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Special relativistic covariant fluctuation theorems

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Contents

- Covariance of physical laws and fluctuation theorems
- Covariant Fluctuation theorems for moving system and/or heat bath
- Example: relativistic particle system
  - heat distribution
- Example: relativistic field system
  - work distribution
- Summary

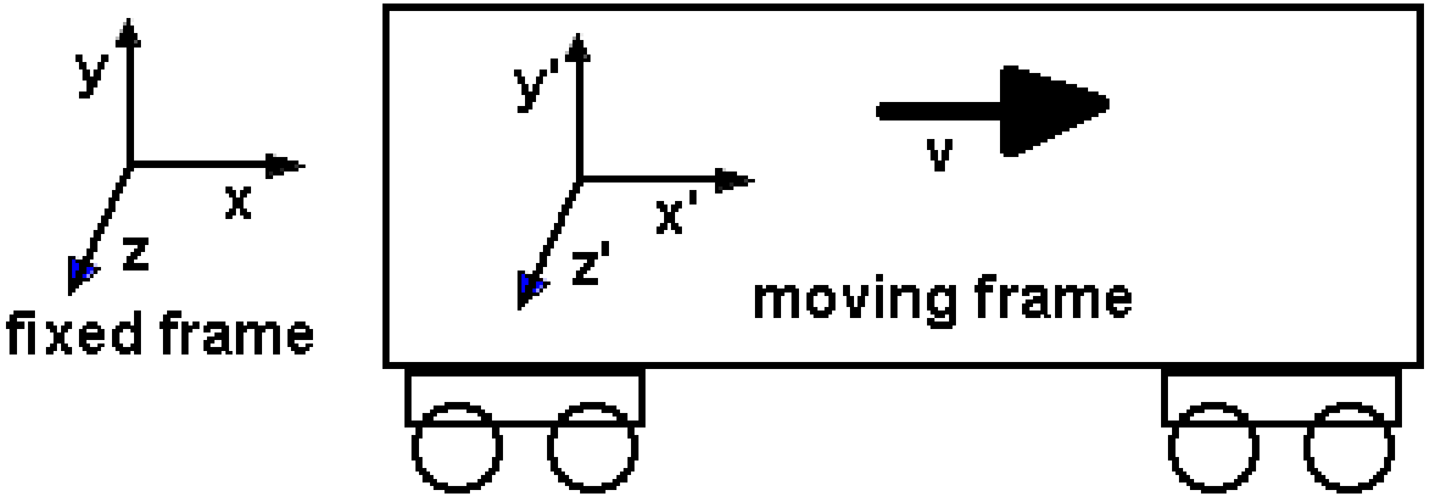
Transformation of frame of reference

Galileo transformation:

$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

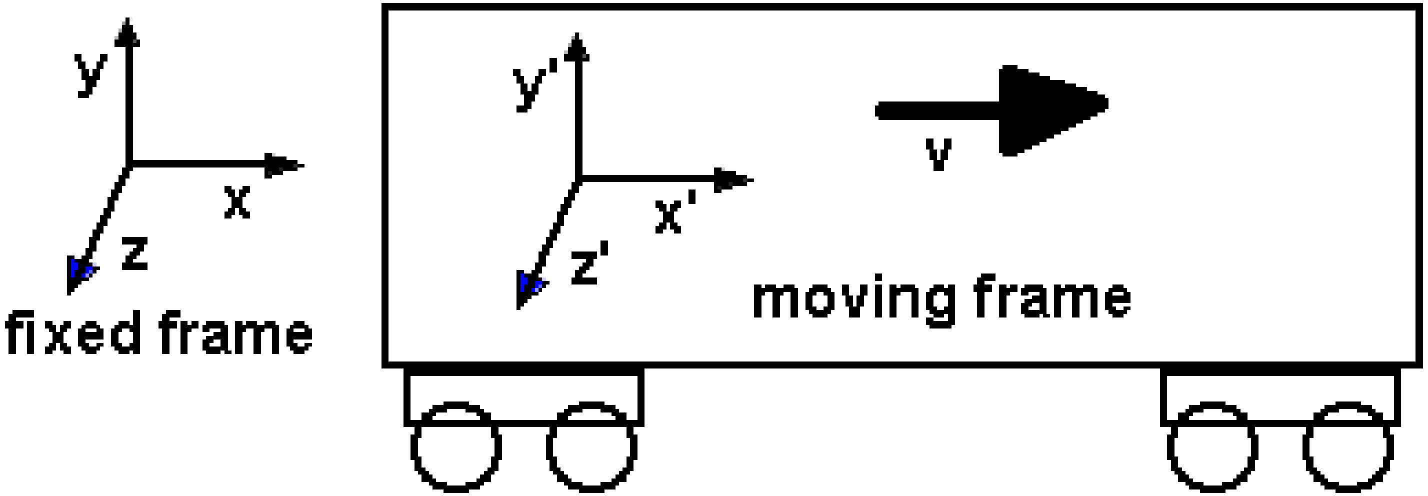
Lorentz transformation:

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma(t - \frac{vx}{c^2})$$



A fundamental requirement: Covariance of physical laws

- Newton’s equation is invariant under Galileo transformation (1687)
- Maxwell’s equations are invariant under Lorentz transformation (1904)

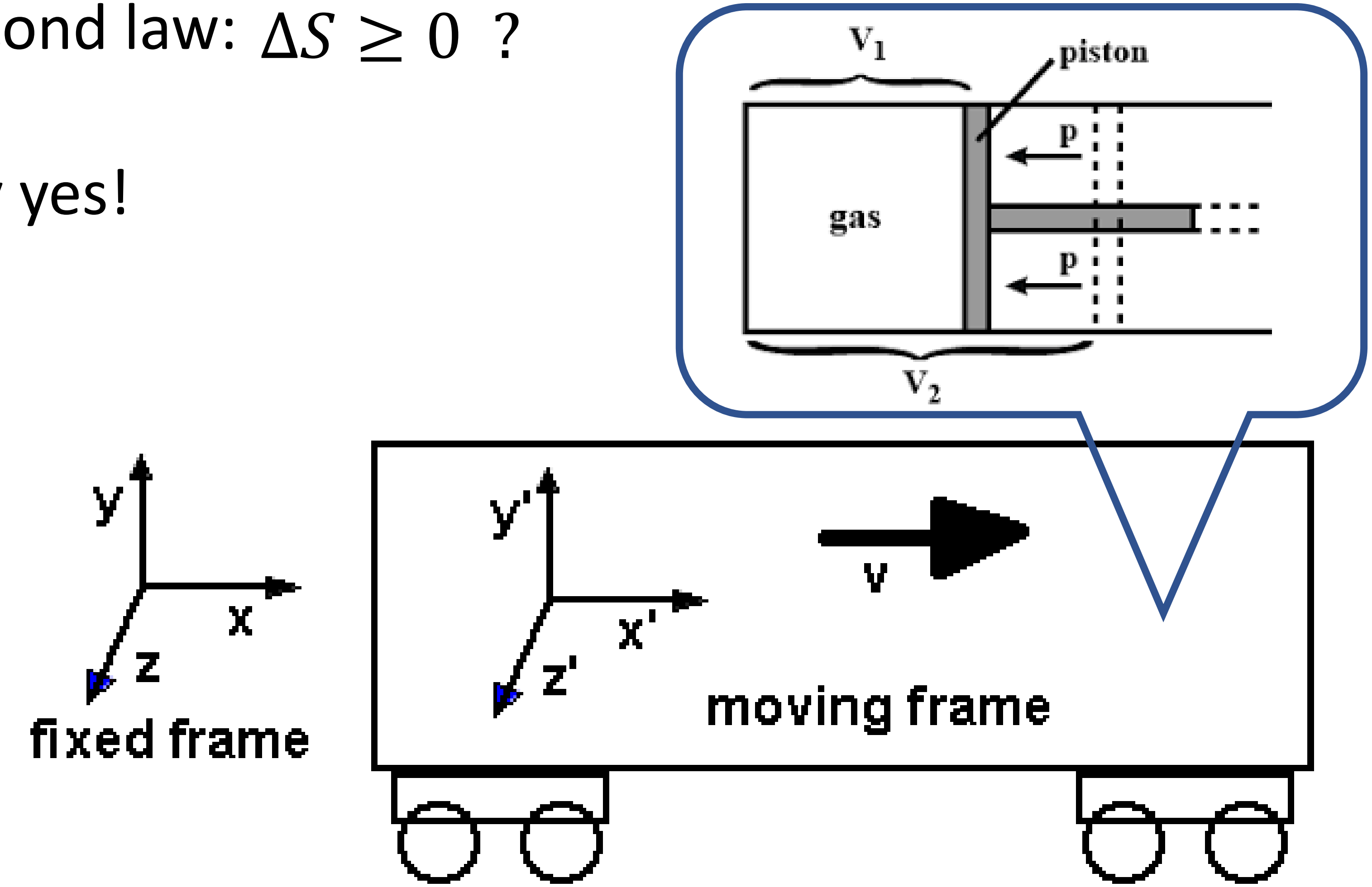


Coordinates do not exist *a priori* in nature, but are only artifices used in describing nature, and hence should play no role in the formulation of fundamental physical laws.----Wikipedia article “general covariance”

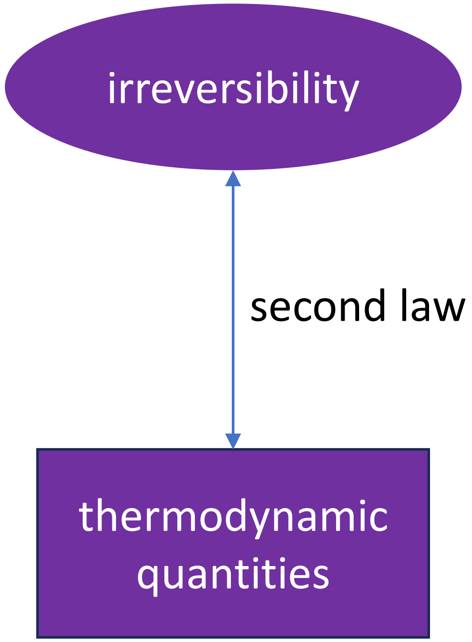
Covariance of thermodynamic laws?

The second law: $\Delta S \geq 0$?

Trivially yes!

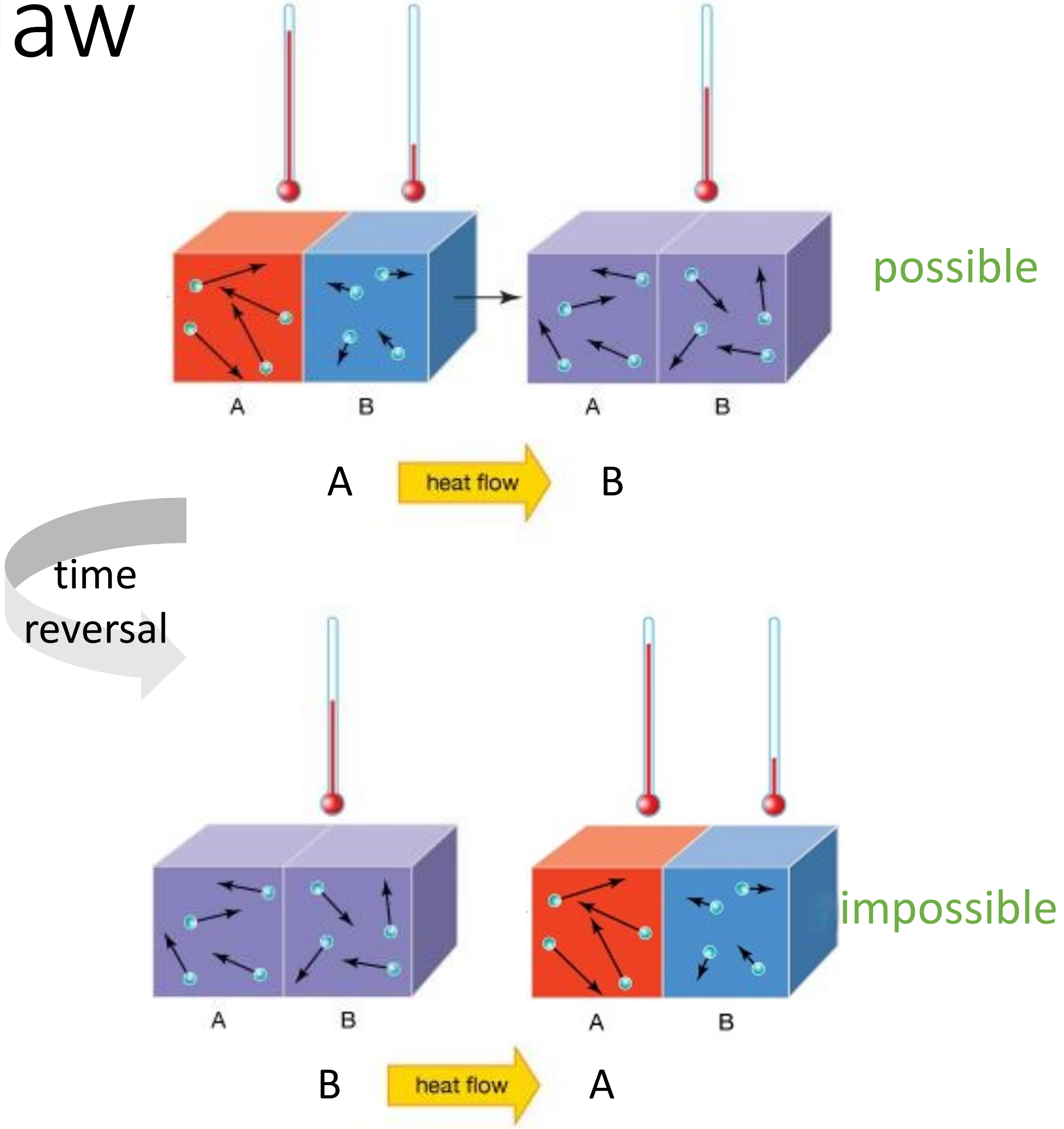


Irreversibility and second law



$$\Delta S \geq 0$$

second law $Q(T_{\text{high}} \to T_{\text{low}}) > 0$



Stochastic thermodynamics at mesoscopic

- For mesoscopic systems, such as a colloidal particle doing Brownian motion, fluctuations play a role.

- The backward event is possible, but with a smaller probability compared to the forward one.

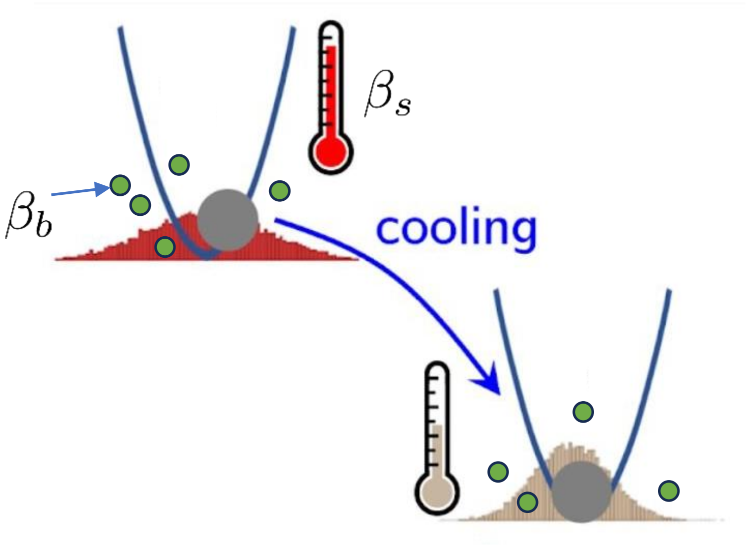
- This probability ratio is characterized by fluctuation theorems

- The underlying mechanics can be either relativistic or non-relativistic

Fluctuation theorems

ω: forward trajectory
$\tilde{\omega}$: backward trajectory

- (Noncovariant) heat exchange Fluctuation Theorem



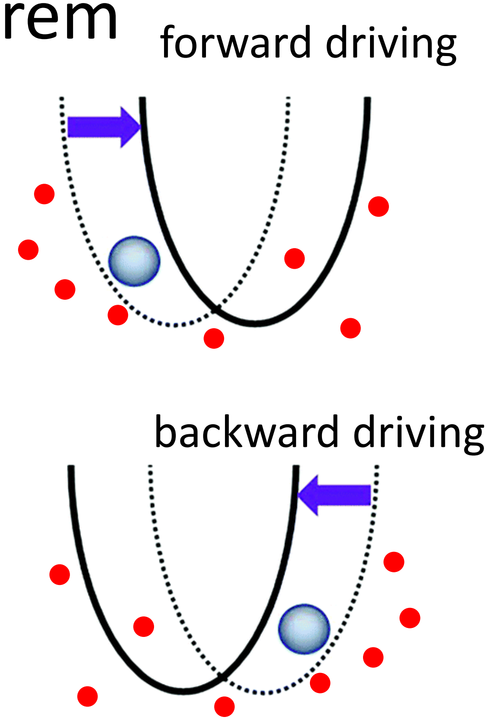
$$\frac{\Pr(\omega)}{\Pr(\tilde{\omega})} = \mathrm{e}^{-(\beta_b - \beta_s)Q[\omega]}$$

$$\Downarrow$$

$$(\beta_b - \beta_s)\langle Q \rangle \geq 0$$

C. Jarzynski and D. K. Wójcik, PhysRevLett 92.230602 (2004).

- (Noncovariant) Crooks fluctuation theorem



$$\frac{\Pr(\omega)}{\tilde{\Pr}(\tilde{\omega})} = \mathrm{e}^{\beta(W[\omega] - \Delta F)}$$

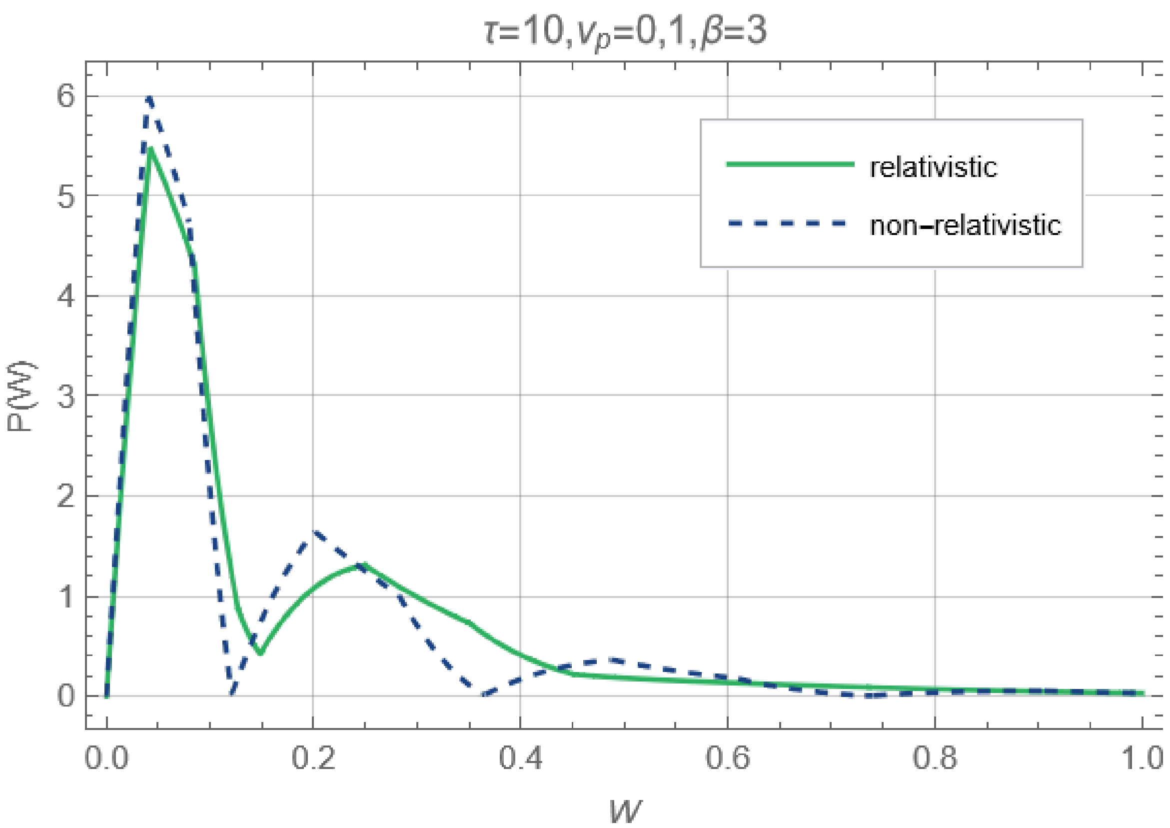
$$\Downarrow$$

$$\langle W \rangle - \Delta F \geq 0$$

G. E. Crooks, Journal of Statistical Physics 90, 1481 (1998).

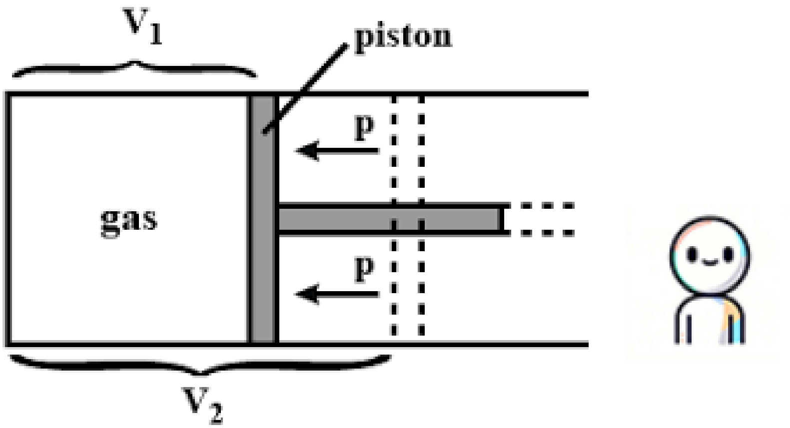
Fluctuation theorems

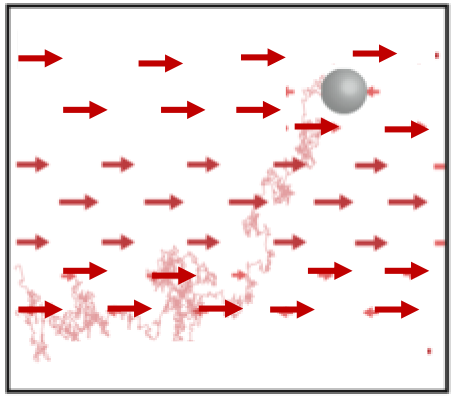
- Nonrelativistic mechanics
- Relativistic mechanics



Underlying assumption

- Precondition: The observer is at rest with the heat bath and initial equilibrium systems (left figure)





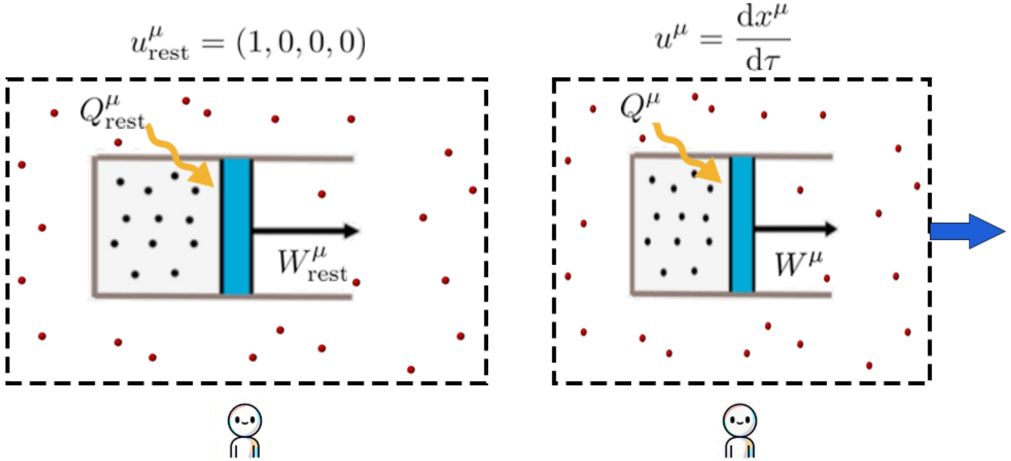
- If precondition is not fulfilled: relaxation between initial rest system and moving heat bath (right figure)

Even if initial rest temperatures are the same, particle will absorb energy from heat bath

The Clausius statement as well as fluctuation theorems need to be modified.

Fluctuation theorems under the transformation of frame of reference

- For a rest observer, the heat bath and/or the system are moving



- For a rest observer, covariant fluctuation theorems for moving heat bath and/or system?

- Difference between Lorentz covariance and Galilean covariance?

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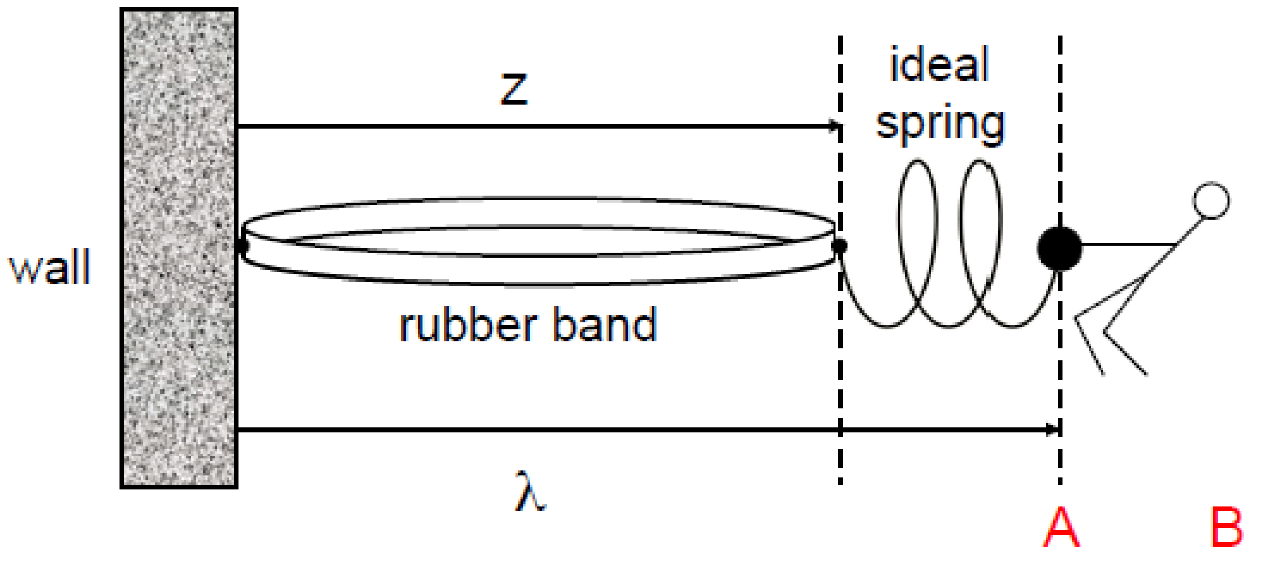
Work and free energy: a macroscopic example …



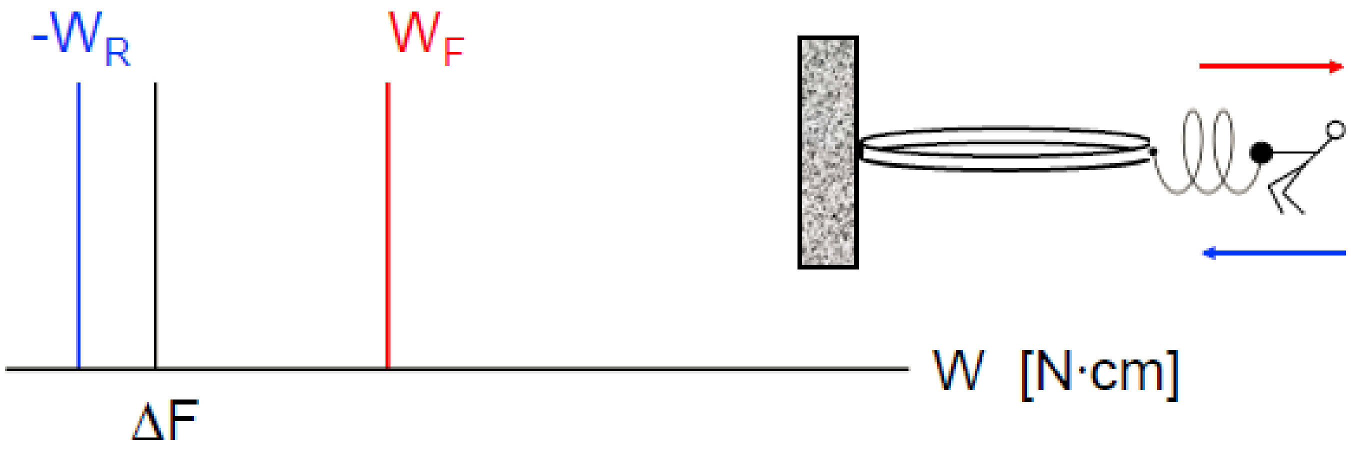
Irreversible process:

1. Begin in equilibrium $\lambda = A$
2. Stretch the rubber band $\lambda : A \rightarrow B$
   W = work performed
3. End in equilibrium $\lambda = B$

Thermodynamic cycles

forward process : A→B $W_F \geq \Delta F$

reverse process : A←B $W_R \geq -\Delta F$



Kelvin-Planck statement of 2nd Law: $W_F + W_R \geq 0$

We perform more work during the forward half-cycle (A→B)
than we recover during the reverse half-cycle (A←B).

(no free lunch)

Equilibrium distribution in relativity

- Relativistic covariant equilibrium distribution (Juttner distribution for a single rest particle system), valid for arbitrary observer

$$\rho_{\text{eq}} \propto \exp(-\beta_\mu P^\mu)$$

4-momentum P^μ

4-inverse temperature $\beta_\mu = \beta u_\mu$

4-velocity of the moving system $u_\mu = \eta_{\mu\nu} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}$

(rest) inverse temperature $\beta = \sqrt{\beta_\mu \beta^\mu}$

metric $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$

- For rest system, $u^\mu = (1,0,0,0)$, it reduces to

$$\rho_{\text{eq}} \propto \exp(-\beta H)$$

D. van Dantzig, Physica 6, 673 (1939)
N.G. van Kampen, Phys. Rev. 173, 295 (1968)
Z. C. Wu, Europhys.Lett., 88, 20005 (2009)

Backward process and trajectory

Lorentz transformation mixes space and time

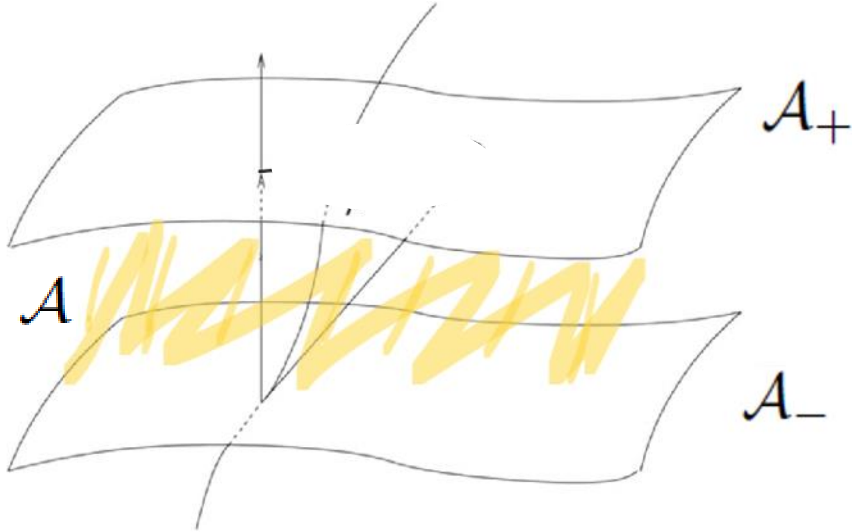
- Backward process: both space and time reversal

driving field

$$h(x) \to \tilde{h}(x) = h(-x)$$

starting and ending hypersurfaces are reversed

$$\tilde{\mathcal{A}}_- = -\mathcal{A}_+ \qquad \tilde{\mathcal{A}}_+ = -\mathcal{A}_-$$

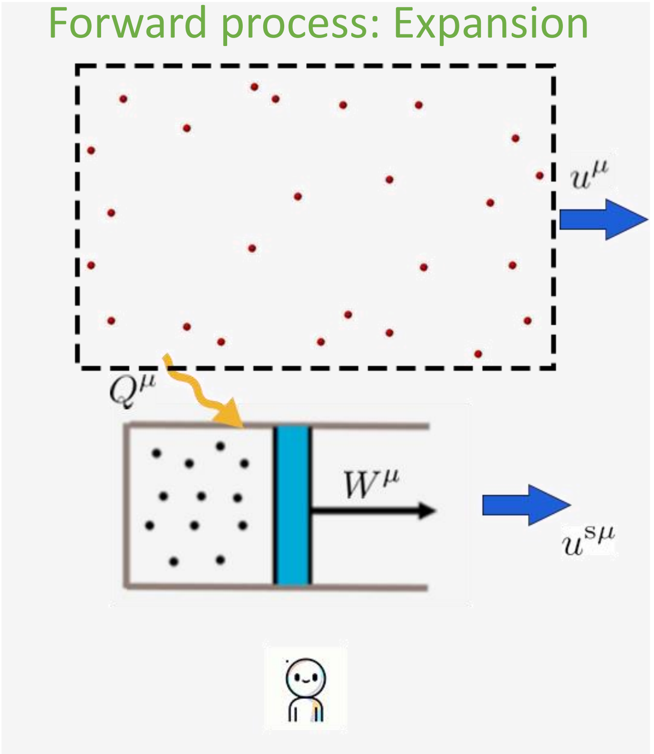


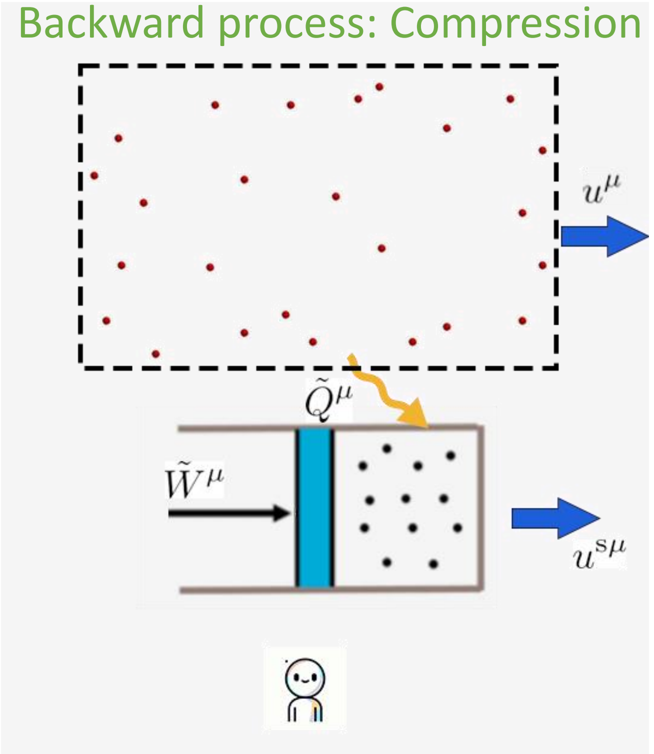
- Backward trajectory: spacetime reversed trajectory in backward process

Backward process

Lorentz transform mixes space and time

- Backward process: both space and time reversal $h(x) \to \tilde{h}(x) = h(-x)$





Covariant thermodynamic quantities

Momentum and energy are equally important for moving systems

- Covariant generalization of trajectory work and trajectory heat:

$$\Delta P^{\mu}[\omega] = W^{\mu}[\omega] + Q^{\mu}[\omega]$$

- 4-work: 4-momentum change due to external driving

- 4-heat: 4-momentum change due to energy exchange with the heat bath

N.G. van Kampen, Phys. Rev. 173, 295 (1968)
K. Sekimoto, Stochastic Energetics, Springer, (2010)

Detailed fluctuation theorems

- Covariant Crooks FT in driving process

$$\frac{\mathrm{Pr}(\omega)}{\tilde{\mathrm{Pr}}(\tilde{\omega})} = \mathrm{e}^{\beta_\mu W^\mu[\omega] - \beta \Delta F}$$

(Lorentz covariant Crooks fluctuation theorem)

initial state of forward process: $\rho_{eq}(\beta_\mu, h_{ini})$
initial state of backward process: $\rho_{eq}(\beta_\mu, h_{fin})$
4-inverse temperatureβ_μ are the same as heat bath
The system and the heat bath are not moving relative to each other

- Covariant heat exchange FT in pure relaxation process

$$\frac{\mathrm{Pr}(\omega)}{\mathrm{Pr}(\tilde{\omega})} = \mathrm{e}^{-(\beta_\mu - \beta_\mu^{\mathrm{s}}) Q^\mu[\omega]}$$

(Lorentz covariant heat exchange fluctuation theorem)

initial state of forward process and initial state of backward process are the same: $\rho_{eq}(\beta_\mu^s)$,
with 4-inverse temperature different from that of the heat bath

Initially, system and bath can move relative to each other

Entropy production

- Total Entropy production is a Lorentz scalar (k=1)

$$\Sigma = \Delta S - \beta_\mu Q^\mu \qquad\qquad \Delta S = \ln \rho_{\mathcal{A}_-} - \ln \rho_{\mathcal{A}_+}$$

- Entropy production fluctuation theorem:

$$\frac{\Pr(\omega)}{\tilde{\Pr}(\tilde{\omega})} = \exp(\Sigma[\omega]).$$

arbitrary initial state of forward process, initial state of the backward process is the final state of initial process

(Lorentz covariant entropy production fluctuation theorem)

More relations

- Integral fluctuation theorems (IFTs)

$$\langle \exp(-\beta_\mu W^\mu) \rangle = \exp(-\beta \Delta F)$$

(Lorentz covariant Jarzynski's equality)

$$\left\langle \exp((\beta^s_\mu - \beta_\mu) Q^\mu) \right\rangle = 1$$

(Lorentz covariant heat exchange fluctuation theorem)

- All statements of second law and FTs should be modified to include momentum components, e.g.

$$\left\langle (\beta^{\mathrm{s}}_\mu - \beta_\mu) Q^\mu \right\rangle \geq 0$$

modified Clausius statement for relaxation process

$$\langle \beta_\mu W^\mu \rangle \geq 0$$

modified Kelvin statement for cyclic driving process

Remarks

$$\frac{\mathrm{Pr}(\omega)}{\tilde{\mathrm{Pr}}(\tilde{\omega})} = \mathrm{e}^{\beta_\mu W^\mu[\omega] - \beta \Delta F}$$

- If the system is initially at rest relative to the heat bath:

The rest frame of the heat bath is special

$$\beta_\mu W^\mu = \beta W^0_{rest}$$

Irreversibility is related only to the energy-related thermal quantities in the rest frame

normal FTs are recovered in the rest frame

- However, a moving observer, to construct quantities in the rest frame, needs to know all the components and the 4-velocity of the heat bath.

Remarks

$$\frac{\mathrm{Pr}(\omega)}{\mathrm{Pr}(\tilde{\omega})} = \mathrm{e}^{-(\beta_\mu - \beta_\mu^{\mathrm{s}})Q^\mu[\omega]}$$

- If initially the system has relative motion to the heat bath:

No special reference frame in which both the system and the heat bath are at rest

No fluctuation theorem is solely about energy-related quantity

Key quantities related to irreversibility is

$$(\beta_\mu - \beta_\mu^{\mathrm{s}})Q^\mu$$

- Possible extension: several systems or baths in relative motion

Non-relativistic limit

- Coordinate transform in nonrelativistic limit

$$W^0 - \sum_{i=1}^{3} v^i W^i = W^0_{\mathrm{rest}} + O(v^2/c^2)$$

- Covariant FTs in nonrelativistic limit

$$\frac{\Pr(\omega)}{\tilde{\Pr}(\tilde{\omega})} = \mathrm{e}^{\beta(W^0 - \sum_{i=1}^{3} v^i W^i) - \beta \Delta F}$$

(Galileo covariant Crooks fluctuation theorem)

$$\frac{\Pr(\omega)}{\Pr(\tilde{\omega})} = \mathrm{e}^{-(\beta - \beta^{\mathrm{s}})Q^0 + \sum_{i=1}^{3} (\beta v^i - \beta^{\mathrm{s}} v^{\mathrm{s}i}) Q^i}$$

(Galileo covariant heat exchange fluctuation theorem)

- Expressions are similar; momentum still plays a role

Non-relativistic limit: example

$$\frac{\Pr(\omega)}{\tilde{\Pr}(\tilde{\omega})} = e^{\beta(W^0 - \sum_{i=1}^{3} v^i W^i) - \beta\Delta F}$$

It can be checked:

$$W^0 = \frac{1}{2}m(v_f - v)^2 - \frac{1}{2}m(v_i - v)^2$$

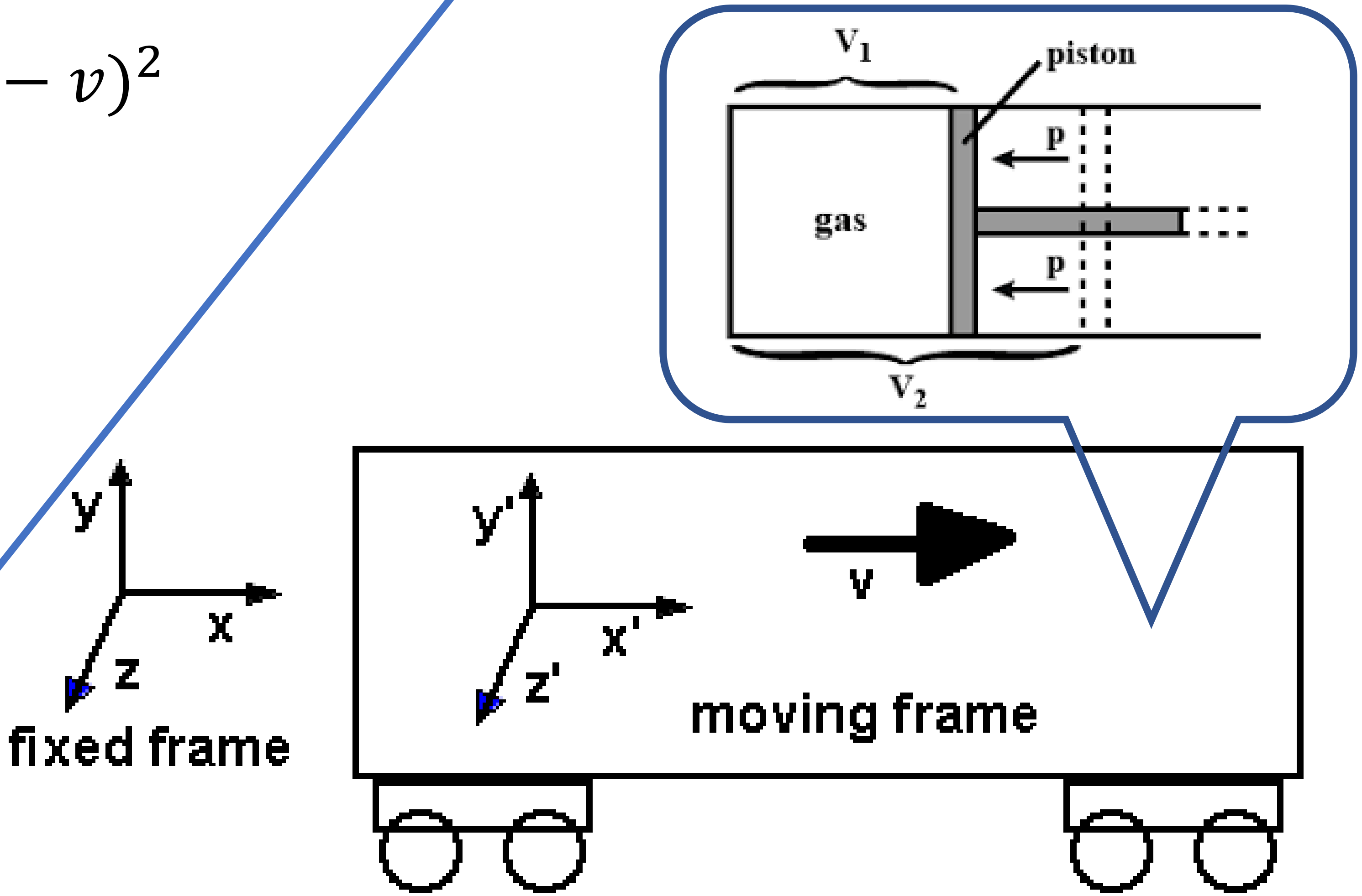
$$W^0_{rest} = \frac{1}{2}m{v_f}^2 - \frac{1}{2}m{v_i}^2$$

$$W^0 - m\left(v_f - v_i\right)v = W^0_{rest}$$

$$W^0 - vW = W^0_{rest}$$

Recover previous result:

$$\frac{\Pr(\omega)}{\widetilde{\Pr}\,(\widetilde{\omega})} = e^{\beta(W^0_{rest} - \Delta F)}$$



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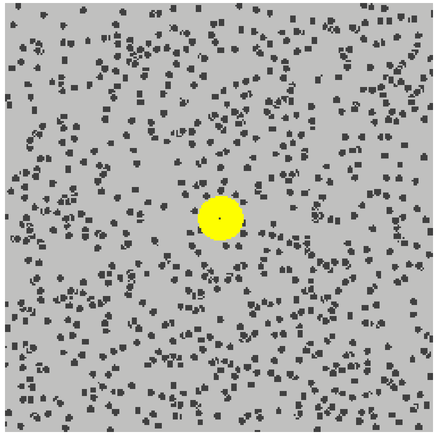
Relativistic Ornstein-Uhlenbeck process



- A kind of Relativistic Brownian motion

- A charged particle driven by electromagnetic field
- In the rest frame of heat bath

$$\frac{\mathrm{d}p^i}{\mathrm{d}t} = f^i - \kappa \frac{p^i}{p^0} + \sqrt{\frac{2\kappa}{\beta}}\xi^i$$

$$\vec{f} = q\vec{E} + q\vec{v} \times \vec{B}$$

$$\langle \xi^i(t)\xi^j(s) \rangle = \delta(t-s)\delta^{ij}$$

friction proportional to the velocity

- Canonical 4-momentum $\quad P^\mu = p^\mu + A^\mu$

J. Dunkel and P. Hanggi, Relativistic brownian motion,Phys. Rep. 471, 1 (2009).

Thermodynamics for relativistic OU particle

- 4-heat corresponds to the effect of friction and fluctuation force

$$Q^0 = \int_{t_\mathrm{i}}^{t_\mathrm{f}} (-\kappa v^i + \sqrt{2\kappa/\beta}\xi^i) v^i \mathrm{d}t \qquad Q^i = \int_{t_\mathrm{i}}^{t_\mathrm{f}} (-\kappa v^i + \sqrt{2\kappa/\beta}\xi^i) \mathrm{d}t$$

- 4-work is the change of canonical momentum minus 4-heat

$$W^\mu = \Delta P^\mu - Q^\mu$$

- Covariant expressions

$$\mathrm{d}Q^\mu = \mathrm{d}p^\mu - F^{\mu\nu}\mathrm{d}x_\nu \qquad \mathrm{d}W^\mu = \partial^\mu A^\nu \mathrm{d}x_\nu$$

K. Sekimoto, Stochastic Energetics, Springer, (2010)

Proof of fluctuation theorems

- Conditional path probability in the rest frame

$$P[x|x(0),p(0)] \propto \exp[-\frac{\beta}{4\kappa}\int \mathrm{d}t(\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} - q\vec{E} - q\vec{v}\times\vec{B} + \kappa\frac{\vec{p}}{p^0})^2]$$

- Take the probability ratio and boost to a generic reference frame,

$$\frac{P[x|x_{\mathcal{A}_-}, p_{\mathcal{A}_-}]}{\tilde{P}[\tilde{x}|\tilde{x}_{\tilde{\mathcal{A}}_-}, \tilde{p}_{\tilde{\mathcal{A}}_-}]} = \exp(-\beta u_\mu Q^\mu)$$

- Multiplying by different initial distributions for different FTs and rearranging terms, we get all differential FTs for trajectory

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Stochastic relativistic scalar field

- Field is another kind of systems in relativity
- Driven relativistic scalar field: covariant equation of motion

$$\partial^\mu \partial_\mu \phi + \kappa^\mu \partial_\mu \phi + m^2 \phi + V'(\phi) = h + \sqrt{\frac{2\kappa}{\beta}}\xi \qquad \langle \xi(x)\xi(y)\rangle = \delta^4(x-y)$$

- higher order potential density $V(\phi) = \frac{1}{4} g\phi^4 + \cdots$

- friction $\kappa^\mu = \kappa u^\mu$ u^μ is the 4-velocity of the heat bath.

Energy-momentum tensor

- Lagrangian $$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - V(\phi) + h\phi.$$

- energy-momentum tensor

$$T^{\mu\nu} = \eta^{\nu\alpha}\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\partial_\alpha\phi - \eta^{\mu\nu}\mathcal{L}.$$

- 4-momentum on hypersurface

$$P^\nu = \int_{\mathcal{A}_\pm} \mathrm{d}\sigma n_\mu T^{\mu\nu}$$

Covariant thermodynamic quantities

- 4-work is related to the change of external driving
- 4-heat is expressed by imposing the first law

$$W^\nu = \int_{\mathcal{A}} \mathrm{d}^4x \frac{\partial T^{\mu\nu}}{\partial h} \frac{\mathrm{d}h}{\mathrm{d}x^\mu}$$

$$Q^\nu = \int_{\mathcal{A}} \mathrm{d}^4x \frac{\mathrm{d}T^{\mu\nu}}{\mathrm{d}x^\mu} - W^\nu$$

- Concrete expressions

$$W^\nu = \int_{\mathcal{A}} \mathrm{d}^4x(-\phi\partial^\nu h)$$

$$Q^\nu = \int_{\mathcal{A}} \mathrm{d}^4x\partial^\nu\phi(\partial^\mu\partial_\mu\phi + m^2\phi + V' - h)$$

K. Sekimoto, Stochastic Energetics, Springer, (2010)

A sketch of Proof of FTs

- Path integral gives the conditional trajectory probability

$$\mathrm{Pr}[\phi|\phi_{\mathcal{A}_-}] \propto \exp(-\int_{\mathcal{A}} \mathrm{d}^4 x \alpha(\phi, \partial\phi)) \qquad \alpha = \frac{\beta}{4\kappa}[\partial^\mu \partial_\mu \phi + \kappa^\mu \partial_\mu \phi + m^2 \phi + V'(\phi) - h]^2$$

- Take the ratio between forward and backward trajectories

$$\frac{\mathrm{Pr}[\phi|\phi_{\mathcal{A}_-}]}{\tilde{\mathrm{Pr}}[\tilde{\phi}|\tilde{\phi}_{\tilde{\mathcal{A}}_-}]} = \mathrm{e}^{-\beta u_\mu Q^\mu[\phi]}$$

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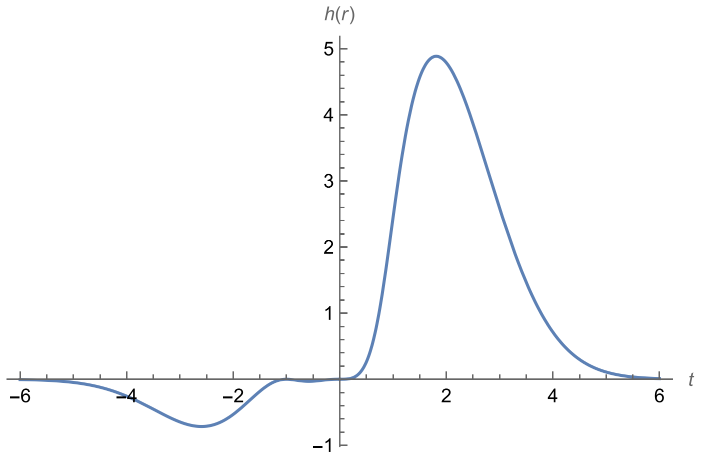
Work statistics



- Lagrangian is quadratic, higher order term V=0
- In the infinite past and the infinite future, h=0

the external field performs work, finally dissipated into the heat bath

- Generating functional can be calculated by path integral

$$\chi[J] = \left\langle \exp(-\int d^4x J(x)\phi(x)) \right\rangle = \frac{\int D\phi \exp(-\int d^4x[\alpha + J(x)\phi(x)])}{\int D\phi \exp(-\int d^4x\alpha)}$$

Work statistics

- Statistics of 4-work: joint Gaussian distribution with

$$\langle W^\mu \rangle = -\int \mathrm{d}^4x \mathrm{d}^4y \partial^\mu h(x) \Delta(x-y)(\partial^\nu \partial_\nu - \kappa^\nu \partial_\nu + m^2) h(y)$$

$$\langle W^\mu W^\nu \rangle - \langle W^\mu \rangle \langle W^\nu \rangle = \int \mathrm{d}^4x \mathrm{d}^4y \frac{2\kappa}{\beta} \partial^\mu h(x) \Delta(x-y) \partial^\nu h(y)$$

$$\Delta(x) = \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{\mathrm{e}^{\mathrm{i}p_\mu x^\mu}}{(-p^\mu p_\mu + m^2)^2 + (\kappa_\mu p^\mu)^2}$$

- It satisfies covariant Jarzynski equality

$$\left\langle \exp(-\int \mathrm{d}^4x \beta_\nu W^\nu(x)) \right\rangle = 1$$

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Summary and Prospects

- Generalization of trajectory covariant 4-work and 4-heat
- Covariant fluctuation theorems for arbitrary moving objects and observer
- Statements should be modified to include momentum-related component
- Formulas for relativistic and nonrelativistic systems are similar
- For system and bath initially in relative motion, no special reference frame exits, and noncovariant fluctuation theorems are insufficient
- Irreversibility characterized by the entropy production is the same for all inertial frames of reference

- Potential value in system under moving heat bath or sheared flow; high-energy nonequilibrium process

Acknowledgements

Ji-Hui Pei

Jin-Fu Chen

Ref: J. H. Pei, J. F Chen, H. T. Quan, arXiv preprint arXiv:2312.17621

Postdoc opening

- Working in the field of quantum thermodynamics, Stochastic thermodynamics, statistical field theory
- Highly motivated
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Thank you for your attention!