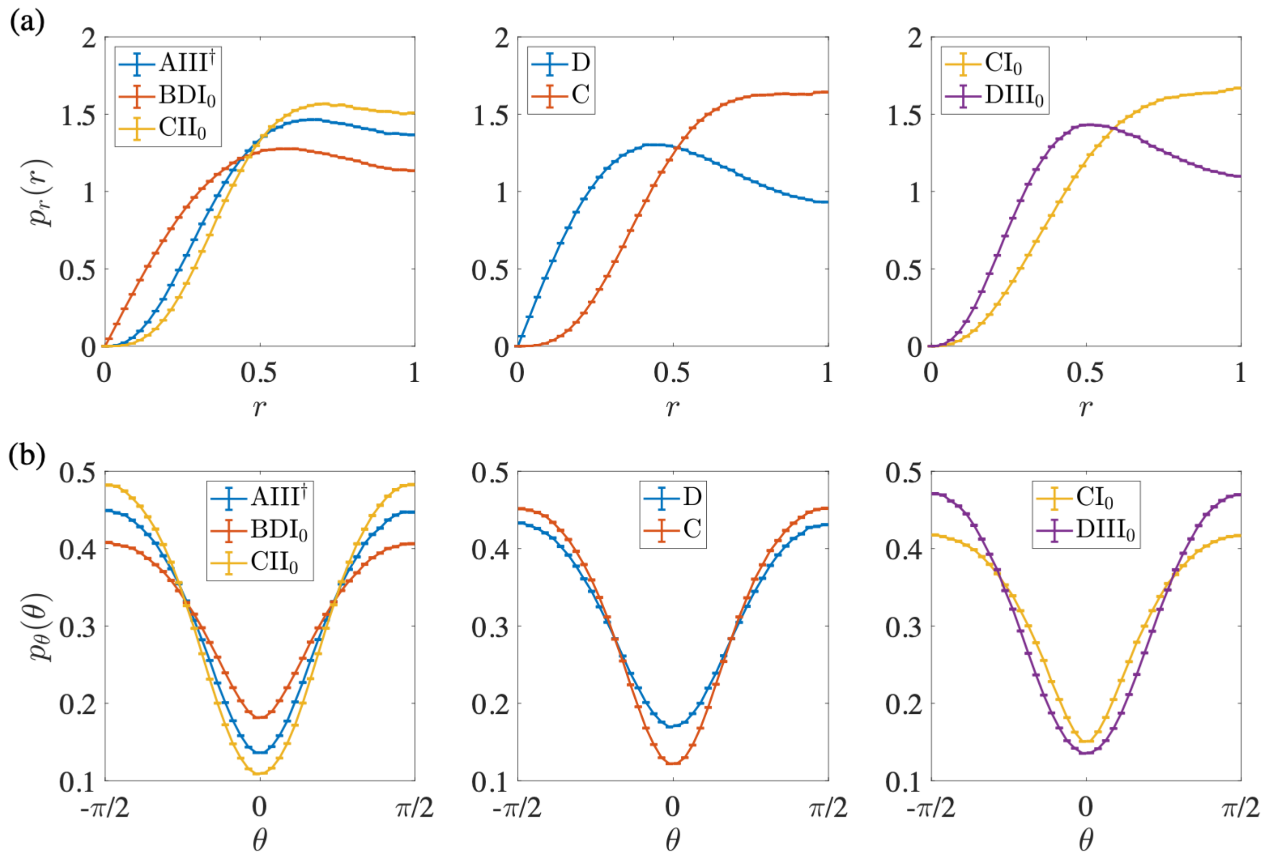
$$
\rho\left(|z|\right) \propto \begin{cases} |z| & \text{(classes D and BDI}_0\text{)}\,; \\ -|z|^3 \log |z| & \text{(classes AIII}^{\dagger}\text{, CI}_0\text{, and DIII}_0\text{)}\,; \\ |z|^3 & \text{(classes CII}_0\text{ and C)}\,, \end{cases}
$$

Complex level ratio

A new measure to quantify the level repulsion around the origin:

$$re^{\mathrm{i}\theta} := z_1/z_2 \quad (|z_1| \leq |z_2| \leq \cdots)$$

Both *r* and θ detect the level repulsion and depend on symmetry!



cf. Poisson statistics

$$p(r) = 2r$$

$$p(\theta) = 1/\pi$$

General classification

More generic symmetry classes are characterized by all the symmetries.

Class (H)	Class ($\mathrm{i}H$)	TRS	PHS	TRS†	PHS†	CS	SLS	pH	$[\Gamma, \mathcal{S}]_\pm = 0$	$\langle \lvert z_{\min} \rvert^2 \rangle$	$\Pr(z_{\min} \in \mathbb{R})$	$\Pr(\mathrm{i}z_{\min} \in \mathbb{R})$
$N=0$												
AIII + $\mathcal{S}_+$	**AIII + $\mathcal{S}_+$**	0	0	0	0	1	1	1	+	1.1680(9)	**0.2279(5)**	**0.2280(5)**
AIII + $\mathcal{S}_-$	**AIII + $\mathcal{S}_-$**	0	0	0	0	1	1	1	−	1.2707(12)	**0.3336(5)**	**0.3334(5)**
$N=2$												
BDI	D + η_+	+1	+1	0	0	1	0	0		1.4488(6)	0.5373(2)	0.1960(2)
CI	C + η_-	+1	−1	0	0	1	0	0		1.2223(4)	0.4247(2)	0.2732(2)
DIII	D + η_-	−1	+1	0	0	1	0	0		1.3390(13)	0	0.5419(5)
CII	C + η_+	−1	−1	0	0	1	0	0		1.0926(7)	0	0.2531(5)
AI + $\mathcal{S}_+$	**AI + $\mathcal{S}_+$**	+1	0	0	+1	0	1	0		1.3094(14)	**0.3885(5)**	**0.3894(5)**
AI + $\mathcal{S}_-$	AII + $\mathcal{S}_-$	+1	0	0	−1	0	1	0		1.2055(10)	0.5788(5)	0
AII + $\mathcal{S}_+$	**AII + $\mathcal{S}_+$**	−1	0	0	−1	0	1	0		1.0623(2)	**0**	**0**
$N=4$												
BDI + $\mathcal{S}_{++}$	**BDI + $\mathcal{S}_{++}$**	+1	+1	+1	+1	1	1	1	+	1.4387(17)	**0.3502(5)**	**0.3504(5)**
BDI + $\mathcal{S}_{--}$	DIII + $\mathcal{S}_{--}$	+1	+1	−1	−1	1	1	1	+	1.2097(10)	0.5293(5)	0.0823(3)
DIII + $\mathcal{S}_{++}$	**DIII + $\mathcal{S}_{++}$**	−1	+1	+1	−1	1	1	1	+	1.1071(7)	**0**	**0**
CI + $\mathcal{S}_{++}$	**CI + $\mathcal{S}_{++}$**	+1	−1	−1	+1	1	1	1	+	1.2134(10)	**0.3655(5)**	**0.3661(5)**
CI + $\mathcal{S}_{--}$	CII + $\mathcal{S}_{--}$	+1	−1	+1	−1	1	1	1	+	1.1692(9)	0.4193(5)	0
CII + $\mathcal{S}_{++}$	**CII + $\mathcal{S}_{++}$**	−1	−1	−1	−1	1	1	1	+	1.0648(6)	**0.1022(4)**	**0.1025(4)**
BDI + $\mathcal{S}_{+-}$	**BDI + $\mathcal{S}_{+-}$**	+1	+1	−1	+1	1	1	1	−	1.3162(14)	**0.3878(5)**	**0.3881(5)**
BDI + $\mathcal{S}_{-+}$	DIII + $\mathcal{S}_{-+}$	+1	+1	+1	−1	1	1	1	−	1.4015(16)	0.6190(5)	0
DIII + $\mathcal{S}_{+-}$	**DIII + $\mathcal{S}_{+-}$**	−1	+1	−1	−1	1	1	1	−	1.2328(11)	**0.2797(5)**	**0.2802(5)**
CI + $\mathcal{S}_{+-}$	**CI + $\mathcal{S}_{+-}$**	+1	−1	+1	+1	1	1	1	−	1.2929(13)	**0.3684(5)**	**0.3692(5)**
CI + $\mathcal{S}_{-+}$	CII + $\mathcal{S}_{-+}$	+1	−1	−1	−1	1	1	1	−	1.1840(9)	0.4316(5)	0.2140(5)
CII + $\mathcal{S}_{+-}$	**CII + $\mathcal{S}_{+-}$**	−1	−1	+1	−1	1	1	1	−	1.0795(6)	**0**	**0**

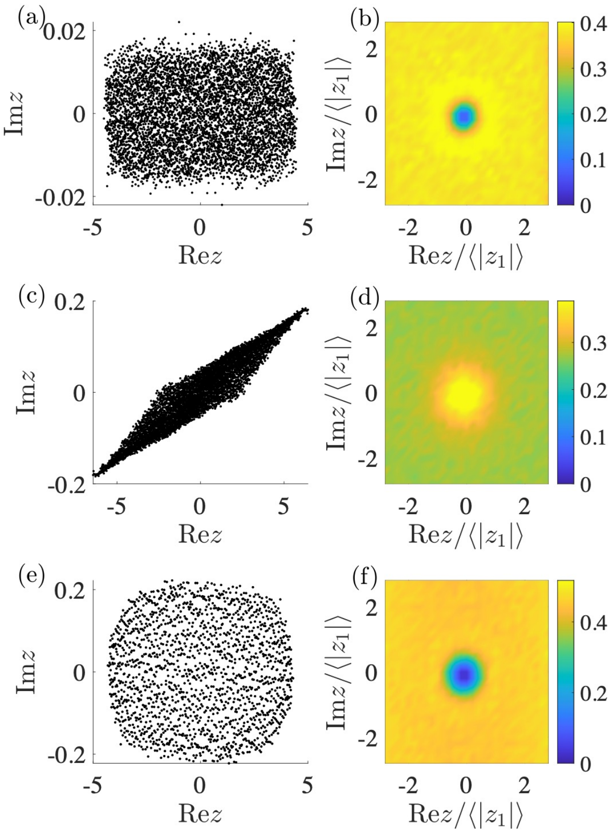
Xiao, Shindou & **Kawabata**, PRR **6**, 023303 (2024)

Quadratic Lindbladians

☆ Level statistics around the origin capture dissipative quantum chaos!

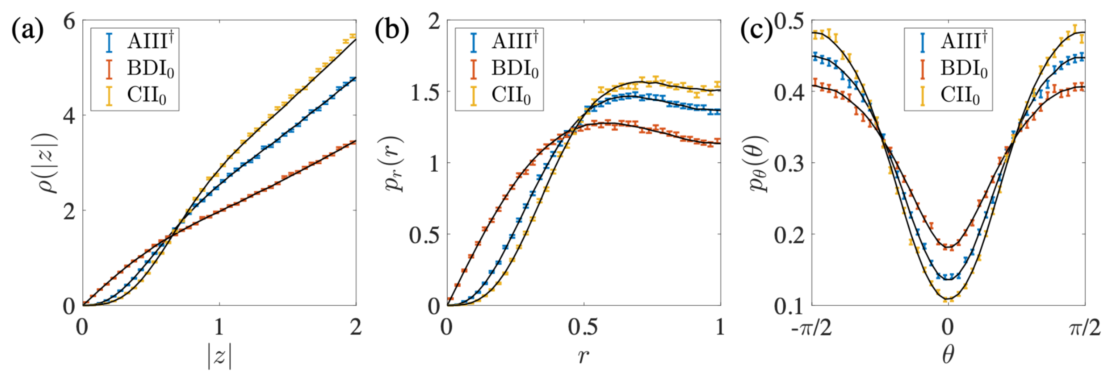
e.g., Lindblad master equation for free fermions

$$\frac{d\rho}{dt} = -\mathrm{i}\left[H, \rho\right] + \sum_n \left(L_n \rho L_n^\dagger - \frac{1}{2}\left\{ L_n^\dagger L_n, \rho \right\} \right)$$

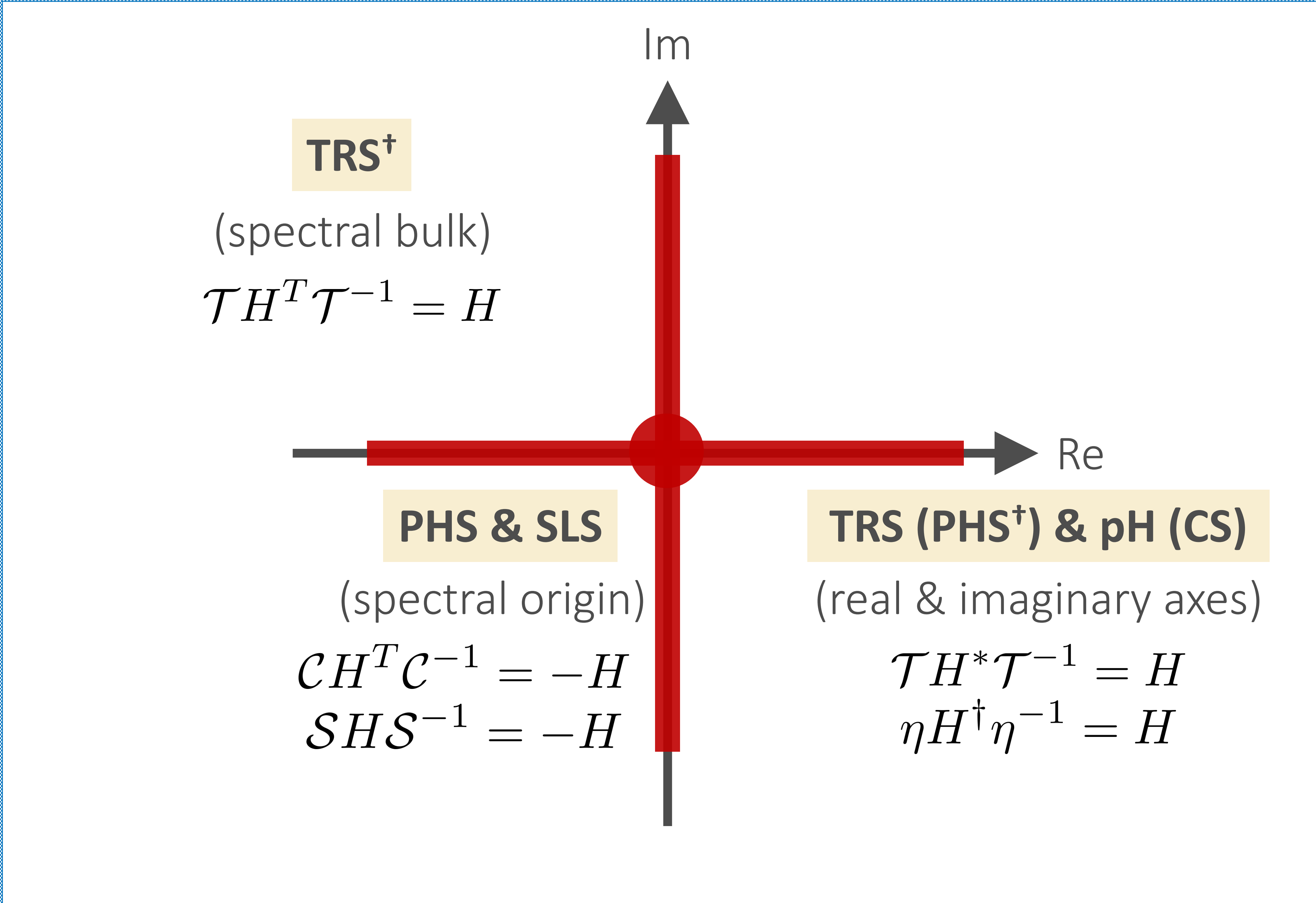


H : disordered free fermions in chiral symmetry classes

L_n: linear dissipators in chiral symmetry classes



Symmetry-enriched random-matrix behavior appears even in the quantum master equation!


Im
Re
TRS†
(spectral bulk)
$\mathcal{T}H^T\mathcal{T}^{-1} = H$
PHS & SLS
(spectral origin)
$\mathcal{C}H^T\mathcal{C}^{-1} = -H$
$\mathcal{S}H\mathcal{S}^{-1} = -H$
TRS (PHS†) & pH (CS)
(real & imaginary axes)
$\mathcal{T}H^*\mathcal{T}^{-1} = H$
$\eta H^\dagger \eta^{-1} = H$

Singular-value statistics of non-Hermitian systems

<u>Kawabata</u>, Xiao, Ohtsuki & Shindou, PRX Quantum **4**, 040312 (2023)

Singular values

So far, we have focused on complex eigenvalues.

→ **Singular values have different information!**

Eigenvalues of $\sqrt{H^\dagger H}$ or $\sqrt{H H^\dagger}$

Always nonnegative even for non-Hermitian matrices

Physical relevance to open systems

e.g., amplification in photonics — Porras & Fernández-Lorenzo, PRL **122**, 143901 (2019)

e.g., random nonunitary quantum dynamics — Bulchandani *et al.*, J. Stat. Phys. **191**, 55 (2024)

How can we classify the statistics of singular values?

Are they relevant to chaotic behavior in open quantum systems?

Hermitization

Girko, Theory Probab. Appl. **29**, 694 (1985)
Feinberg & Zee, Nucl. Phys. B **504**, 579 (1997)

H **: non-Hermitian matrix**

→ **Hermitized matrix:** $\tilde{H} := \begin{pmatrix} 0 & H \\ H^\dagger & 0 \end{pmatrix}$

☆ Singular values of non-Hermitian matrices H coincide with nonnegative eigenvalues of Hermitized matrices $\tilde{H}$

Hermitization leads to additional chiral symmetry: $\sigma_z \tilde{H} \sigma_z = -\tilde{H}$

e.g., real non-Hermitian random matrix
(Ginibre orthogonal ensemble; class AI)

→ Hermitian random matrix with time-reversal and chiral symmetries
(class BDI)

Classification

☆ Using Hermitization, we classify the singular-value statistics of non-Hermitian random matrices in all the 38 symmetry classes!

β = 1, 2, 4: level statistics in the spectral bulk (Wigner-Dyson)

α = 0, 1, 2, 3: level statistics around the spectral origin (chiral & BdG)

Class	CS	SLS	Classifying space	Hermitization	β	α
A	0	0	$\mathcal{C}_1$	AIII	2	1
AIII = A + η	1	0	$\mathcal{C}_0$	A	N/A (A)	N/A (A)
AIII $+\mathcal{S}_+$	1	1	$\mathcal{C}_1$	AIII	2	1
A $+\mathcal{S}$ = AIII†	0	1	$\mathcal{C}_1 \times \mathcal{C}_1$	AIII $\times$ AIII	N/A (A)	1
AIII $+\mathcal{S}_-$	1	1	$\mathcal{C}_0 \times \mathcal{C}_0$	A $\times$ A	N/A (A)	N/A (A)

Class	TRS	PHS	CS	Classifying space	Hermitization	β	α
AI = D†	+1	0	0	$\mathcal{R}_1$	BDI	1	0
BDI	+1	+1	1	$\mathcal{R}_2$	D	2	0
D	0	+1	0	$\mathcal{R}_3$	DIII	4	1
DIII	-1	+1	1	$\mathcal{R}_4$	AII	N/A (AII)	N/A (AII)
AII = C†	-1	0	0	$\mathcal{R}_5$	CII	4	3
CII	-1	-1	1	$\mathcal{R}_6$	C	2	2
C	0	-1	0	$\mathcal{R}_7$	CI	1	1
CI	+1	-1	1	$\mathcal{R}_0$	AI	N/A (AI)	N/A (AI)

Class	TRS†	PHS†	CS	Classifying space	Hermitization	β	α
AI†	+1	0	0	$\mathcal{R}_7$	CI	1	1
BDI†	+1	+1	1	$\mathcal{R}_0$	AI	N/A (AI)	N/A (AI)
D† = AI	0	+1	0	$\mathcal{R}_1$	BDI	1	0
DIII†	-1	+1	1	$\mathcal{R}_2$	D	2	0
AII†	-1	0	0	$\mathcal{R}_3$	DIII	4	1
CII†	-1	-1	1	$\mathcal{R}_4$	AII	N/A (AII)	N/A (AII)
C† = AII	0	-1	0	$\mathcal{R}_5$	CII	4	3
CI†	+1	-1	1	$\mathcal{R}_6$	C	2	2

Class	Classifying space	Hermitization	β	α
BDI + $\mathcal{S}_{++}$	$\mathcal{R}_1$	BDI	1	0
DIII + $\mathcal{S}_{--}$ = BDI + $\mathcal{S}_{--}$	$\mathcal{R}_3$	DIII	4	1
CII + $\mathcal{S}_{++}$	$\mathcal{R}_5$	CII	4	3
CI + $\mathcal{S}_{--}$ = CII + $\mathcal{S}_{--}$	$\mathcal{R}_7$	CI	1	1
AI + $\mathcal{S}_{-}$ = AII + $\mathcal{S}_{-}$	$\mathcal{C}_1$	AIII	2	1
BDI + $\mathcal{S}_{-+}$ = DIII + $\mathcal{S}_{-+}$	$\mathcal{C}_0$	A	N/A (A)	N/A (A)
D + $\mathcal{S}_{+}$	$\mathcal{C}_1$	AIII	2	1
DIII + $\mathcal{S}_{-+}$ = BDI + $\mathcal{S}_{-+}$	$\mathcal{C}_0$	A	N/A (A)	N/A (A)
AII + $\mathcal{S}_{-}$ = AI + $\mathcal{S}_{-}$	$\mathcal{C}_1$	AIII	2	1
CII + $\mathcal{S}_{-+}$ = CI + $\mathcal{S}_{-+}$	$\mathcal{C}_0$	A	N/A (A)	N/A (A)
C + $\mathcal{S}_{+}$	$\mathcal{C}_1$	AIII	2	1
CI + $\mathcal{S}_{-+}$ = CII + $\mathcal{S}_{-+}$	$\mathcal{C}_0$	A	N/A (A)	N/A (A)
BDI + $\mathcal{S}_{--}$ = DIII + $\mathcal{S}_{--}$	$\mathcal{R}_3$	DIII	4	1
DIII + $\mathcal{S}_{++}$	$\mathcal{R}_5$	CII	4	3
CII + $\mathcal{S}_{--}$ = CI + $\mathcal{S}_{--}$	$\mathcal{R}_7$	CI	1	1
CI + $\mathcal{S}_{++}$	$\mathcal{R}_1$	BDI	1	0
AI + $\mathcal{S}_{+}$	$\mathcal{R}_1 \times \mathcal{R}_1$	BDI $\times$ BDI	N/A (AI)	0
BDI + $\mathcal{S}_{+-}$	$\mathcal{R}_2 \times \mathcal{R}_2$	D $\times$ D	N/A (A)	0
D + $\mathcal{S}_{-}$	$\mathcal{R}_3 \times \mathcal{R}_3$	DIII $\times$ DIII	N/A (AII)	1
DIII + $\mathcal{S}_{+-}$	$\mathcal{R}_4 \times \mathcal{R}_4$	AII $\times$ AII	N/A (AII)	N/A (AII)
AII + $\mathcal{S}_{+}$	$\mathcal{R}_5 \times \mathcal{R}_5$	CII $\times$ CII	N/A (AII)	3
CII + $\mathcal{S}_{+-}$	$\mathcal{R}_6 \times \mathcal{R}_6$	C $\times$ C	N/A (A)	2
C + $\mathcal{S}_{-}$	$\mathcal{R}_7 \times \mathcal{R}_7$	CI $\times$ CI	N/A (AI)	1
CI + $\mathcal{S}_{+-}$	$\mathcal{R}_0 \times \mathcal{R}_0$	AI $\times$ AI	N/A (AI)	N/A (AI)

Kawabata, Xiao, Ohtsuki & Shindou, PRX Quantum **4**, 040312 (2023)

Many-body Lindbladians

☆ Singular-value statistics also capture dissipative quantum chaos!

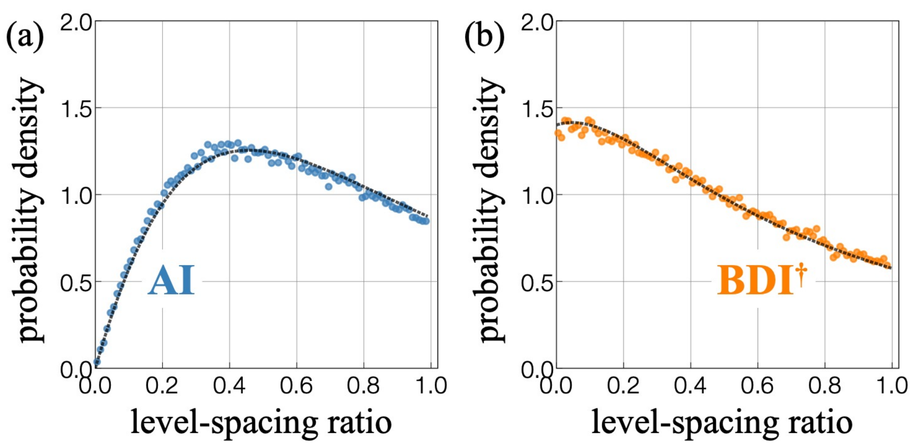
e.g., Lindblad master equation for interacting spins

$$\frac{d\rho}{dt} = -\mathrm{i}\,[H,\rho] + \sum_n \left(L_n \rho L_n^\dagger - \frac{1}{2}\left\{ L_n^\dagger L_n, \rho \right\} \right)$$

Ising model: $H = -J \sum_{n=1}^{L-1} (1+\varepsilon_n)\,\sigma_n^z \sigma_{n+1}^z - \sum_{n=1}^{L} (h_x \sigma_n^x + h_z \sigma_n^z)$

damping: $L_n = \sqrt{\gamma}\sigma_n^-$ **(class AI)**

dephasing: $L_n = \sqrt{\gamma}\sigma_n^z$ **(class BDI)**



Level-spacing-ratio distributions of singular values follow the random-matrix behavior in the chaotic regime!

Summary

- We develop the 38-fold symmetry classification of non-Hermitian random matrices and classify their universal spectral statistics.
- Symmetry can manifest itself in the spectral bulk, real and imaginary axes, and spectral origin in different manners.
- The level statistics characterize dissipative quantum chaos.