Dynamics of large oscillator populations with random interactions

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Populations of globally coupled oscillators appear in different fields of physics, engineering, and life sciences. In many situations, there is disorder in the coupling, and the coupling terms are not identical but vary, for example, due to different coupling strengths and phase shifts. While the phenomenon of collective synchronization in oscillator populations which attracted much interest in the last decades is well-understood in a regular situation, the influence of disorder remains a subject of intensive current studies. The disordered case is relevant for many applications, especially in neuroscience, where in the description of the correlated activity of neurons, one can hardly assume the neurons themselves to be identical and the coupling between them to be uniform.

We explore large populations of rorators $\varphi_k(t)$ $(k = 1, \dots, N)$ interacting via random coupling functions:

$$\mu\ddot{\varphi}_k + \dot{\varphi}_k = \omega_k + \sigma\xi_k(t) + H\big(\{\varphi_j(t)\}\big), \tag{1}$$

where each $\varphi_k(t)$ is assumed to be a phase or an angle variable with a first-order ($\mu = 0$) or a second-order($\mu \neq 0$) in time dynamics, respectively. Here, we assume that the individual phase dynamics of an oscillator is described within the "standard" model as rotations with a natural frequency ω_k, possibly with individual Gaussian white noises $\sigma\xi_k(t)$. In Eq. (1), we separate this individual dynamics and the coupling terms $H\big(\varphi_1(t), \varphi_2(t), \dots, \varphi_N(t)\big) = H\big(\{\varphi_j(t)\}\big)$. Note that the model (1) with $\mu = 0$ corresponds to the model of coupled phase oscillators which is most popular because it can be directly derived for generic coupled oscillators from the original equations governing the oscillator dynamics, in the first order in the small parameter describing the coupling. The model (1) with $\mu \neq 0$ are discussed in the literature, for example, in the context of modeling power grads.

Next, we specify the coupling terms $H\big(\{\varphi_j(t)\}\big)$ according to the Kuramoto-Daido and the Winfree approaches. In both two cases, we assume that all the pairwise coupling terms are different, taken from some random distribution of random functions. In this assumption that all the coupling terms are generally different, the coupling function in the Kuramoto-Daido form as a function of phase differences $(\varphi_j - \varphi_k)$ reads

$$H_{KD}\big(\{\varphi_j(t)\}\big) = \frac{1}{N}\sum_{j=1}^{N} F_{jk}(\varphi_j - \varphi_k). \tag{2}$$

For the Winfree-type model, in a such case of the general randomness case, the action on the oscillator k from the oscillator j is proportional to the product $S_{jk}(\varphi_k)Q_{jk}(\varphi_j)$, where $S_{jk}(\varphi_k)$ is the j-th phase sensitivity function of the unit k, and $Q_{jk}(\varphi_j)$ describe the force with which the element j is acting on the oscillator k:

$$H_W\big(\{\varphi_j(t)\}\big) = \frac{1}{N}\sum_{j=1}^{N} S_{jk}(\varphi_k)Q_{jk}(\varphi_j). \tag{3}$$

It is well known that, in the regular setups, the Kuramoto-Daido and the Winfree coupling functions can be reformulated in terms of the Kuramoto-Daido order parameters $Z_m(t)$ which are defined as

$$Z_m(t) = \frac{1}{N}\sum_{j=1}^{N} e^{\mathrm{i}m\varphi_j(t)} = \left\langle e^{\mathrm{i}m\varphi_j(t)} \right\rangle. \tag{4}$$

One can obtain these representations representing the 2π-periodic coupling functions as Fourier series. We use these expressions as "templates" for identifying the effective coupling functions in the case of random interactions.

Thus, we represent the functions $F_{jk}(x)$, $S_{jk}(x)$ and $Q_{jk}(x)$ describing random pairwise interactions in the Kuramoto-Daido and the Winfree models via random complex Fourier coefficients $f_{m,jk}$, $s_{m,jk}$ and $q_{m,jk}$, respectively:

$$F_{jk}(x) = \sum_m f_{m,jk}e^{\mathrm{i}mx}\,, \quad S_{jk}(x) = \sum_m s_{m,jk}e^{\mathrm{i}mx}\,, \quad Q_{jk}(x) = \sum_m q_{m,jk}\,e^{\mathrm{i}mx}\,, \tag{5}$$

$$f_{m,jk} = \frac{1}{2\pi}\int_0^{2\pi} dx\, F_{jk}(x)e^{-\mathrm{i}mx}\,, \quad s_{m,jk} = \frac{1}{2\pi}\int_0^{2\pi} dx\, S_{jk}(x)e^{-\mathrm{i}mx}\,, \quad q_{m,jk} = \frac{1}{2\pi}\int_0^{2\pi} dx\, Q_{jk}(x)\,e^{-\mathrm{i}mx}\,. \tag{6}$$

Next, we assume statistical independence of the phases and the corresponding Fourier coefficients. We expect this independence to be valid for a large population, where many different couplings influence each phase. This assumption

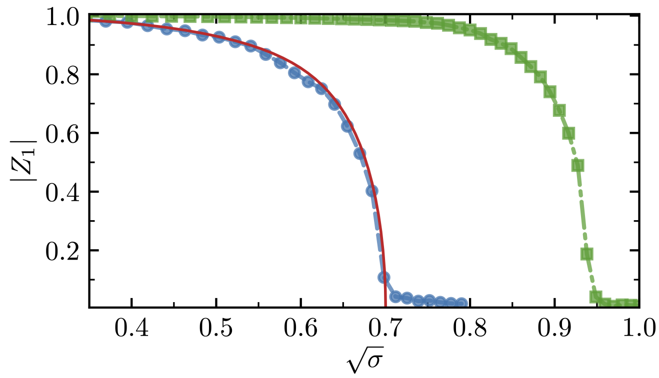

Рис. 1: Behavior of the first order parameter $\langle|Z_1|\rangle$ an ensemble of $N=12\times10^3$ noisy rotators (1) with equal natural frequencies ($\omega_k=\Omega$) and coupling function $F(x)=K\big(\sin(x)+4\sin(2x)\big)$ with $K=1$ in dependence on $\sqrt{\sigma}$ for the moment of inertia $\mu=0.5$. Green squares and blue circles are simulations without and with phase shifts, respectively. We consider random phase shifts α_{jk} distributed according to $G(\alpha)=(1+\cos M\alpha)/2\pi$ with $M=1$. Thus, the effective coupling function is $\mathcal{F}(x)=0.5\,K\sin(x)$. For such coupling, the analytical expression (solid red line) for the order parameter in dependence on the noise intensity σ^2 can be written in the parametric (parameter R) form: $|Z_1|=2\pi RI_0^2(R)I_1(R)\big/\big(2\pi RI_0^2(R)+\mu KI_1(R)\big)$, $\sigma^2=K|Z_1|/2R$, where $I_0(R)$ and $I_1(R)$ are the principal branches of the modified Bessel functions of the first kind with orders 0 and 1, respectively.

allows us to obtain the reduced coupling terms and conclude that the interaction is described with an effective deterministic coupling. For the Kuramoto-Daido-type model, we arrive at the effective averaged coupling function, Fourier modes of which are just $\langle f_{m,jk}\rangle$:

$$\frac{1}{N}\sum_{j=1}^{N}F_{jk}(\varphi_j-\varphi_k)\;\Rightarrow\;\frac{1}{N}\sum_{j=1}^{N}\mathcal{F}(\varphi_j-\varphi_k)=\frac{1}{N}\sum_{j=1}^{N}\langle F_{jk}(\varphi_j-\varphi_k)\rangle=\sum_{m}\langle f_{m,jk}\rangle\, e^{-\mathrm{i}m\varphi_k}Z_m. \tag{7}$$

For the random Winfree-type model, we have

$$\frac{1}{N}\sum_{j=1}^{N}S_{jk}(\varphi_k)Q_{jk}(\varphi_j)\;\Rightarrow\;\mathcal{S}(\varphi_k)\frac{1}{N}\sum_{j}\mathcal{Q}(\varphi_j)=\langle S_{jk}(\varphi_k)\rangle\frac{1}{N}\sum_{j=1}^{N}\langle Q_{jk}(\varphi_j)\rangle=\sum_{m}\langle s_{m,jk}\rangle\, e^{\mathrm{i}m\varphi_k}\sum_{m'}\langle q_{m',jk}\rangle\, Z_{m'}\,. \tag{8}$$

It is worth mentioning that because the Fourier transform is a linear operation, averaging the Fourier coefficients is the same as averaging the functions. Thus, our main theoretical result is that one can reduce the dynamics of a large population with random coupling functions to an effective ensemble without disorder, where the effective coupling functions are averages of the original random coupling functions.

The relations (7) and (8) are derived in the case of general randomness of interactions, which includes a situation where different coupling functions have different shapes. For example, some oscillators can be coupled via the first harmonic coupling function, while others are coupled with the second harmonic coupling function. A particular situation is one where all the shapes are the same, but the interactions differ in their coupling strengths and the phase shifts. Using (7) and (8) in the case where the randomness is restricted to coupling strength and phase shifts, one can see that the randomness of coupling strengths renormalizes the total coupling strength, but does not influence the shape of the coupling function. In contradistinction, the randomness of the phase shifts changes the form of the coupling function and the effective coupling function is the convolution operator of the original one with the phase shift distribution density. Our exhaustive numerical simulations confirm this theoretical prediction (e.g., see Fig. 1).

Summarizing the results, we have considered different models of globally coupled phase oscillators and rotators. In the case of a “maximal disorder”, all the coupling functions are distinct and random, sampled from some distribution. Based on the assumption of independence of the phases and the coupling functions in the thermodynamic limit, we derived the averaged equations for the phases, where effective deterministic coupling functions enter. A more detailed consideration was devoted to the case where the shapes of the random coupling functions are the same, but the amplitudes and the phase shifts are random. Then, the effective functions are renormalized convolutions of the original coupling functions and the distribution densities of the phase shifts. In particular, if the distribution of the phase shifts possesses just one Fourier mode, the effective coupling function will possess only this mode, too. This property allows us to check the validity of the approach numerically because, for the one-mode coupling function, there are theoretical predictions for the behavior of the order parameters.

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