Evaluating the instability & the inertial manifold dimension of space-time chaos from time series data

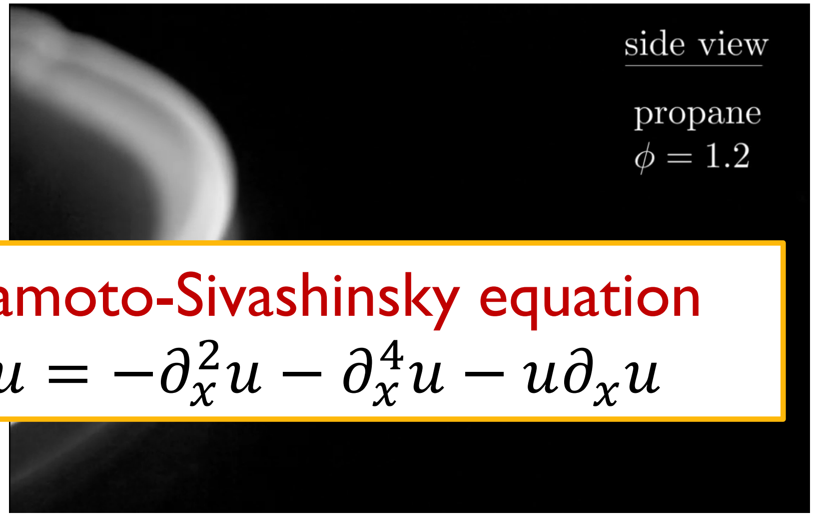
Kazumasa A. Takeuchi
(Univ. Tokyo)

Takeuchi Lab

Collaboration with
Taro P. Shimizu (formerly Tokyo Tech)
Daiki Nishiguchi (Univ. Tokyo)

Space-Time Chaos

fire front

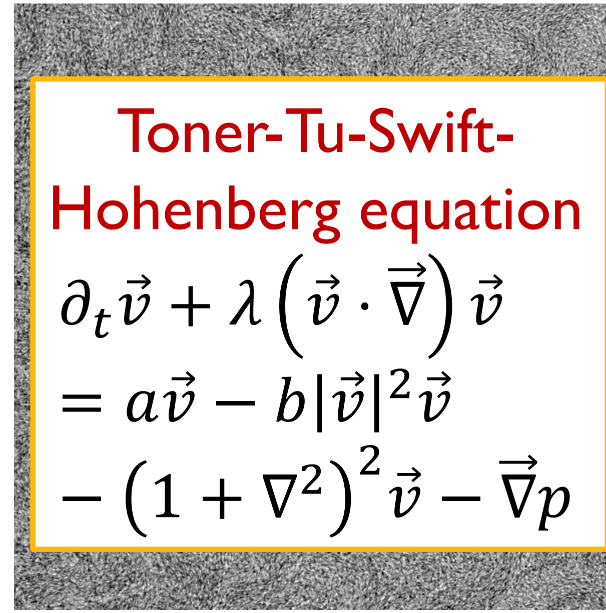


Kuramoto-Sivashinsky equation

$$\partial_t u = -\partial_x^2 u - \partial_x^4 u - u\partial_x u$$

[Martínez-Ruiz et al., APS Gallery of fluid motion 2018]

bacterial turbulence



Toner-Tu-Swift-Hohenberg equation

$$\partial_t \vec{v} + \lambda\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} = a\vec{v} - b|\vec{v}|^2\vec{v} - \left(1+\nabla^2\right)^2\vec{v} - \vec{\nabla}p$$

[Nishiguchi et al., Nat Comm 2018]

Exciton-polariton condensate

dissipative Gross-Pitaevskii equation

$$i\partial_t\psi = \left[-\nabla^2 + g_c|\psi|^2 + \frac{g_R P}{\gamma_R + R|\psi|^2} + \frac{i}{2}\left(\frac{RP}{\gamma_R + R|\psi|^2} - \gamma_C\right)\right]\psi$$

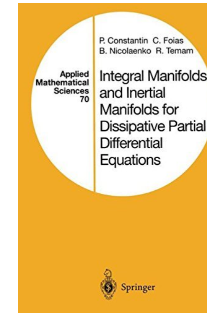
Nature 2022]

How many variables needed? (degrees of freedom, DOFs)

Infinitely many DOFs?

Physically, finite DOFs should do.

How Many Variables are Needed?



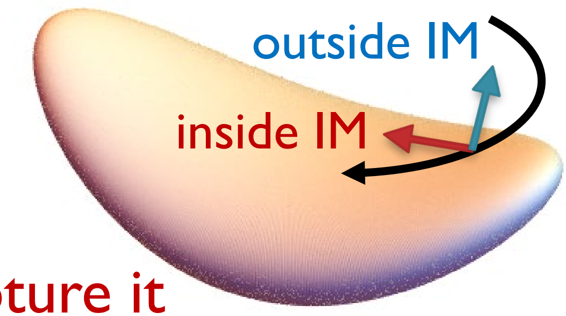
Constantin et al., Springer (1989)

- Solutions of PDEs ∈ ∞-dimensional phase space

  ⇕ but, for dissipative systems,

  Trajectories are believed to be attracted to "inertial manifold" (IM) w/ finite integer dimension.

  If so, solutions are described in principle by a finite # of variables.



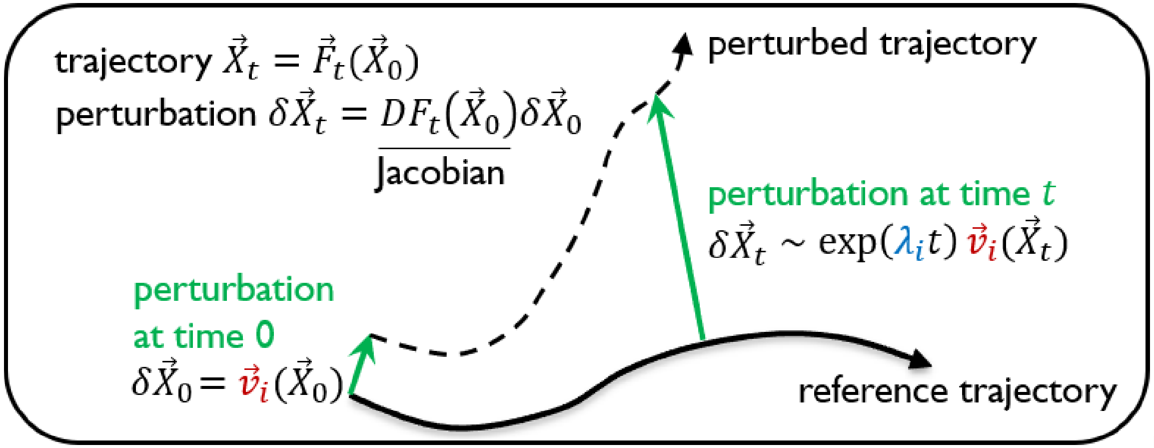
- Though IM had been a formal object, we developed a numerical method to capture it by Lyapunov analysis. [Yang, Takeuchi, …, PRL **102**, 074102; PRE **84**, 046214; …]

Basic idea

- Perturbation inside IM → relevant effect; perturbed trajectory will separate exponentially from the unperturbed one.
- Perturbation outside IM → decay exponentially w/o affecting others

Lyapunov Exponents & Vectors

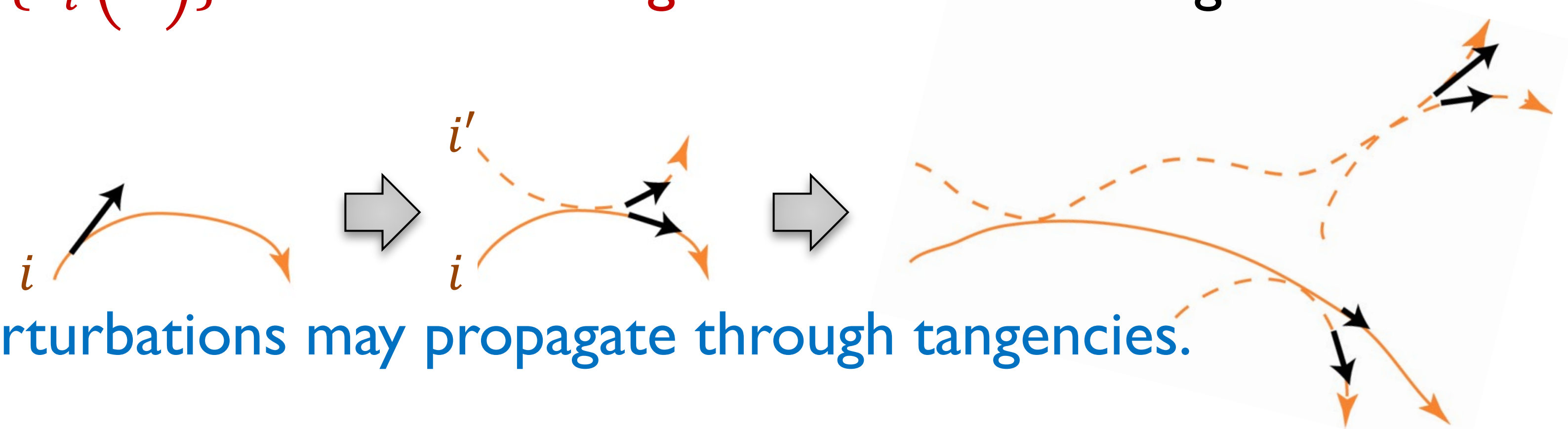
[review: Eckmann & Ruelle, Rev. Mod. Phys. 1985]



Lyapunov exponent λ_i
exponential growth rate

Lyapunov vector $\vec{v}_i(\vec{X})$
proper direction of perturbations growing at λ_i

※ $\{\vec{v}_i\left(\vec{X}\right)\}$ is NOT an orthogonal basis. Can be tangent sometimes.



Perturbations may propagate through tangencies.

Perturbations w/o tangency just decay like $e^{\lambda_i t}$ (if $\lambda_i < 0$)
Perturbations outside IM → no tangency, $\lambda_i < 0$

Results for Kuramoto-Sivashinsky (KS) Equation

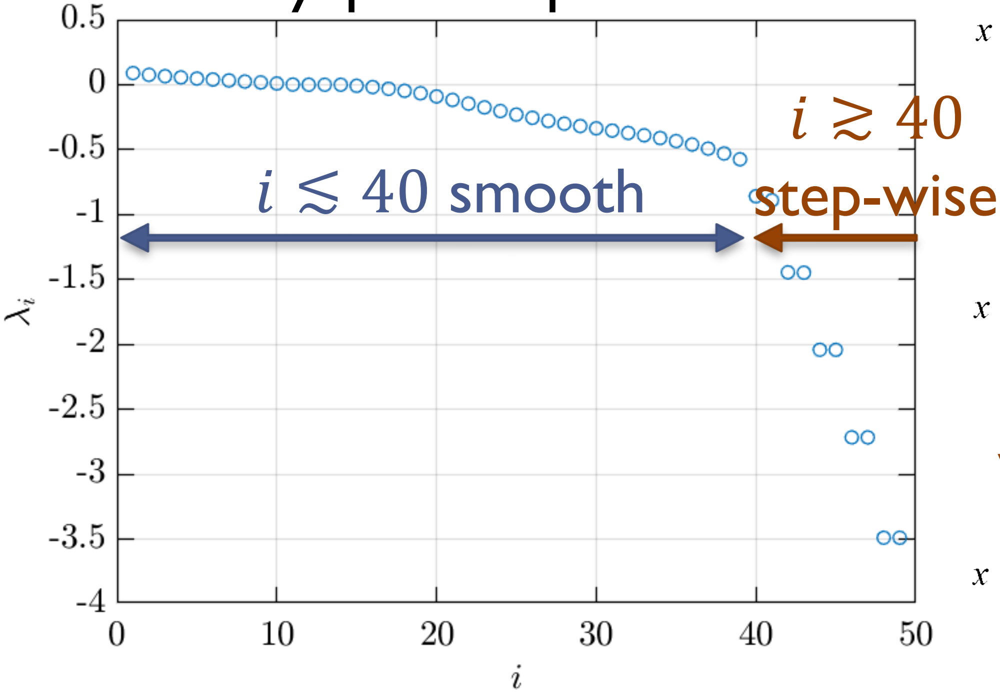
PRL 102, 074102 (2009)
PRE 84, 046214 (2011)

$$\partial_t u = -\partial_x^2 u - \partial_x^4 u - u\partial_x u, \qquad x \in [0, L]$$

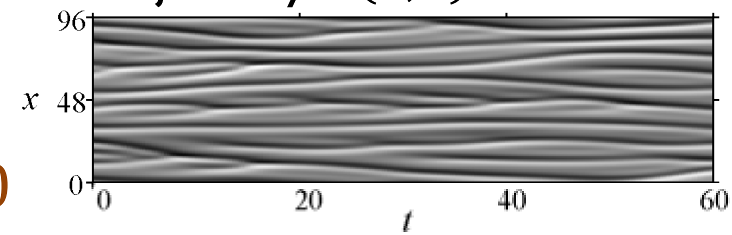
$L = 96$
periodic boundary

negative diffusion → instability

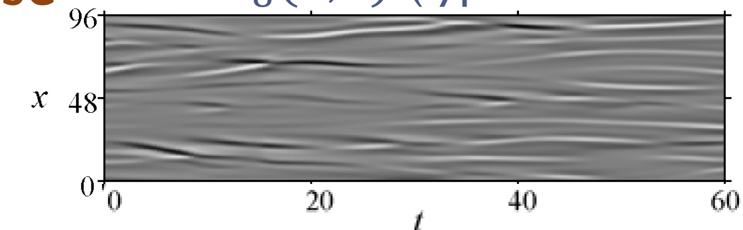
Lyapunov spectrum



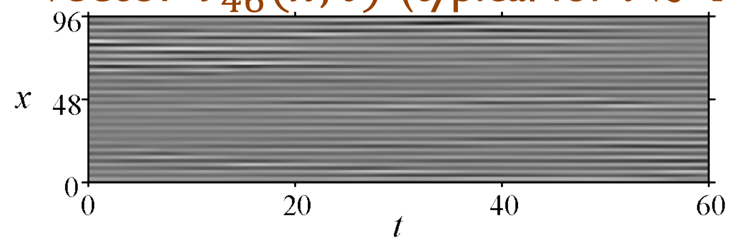
trajectory $u(x,t)$: chaotic



vector $v_8(x,t)$ (typical for $i \lesssim 40$)



vector $v_{46}(x,t)$ (typical for $i \gtrsim 40$)



※Steps appear only for periodic boundaries.
Step-wise structure can't be a criterion.

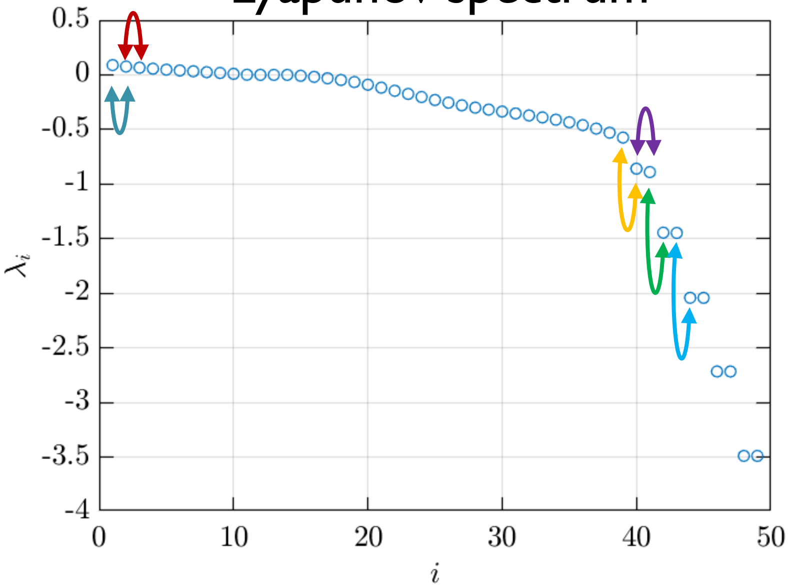
→ Just sinusoidal..
no relation to trajectory?

KS Inertial Manifold Dimension

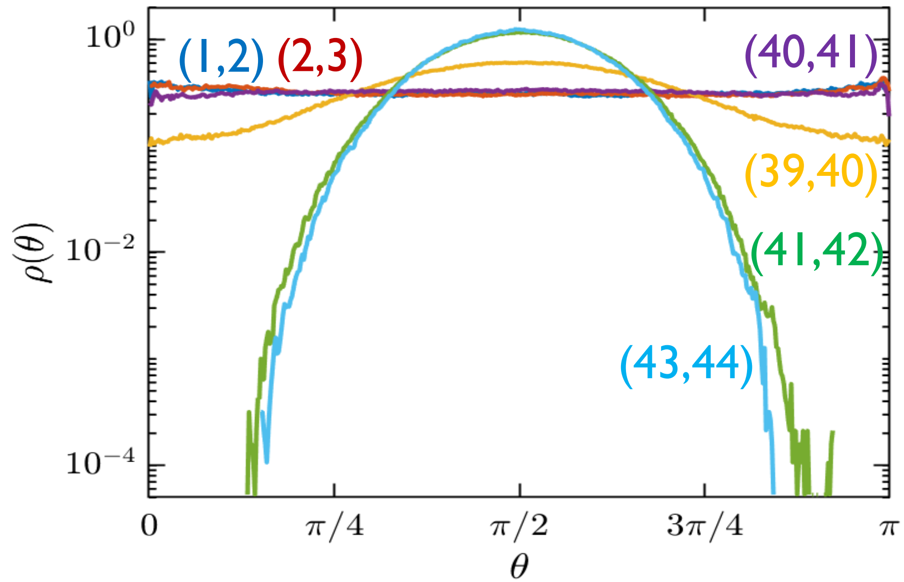
Yang, Takeuchi, ...,
PRL **102**, 074102 (2009)
PRE **84,** 046214 (2011)

Criterion is given by vector tangencies!

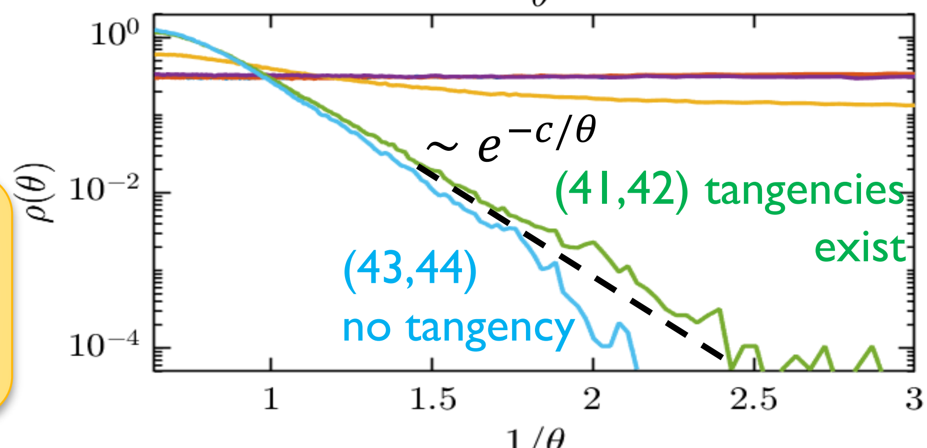
Lyapunov spectrum



Distribution of angle θ between vectors



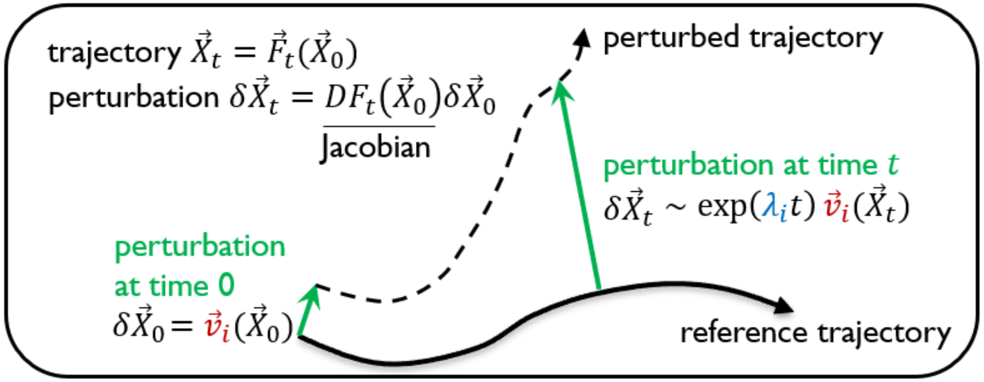
Decay & no tangency for $i \geq 44$
→ perturbations outside IM!



IM dimension
(necessary # of variables)
is determined! $N = 43$

This Opens the Door of IM to Numerics … but not to Experiments

- Need to know the time evolution equation!
  Even the measurement of the exponents is limited to tiny systems.
  "Butterfly effect" is unobservable for large experimental systems!



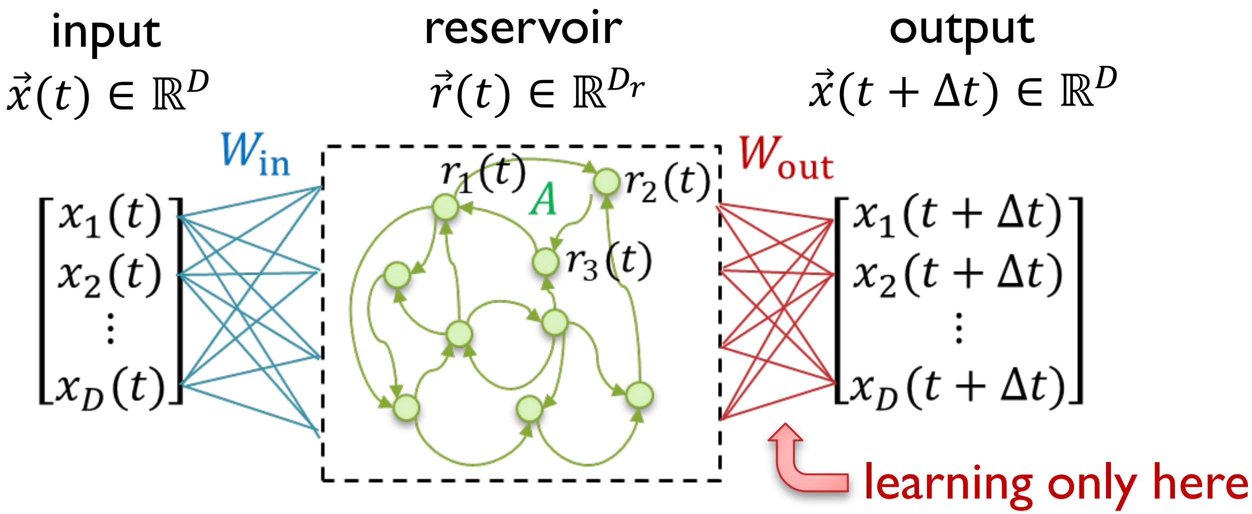
- Edward Ott group (2017) [Pathak et al. Chaos 27, 121102 (2017); PRL 2018] proposed a machine learning method (reservoir computing) to evaluate Lyapunov exponents from time-series data
  (they demonstrated with KS, but not with experiments)

- Here: by measuring not only the exponents but the vectors, we test if reservoir computing can capture IM from time series data.

Reservoir Computing

[Maass et al. Neural Comput. 2002;
Jaeger & Haas, Science 2004]
Also recall Nakajima-san's talk yesterday.

A variant of recurrent neural network (RNN)



$$\vec{r}(t+\Delta t) = \tanh[A\vec{r}(t) + W_{\text{in}}\vec{x}(t)], \qquad \vec{x}(t+\Delta t) = W_{\text{out}}\vec{r}\,'(t+\Delta t)$$

Trick to break the sign-reversal symmetry → with $r_i' = \begin{cases} r_i & (i \text{ is odd}) \\ r_i^2 & (i \text{ is even}) \end{cases}$
[Pathak et al., Chaos 27, 121102 (2017)]

We applied this to KS & computed both exponents & vectors.

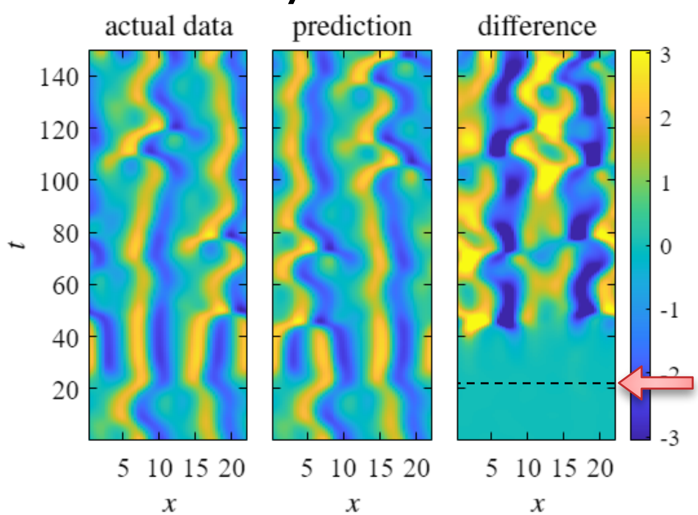
(Ginelli et al. PRL 2007's standard vector computation method can be used)

$D_r = 5000,\ D = 64$ (sampling intervals: $\Delta x = \frac{L}{D} \approx 0.34, \Delta t = 0.005$)

A: directed Erdős-Rényi graph, W_{in}: each r_i is linked to a single x_j

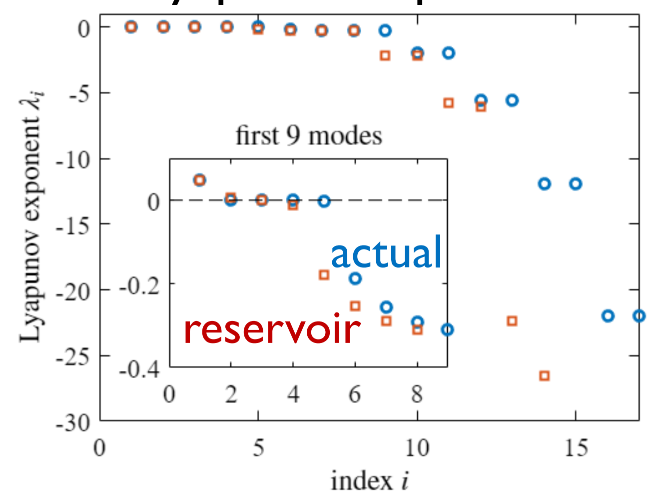
Results (KS time series + Reservoir Computing)

dynamics



→ trained successfully

Lyapunov exponents

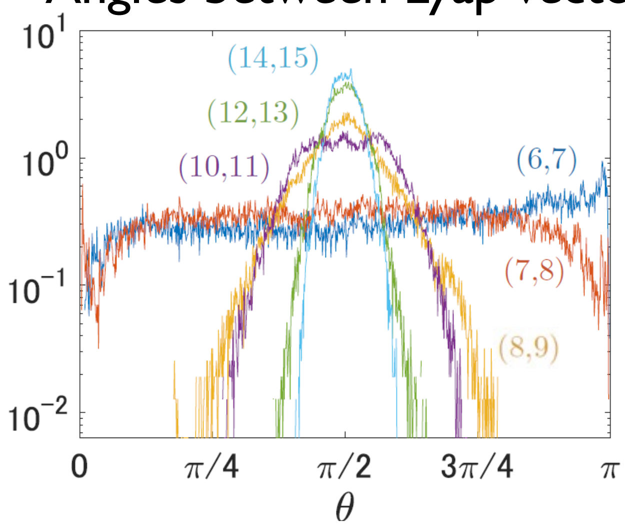


→ spectrum is essentially reproduced

apart from 1 missing neutral exponent

[similar report in Pathak et al., Chaos 2017]

Angles between Lyap vectors



→ determined the IM dimension at

$N_{\text{reservoir}} = 8$ vs $N_{\text{actual}} = 9$ for $L = 22$ [Ding et al. PRL 2016]

Reservoir successfully captured the IM dimension from time series apart from the missed exponent

What is the Missing Neutral Exponent?

- KS has 3 neutral exponents:

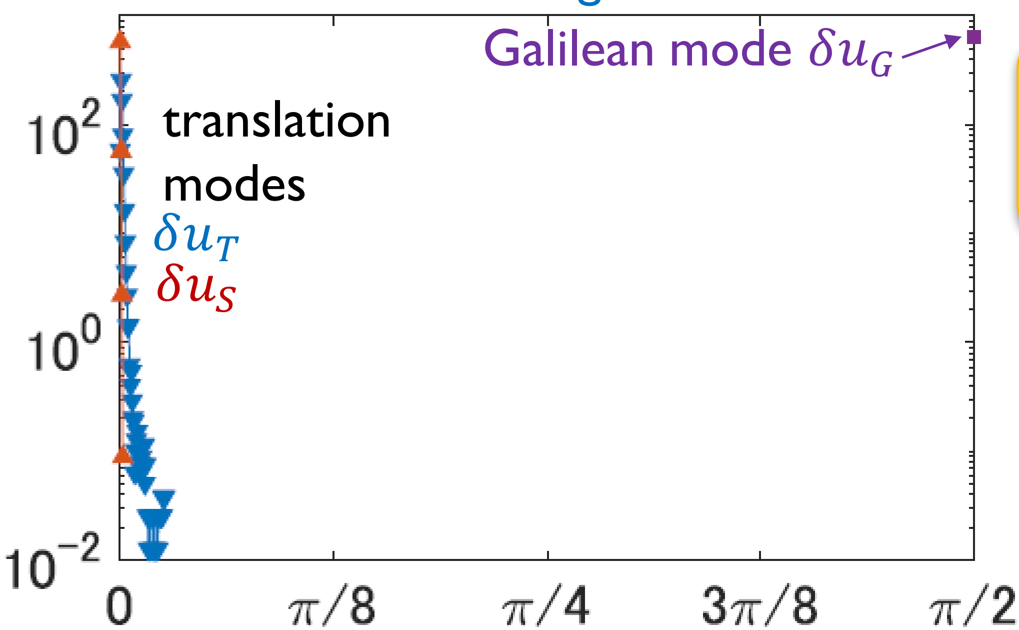
  time trans. $\delta u_T \propto \frac{\partial u}{\partial t}$, space trans. $\delta u_S \propto \frac{\partial u}{\partial x}$, Galilean $\delta u_G \propto 1$

  but only 2 of them were detected.

- If mode δu is detected → $\delta u \in \mathcal{S} \equiv \mathrm{Span}[v_1, v_2, \cdots, v_{N=8}]$

tangent space counterpart of IM
(local approximation of IM)

Distribution of angle b/w δu & $\mathcal{S}$



δu_T & δu_S are detected.
δu_G is undetected!

Presumably because δu_G is linked to conservation of $\int u dx$.

Reservoir can't learn how a perturbation of a conserved quantity evolves.

Summary

[Shimizu, Nishiguchi, Takeuchi,
to appear soon]



We realized Lyapunov analysis of large chaotic systems

→ Now we can measure, e.g.

Lyapunov exponents (the butterfly effect)

inertial manifold (IM) dimension (minimum # of var's)

- IM dimension [Yang, Takeuchi, …, PRL **102**, 074102 (2009); PRE **84**, 046214 (2011)]
  - can be determined *exactly* by inspecting tangencies of Lyapunov vectors
  - Numerically demonstrated for Kuramoto-Sivashinsky eq. (& others)
- Time-series data + reservoir computing [Shimizu, Nishiguchi, Takeuchi, to appear soon]
  - We can evaluate Lyapunov vectors too!
  - For Kuramoto-Sivashinsky, this correctly estimated the IM dimension & identified what neutral exponent was missing.