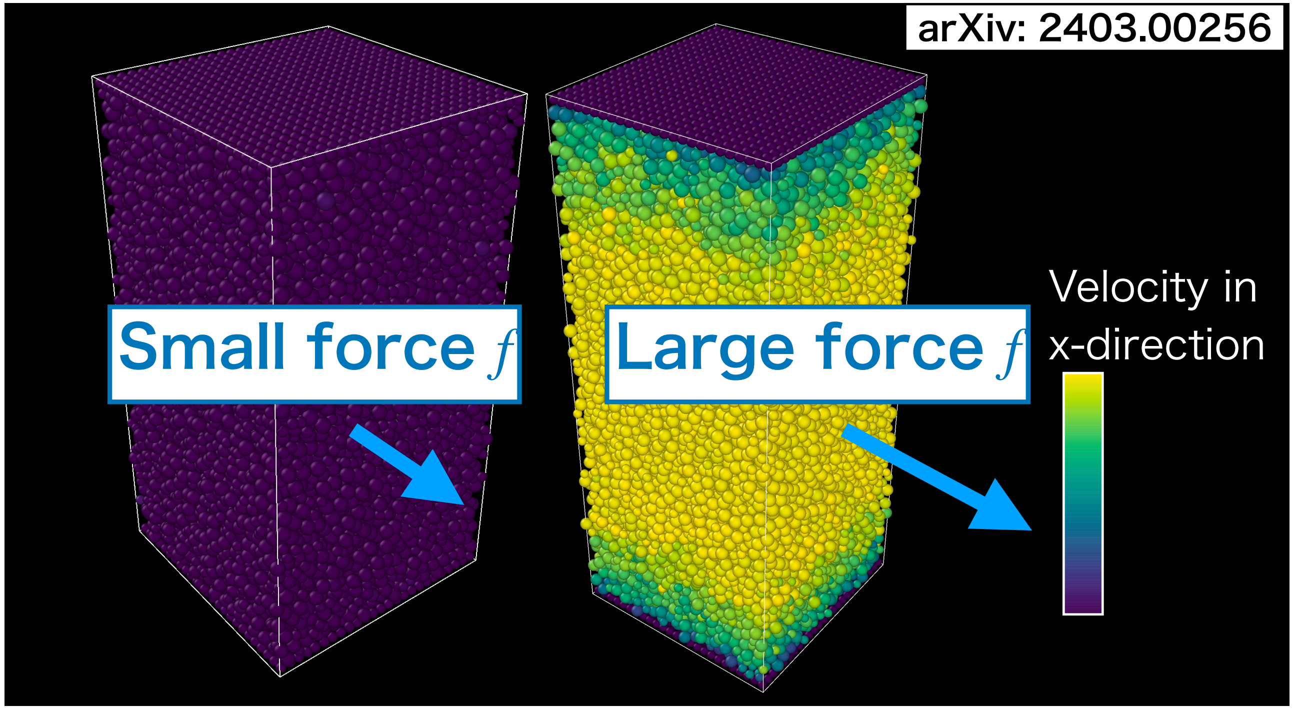
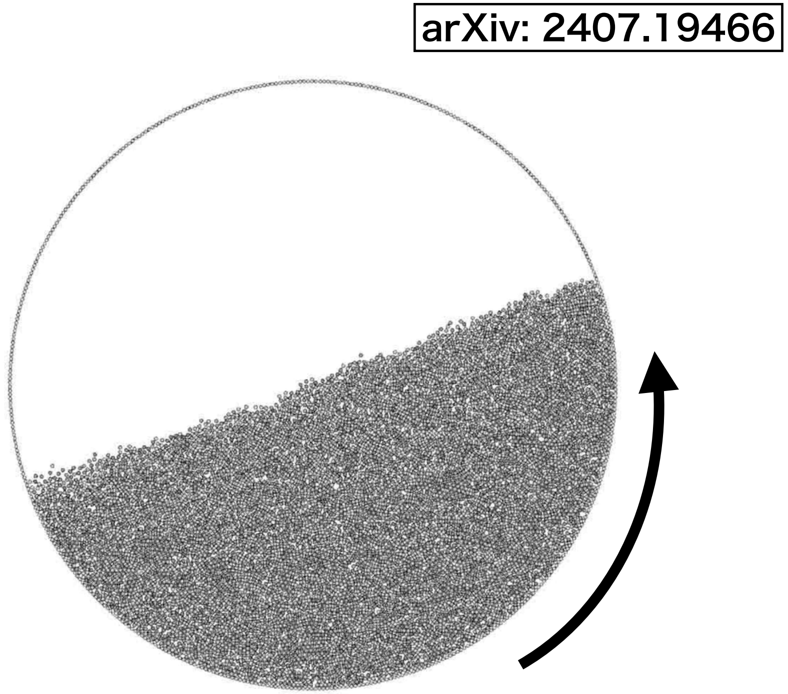
Flow of jammed granular materials

Michio Otsuki (Osaka Univ.)

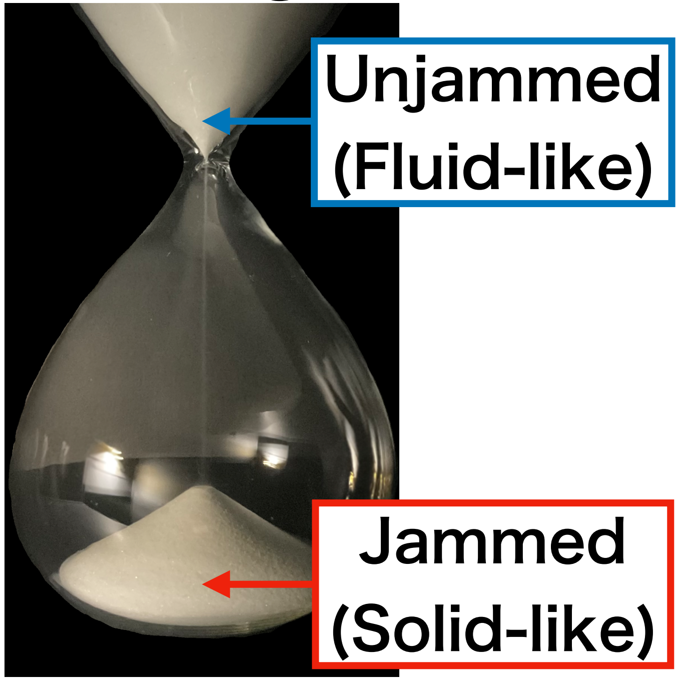




Collaborators: Kiwamu Yoshii, Kenta Hayashi, Hiroki Oba

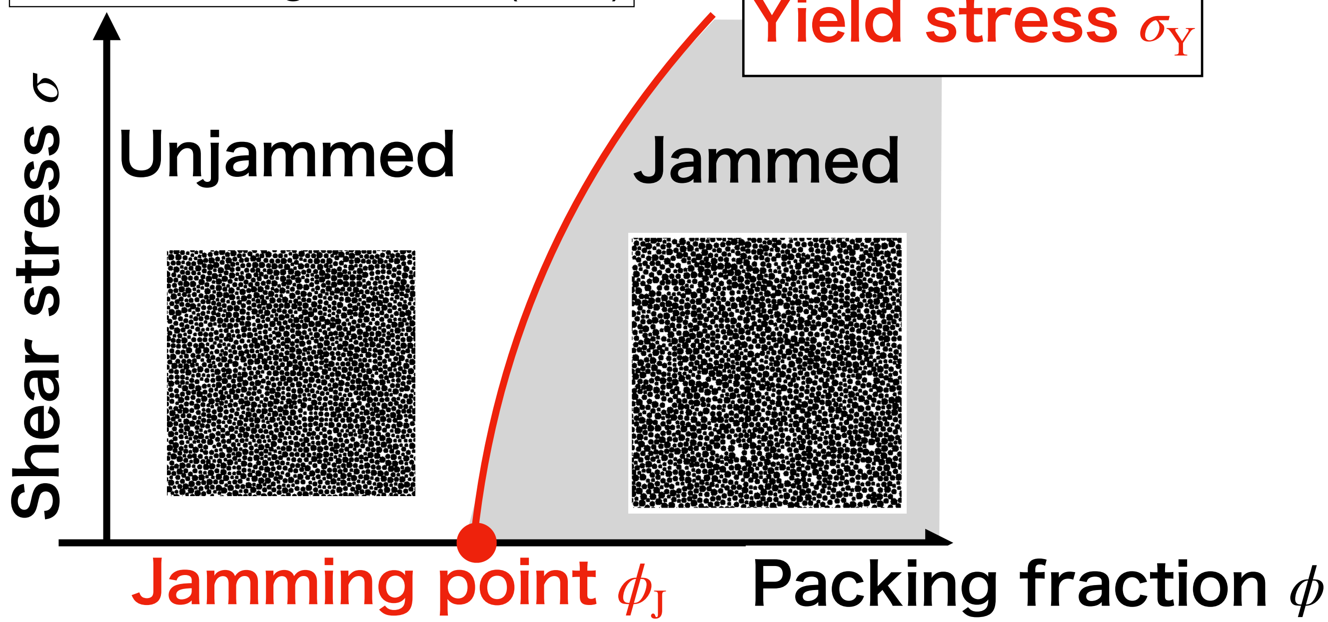
Jamming of granular materials

- Hourglass



- Jamming phase diagram

A. Liu & S. Nagel, Nature (1998)

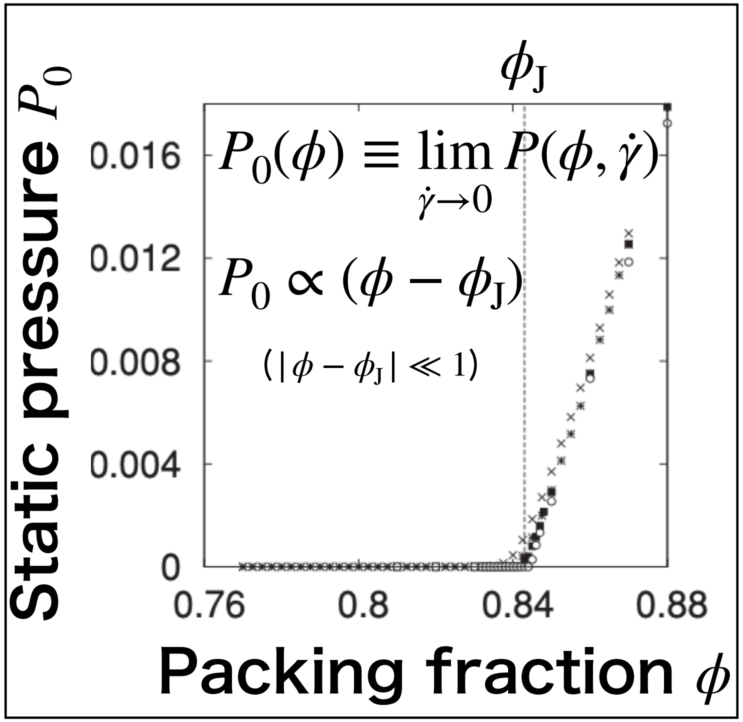


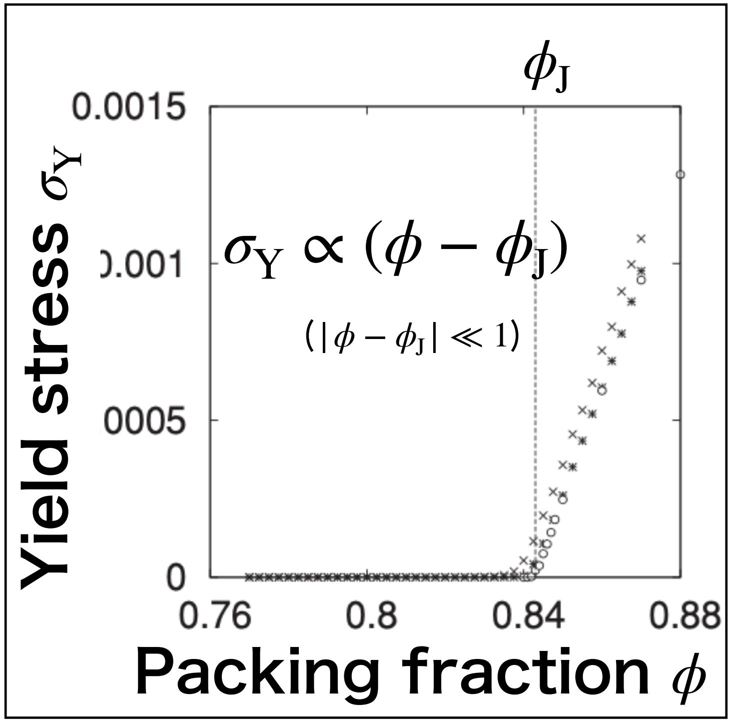
- Unjammed: Fluid-like state with flow ($\phi < \phi_J$ or $\sigma > \sigma_Y$)
- Jammed: Solid-like state without flow ($\phi > \phi_J$ and $\sigma < \sigma_Y$)

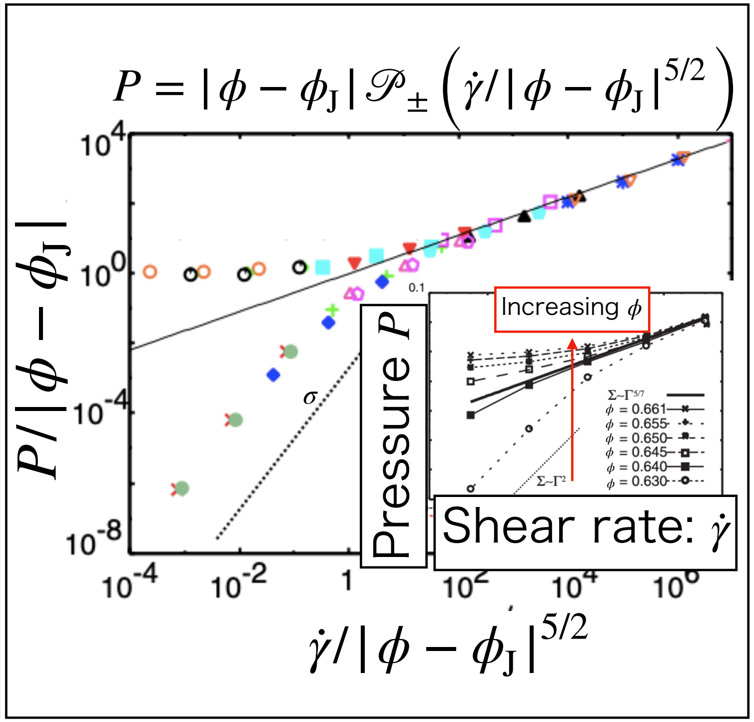
Scaling laws near jamming

- Homogeneous sheared systems with constant ϕ and $\dot{\gamma}$

Otsuki, Hayakawa PRE (2009): Linear repulsive interaction



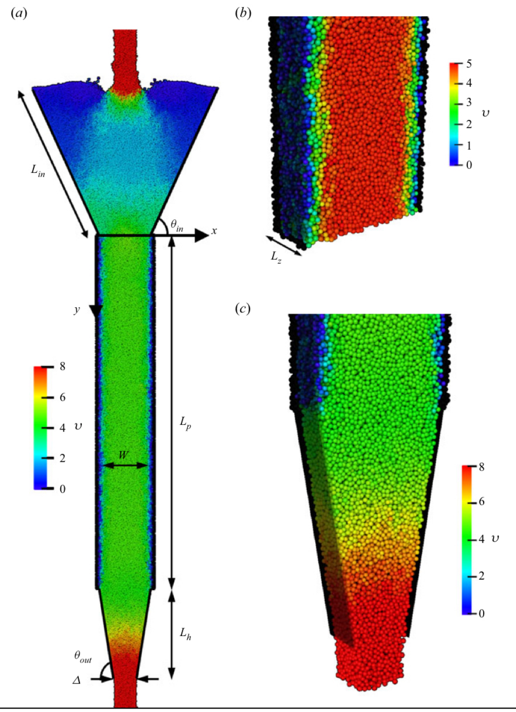




Critical behaviors similar to those for conventional phase transition

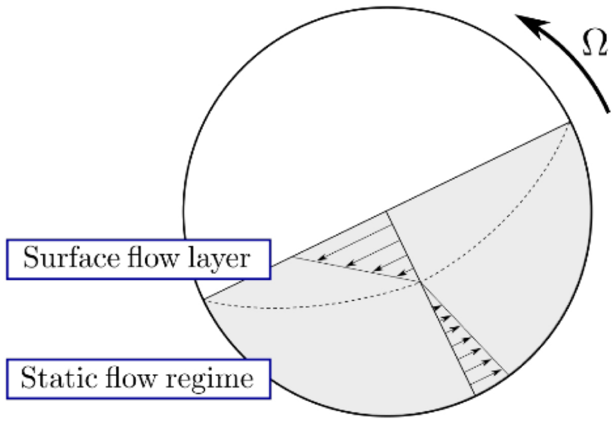
Jamming in inhomogeneous systems

- Plug region in chute



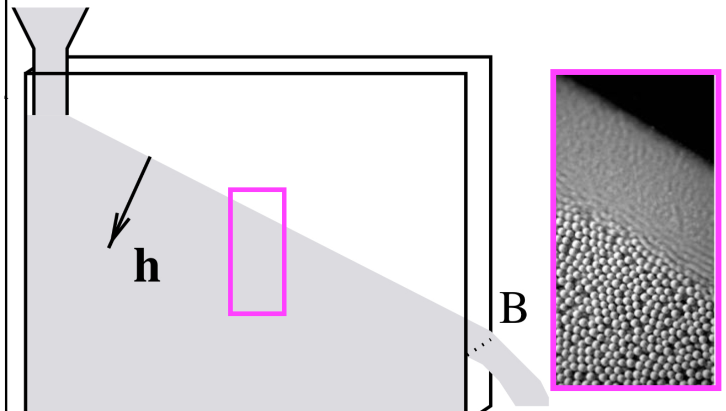
T. Baker et al., J. Fluid Mech. (2022).

- Static region in rotating drum



J. Mellmann , Powder Technol. (2001).

- Static region in sand pile



T. S. Komatsu et al., PRL (2002).

We theoretically derive velocity profile and scaling laws for inhomogeneous flows.

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DEM and continuum equation

Discrete Element Method (DEM)

Position $\boldsymbol{r}_i$, Force $\boldsymbol{F}_{ij}$

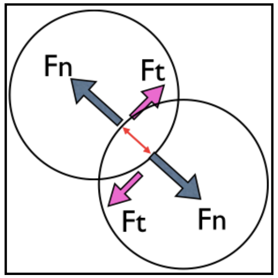
$$m\ddot{\boldsymbol{r}}_i = \sum_j \boldsymbol{F}_{ij}$$

Linear repulsive force F_n

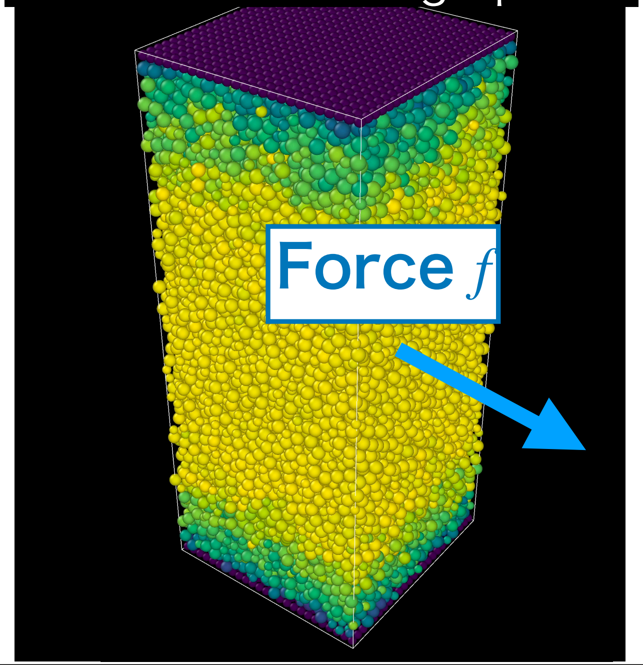
Tangential friction F_t

Coulomb law $|F_\mathrm{t}| \leq \mu_\mathrm{p} F_\mathrm{n}$

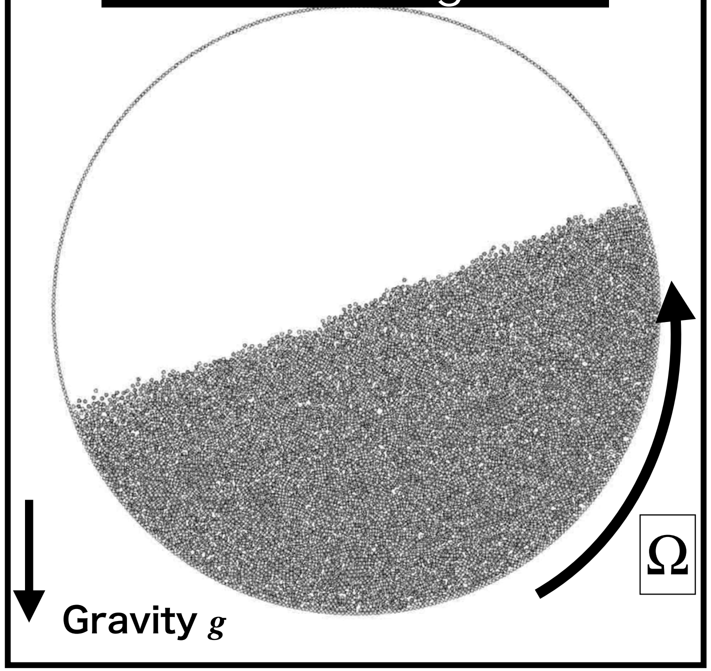
Friction coefficient μ_p



Flow between rough plates



Flow in rotating drum



Continuum equations for theoretical analysis

$$\rho(\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}) = -\nabla p + \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{K}$$

Velocity $\boldsymbol{u}$, Density ρ, Pressure p, Stress $\boldsymbol{\sigma}$, Body force $\boldsymbol{K}$

Constitutive equation for σ: $\mu(I)$ rheology

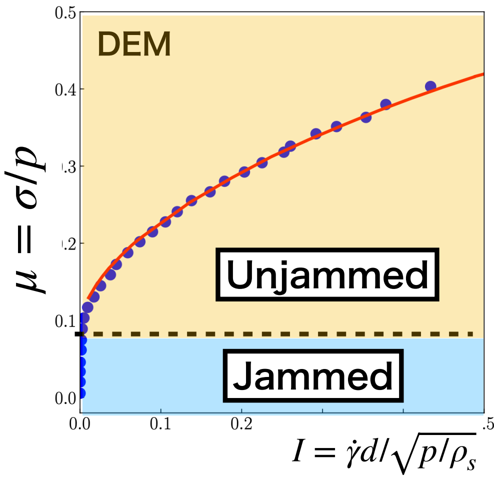
▸ Simple shear

Bulk friction $\mu = \sigma/p$

Inertia number $I = \dot{\gamma} d/\sqrt{p/\rho_s}$

Shear stress σ, Pressure p, Shear rate $\dot{\gamma}$

Particle diameter d, Particle density ρ_s



$$\sigma(\dot{\gamma}) = \mu(I(\dot{\gamma}))p$$

Empirical law

$$\mu(I) = \mu_s + \frac{\mu_2 - \mu_s}{I_0/I + 1}$$

Parameters: μ_s, μ_2, I_0

GDR MiDi (2004)

(Yield stress fluids with $\sigma_Y \propto p$)

Consistent with scaling laws near jamming for $\dot{\gamma} \to 0$: σ_Y, p

▸ Generalized form

Stress tensor $\boldsymbol{\sigma} = \mu(I)p\frac{\dot{\boldsymbol{\gamma}}}{|\dot{\boldsymbol{\gamma}}|} = \eta\dot{\boldsymbol{\gamma}}$

Shear rate tensor $\dot{\boldsymbol{\gamma}} = \nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}}$

$$\mu(I) = \mu_s + \frac{\mu_2 - \mu_s}{I_0/I + 1}, \quad I = |\dot{\boldsymbol{\gamma}}| d/\sqrt{p/\rho_s}$$

Viscosity $\eta = \dfrac{\mu(I)p}{|\dot{\boldsymbol{\gamma}}|}$

Parameters (μ_s, μ_2, I_0) can be estimated from DEM under simple shear.

P. Jop et al. Nature (2006)

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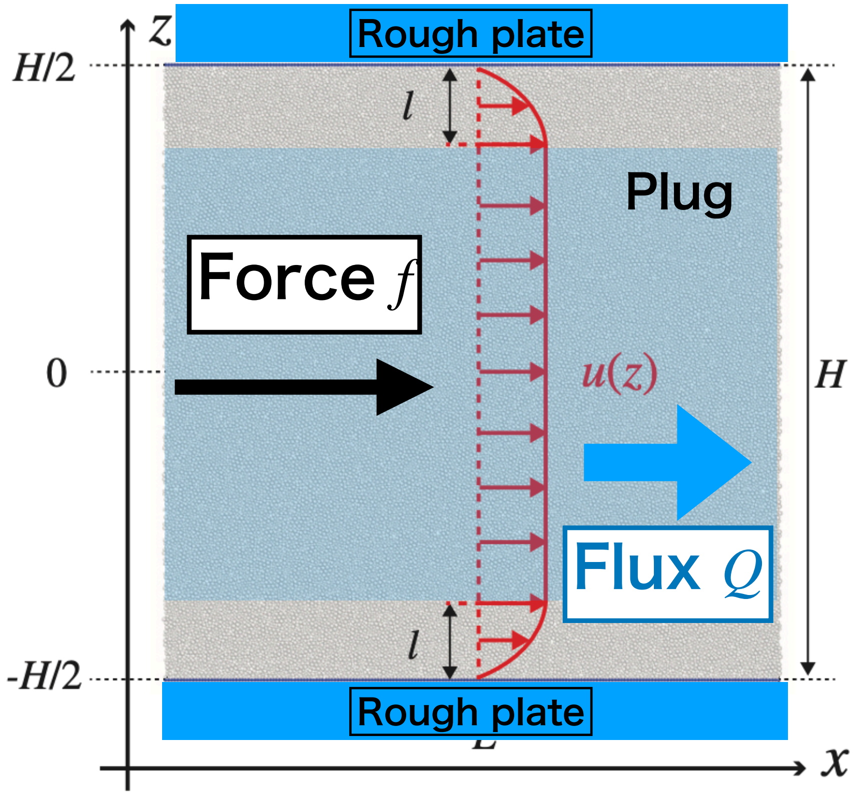
Setup: 2D flow between rough parallel plates

- Constant volume, periodic B.C along x-direction



Frictionless N grains

Grain diameter: d

Distance between plates: $H \gg d$

Velocity $u(z)$

Local packing fraction: $\phi(z)$

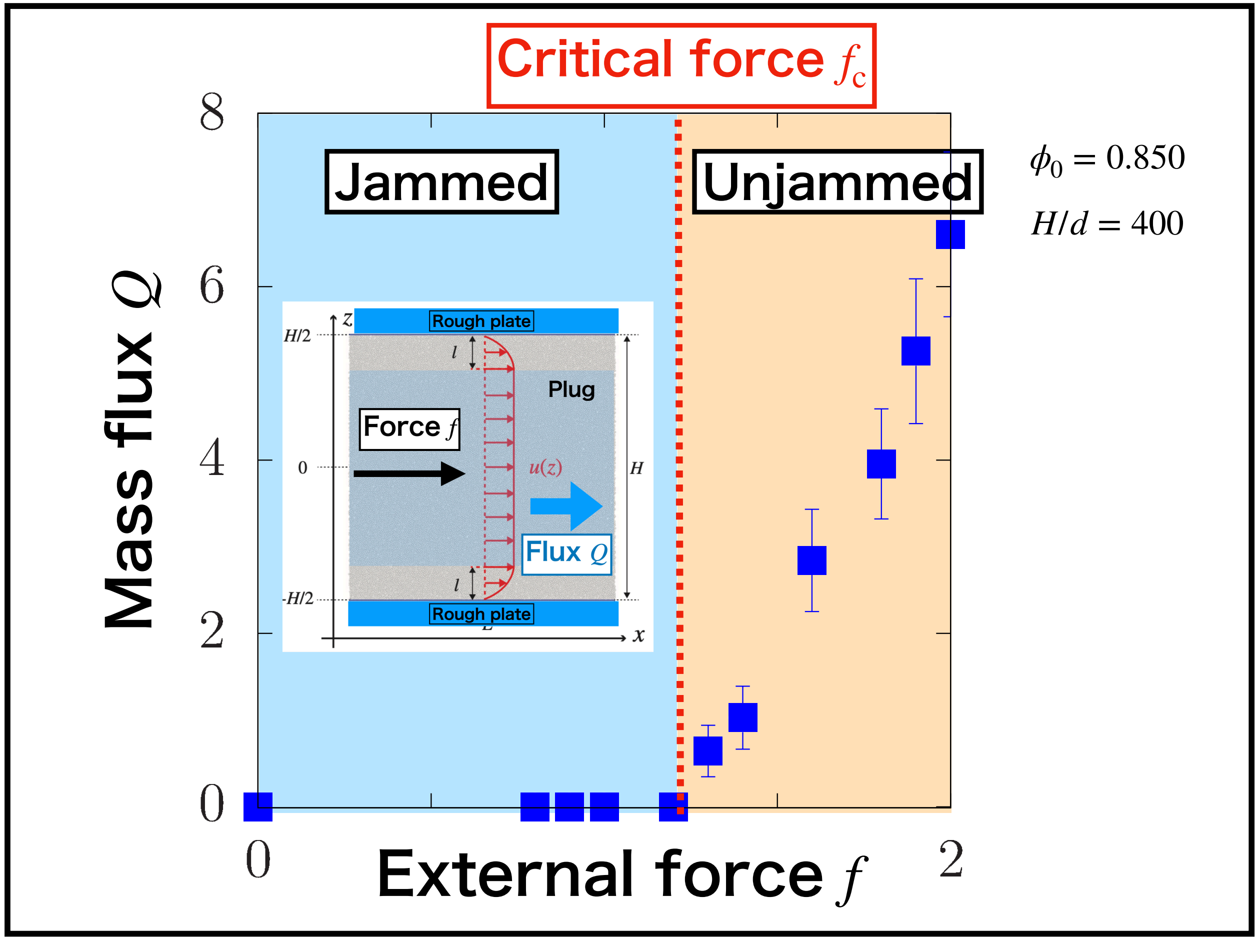
Average packing fraction: $\phi_0 \geq \phi_J$

External force per unit mass: f

* Constant p, friction, 3D → Same scaling law

- Solid-like plug region with $\dot{\gamma} = 0$ appears.

Mass flux Q and critical force f_c



Mass flux: Q

External force: f

Average packing fraction: ϕ_0

Distance: H

Critical force: $f_c(\phi_0, H)$

- Jamming transition

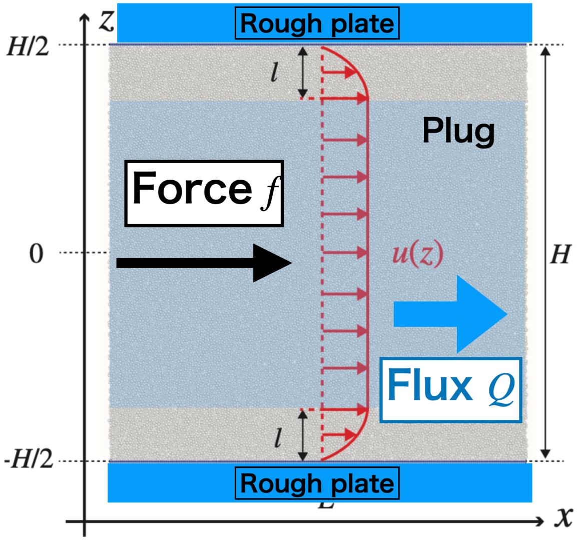
Jammed ($f < f_c(\phi_0, H)$): $Q = 0$

Unjammed ($f > f_c(\phi_0, H)$): $Q > 0$

Critical behaviors for $f \sim f_c(\phi_0, H)$, $\phi_0 \sim \phi_J$?

Theoretical analysis

arXiv: 2403.00256



Steady parallel flow

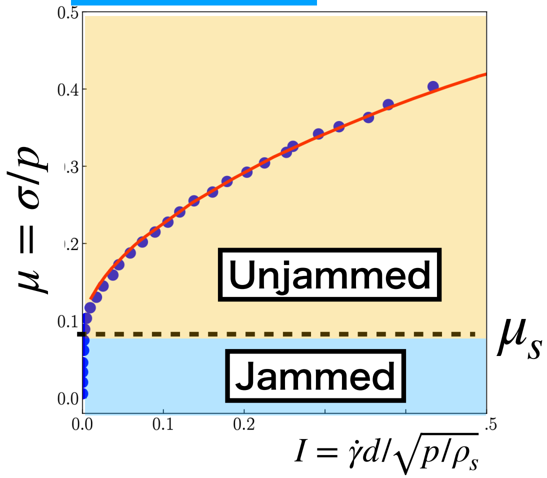
Velocity $\boldsymbol{u} = (u(z), 0)$

Continuum equations

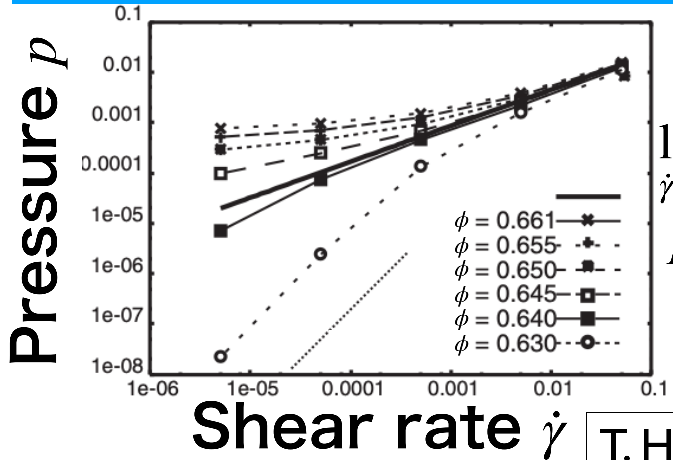
$$\rho(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}) = -\nabla p + \nabla\cdot\boldsymbol{\sigma} + \rho \boldsymbol{f}$$

External force f

$\mu(I)$-rheology



Equation of state for p near jamming



T. Hatano et al., (2007).

$$p = \Pi(\dot\gamma, \phi)$$

$$\lim_{\dot\gamma\to 0} \Pi(\dot\gamma, \phi) = P_0(\phi)$$

$$P_0 \simeq B\,|\phi - \phi_J|$$

(for $\phi > \phi_J$)

B: constant

↓ Analytical results for $f \sim f_c(\phi_0, H)$, $\phi_0 \sim \phi_J$ using perturbation theory

Critical force

$$f_c(\phi_0, H) = \frac{2\mu_s P_0(\phi_0)}{\rho_s H}, \quad P_0(\phi) = B\,|\phi - \phi_J|$$

$(|\phi - \phi_J| \ll 1)$

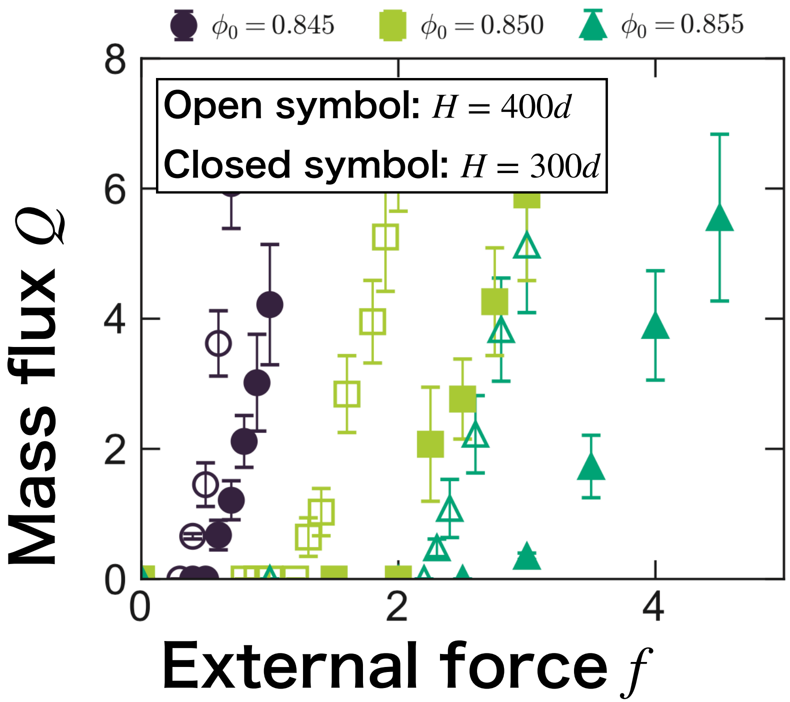
Mass flux

$$Q(f, \phi_0, H) = \frac{\mu_s \phi_0 \sqrt{\rho_s P_0(\phi_0)}\,H^2}{4bd}\left(\frac{f}{f_c(\phi_0, H)} - 1\right)^2$$

Scaling laws for critical force f_c

Jammed ($f < f_c$): $Q = 0$

Unjammed ($f > f_c$): $Q > 0$

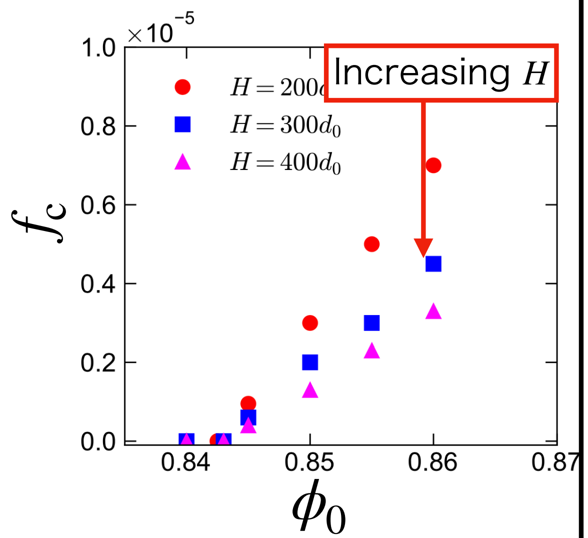


▸ **Critical force** ($|\phi - \phi_J| \ll 1$)

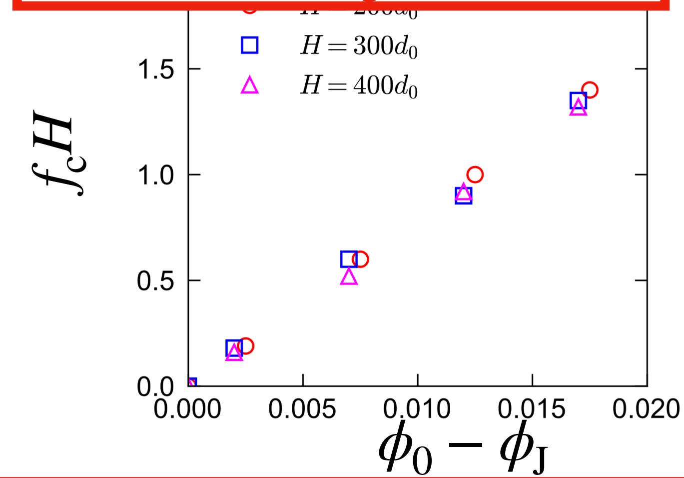
$$f_c(\phi_0, H) = \frac{2\mu_s P_0(\phi_0)}{\rho_s H} = \frac{2\mu_s B|\phi_0 - \phi_J|}{\rho_s H}$$

f_c increases with ϕ_0. f_c decreases with H.

▸ **Scaling law:** $f_c H = \mathscr{F}\left(\phi_0 - \phi_J\right)$



Numerically verified

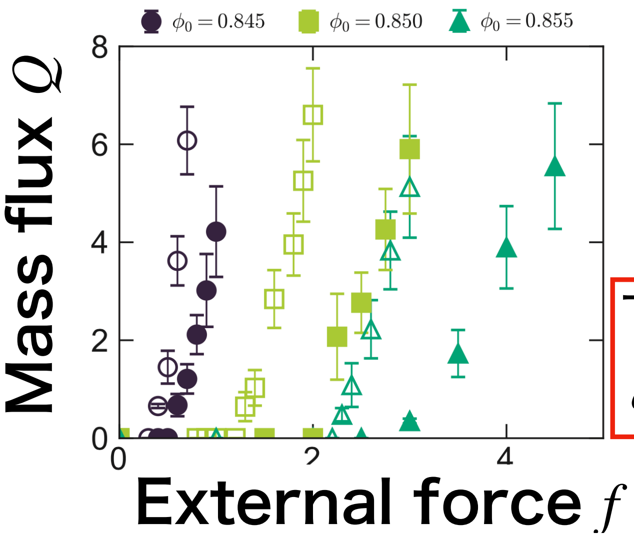


Scaling laws for mass flux Q

▸ Theoretical results for $f > f_c$

$$Q(f, \phi_0, H) = \frac{\mu_s \phi_0 \sqrt{\rho_s P_0(\phi_0)} H^2}{4bd} \left(\frac{f}{f_c(\phi_0, H)} - 1 \right)^2$$

$$f_c(\phi_0, H) = \frac{2\mu_s P_0(\phi_0)}{\rho_s H}, \qquad P_0(\phi) = B|\phi - \phi_J| \quad (|\phi - \phi_J| \ll 1)$$



Open symbol: $H = 400d$

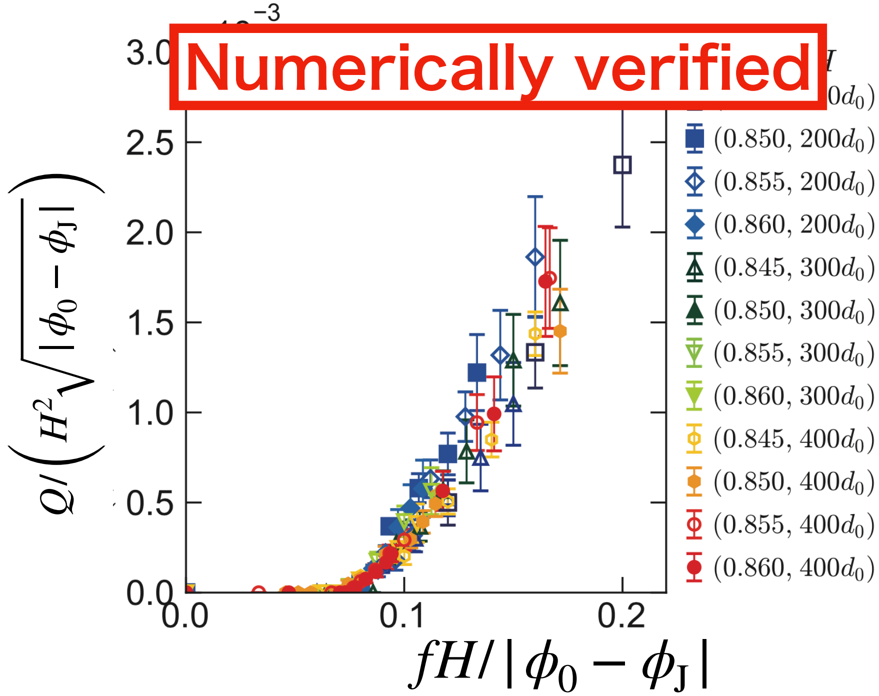
Closed symbol: $H = 300d$

Theoretical results

$\phi_0 \to \phi_J, \quad H \to \infty$

▸ Critical scaling law

$$\frac{Q}{H^2 \sqrt{|\phi_0 - \phi_J|}} = \mathcal{Q}\left(\frac{fH}{|\phi_0 - \phi_J|} \right)$$



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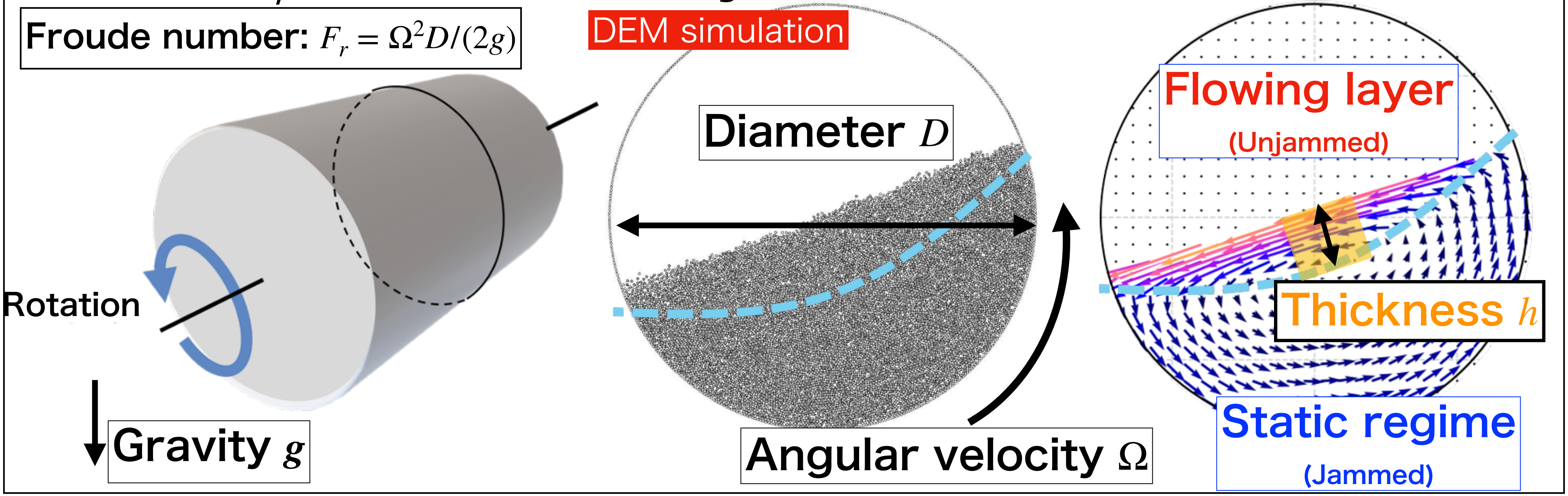
Setup: granular materials in rotating drum

- $10^{-4} < F_r < 10^{-2} \rightarrow$ **steady flows**



- Flowing layer (parallel flow)　Static regime (rigid rotation)
- Previous studies : layer thickness $h(\Omega)$????

D. J. Parker et al.(1997) : $h \sim \text{const.}$, G. Felix et al. (2007) :$h \sim \Omega^{\alpha}$

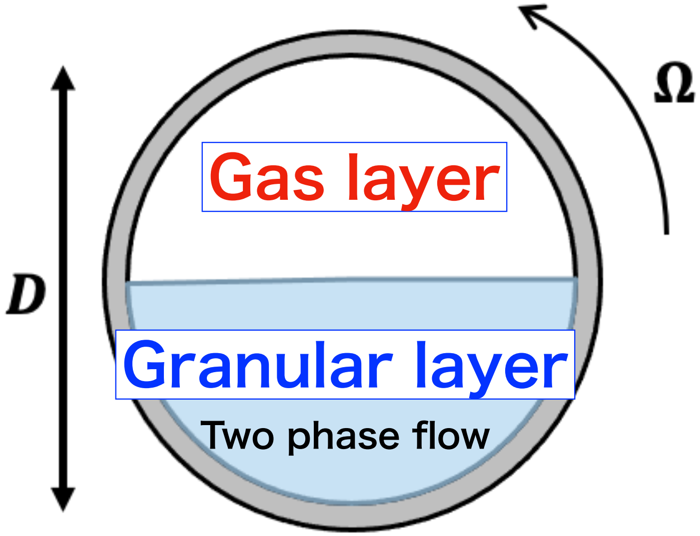
Simulation: 2D-CFD (Computational Fluid Dynamics)

Time evolution: $\rho(\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}) = -\nabla p + \nabla \cdot \boldsymbol{\sigma} + \rho \boldsymbol{g}$

$\mu(I)$-rheology: $\boldsymbol{\sigma} = \eta \dot{\boldsymbol{\gamma}}$, $\eta = \dfrac{\mu(I) p}{|\dot{\boldsymbol{\gamma}}|}$, $\mu(I) = \mu_s + \dfrac{\mu_2 - \mu_s}{I_0/I + 1}$, $I = \dfrac{|\dot{\boldsymbol{\gamma}}| d}{\sqrt{p/\rho_s}}$

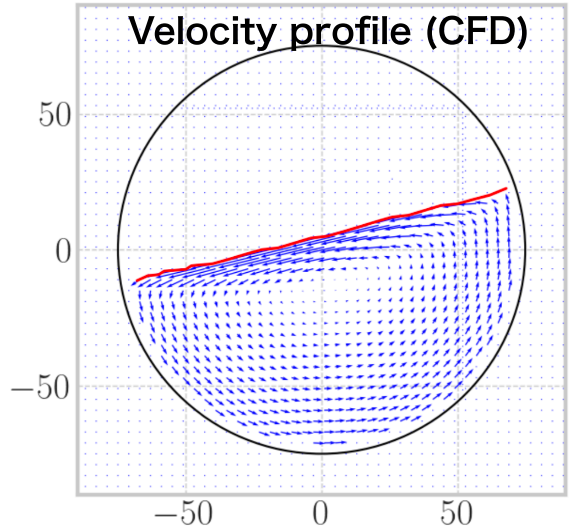
Incompressible condition: $\nabla \cdot \boldsymbol{u} = \boldsymbol{0}$

Density ρ, Pressure p, Stress σ, Gravity g

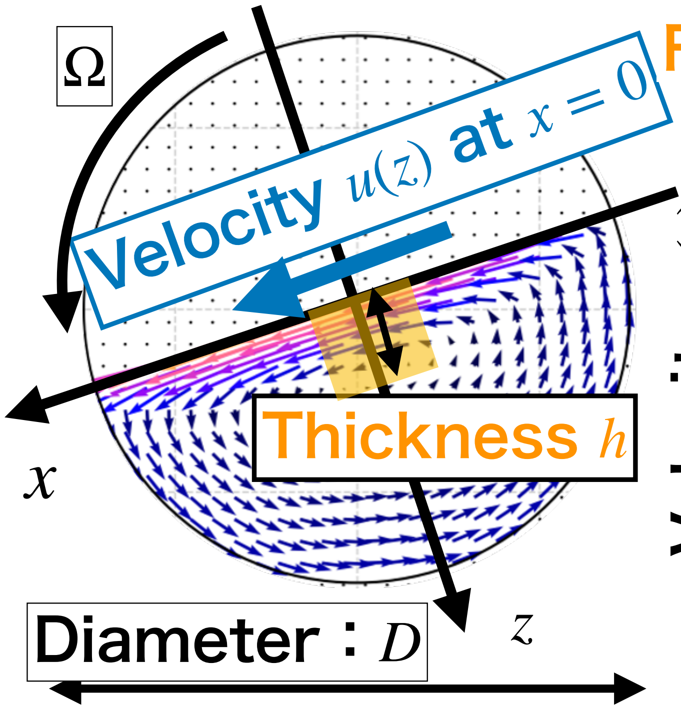


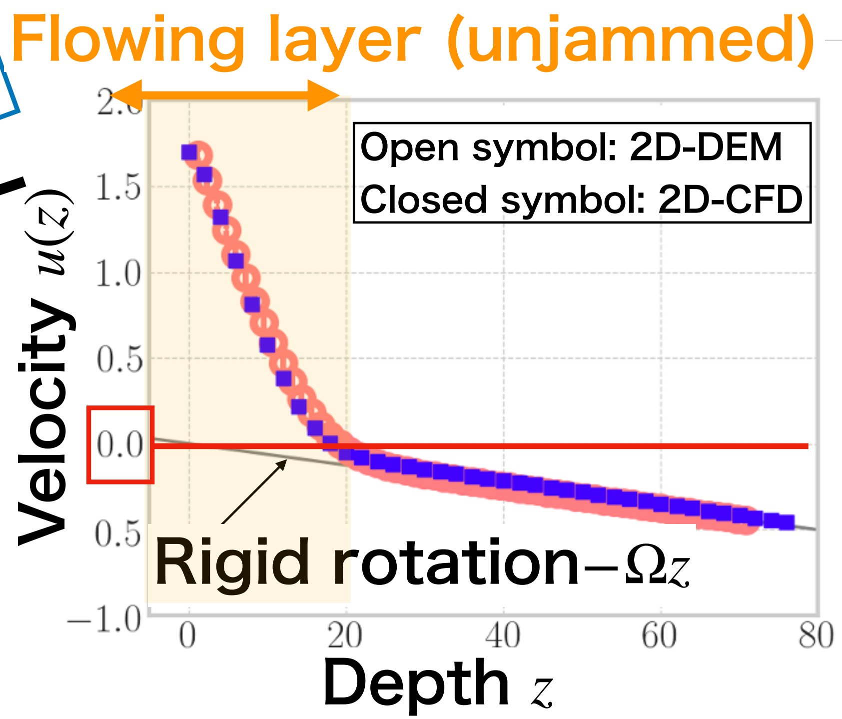
- Free surface: VOF method
- Wall boundary: BDI method

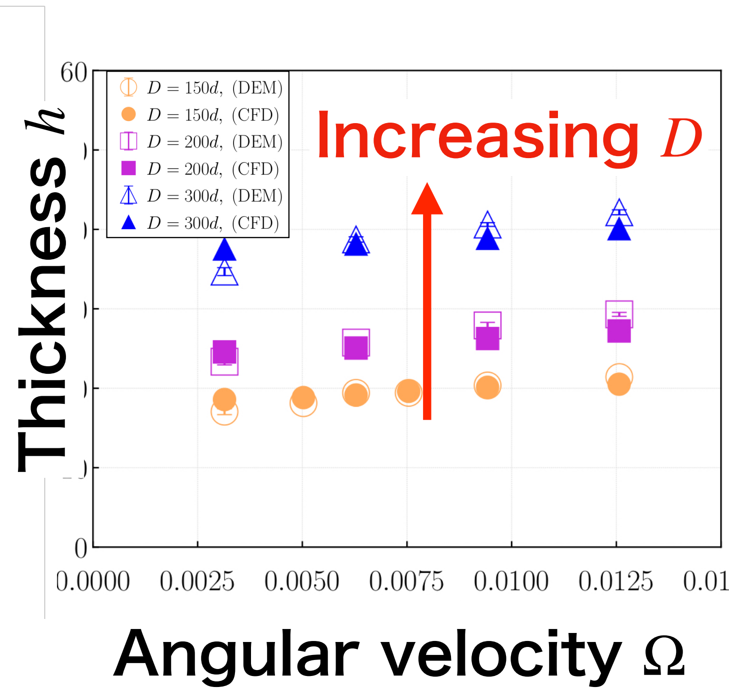
Newtonian fluids: D. Watanabe and S. Goto,(2022)



Velocity $u(z)$ and thickness h







- DEM and CFD results are quantitatively consistent.
- Flowing layer : $u(z) > 0$, Thickness h : $u(z = h) \simeq 0$
- Thickness h increases with D, and has a slight dependence on Ω.

Nondimensionalized continuum eq.

The continuum model reproduces DEM results.

$$\nabla \cdot \boldsymbol{u} = \boldsymbol{0}$$

$$\rho(\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}) = - \nabla p + \nabla \cdot \boldsymbol{\sigma} + \rho \boldsymbol{g}$$

$$\boldsymbol{\sigma} = \eta \dot{\boldsymbol{\gamma}},\ \eta = \frac{\mu(I)p}{|\dot{\gamma}|},\ \mu(I) = \mu_{\mathrm{s}} + \frac{\mu_2 - \mu_{\mathrm{s}}}{I_0/I + 1},\ I = \frac{|\dot{\gamma}|d}{\sqrt{p/\rho_{\mathrm{s}}}}$$

Steady flow

$\tilde{}$: normalized variable

Diameter of drum D,

Angular velocity Ω,

Density ρ

$$\nabla \cdot \tilde{\boldsymbol{u}} = \boldsymbol{0}$$

Scaled velocity $\tilde{\boldsymbol{u}}(\tilde{\boldsymbol{r}})$, Scaled position $\tilde{\boldsymbol{r}}$

$$\tilde{\boldsymbol{u}} \cdot \tilde{\nabla} \tilde{\boldsymbol{u}} = - \tilde{\nabla} \tilde{p} + \tilde{\nabla} \cdot \left(\tilde{\eta} \tilde{\dot{\boldsymbol{\gamma}}} \right) + \left(g/\Omega^2 D \right) \boldsymbol{e}_z$$

$$\tilde{\eta} = \frac{\mu_{\mathrm{s}} \tilde{p}}{|\tilde{\dot{\gamma}}|} + \frac{(\mu_2 - \mu_{\mathrm{s}})\tilde{p}(d/D)}{I_0 \sqrt{\phi \tilde{p}} + |\tilde{\dot{\gamma}}|(d/D)},\ \tilde{\dot{\boldsymbol{\gamma}}} = \tilde{\nabla} \tilde{\boldsymbol{u}} + (\tilde{\nabla} \tilde{\boldsymbol{u}})^{\mathrm{T}}$$

Packing fraction ϕ

Scaling law for velocity profile

$\tilde{}$: normalized variable

Diameter of drum D,

Angular velocity Ω,

Density ρ, Gravity g

Particle diameter d

$$\nabla \cdot \tilde{\boldsymbol{u}} = 0$$

Scaled velocity $\tilde{u}(\tilde{r})$, Scaled position $\tilde{r}$

$$\tilde{\boldsymbol{u}} \cdot \tilde{\nabla} \tilde{\boldsymbol{u}} = -\tilde{\nabla}\tilde{p} + \tilde{\nabla} \cdot \left(\tilde{\eta}\tilde{\dot{\gamma}}\right) + \left(g/\Omega^2 D\right) \boldsymbol{e}_z$$

$$\tilde{\eta} = \frac{\mu_s \tilde{p}}{|\tilde{\dot{\gamma}}|} + \frac{(\mu_2 - \mu_s)\tilde{p}(d/D)}{I_0\sqrt{\phi\tilde{p}} + |\tilde{\dot{\gamma}}|(d/D)}, \quad \tilde{\dot{\gamma}} = \tilde{\nabla}\tilde{\boldsymbol{u}} + (\tilde{\nabla}\tilde{\boldsymbol{u}})^{\mathrm{T}}$$

$d/D < 0.006 \ll 1$

$$\nabla \cdot \tilde{\boldsymbol{u}} = \boldsymbol{0}, \quad \tilde{\boldsymbol{u}} \cdot \tilde{\nabla} \tilde{\boldsymbol{u}} = -\tilde{\nabla}\tilde{p} + \tilde{\nabla} \cdot \left(\frac{\mu_s \tilde{p}}{|\tilde{\dot{\gamma}}|}\tilde{\dot{\gamma}}\right) + \frac{g}{\Omega^2 D}\boldsymbol{e}_z$$

Froude number Fr : $Fr^{-1} = 2g/\Omega^2 D$

External parameter

$$\tilde{\boldsymbol{u}} = \boldsymbol{u}/(\Omega D) = \boldsymbol{V}(\tilde{\boldsymbol{r}}; Fr)$$

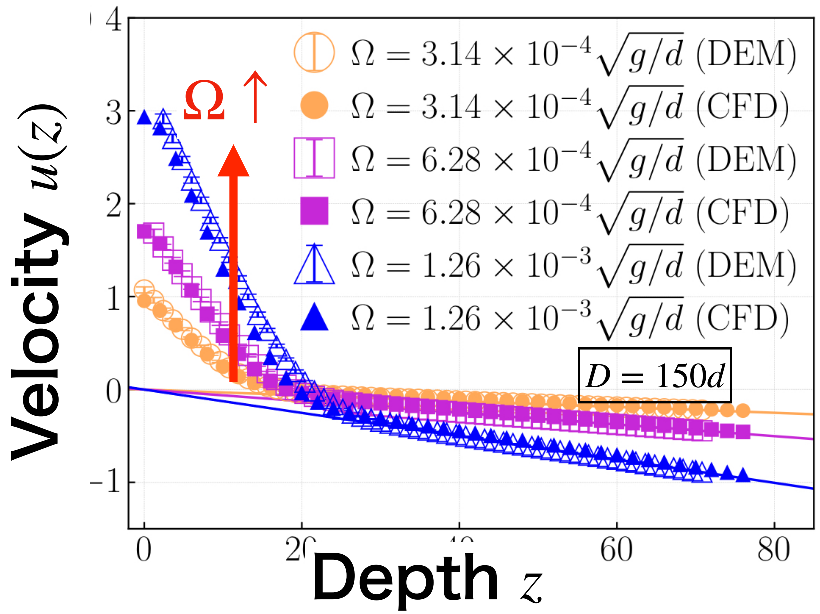
$$\tilde{\boldsymbol{r}} = \boldsymbol{r}/D$$

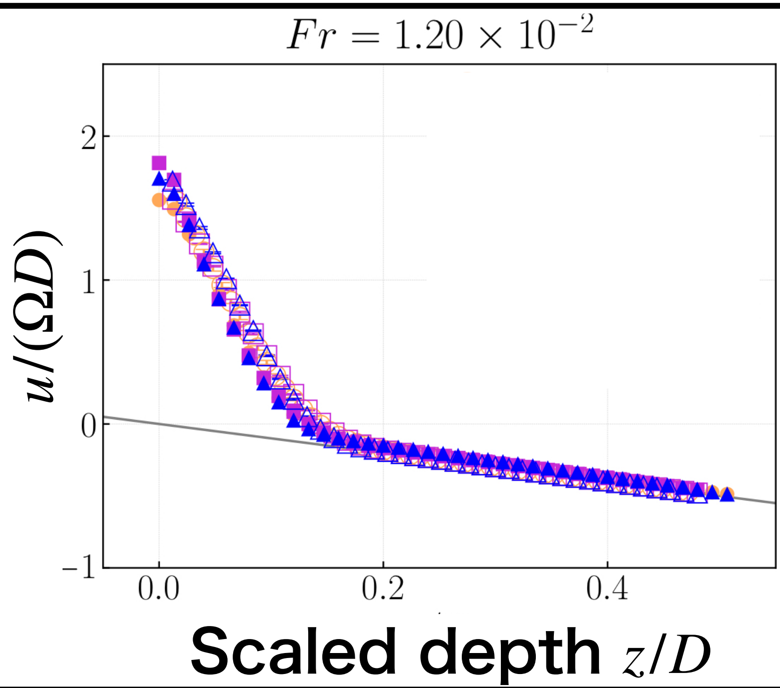
$u(z)$: Velocity at $x = 0$

Scaling law : $\dfrac{u(z; D, \Omega)}{\Omega D} = U\left(\dfrac{z}{D}; Fr\right)$

Validity of scaling law

Scaling law : $\dfrac{u(z;D,\Omega)}{\Omega D} = U\left(\dfrac{z}{D}; Fr\right)$ **Froude number :** $Fr = \Omega^2 D/(2g)$



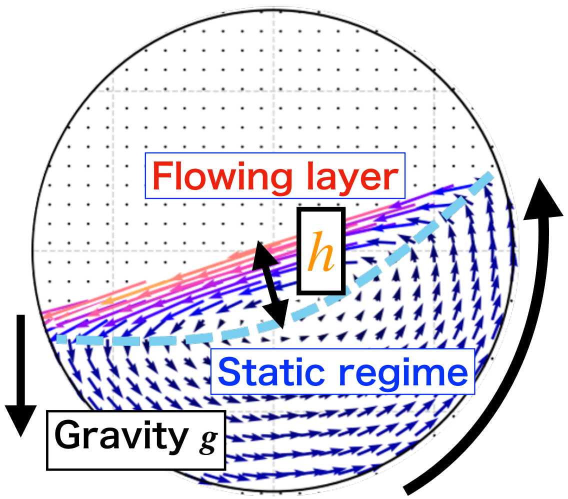


- The scaling law is confirmed by simulations.

Scaling law for thickness of flowing layer

Scaling law : $\dfrac{u(z; D, \Omega)}{\Omega D} = U\left(\dfrac{z}{D}; Fr\right)$

Thickness h **:** $u(z = h) \simeq 0$

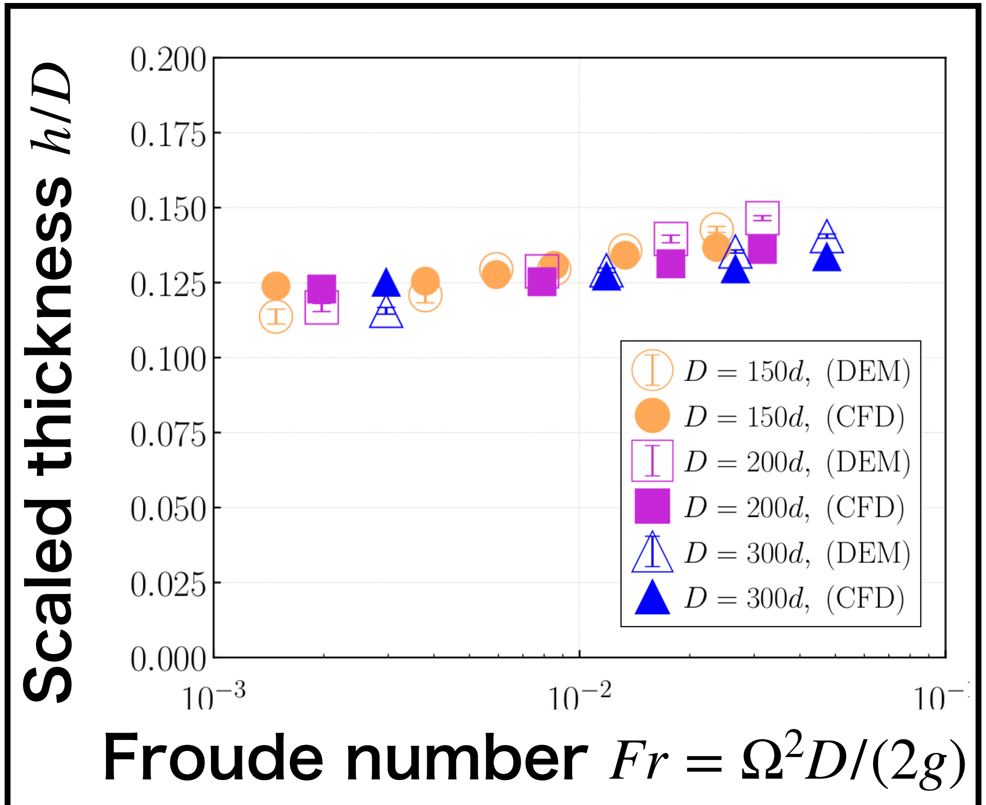


$$\frac{h(D, \Omega)}{D} = \mathscr{H}(Fr)$$

$$\mathscr{H}(Fr) \simeq \tilde{H}^{(0)} + Fr\tilde{H}^{(1)}$$

Derived from conservation of flux

$$Fr = \Omega^2 D/(2g)$$



- The scaling law for h is verified. cf. long drum exp. [D. J. Parker et al. (1997)]
- h is proportional to D and slowly increases with Ω.

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Summary

- **Topic:** Flow of jammed granular materials
- **Theoretical analysis:** Continuum eq. with $\mu(I)$-rheology
- **Result 1:** Scaling law for mass flux in parallel flows.

arXiv: 2403.00256

- **Result 2:** Scaling law for flowing layer thickness h in rotating drums.

arXiv: 2407.19466