Dynamics Days Asia Pacific 13 / YKIS2024

Exact solutions in the multi-dimensional ASEP

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Contents

1. Multi-dimensional ASEP [arXiv:2403.01934] (to be published in Phys. Rev. Research)
2. 2D ASEP with Langmuir kinetics [arXiv:2405.09261]

ASEP (1D)

Asymmetric random walk of particles with hardcore interactions on a lattice

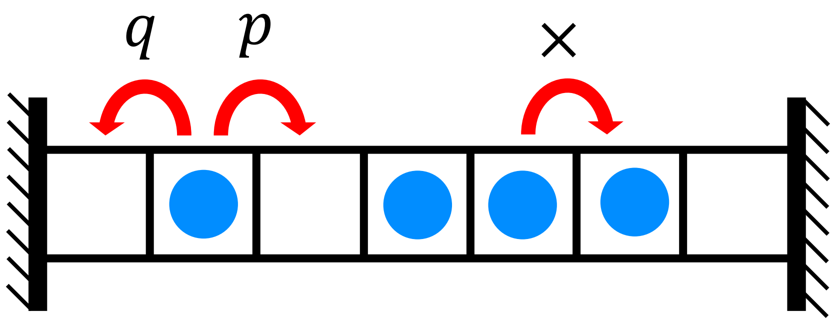
- **Rules**

1. Asymmetric hopping :

   Particles move to the right (left) site with a rate $\boldsymbol{p}$ ($\boldsymbol{q}$).

2. Hardcore interactions :

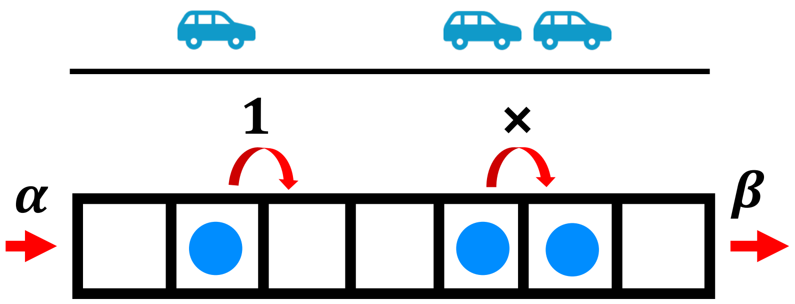
   Each site contains a single particle at most.



- **Applications**
  - Vehicular traffic flow

    [Physica A 285, 101]
  - Biological transportation

    [Biopolymers 6, 1]



- **Rich physics**
  - KPZ universality class

    [Commun. Math. Phys. 183, 571]
  - Boundary-induced phase transition

    [J. Phys. A: Math. Theor. 40, R333]

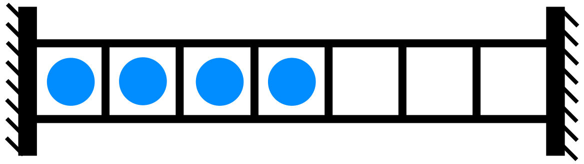
I. Introduction: What does "exactly solvable" mean ?

[arXiv:2403.01934]

1D ASEP is an exactly solvable model.

The physics of ASEP

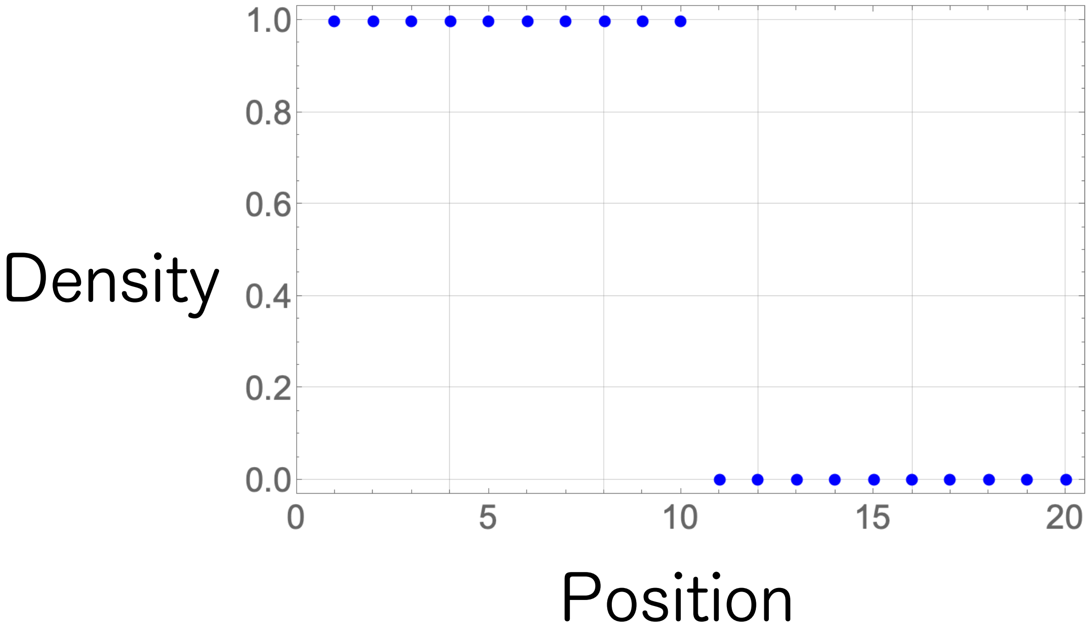
Initial state

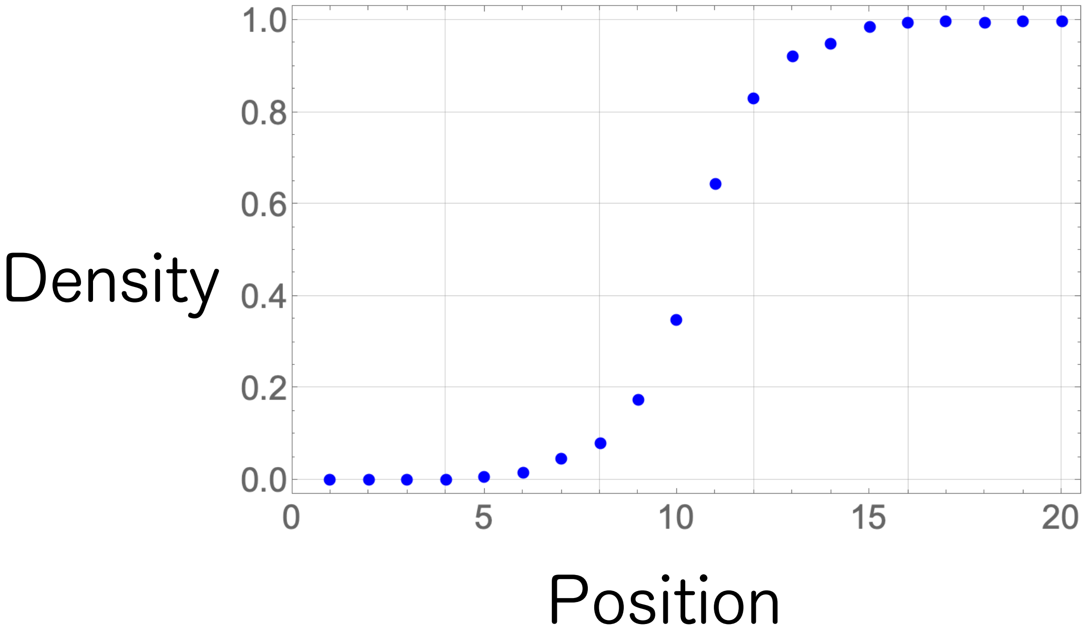


Time evolution
(Dynamics)

Steady state

?





Ⅰ. Introduction: What does “exactly solvable” mean ?

[arXiv:2403.01934]

1D ASEP is an exactly solvable model.

ASEP is formulated like a quantum mechanics.

- **Time evolution**

Master eq. $$\frac{d}{dt}|P(t)\rangle = \mathcal{H}|P(t)\rangle$$

- $\mathcal{H}$: Hamiltonian (Markov matrix)

$$\mathcal{H} = \sum_{j=1}^{L}\left[pS_j^+S_{j+1}^- + qS_j^-S_{j+1}^+ + S_j^zS_{j+1}^z - \frac{1}{4}\right]$$

- State vector $|P(t)\rangle = \sum_n P(n,t)|n\rangle$

$|n\rangle$: configuration of particles

$P(n,t)$: probability of being n

- **Steady state**

Eigenstate with a zero eigenvalue of $\mathcal{H}$

$$\left(\frac{d}{dt}|P(n)\rangle = 0 \longleftrightarrow \mathcal{H}|P_{st}\rangle = 0\right)$$

cf) Matrix product ansatz, Quantum group

[J. Phys. A: Math. Theor. 40, R333] [Europhys. Lett. 26, 7]

- **Dynamics**

All eigenstates of $\mathcal{H}$

cf) Bethe ansatz

[J. Phys. A: Math. Gen. 39, 12679]

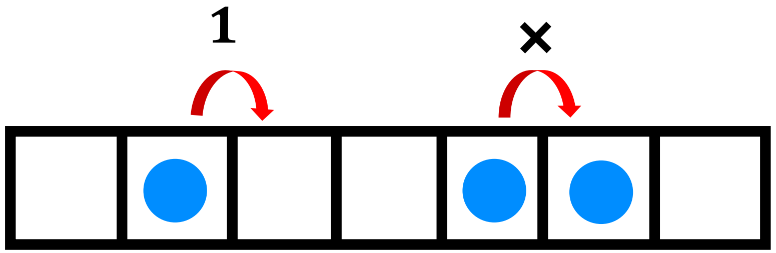
[Y. Ishiguro, et al. Phys. Rev. Res. 5, 033102]

2. Motivation: Exactly solvable model beyond 1D

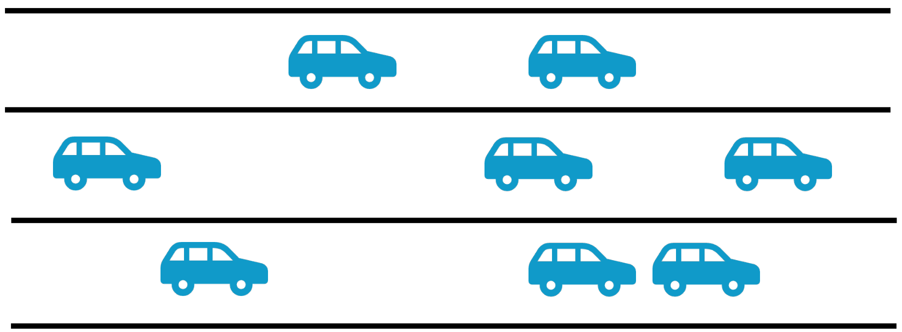
[arXiv:2403.01934]

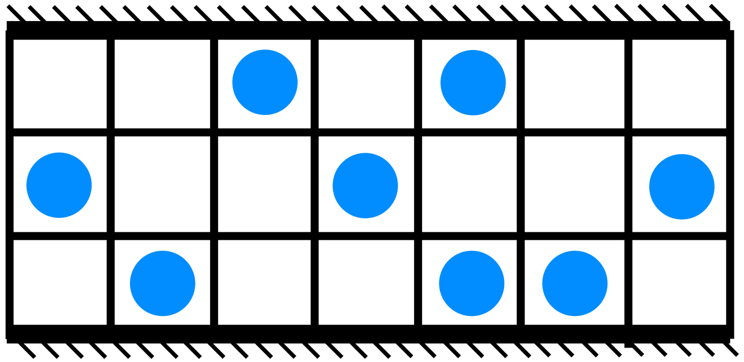
Motivation

- Most of the theory of exact solvable models are limited to 1D systems.



- Many natural phenomena in the real world occur in systems beyond one dimension.



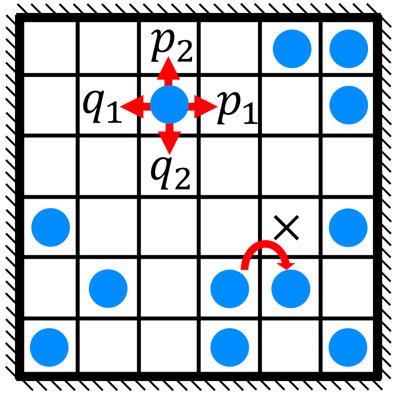


Pioneering the theory of exactly solvable models beyond 1D systems.

Steady states → Dynamics → Theory

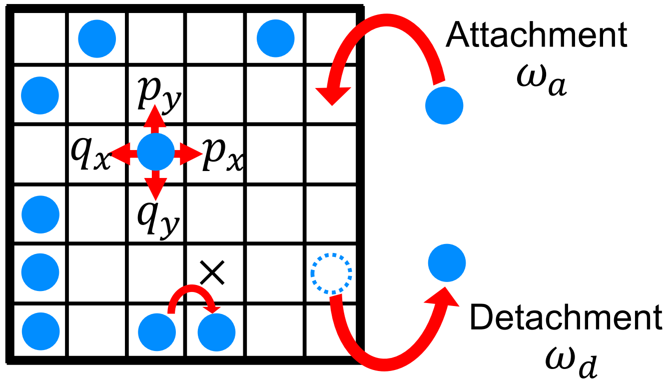
In this talk, we present

1. Exact steady states in the ASEP in an arbitrary dimensional lattice.
[arXiv:2403.01934]



The number of particles is conserved.

2. Exact steady states in the 2D ASEP with Langmuir kinetics.
[arXiv:2405.09261]



The number of particles is not conserved.

2. Motivation: Exactly solvable model beyond 1D

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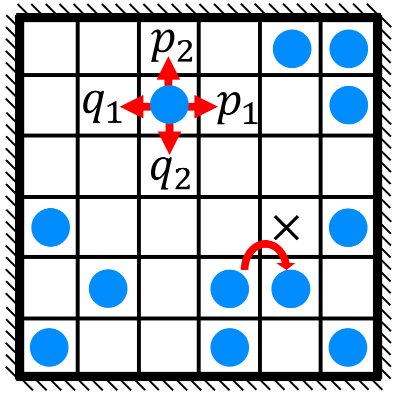


The number of particles is conserved.

2. Exact steady states in the 2D ASEP with Langmuir kinetics. [arXiv:2405.09261]

The number of particles is not conserved.

3. Model: Multidimensional ASEP

[arXiv:2403.01934]

Multidimensional ASEP

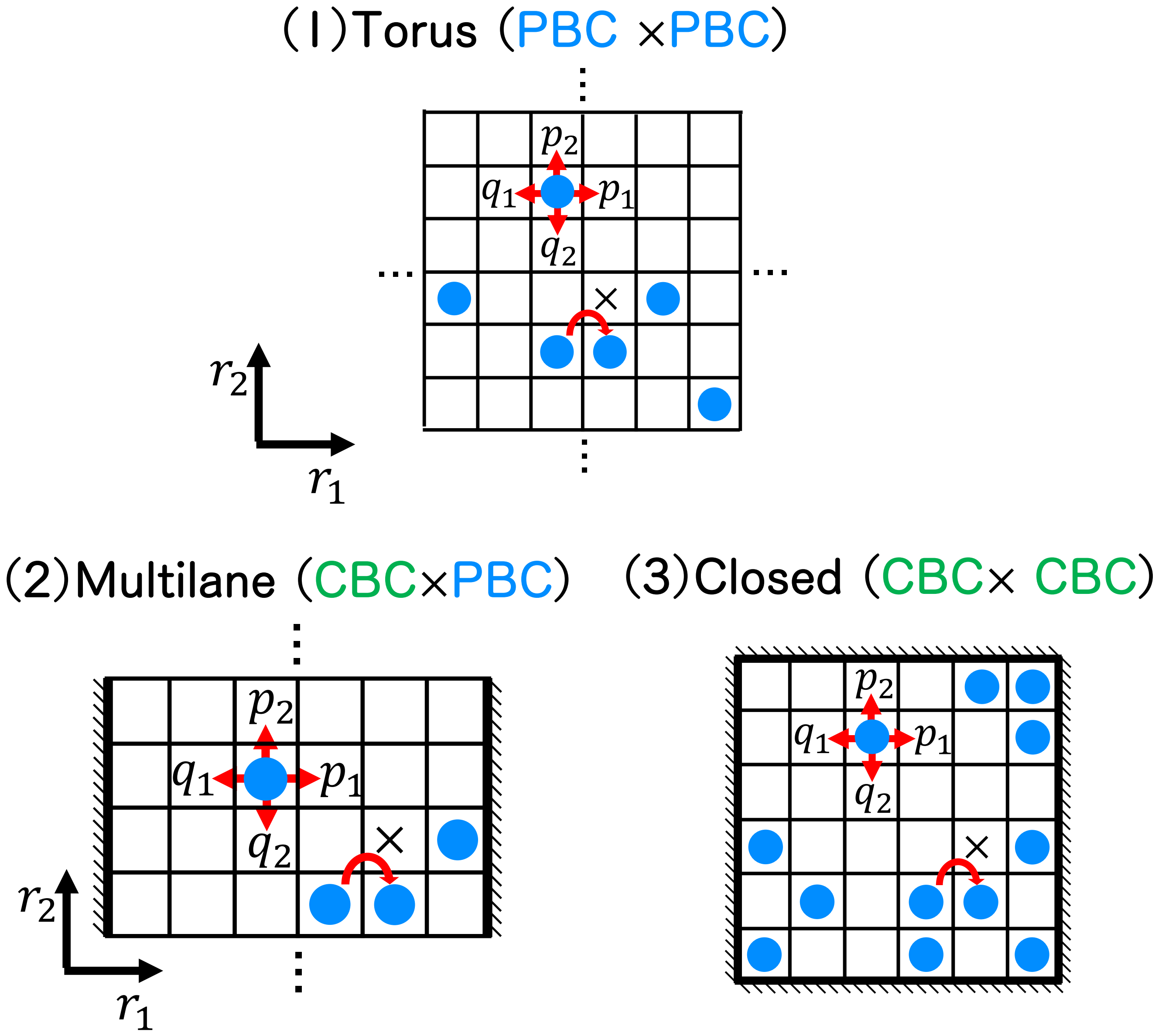
- Settings
  - d-dimensional lattice $(L_1 \times L_1 \times \cdots \times L_d)$
  - Boundary condition is Closed (CBC) and Periodic (PBC).

    CBC for r_i $(1 \leq i \leq \ell)$ directions
    PBC for r_i $(\ell < i \leq d)$ directions

- Rules

1. Asymmetric hopping :
   Particles move to the forward (backward) site in the r_i direction with a rate $\boldsymbol{p_i}$ $(\boldsymbol{q_i})$.
2. Hardcore interactions :
   Each site contains a single particle at most.

ex) 2D ASEP



4. Result : Exact steady states

[arXiv:2403.01934]

We constructed the exact steady states of the ASEP in arbitrary dimension.

- Steady state of the d-dimensional ASEP

$$|S_N\rangle = \frac{1}{Z_N}\sum_{n_N} P_{\rm st}(n_N)|n_N\rangle$$

$$P_{\rm st}(n_N) = \prod_{i=1}^{\ell}\left(\frac{p_i}{q_i}\right)^{\sum_{j=1}^{N} r_{j;i}} \qquad \left[Z_N := \sum_{n_N} P_{\rm st}(n_N)\right]$$

($r_{j;i}$: position r_i of the jth particle)

that satisfies

$$\mathcal{H}_{\rm ASEP}|S_N\rangle = 0$$

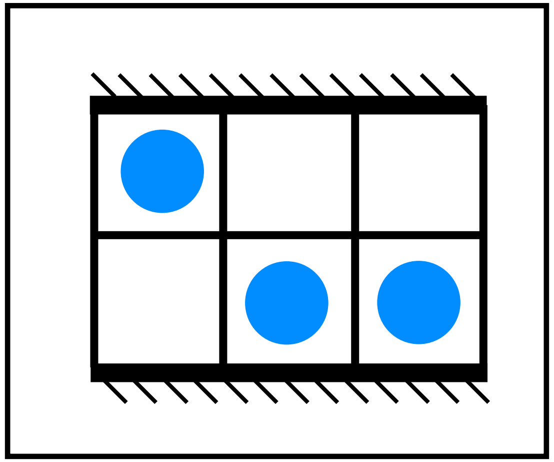
where the Hamiltonian is given by

$$\mathcal{H}_{\rm ASEP} = \left(\sum_{i=1}^{\ell}\sum_{r_i=1}^{L_i-1}\sum_{\{r_1,\cdots,r_d\}\backslash r_i} + \sum_{i=\ell+1}^{d}\sum_{\{r_1,\cdots,r_d\}}\right)\left[p_i\left\{\hat{S}^+_{\boldsymbol{r}}\hat{S}^-_{\boldsymbol{r}+\boldsymbol{e}_i} - \hat{n}_{\boldsymbol{r}}\left(1-\hat{n}_{\boldsymbol{r}+\boldsymbol{e}_i}\right)\right\} + q_i\left\{\hat{S}^-_{\boldsymbol{r}}\hat{S}^+_{\boldsymbol{r}+\boldsymbol{e}_i} - \left(1-\hat{n}_{\boldsymbol{r}}\right)\hat{n}_{\boldsymbol{r}+\boldsymbol{e}_i}\right\}\right]$$

The key idea is "transition decomposition"

ex) 2D multilane ASEP (CBC×PBC)

- What state does it transition to?



 [arXiv:2403.01934]

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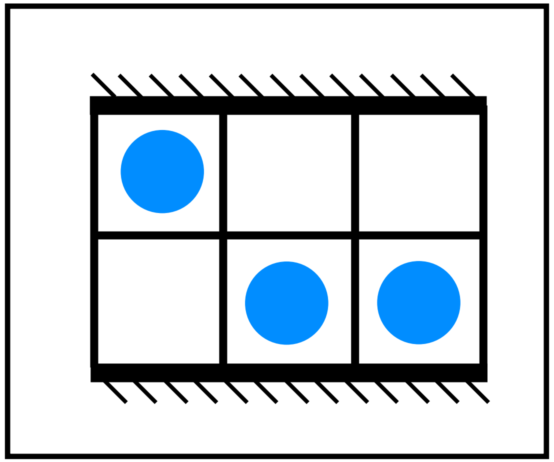

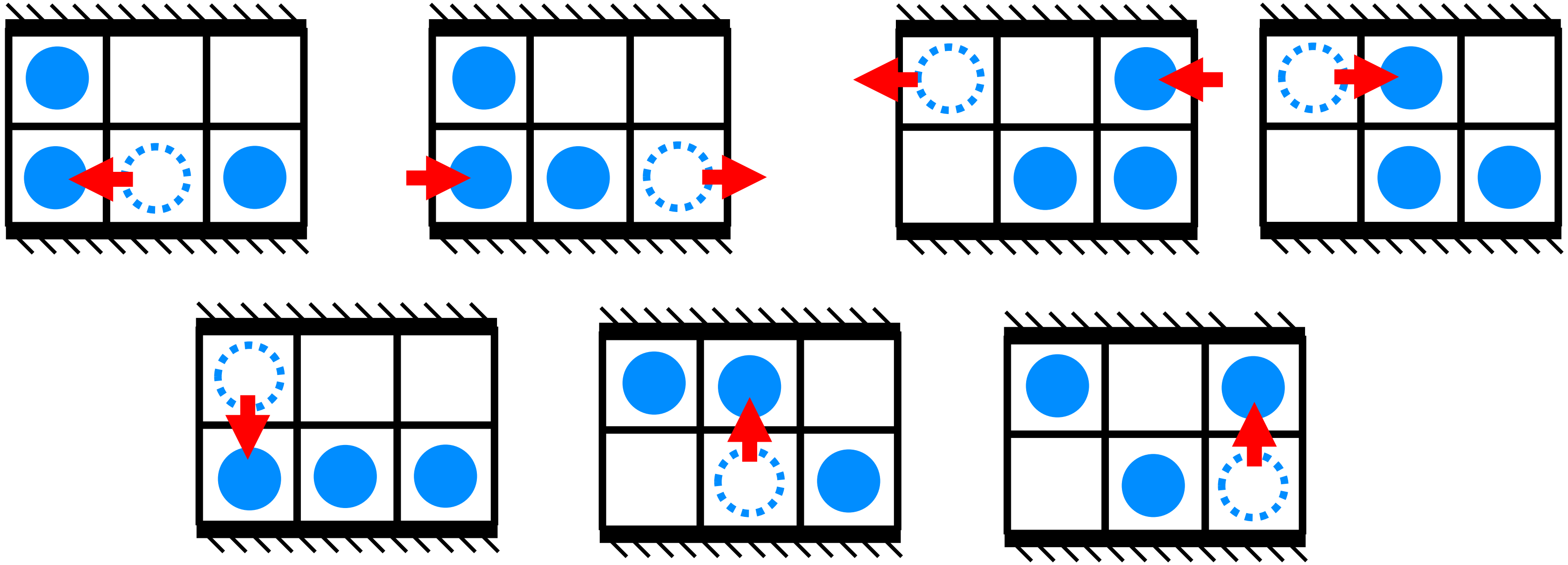

 [arXiv:2403.01934]

The key idea is ”transition decomposition”

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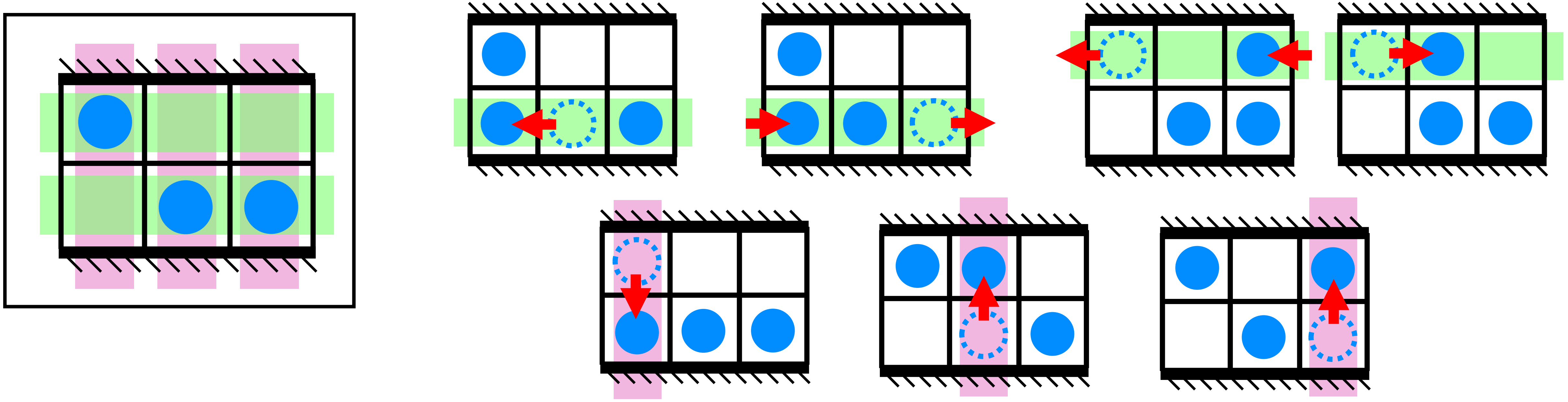


The transitions of multidimensional ASEP can be decomposed into those of 1D ASEPs.

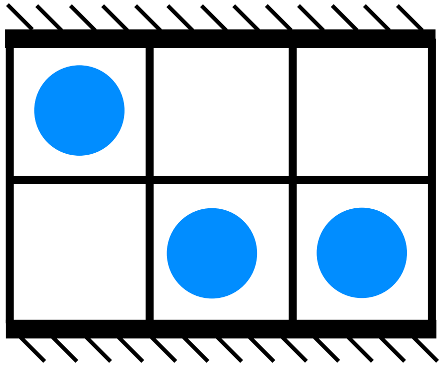
4. Result : Exact steady states

[arXiv:2403.01934]

The key idea is ”transition decomposition”

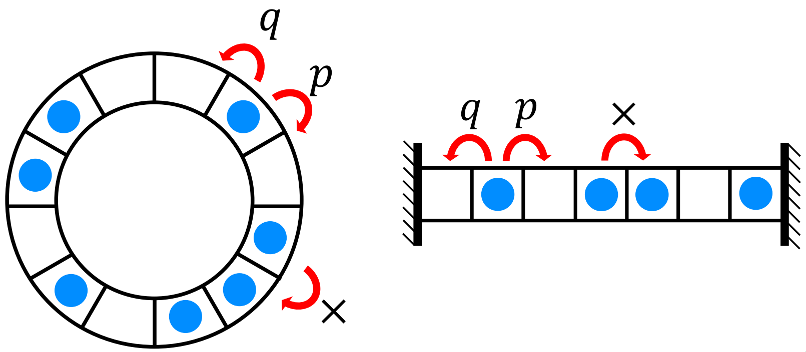
Outline of the proof

Multidimensional ASEP



“decompose” →

Combination of 1D ASEPs



The steady state of the 1D ASEP can be constructed exactly.

The steady state of the multidimensional ASEP can be constructed by combining those of the 1D ASEPs.

5. Application：Quasi-1D-flow of the multilane ASEP

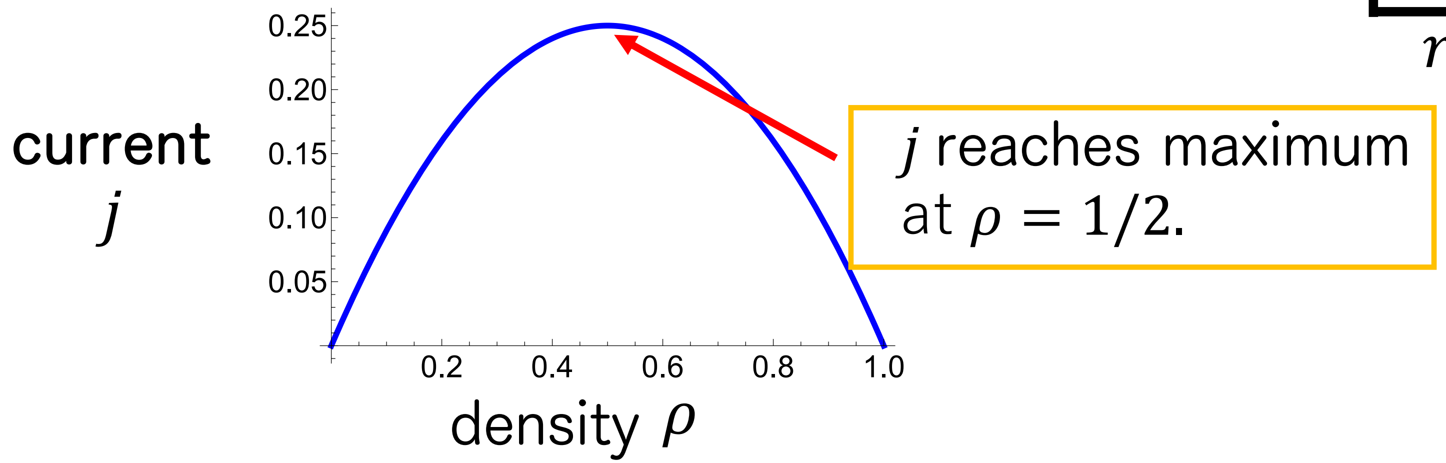
As an example, we investigated the effect of the two-dimensionality on the quasi-1D-flow.

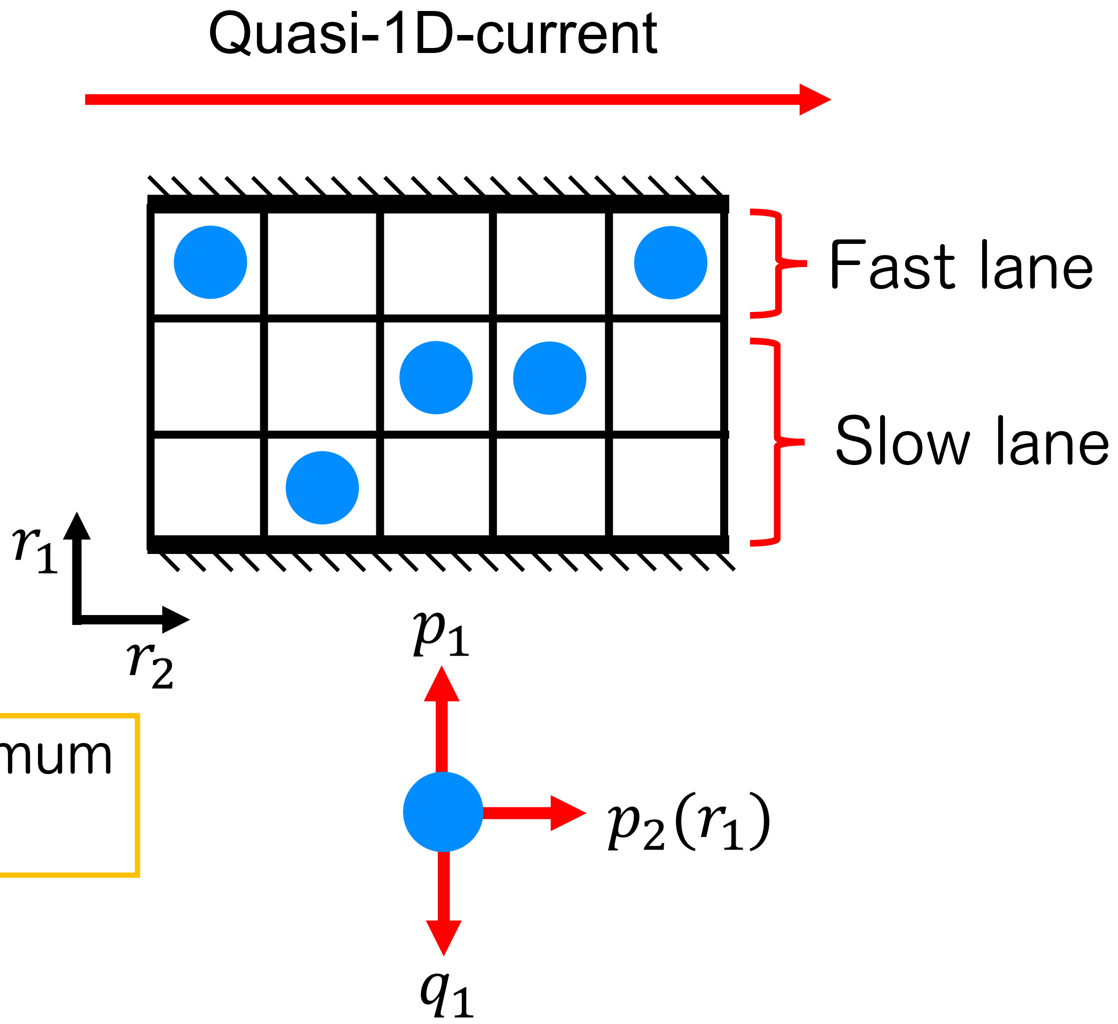
ex) 2D multilane ASEP (CBC×PBC)

- Two types of lanes
  - Slow lane $p_2(r_1) = 0.3$
  - Fast lane $p_2(r_1) = 1.0$

Q：How does the current change compared to the one-dimensional case?

cf) Current vs density in the 1D ASEP





5. Application : Quasi-1D-flow of the multilane ASEP

[arXiv:2403.01934]

As an example, we investigated the effect of the two-dimensionality on the quasi-1D-flow.

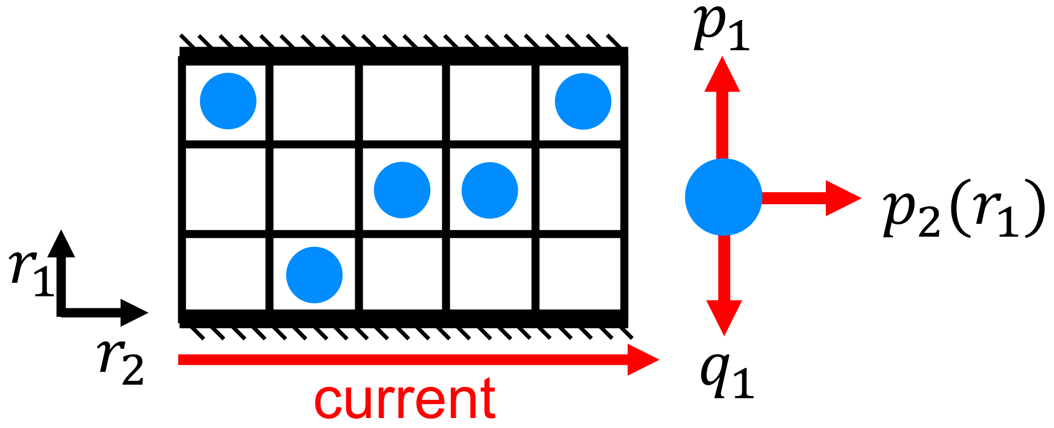
ex) 2D multilane ASEP (CBC×PBC)

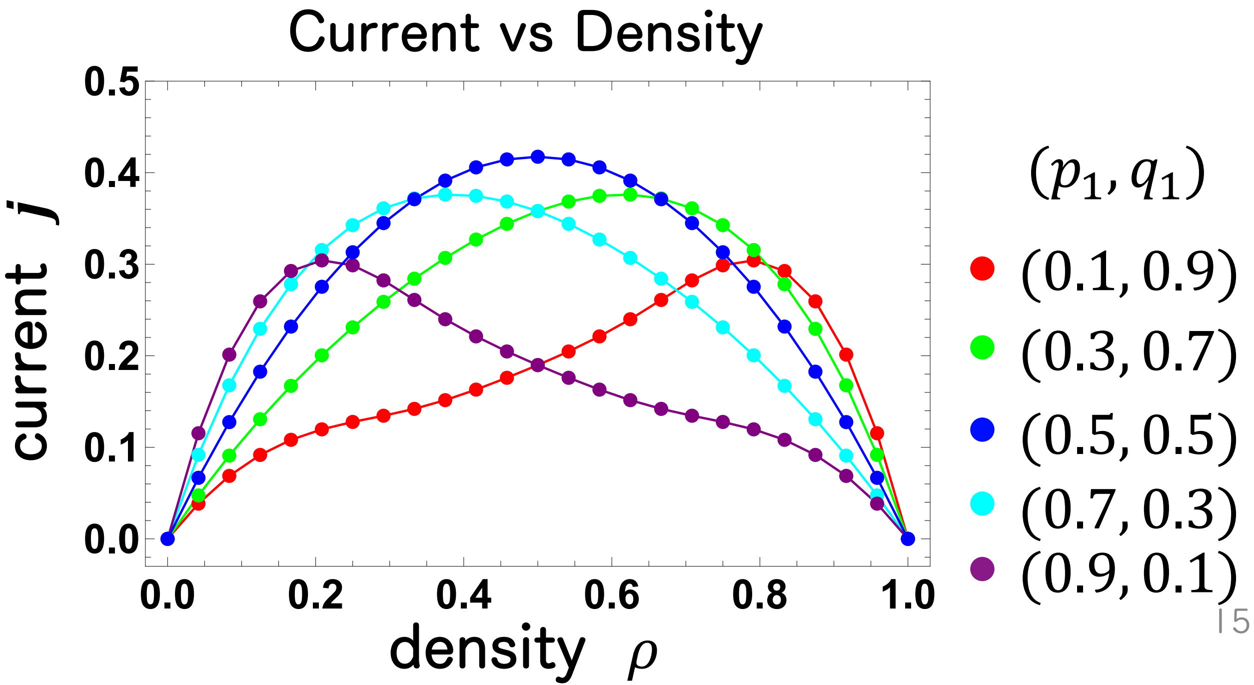
- $p_1 = q_1$ (blue line)

  Equivalent to the 1D case.

- $p_1 \neq q_1$

  Two-dimensionality changes the property of the quasi-1D-flow.





- Quasi-1D-current (r_2 direction)

$$j = \frac{1}{Z} \sum_{r_1=1}^{L_1} \sum_{\tilde{n}} p_2(r_1) \left(\frac{p_1}{q_1}\right)^{\sum_{j=1}^{N} r_{j;1}}$$

6. Introduction: ASEP with Langmuir kinetics

[arXiv:2405.09261]

Pioneering the theory of exactly solvable models beyond 1D systems.

Steady states → Dynamics → Theory

In this talk, we present

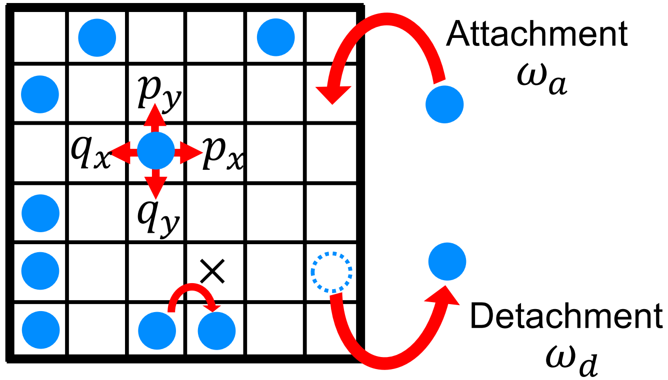
1. Exact steady states in the ASEP in an arbitrary dimensional lattice.

[arXiv:2403.01934]

The number of particles is conserved.

2. Exact steady states in the 2D ASEP with Langmuir kinetics.

[arXiv:2405.09261]

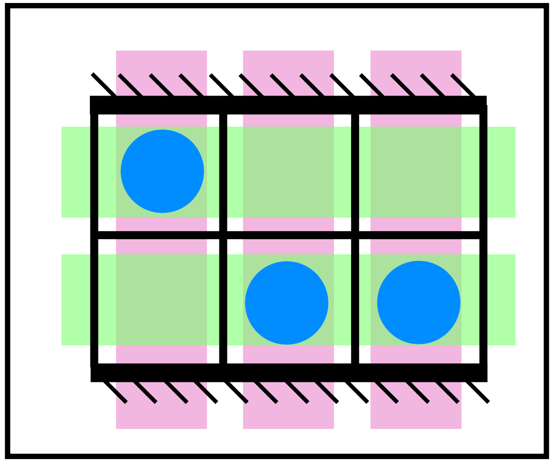


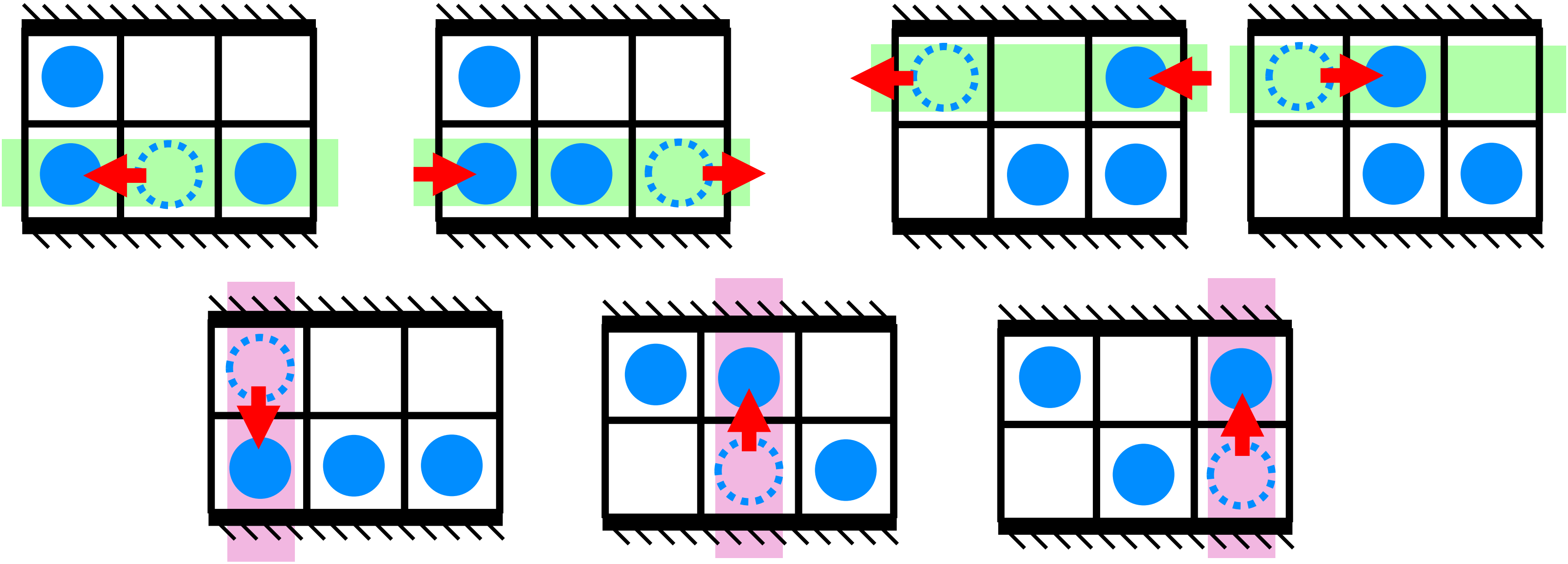
The number of particles is not conserved.

"Transition decomposition" is applicable due to the particle number conservation.

ex) 2D multilane ASEP (CBC×PBC)

- What state does it transition to?





Is it possible to construct the steady state where the particle number conservation is violated?

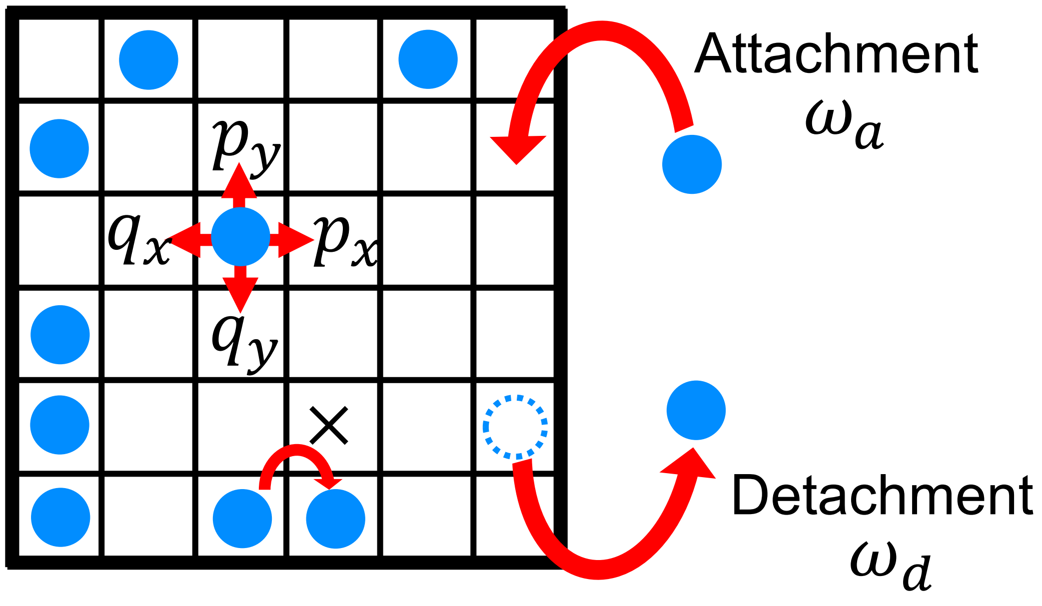
Violate the particle number conservation by introducing attachment and detachment of particles.

2D ASEP with Langmuir kinetics (ASEP-LK)

- **Settings**
  - 2-dimensional lattice ($L_T = L_x \times L_y$)
  - Boundary condition is CBC and PBC.

- **Rules**

1. Asymmetric hopping :
   Particles move to the forward (backward) site in the r_i direction with a rate $\boldsymbol{p_i}$ $(\boldsymbol{q_i})$.
2. Hardcore interactions :
   Each site contains a single particle at most.
3. Langmuir kinetics
   Particles attach (detach) a site with a rate $\boldsymbol{\omega_a}$ $(\boldsymbol{\omega_d})$.



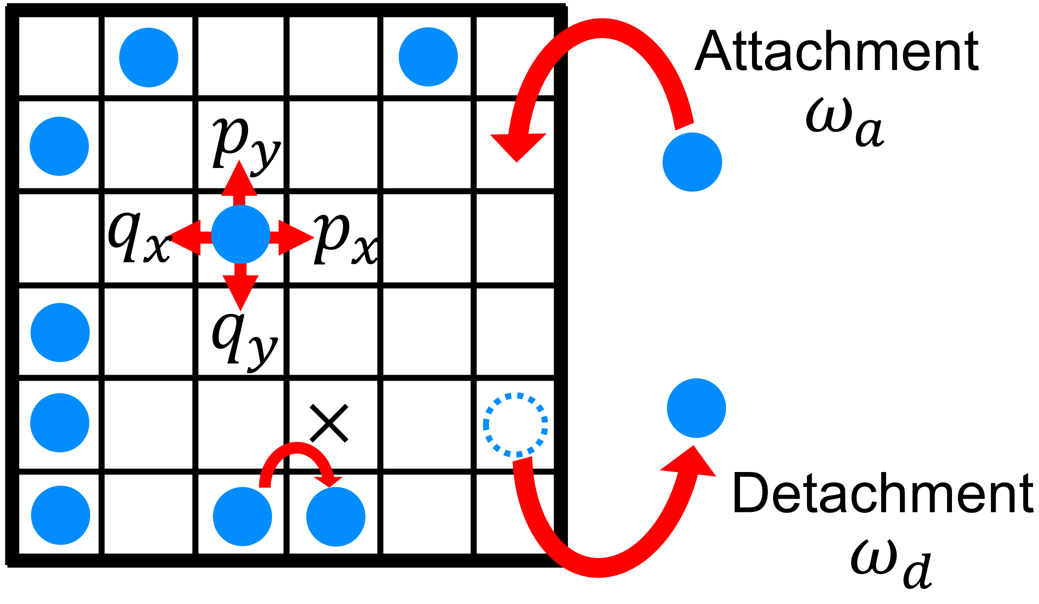
Violate the particle number conservation by introducing attachment and detachment of particles.

2D ASEP with Langmuir kinetics (ASEP-LK)

- Master equation

$$\frac{d}{dt}|P(t)\rangle = \mathcal{H}_{\text{ASEP-LK}}|P(t)\rangle$$

$$\mathcal{H}_{\text{ASEP-LK}} = \underbrace{\sum_{i\in\{x,y\}}\sum_{x,y}\mathcal{M}_{\boldsymbol{r},\boldsymbol{r}+\boldsymbol{e}_i}}_{\text{hopping}} + \underbrace{\sum_{\boldsymbol{r}} h_{\boldsymbol{r}}}_{\text{Langmuir kinetics}}$$



where

$$h_{\boldsymbol{r}} = \omega_{\text{a}}\left[\hat{S}^-_{\boldsymbol{r}} - (1-\hat{n}_{\boldsymbol{r}})\right] + \omega_{\text{d}}\left[\hat{S}^+_{\boldsymbol{r}} - \hat{n}_{\boldsymbol{r}}\right]$$

$$\mathcal{H}_{\text{ASEP}} = \sum_{i\in\{x,y\}}\sum_{x,y}\left[p_i\left\{\hat{S}^+_{\boldsymbol{r}}\hat{S}^-_{\boldsymbol{r}+\boldsymbol{e}_i} - \hat{n}_{\boldsymbol{r}}\left(1-\hat{n}_{\boldsymbol{r}+\boldsymbol{e}_i}\right)\right\} + q_i\left\{\hat{S}^-_{\boldsymbol{r}}\hat{S}^+_{\boldsymbol{r}+\boldsymbol{e}_i} - \left(1-\hat{n}_{\boldsymbol{r}}\right)\hat{n}_{\boldsymbol{r}+\boldsymbol{e}_i}\right\}\right]$$

$$= \sum_{i\in\{x,y\}}\sum_{x,y}\mathcal{M}_{\boldsymbol{r},\boldsymbol{r}+\boldsymbol{e}_i}.$$

7. Result: Exact steady state of the 2D ASEP-LK

Steady state of the ASEP-LK is constructed by superposing those of the ASEP without LK.

● Steady state

$$|S_{LK}\rangle = \frac{1}{(1+\alpha)^{L_\mathrm{T}}} \sum_{N=0}^{L_\mathrm{T}} P_\mathrm{T}(N) |S_N\rangle \qquad \left(P_\mathrm{T}(N) = \binom{L_\mathrm{T}}{N} \alpha^N \right)$$

$|S_N\rangle$: Steady state of the ASEP with N particles

where

- Strength of Langmuir kinetics $\omega := \omega_\mathrm{a} + \omega_\mathrm{d}$
- Ratio of the Langmuir kinetics $\alpha := \dfrac{\omega_\mathrm{a}}{\omega_\mathrm{d}}$

(Proven result)

$|S_\mathrm{LK}\rangle$ is the steady state for any ω in the torus case.

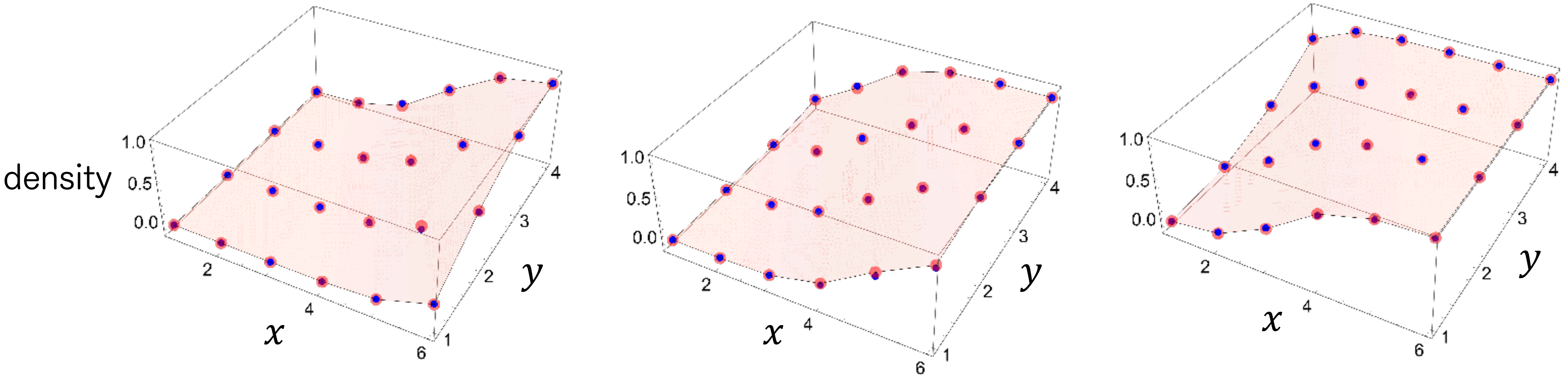
(Conjecture)

$|S_\mathrm{LK}\rangle$ is the steady state for infinitesimal ω with fixed α in the multilane and closed case.

Results from the analytical expressions corresponds those from Monte Carlo simulations.

ex) 2D closed ASEP-LK (CBC× CBC)

$\alpha = 0.3$ $\alpha = 1.0$ $\alpha = 3.0$



- Monte Carlo simulation
- Exact steady state

$$(p_x, q_x, p_y, q_y) = (1.0, 0.1, 0.8, 0.1)$$

8. Summary

[arXiv:2403.01934] · [arXiv:2405.09261]

- Although the ASEP is known as the exactly solvable models, most of the exact results are limited to the 1D systems.
- We constructed the exact steady state of the multidimensional ASEP through the concept of "transition decomposition".
- By employing the result, we also constructed the steady state of the 2D ASEP-LK where the particle number conservation is violated.
- The results correspond to a range of situations, such as asymmetric diffusion in a box and quasi-one-dimensional flow in a tube.
- As an example, we considered the 2D multilane ASEP with inhomogeneous lanes, and revealed the effect of the two-dimensionality on the qusi-1D-current.