Exact Work Distribution and Jarzynski's Equality of a Relativistic Particle in an Expanding Piston

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Xianghang Zhang, Tingzhang Shi, H. T. Quan, arXiv:2403.15986

Outline

- Background and motivation
- Setup of the relativistic piston model
- Trajectory of a particle and verification of the Jarzynski's equality
- Relativistic work distribution and its non-relativistic limit
- Summary

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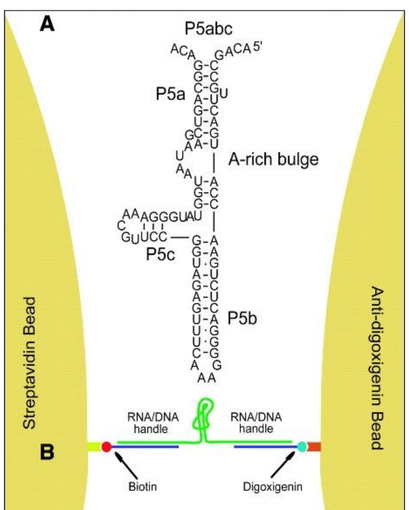
Background and motivation

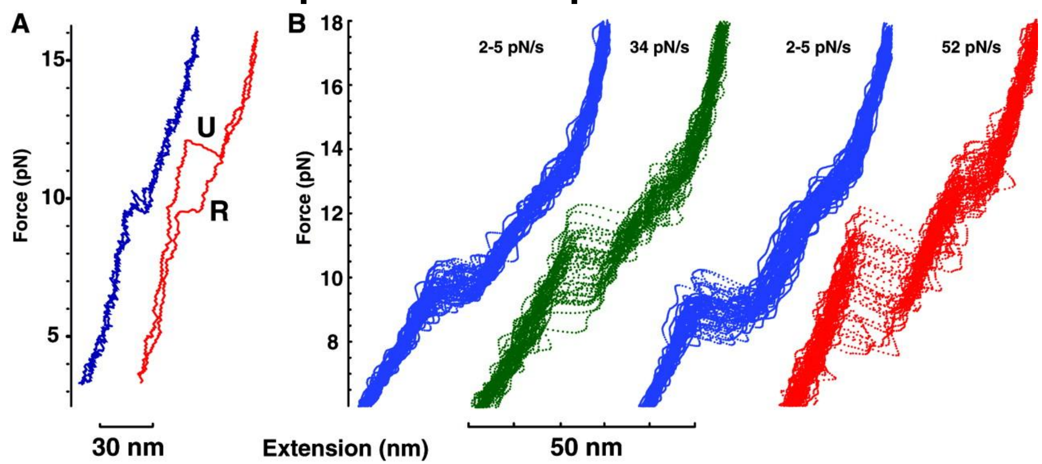
The Jarzynski's equality

- Second law of thermodynamics: $\langle W \rangle \geq \Delta F$
- Jarzynski's equality：

$$\langle e^{-\beta W} \rangle = e^{-\beta \Delta F}$$

- It is possible to obtain equilibrium thermodynamic parameters from non-equilibrium processes.



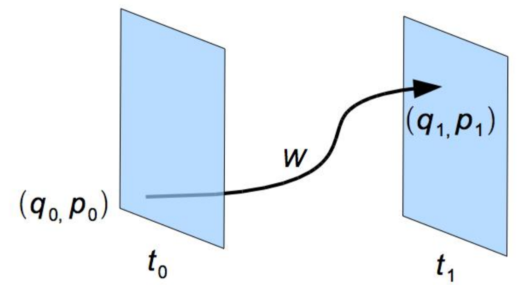


Liphardt, et al., Science 296, 1832 (2002)

Background and motivation

The Jarzynski's equality

- Trajectory work:



$$W(q_0, p_0) = H(q_1, p_1; \lambda_1) - H(q_0, p_0; \lambda_0)$$

- The proof of Jarzynski's equality:

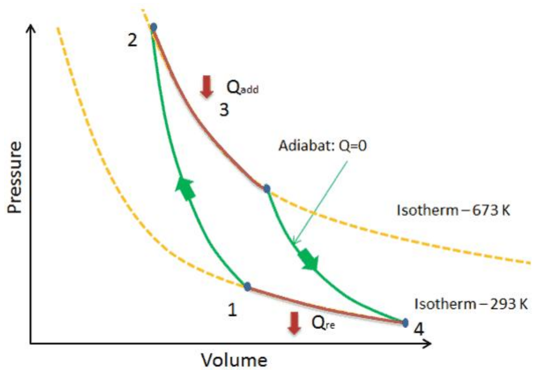
$$\begin{aligned}
&\left\langle e^{\beta W} \right\rangle \\
&= \int \mathrm{d}p_{i,0}\mathrm{d}q_{i,0}\, \rho(p_{i,0}, q_{i,0}) e^{\beta W(p_{i,0}, q_{i,0}, \tau)} \\
&= \frac{1}{Z_0} \int \mathrm{d}p_{i,0}\mathrm{d}q_{i,0}\, e^{-\beta H_{\lambda(\tau)}(p_{i,\tau}, q_{i,\tau})} \\
&= \frac{1}{Z_0} \int \mathrm{d}p_{i,\tau}\mathrm{d}q_{i,\tau}\, e^{-\beta H_{\lambda(\tau)}(p_{i,\tau}, q_{i,\tau}))} \\
&= \frac{Z_\tau}{Z_0},
\end{aligned}$$

C. Jarzynski, Phys. Rev. Lett. 78, 2690 (1997).

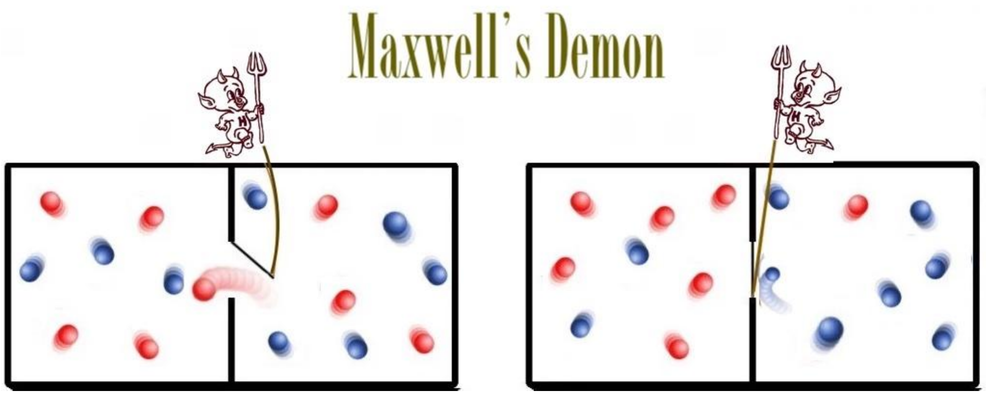
Background and motivation

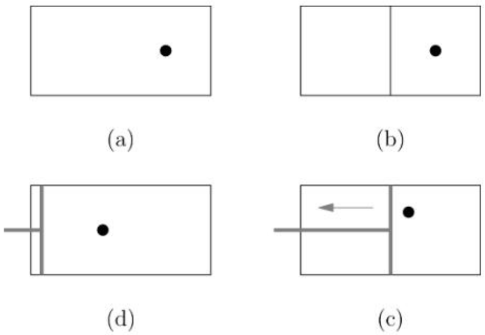
Piston model

- Piston model: Paradigmatic



Carnot cycle





Landauer Principle

Background and motivation

Piston model

- Piston model: Exact solvable
- Previous work:

Classical Newtonian piston [R. C. Lua and A. Y. Grosberg, 2005]

Quantum non-relativistic piston [H. T. Quan and C. Jarzynski, 2012]

Classical relativistic piston (single kick limit) [R. Nolte and A. Engel, 2009]

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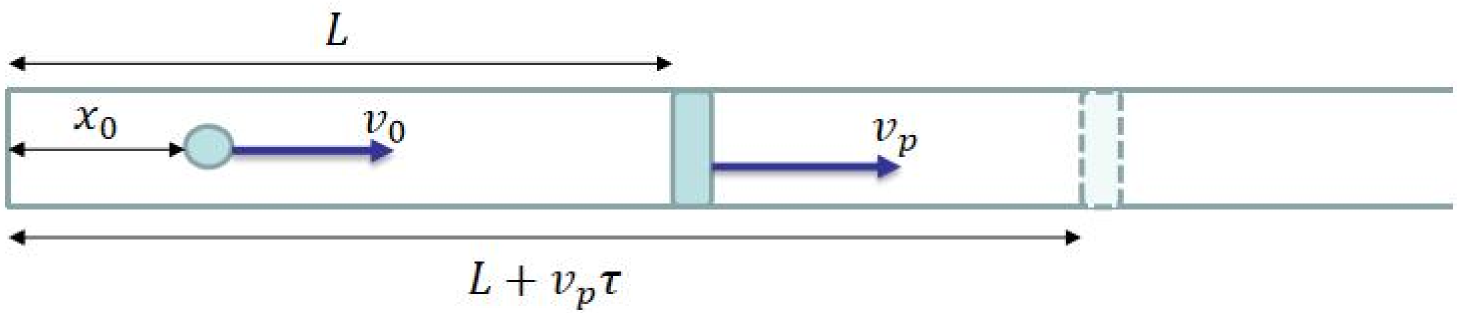
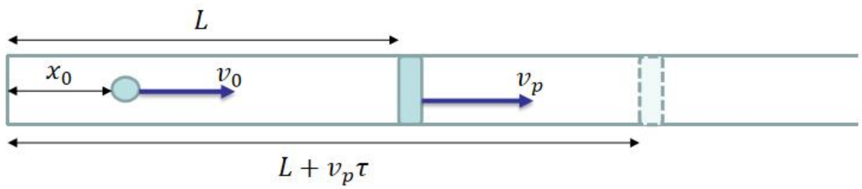
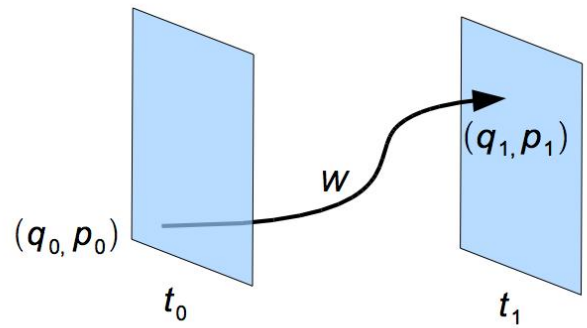
Setup of the relativistic piston model



- A single particle inside a one-dimensional cylinder with a moving piston
- The collision with the piston is elastic.
- The gas is of the inverse temperature β.

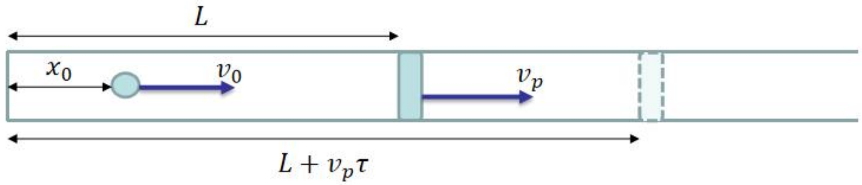
Setup of the relativistic piston model

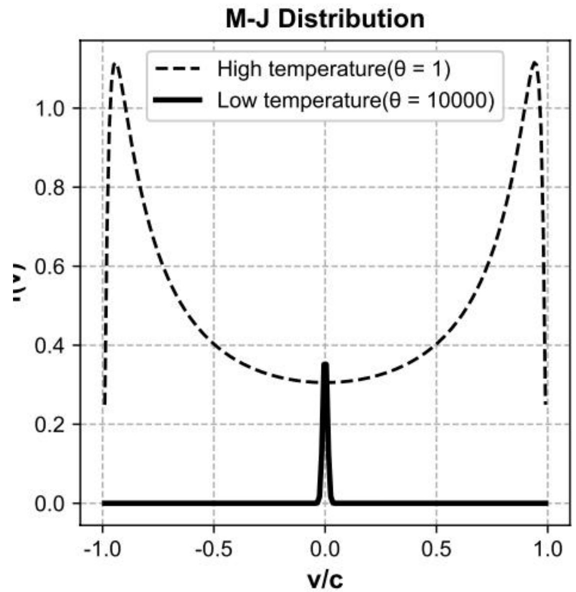




- Elastic collision: in the piston frame, the speed of the particle doesn't change during a collision.
- Trajectory work: $W(q_0, p_0) = H(q_1, p_1; \lambda_1) - H(q_0, p_0; \lambda_0)$
- Time slides are defined in the laboratory frame.

Setup of the relativistic piston model





- Classical
- Relativistic
- Adiabatic

$$f(x,p) = \frac{1}{2K_1(\frac{mc^2}{k_BT})mcL} e^{-\frac{mc^2}{k_BT}\sqrt{1+(\frac{p}{mc})^2}},$$

$$F(x,v) = \frac{1}{2K_1(\frac{mc^2}{k_BT})cL} e^{-\frac{mc^2}{k_BT}\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}} \left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}.$$

F. Jüttner, Annalen der Physik 339, 856 (1911).

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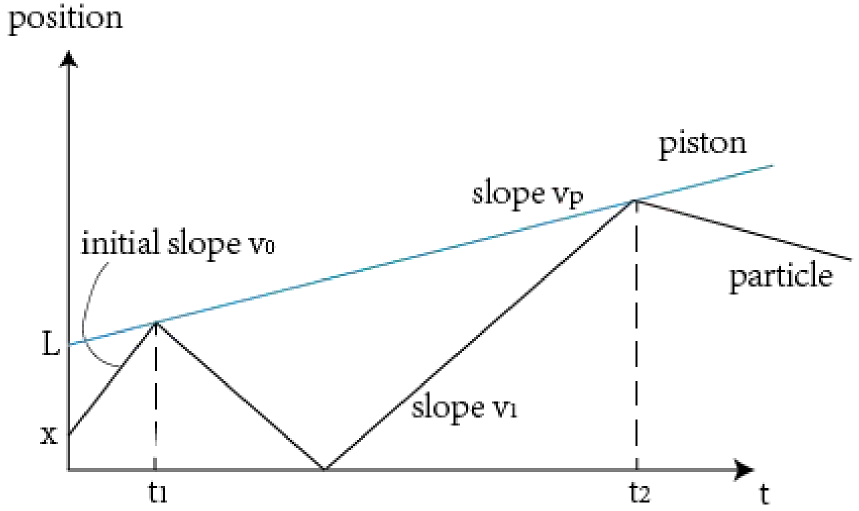
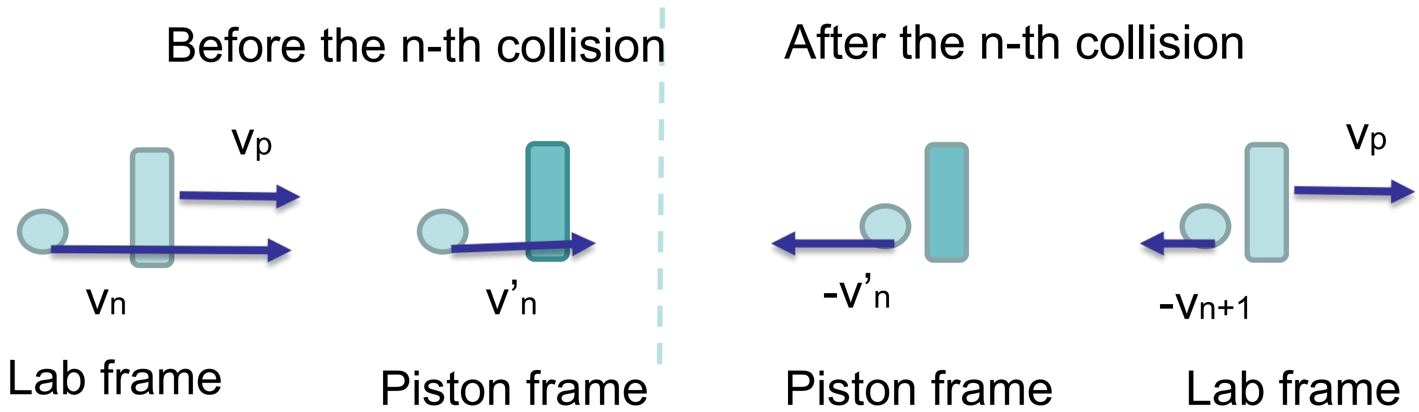
Trajectory of a particle and verification of the Jarzynski's equality

- Trajectory of a particle:



- The key is to obtain the speed after the n-th collision and the time of the n-th collision with the moving piston.

Trajectory of a particle and verification of the Jarzynski's equality



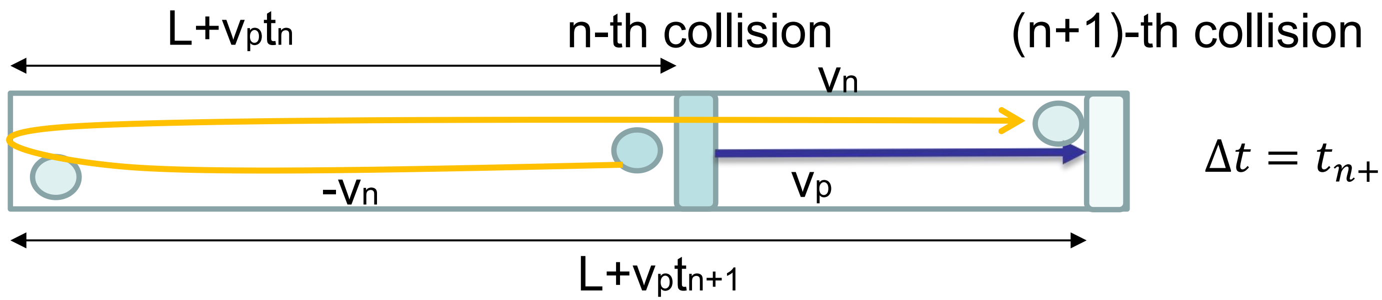
- Recursion relation of the speed :

$$v_{n+1} = \frac{(c^2 + v_p^2)v_n - 2v_p c^2}{c^2 + v_p^2 - 2v_p v_n}.$$

- Solution to the recursion relation:

$$v_n = \frac{(c+v)\alpha_p^{2n} - c + v}{(c+v)\alpha_p^{2n} + c - v} \cdot c, \quad \text{with} \quad \alpha_p = \frac{c - v_p}{c + v_p}.$$

Trajectory of a particle and verification of the Jarzynski's equality

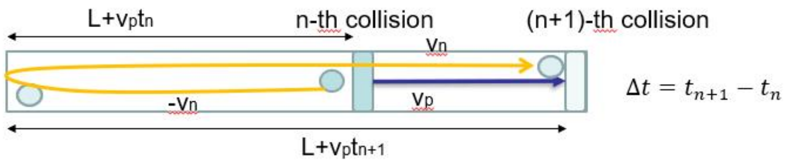


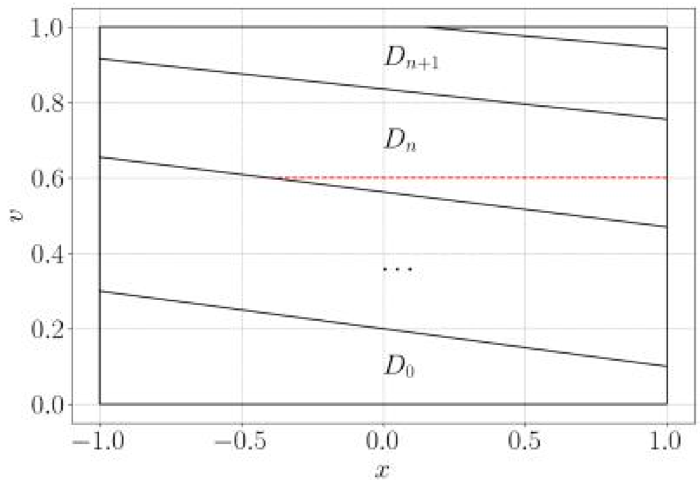
$$\Delta t = t_{n+1} - t_n$$

- Recursion relation of the time of the n-th collision with the piston: $$t_{n+1} = \frac{2L + (v_p + v_n)t_n}{v_n - v_p},$$
- Solution to the recursion relation:

$$t_n = \left[(-\alpha_p^{2n} - \alpha_p + \alpha_p^{n+1} + \alpha_p^n)\frac{v}{c} + (-\alpha_p^{n+1} + \alpha_p^n)\frac{x}{L} + (\alpha_p - \alpha_p^{2n})\right]$$
$$\left[-\alpha_p + \alpha_p^{2n} + (\alpha_p + \alpha_p^{2n})\frac{v}{c}\right]^{-1} \cdot \frac{L(1+\alpha_p)}{c(1-\alpha_p)},$$

Trajectory of a particle and verification of the Jarzynski's equality



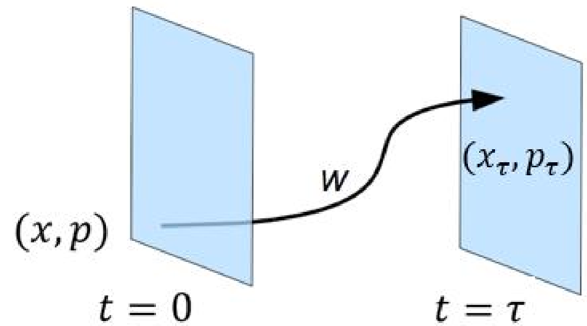


- Solution to the recursion relation:

$$t_n = \left[(-\alpha_p^{2n} - \alpha_p + \alpha_p^{n+1} + \alpha_p^n)\frac{v}{c} + (-\alpha_p^{n+1} + \alpha_p^n)\frac{x}{L} + (\alpha_p - \alpha_p^{2n})\right]$$
$$\left[-\alpha_p + \alpha_p^{2n} + (\alpha_p + \alpha_p^{2n})\frac{v}{c}\right]^{-1} \cdot \frac{L(1+\alpha_p)}{c(1-\alpha_p)},$$

- Final collision number: $n = \max\{N | t_N \leq \tau\}$

Trajectory of a particle and verification of the Jarzynski's equality



- Final position and momentum:

$$(x_\tau, p_\tau) = (|L - v_n\tau + (v_n + v_p)t_n|, p_n) \quad \text{with} \quad p_n = \frac{mv_n}{\sqrt{1-\left(\frac{v_n}{c}\right)^2}}.$$

- Liouville's theorem can be proved:

$$\left|\frac{\partial(x_\tau, p_\tau)}{\partial(x, p)}\right| = 1,$$

which verifies the Jarzynski's equality.

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Relativistic work distribution and its non-relativistic limit

- Non-dimenssionalize: m, L, k_B, c=1.
- Definition of the work distribution:

$$P(W) = \int_{-1}^{1} \mathrm{d}x \int_{0}^{1} \mathrm{d}v \frac{e^{-\frac{\beta}{\sqrt{1-v^2}}} \delta(W - W_\tau(x,v))}{2K_1(\beta)(1-v^2)^{\frac{3}{2}}},$$

- After some tedious calculations, the distribution function of W can be analytically expressed as

$$P(W) = P_0\delta(W) + \frac{1}{2K_1(\beta)} \sum_{n=1}^{N} \varphi_n(v_n(W)) \times \frac{e^{-\beta/\sqrt{1-v_n(W)^2}}}{(\alpha_p^{-n} - 1)\left[1 + \alpha_p^n - v_n(W)(1-\alpha_p^n)\right]}.$$

Relativistic work distribution and its non-relativistic limit

- The work distribution P(W) is:

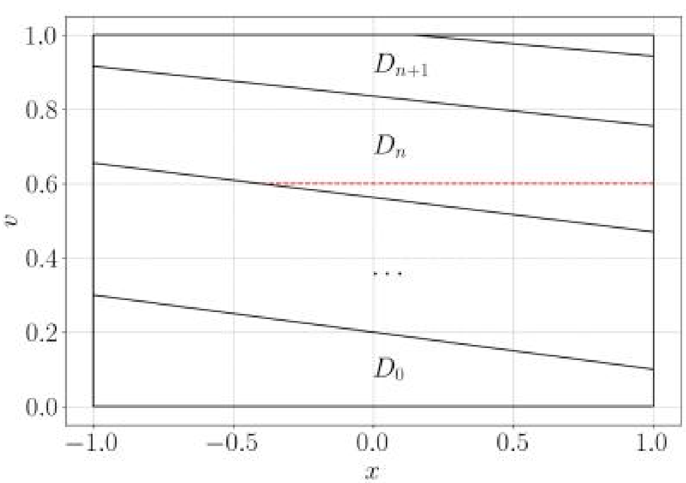
$$P(W) = P_0\delta(W) + \frac{1}{2K_1(\beta)}\sum_{n=1}^{N}\varphi_n(v_n(W)) \times \frac{e^{-\beta/\sqrt{1-v_n(W)^2}}}{(\alpha_p^{-n}-1)\left[1+\alpha_p^n - v_n(W)(1-\alpha_p^n)\right]}.$$

- With

$$v_n(W) = \frac{(1-\alpha_p^n)^3(1+\alpha_p^n) + 4W\sqrt{\alpha_p^{3n}\left((1-\alpha_p^n)^2+\alpha_p^n W^2\right)}}{(1-\alpha_p^{2n})^2 + 4\alpha_p^{2n}W^2},$$

$$\varphi_n(v) = \begin{cases} 1-\xi_n(v), & \frac{X_n-1}{T_n} < v_n(W) \le \frac{X_n+1}{T_n} \\ 2, & \frac{X_n+1}{T_n} < v_n(W) \le \frac{X_{n+1}-1}{T_{n+1}} \\ 1+\xi_{n+1}(v), & \frac{X_{n+1}-1}{T_{n+1}} < v_n(W) \le \frac{X_{n+1}+1}{T_{n+1}} \end{cases},$$

- Overlap factor:



Relativistic work distribution and its non-relativistic limit

- We may recover the dimension of the expressions and let $c \to \infty$, then we have the non-relativistic limit of the work distribution.

$$P(W) = P_0\delta(W) + \frac{\sqrt{\beta}}{\sqrt{2\pi}nv_p} e^{-\frac{\beta}{2}\left(\frac{W}{2nv_p}+nv_p\right)^2} f(W),$$

$$f(W) = \begin{cases} -(n-1)(1+\frac{v_p}{2}) + \frac{W}{4nv_p}, & (n-1)(v_p+2) < \frac{W}{2nv_p} \leq (n-1)(v_p+2)+2 \\ 1, & (n-1)(v_p+2)+2 < \frac{W}{2nv_p} \leq (n-1)(v_p+2)+2+2v_p \\ (n+1)(1+\frac{v_p}{2}) - \frac{W}{4nv_p}, & (n-1)(v_p+2)+2+2v_p < \frac{W}{2nv_p} \leq (n+1)(v_p+2) \end{cases}$$

R. C. Lua and A. Y. Grosberg,
J. Phys. Chem. B 109, 6805 (2005)

The comparison between relativistic work distribution and its non-relativistic limit

$$P(W) = P_0\delta(W) + \frac{1}{2K_1(\beta)}\sum_{n=1}^{N}\varphi_n(v_n(W)) \times \frac{e^{-\beta/\sqrt{1-v_n(W)^2}}}{(\alpha_p^{-n}-1)\left[1+\alpha_p^n - v_n(W)(1-\alpha_p^n)\right]}.$$

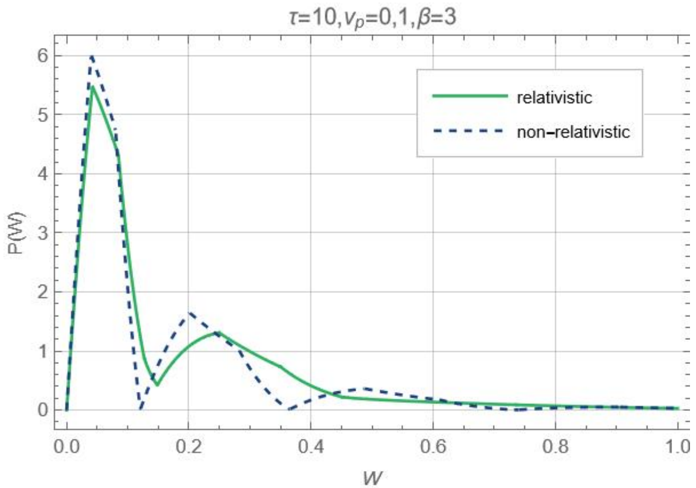
$$v_n(W) = \frac{(1-\alpha_p^n)^3(1+\alpha_p^n) + 4W\sqrt{\alpha_p^{3n}\left((1-\alpha_p^n)^2+\alpha_p^n W^2\right)}}{(1-\alpha_p^{2n})^2 + 4\alpha_p^{2n}W^2},$$

$$\varphi_n(v) = \begin{cases} 1-\xi_n(v), & \frac{X_n-1}{T_n} < v_n(W) \le \frac{X_n+1}{T_n} \\ 2, & \frac{X_n+1}{T_n} < v_n(W) \le \frac{X_{n+1}-1}{T_{n+1}} \\ 1+\xi_{n+1}(v), & \frac{X_{n+1}-1}{T_{n+1}} < v_n(W) \le \frac{X_{n+1}+1}{T_{n+1}} \end{cases},$$

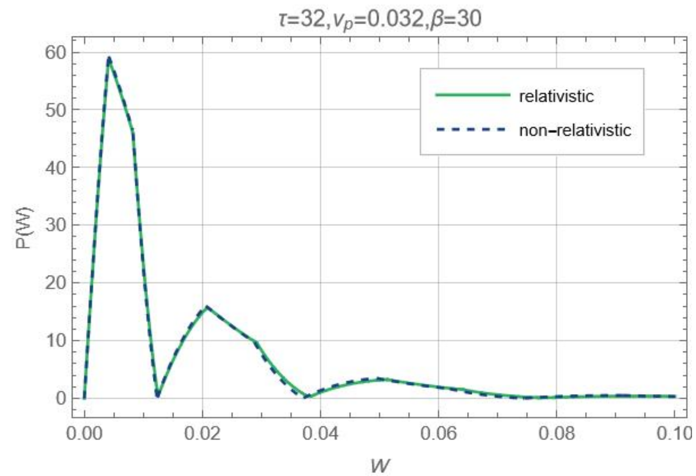
$$P(W) = P_0\delta(W) + \frac{\sqrt{\beta}}{\sqrt{2\pi}nv_p}e^{-\frac{\beta}{2}\left(\frac{W}{2nv_p}+nv_p\right)^2}f(W),$$

$$f(W) = \begin{cases} -(n-1)(1+\frac{v_p}{2}) + \frac{W}{4nv_p}, & (n-1)(v_p+2) < \frac{W}{2nv_p} \le (n-1)(v_p+2)+2 \\ 1, & (n-1)(v_p+2)+2 < \frac{W}{2nv_p} \le (n-1)(v_p+2)+2+2v_p \\ (n+1)(1+\frac{v_p}{2}) - \frac{W}{4nv_p}, & (n-1)(v_p+2)+2+2v_p < \frac{W}{2nv_p} \le (n+1)(v_p+2) \end{cases}$$

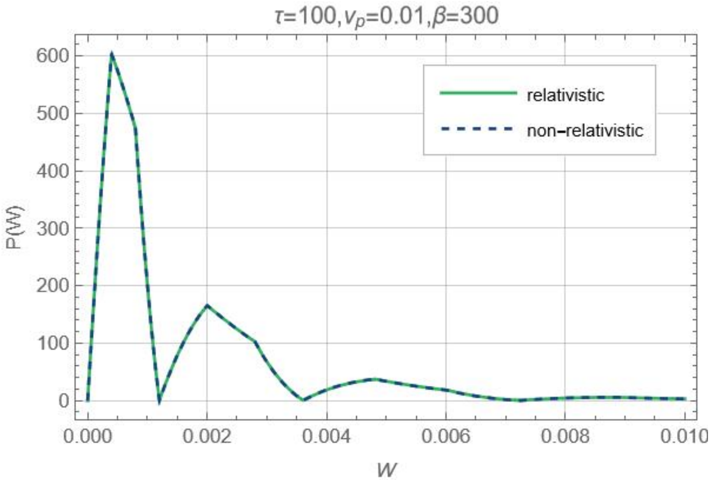
The comparison between relativistic work distribution and its non-relativistic limit



(a)



(b)



(c)

Relativistic and non-relativistic work distribution with different parameters. The initial length is $1cm$. The protocol is:

(a)$\tau = 0.3ns,\ v_p = 3 \times 10^7 m/s, T = 3 \times 10^{12} K$;

$(b)\tau = 1ns,\ v_p = 1 \times 10^7 m/s, T = 3 \times 10^{11} K$;

$(c)\tau = 3ns,\ v_p = 3 \times 10^6 m/s, T = 3 \times 10^{10} K$.

Relativistic work distribution has no zeros, which might be detected.

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Summary

- 1D classical relativistic piston model is an exact solvable model. We analytical solve the trajectory and the work distributuion of the relativistic piston model, and verify the Jarzynski's equality.
- In the non-relativistic limit, our results recover the non-relativistic results [Lua, & Grosberg, J. Phys. Chem. B, 109, 6805(2005)].
- We also find that, unlike the non-relativistic case, the maximun number of collisions in this relativistic gas model is finite, and the relativistic work distribution no longer has zeros.
- It is difficult to detect the relativistic effects of the work distribution of the ideal gas in a piston system with the current experimental techniques.

Thank you!