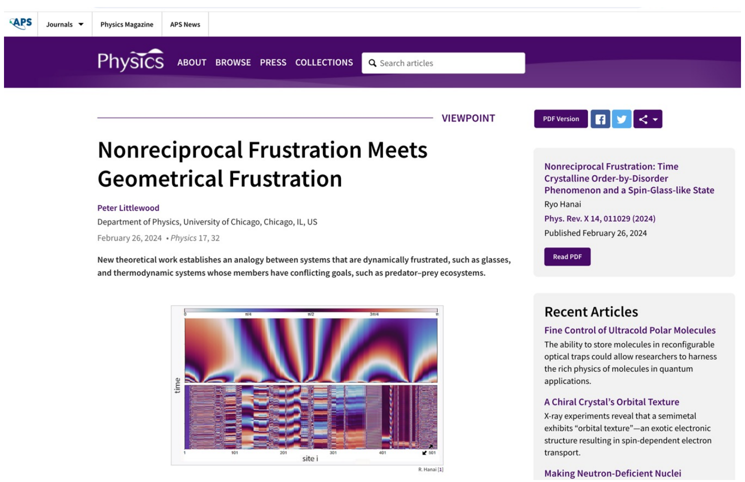
Non-equilibrium physics *Many-body (equilibrium) physics*

Non-reciprocal frustration physics

Ryo Hanai

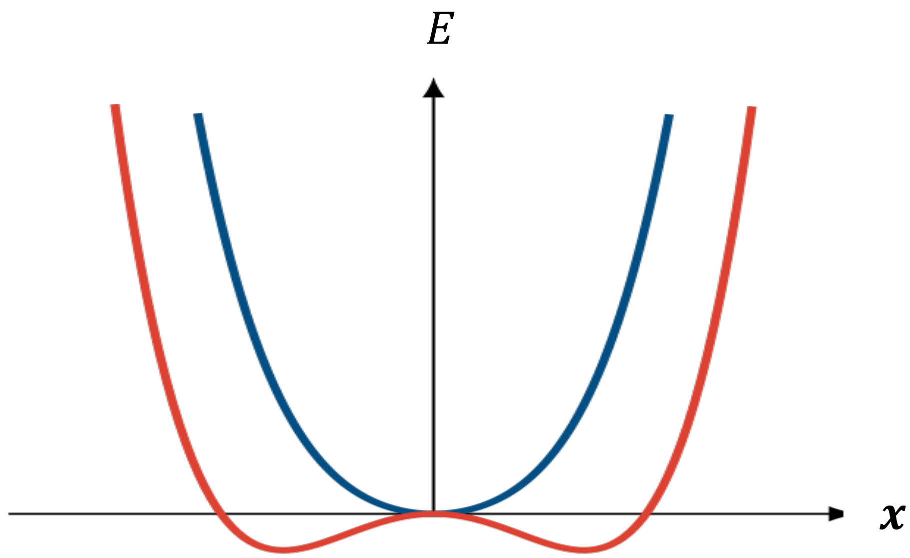
Yukawa Institute for Theoretical Physics (YITP), Kyoto University

Physics "Viewpoint"



R. Hanai, Phys. Rev. X **14**, 011029 (2024).

Equilibrium paradigm:
(Free) energy minimization principle



Energy $$E = -\sum_{i,j} J_{ij} \cos(x_j - x_i)$$

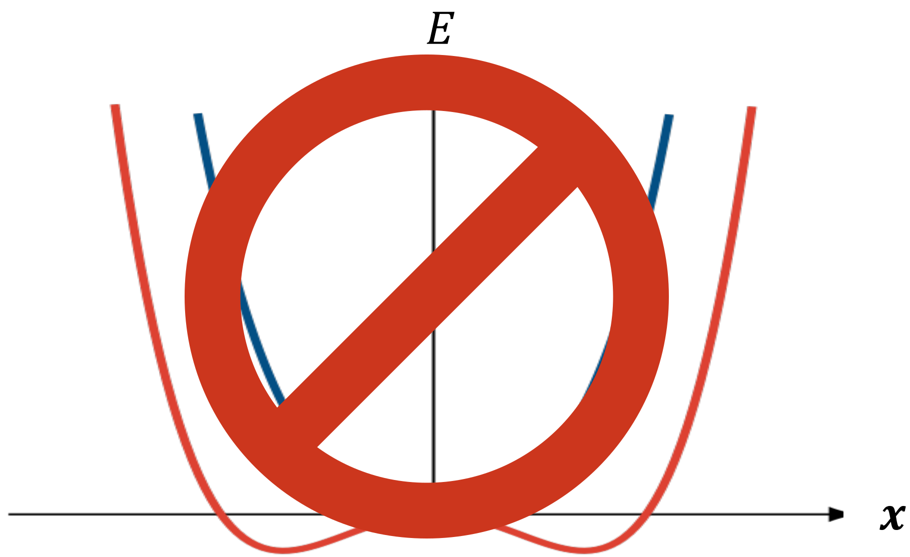
Ground state: $x = \{x_i\}$ that minimizes the energy E

Reciprocal coupling

$$J_{ij} = J_{ji}$$

~~Equilibrium~~ paradigm:

Nonequilibrium



Energy $E = -\sum_{i,j} J_{ij} \cos(x_j - x_i)$

Ground state: $x = \{x_i\}$ that minimizes the energy E

Reciprocal coupling

$J_{ij} = J_{ji}$

Non-reciprocal coupling

$J_{ij} \neq J_{ji}$

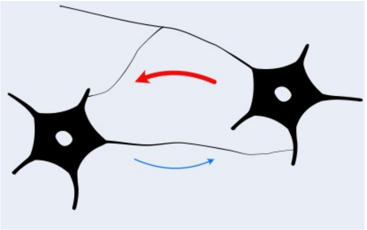
can arise!

Non-reciprocally interacting systems

Ecological systems

http://animal.memozee.com/view.php?tid=3&did=3513

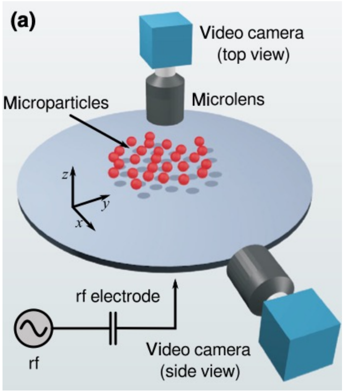
Neural networks



e.g., J. W. Krakauer, et al., Neuron 93, 480 (2017).

Driven systems

(Complex plasma)



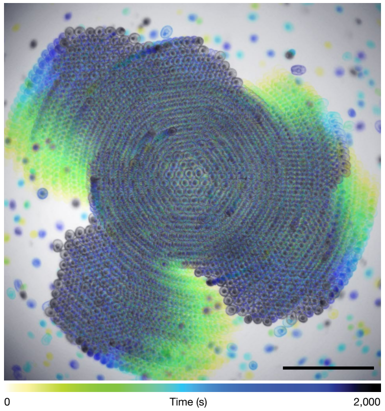
Ivlev, et al., PRX 5, 011035 (2015).

Social network

e.g., A. Pluchino, et al., Inter. J. Mod. Phys. C 16, 515 (2005).

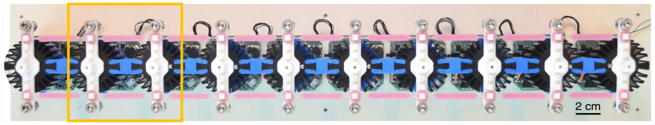
Living matter

(Starfish embryos)



T. H. Tan, et al., , Nature 2022

Robotic metamaterial



Brandenbourger, et al., Nat. Comm. 2019

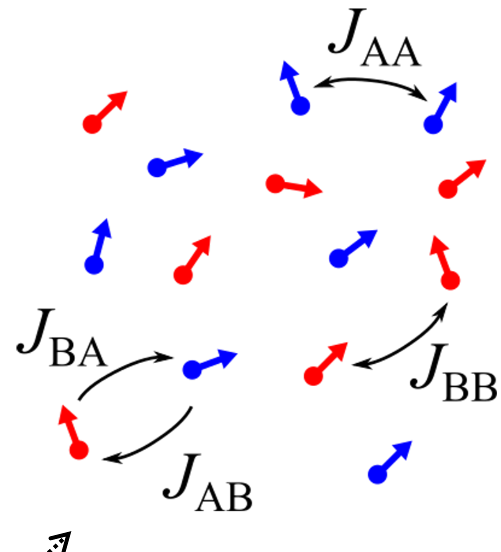
Ghatak ,et al., PNAS 2020

Baconnier, et al., Nat. Phys. 2022

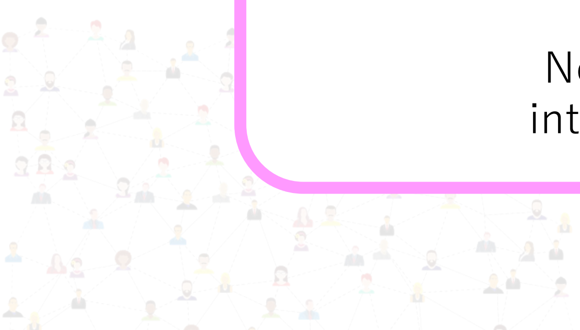
Non-reciprocally interacting systems

"Non-reciprocal Matter"

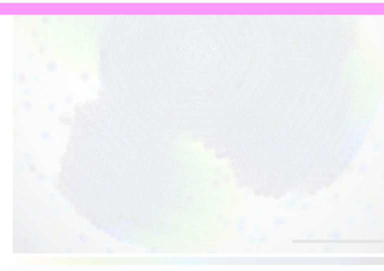
What are the emergent collective properties of **non-reciprocally interacting many-body systems**?



Non-reciprocally interacting agents



e.g., A. Pluchino, et al., Inter. J. Mod. Phys. C 16, 515 (2005).



T. H. Tan, et al., , Nature 2022

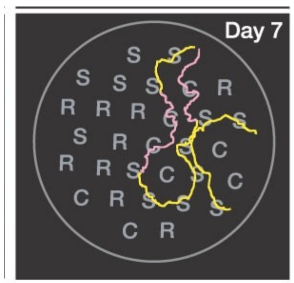
Brandenbourger, et al., Nat. Comm. 2019

Ghatak ,et al., PNAS 2020

Baconnier, et al., Nat. Phys. 2022

Collective phenomena in non-reciprocal many-body systems

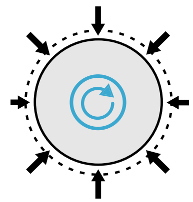
Biodiversity in ecosystems

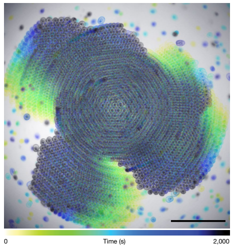


B. Kerr, et al., Nature 2002

S. Allesina and S. Tang, Nature 2012

Odd elasticity

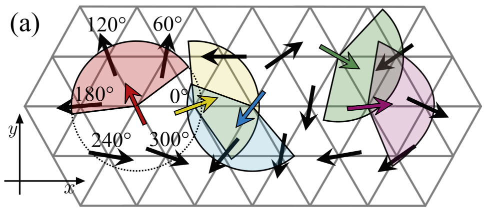




C. Scheibner, et al., , Nat. Phys. 2020

T. H. Tan, et al., , Nature 2022

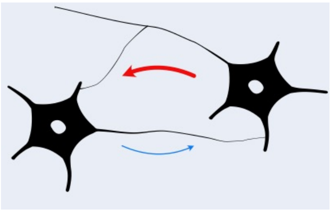
Long-ranged order in 2D



Loos, Klapp, and Martynec, arXiv:2206.10519

Dadhichi, et al., Phys. Rev. E 2020

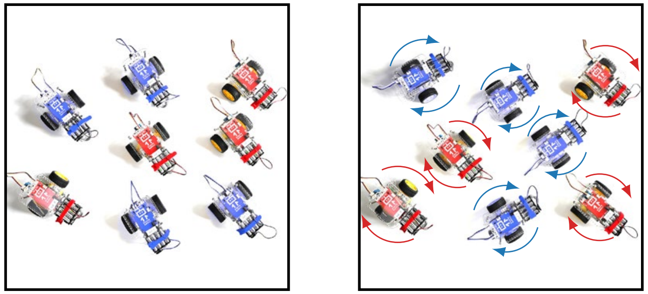
Controlling neuron dynamics



H. Wilson and J. Cowan, Biophys. J. 1972

G. Parisi, J. Phys. A 1986

Non-reciprocal phase transition

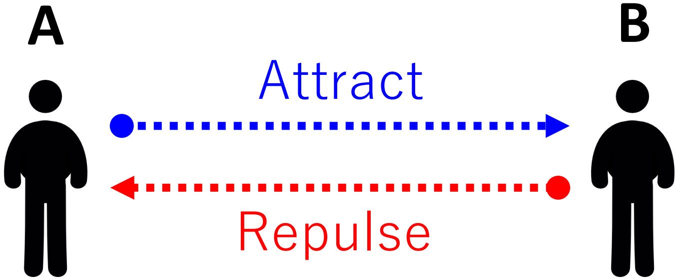


M. Fruchart*, RH*, P. B. Littlewood, and V. Vitelli, Nature 2021

You, Baskaran, Marchetti, PNAS 2020

Saha, Agudo-Canalejo, Golestanian, PRX 2020

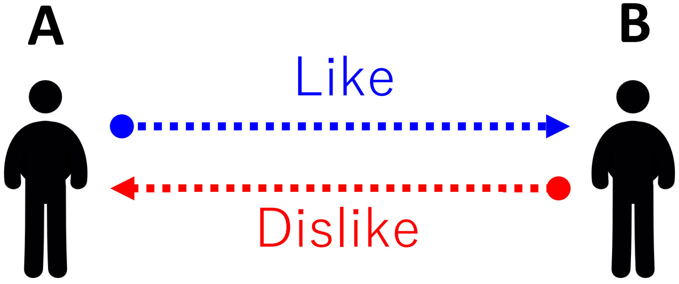
Non-reciprocal interaction ≈Frustration physics



RH, PRX2024

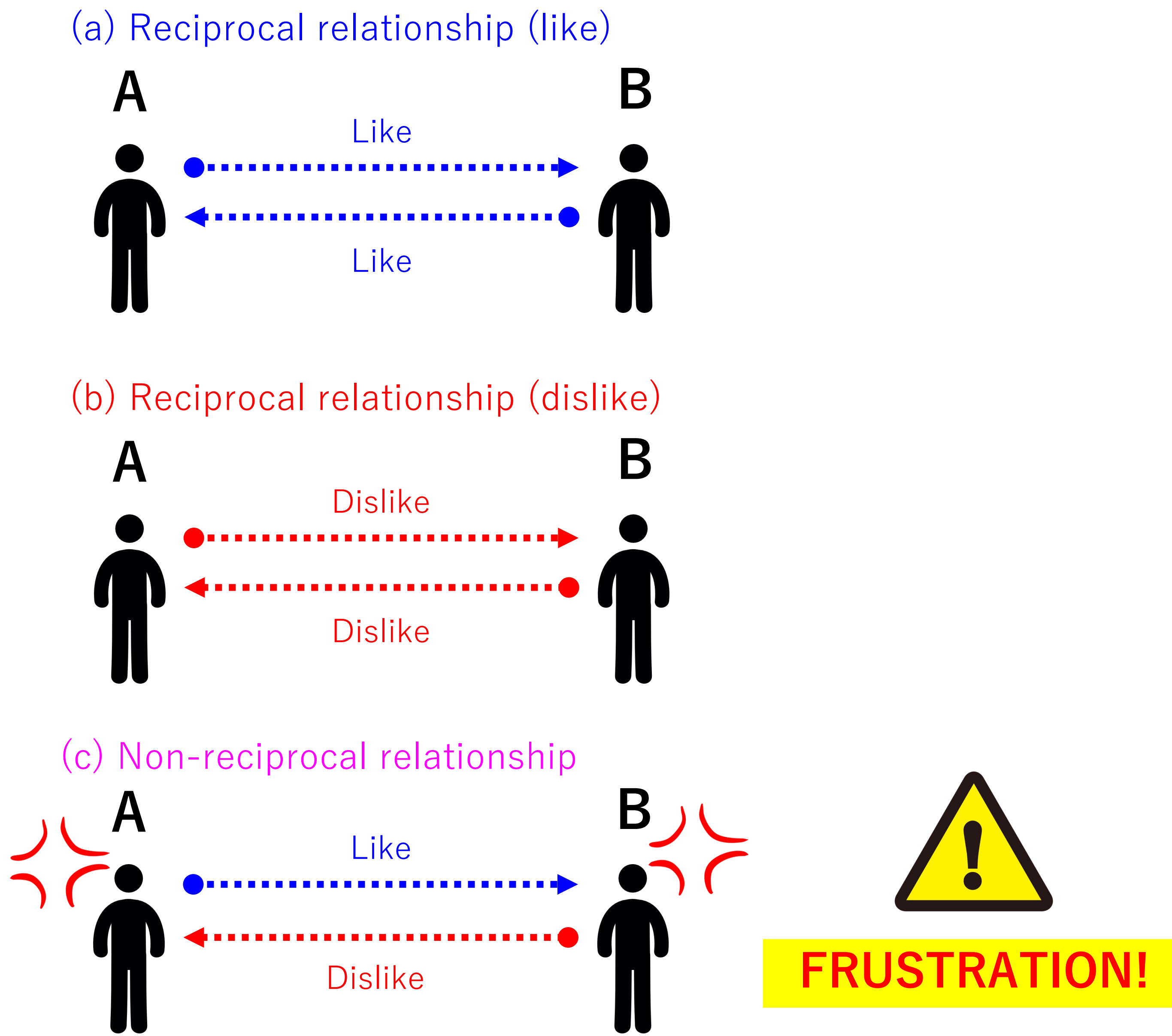
Non-reciprocal friendship

≈Frustration physics



RH, PRX2024

Non-reciprocal friendship: source of frustration



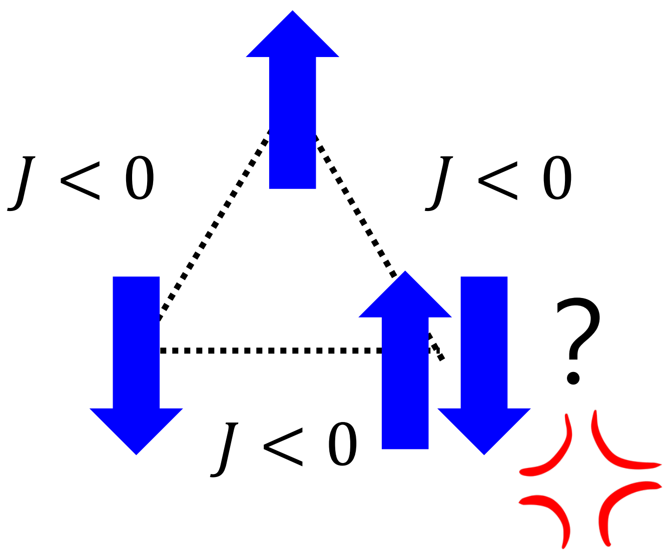
No configuration that makes both happy

RH, PRX2024

Geometrical frustration

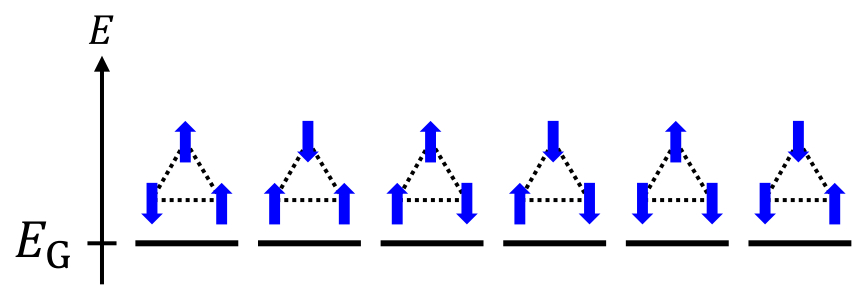
Geometrically frustrated system

A system that cannot satisfy all the constitutes' "desire" to minimize all interaction energies.



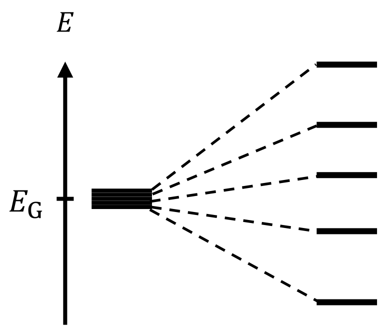
No configuration can make all spins happy

Accidental degeneracy of ground states



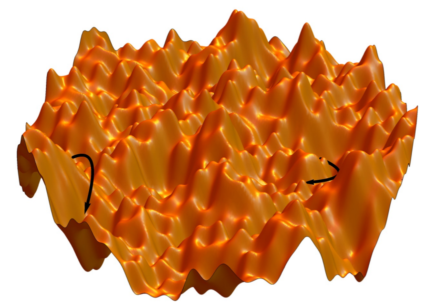
Accidental degeneracy induces exotic phenomena in geometrical frustrated systems

- Order-by-disorder



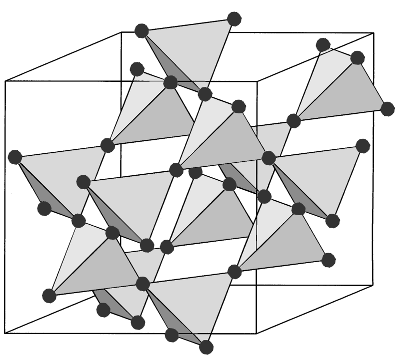
Villain, et al., J. Physique (1980)

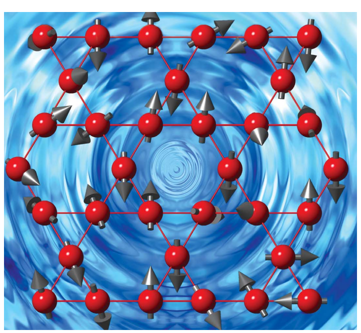
- Spin glass



(Image from https://scglass.uchicago.edu/)

- Quantum/Classical spin liquid

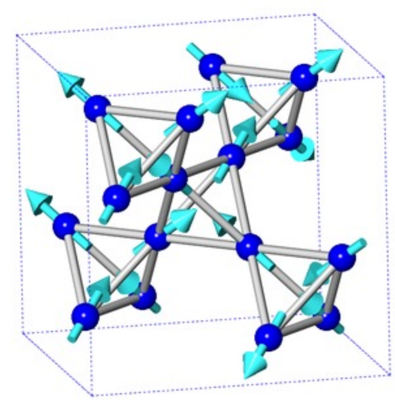




R. Mossener and J. T. Chalker, PRL 1998

T. Imai and Y. Lee, Physics Today 2016

- Spin ice

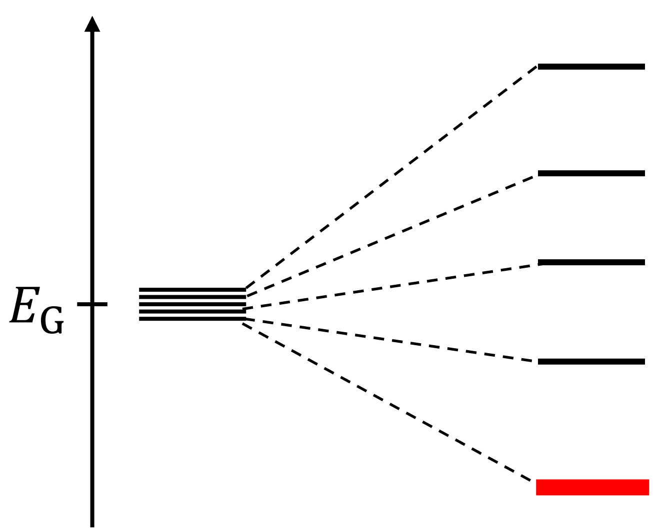


(Image from Wikipedia)

Order by disorder phenomena

Villain, et al., J. Physique (1980)

Accidental degeneracy: Not protected by symmetry nor topology



"Order by disorder"

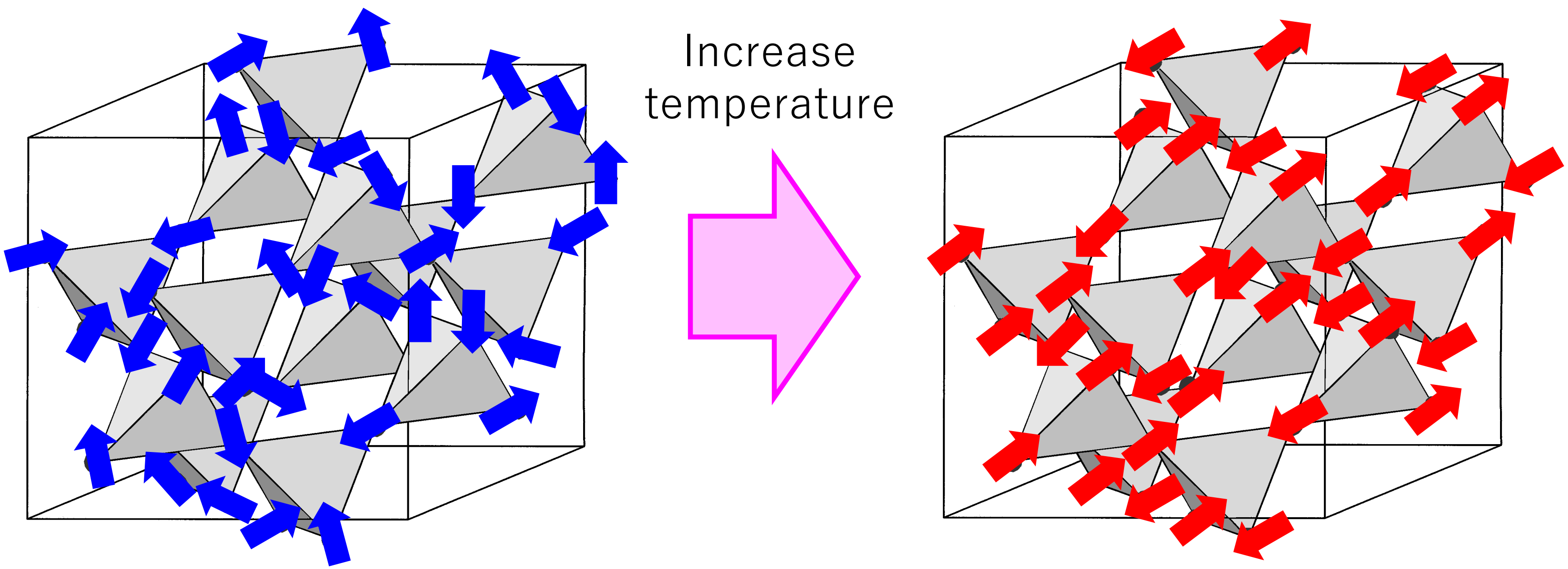
Order by disorder phenomena

(Example)

Moessner and Chalker, PRL1998, PRB1998

<u>Antiferromagnetic XY spins on a pyrochlore lattice</u>

Ground state | Finite temperature

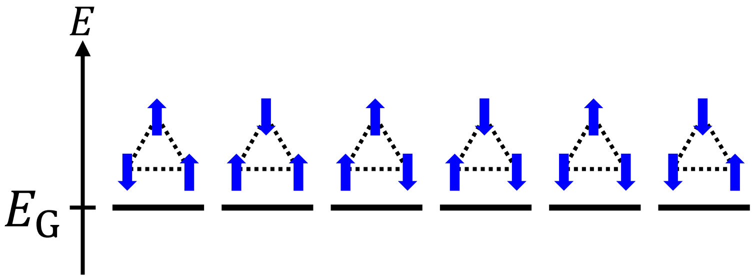


Disordered state | Long-ranged order

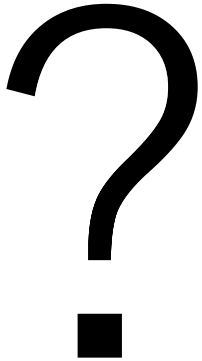
Geometrical vs Non-reciprocal frustration

Geometrical frustration

Accidental degeneracy of
ground state



Non-reciprocal frustration



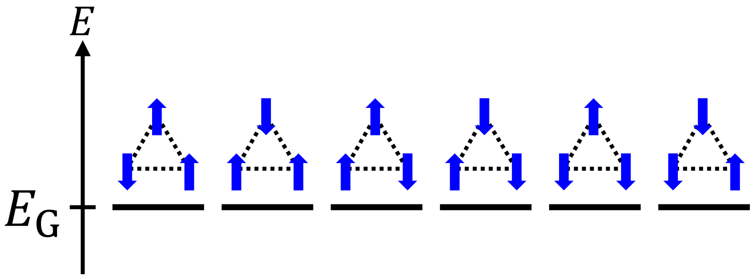
Energy cannot be defined ...

May not even converge to
a static state...

Geometrical vs Non-reciprocal frustration

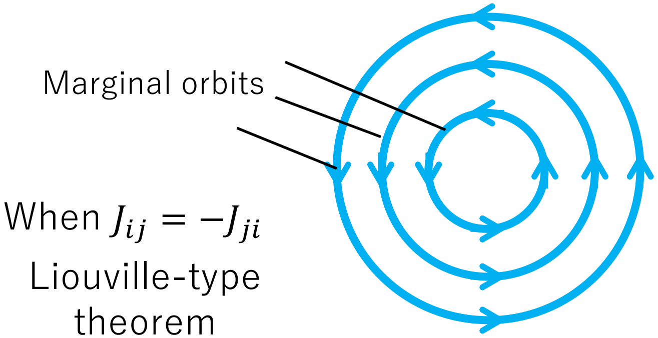
Geometrical frustration

Accidental degeneracy of ground state



Non-reciprocal frustration

"Accidental degeneracy" of orbits



➢ *Dynamical counterpart* of order-by-disorder and spin glass(-like) state occurs!

This talk: order by disorder

RH, PRX2024

Dissipative XY spin dynamics

$$\dot{\theta}_i = \sum_j J_{ij} \sin(\theta_j - \theta_i)$$

Here, couplings are non-reciprocal in general $J_{ij} \neq J_{ji}$

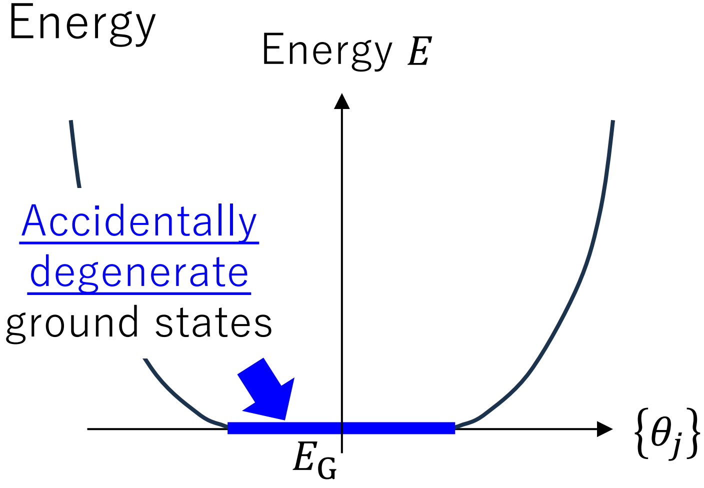
Dissipative XY spin dynamics

- Reciprocal case ($J_{ij} = J_{ji}$)：Energy minimization problem

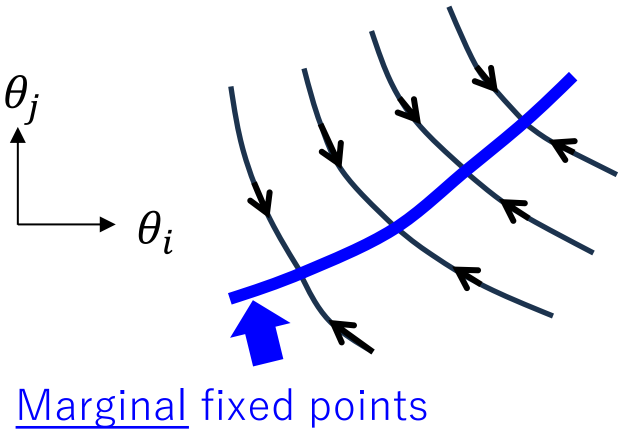
$$\dot{\theta}_i = -\frac{\partial E(\theta)}{\partial \theta_i} \quad \text{with} \quad E(\theta) = -\sum_{i,j} J_{ij} \cos(\theta_j - \theta_i)$$

Potential with geometrically frustration ➡ Accidentally degenerate ground states

- Energy



- Orbits



"Accidental degeneracy" of *orbits*

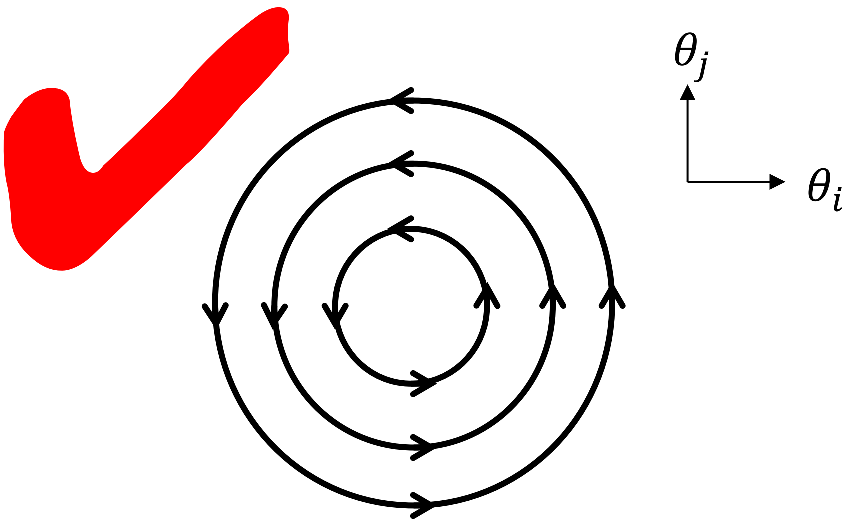
- Anti-symmetric case ($J_{ij} = -J_{ji}$)

Liouville-type theorem

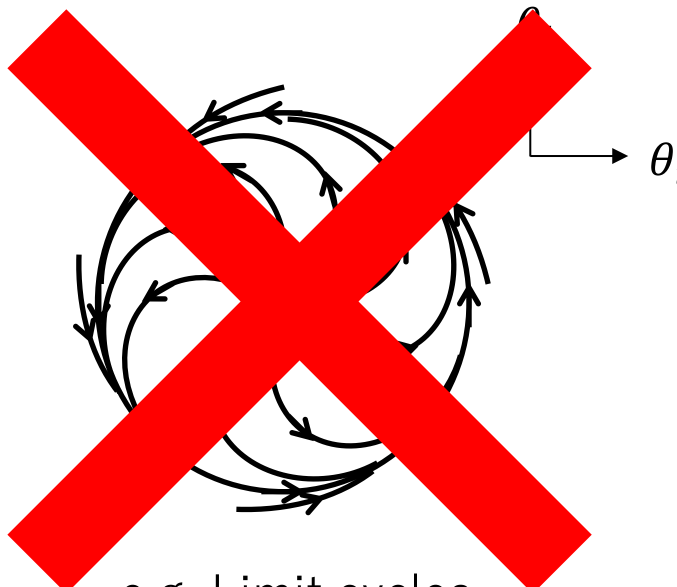
$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \sum_i \frac{\partial\rho}{\partial\theta_i}\dot{\theta}_i = 0$$

RH, PRX2024

Conservation of phase volume = **Non-dissipative** dynamics



Marginal



e.g. Limit cycles

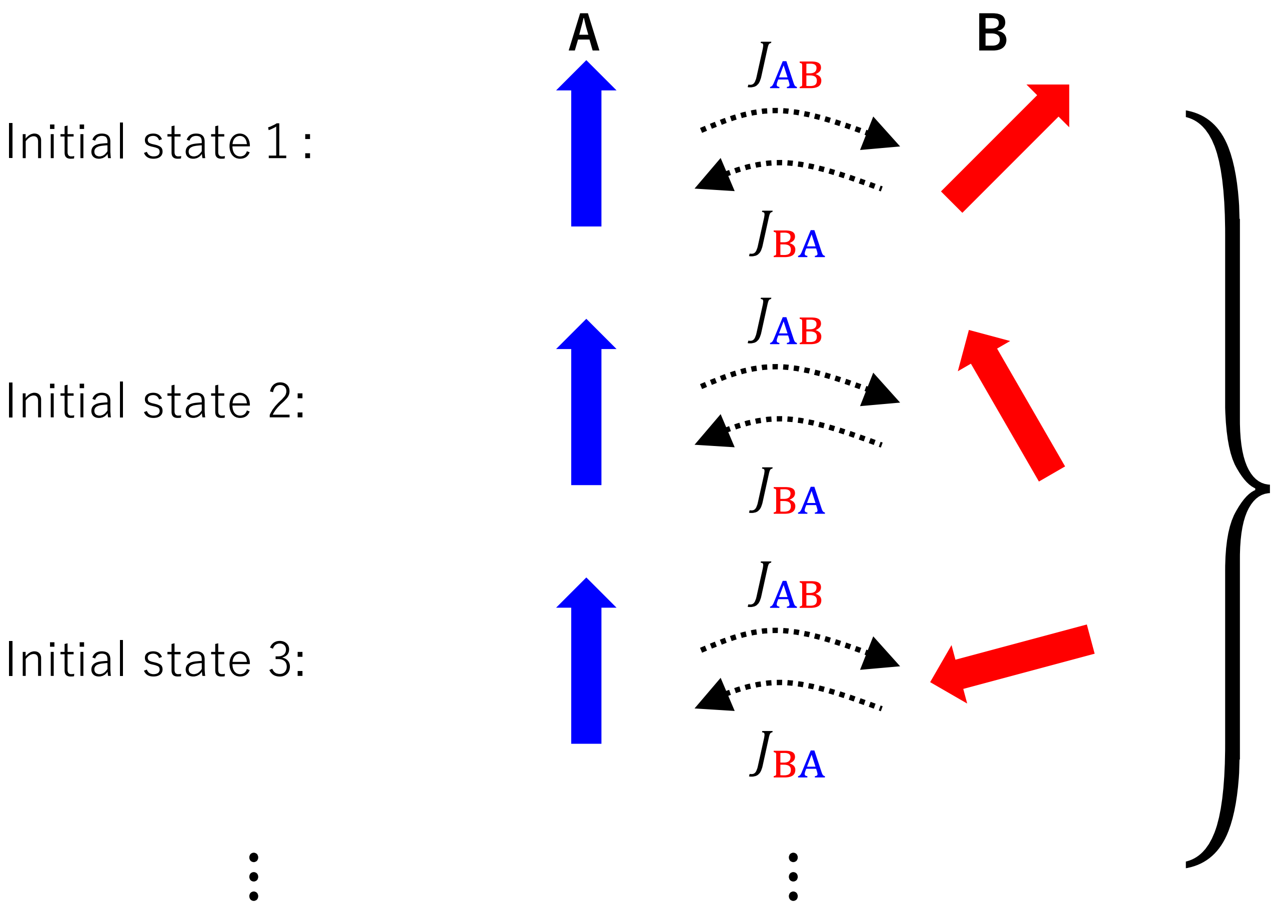
"Accidental degeneracy" of *orbits*

(e.g., two XY spin system)

$$J_{AB} = -J_{BA}$$

$$\dot{\theta}_A = J_{AB} \sin(\theta_B - \theta_A)$$

$$\dot{\theta}_B = J_{BA} \sin(\theta_A - \theta_B)$$



Non-reciprocal frustration induced

"accidental degeneracy" of orbits

Applies for arbitrary number of spins!

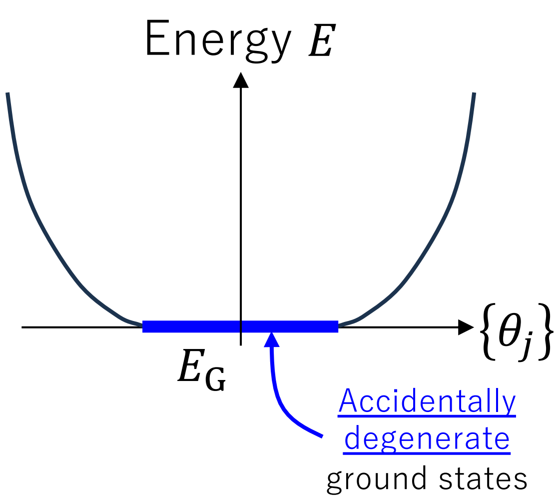
Common feature between geometrically frustrated and non-reciprocal system:

Marginal orbits!

Order-by-disorder phenomena in geometrically frustrated systems

- Zero temperature

Minimize energy E

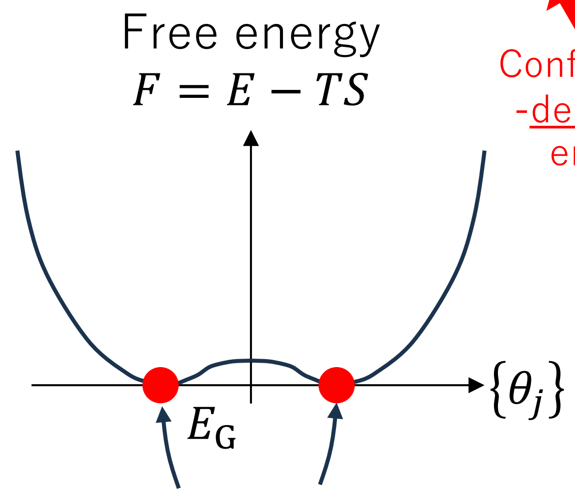


Increase temperature $T > 0$

- Finite temperature

Minimize free energy $F = E - TS(\theta)$

Configuration-dependent entropy



Entropically "selected" ground state

- Dynamics $(T > 0)$:

$$\dot{\theta} = -\frac{\partial F}{\partial \theta} = -\frac{\partial E}{\partial \theta} + T\frac{\partial S}{\partial \theta}$$

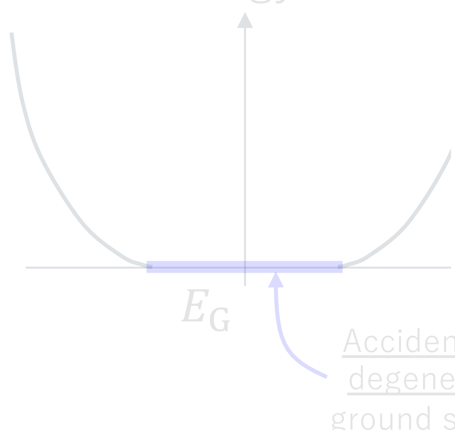
Marginal   Entropic force

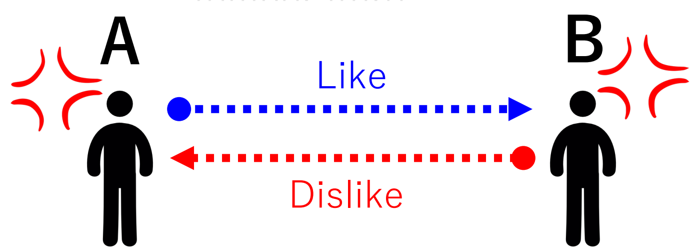
Entropic force selects the orbit!
= order by disorder

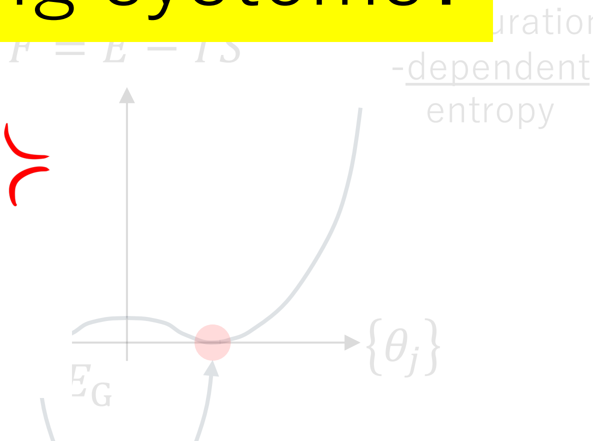
Order-by-disorder phenomena in geometrically frustrated systems

- Zero temperature
- Finite temperature

Q: Does the same physics occur in **non-reciprocally** interacting systems?







Entropically "selected" ground state

- Dynamics ($T > 0$):

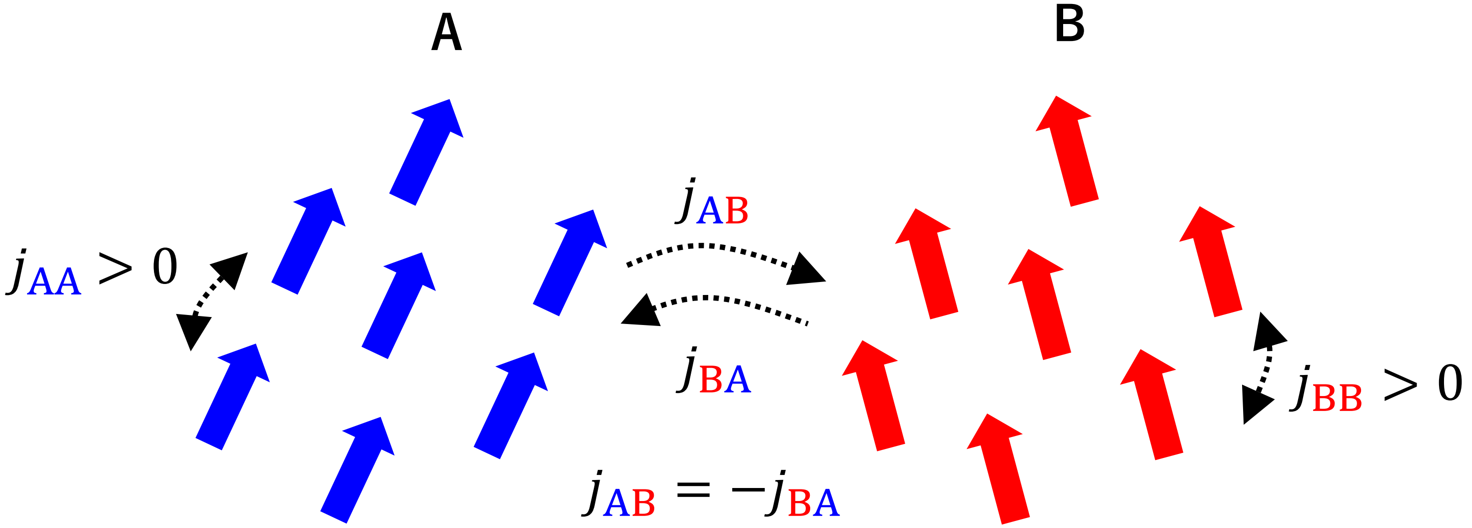
$$\dot{\theta} = -\frac{\partial F}{\partial \theta} = -\frac{\partial E}{\partial \theta} + T\frac{\partial S}{\partial \theta}$$

Entropic force

Entropic force selects the orbit! = order by disorder

Order-by-disorder phenomena

$$\dot{\theta}_i^a = \sum_b \sum_{i=1}^{N_b} \frac{j_{ab}}{N_b} \sin(\theta_j^b - \theta_i^a)$$

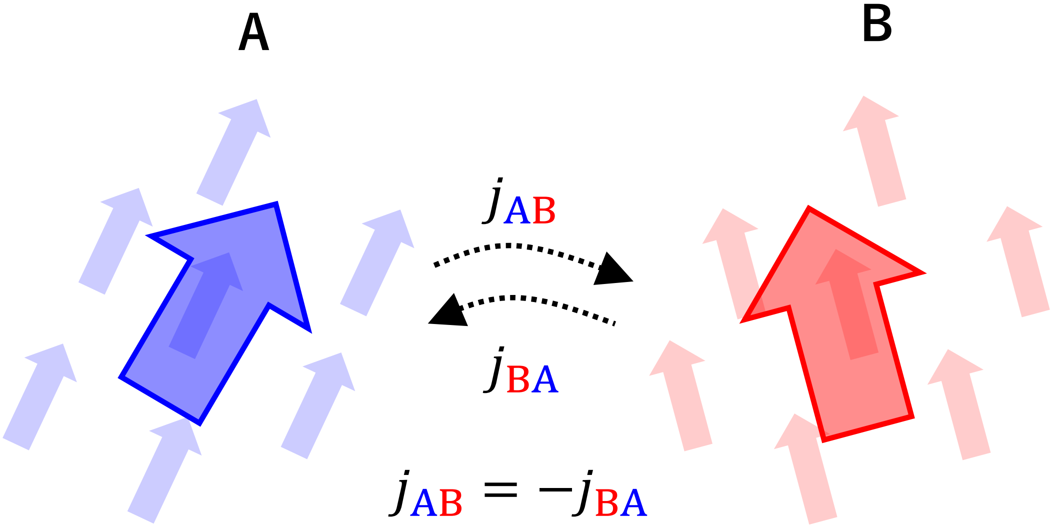


RH, PRX2024

Order-by-disorder phenomena

$$\dot{\theta}_i^a = \sum_b \sum_{i=1}^{N_b} \frac{j_{ab}}{N_b} \sin(\theta_j^b - \theta_i^a)$$

"Accidental degeneracy" parameterized by $\Delta\phi = \phi_{\mathrm{A}} - \phi_{\mathrm{B}}$



j_{AB}

j_{BA}

$j_{\mathrm{AB}} = -j_{\mathrm{BA}}$

Macroscopic spin (No noise)

$$\dot{\phi}_a = \sum_b j_{ab} \sin(\phi_b - \phi_a)$$

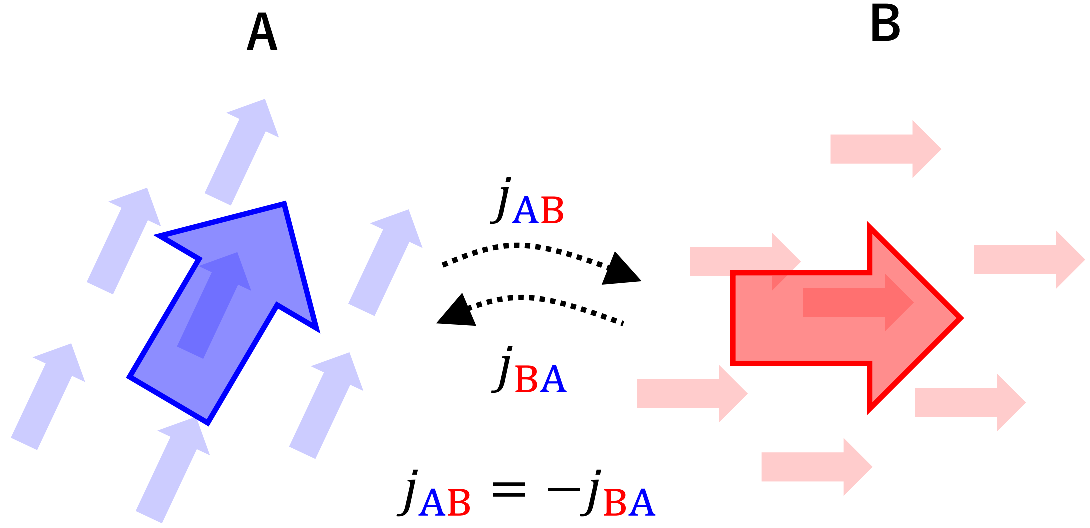
RH, PRX2024

Order-by-disorder phenomena

$$\dot{\theta}_i^a = \sum_b \sum_{i=1}^{N_b} \frac{j_{ab}}{N_b} \sin(\theta_j^b - \theta_i^a) + \boldsymbol{\eta_i^a}$$

Noise

"Accidental degeneracy" parameterized by $\Delta\phi = \phi_{\mathrm{A}} - \phi_{\mathrm{B}}$



Macroscopic spin (No noise)

$$\dot{\phi}_a = \sum_b j_{ab} \sin(\phi_b - \phi_a)$$

RH, PRX2024

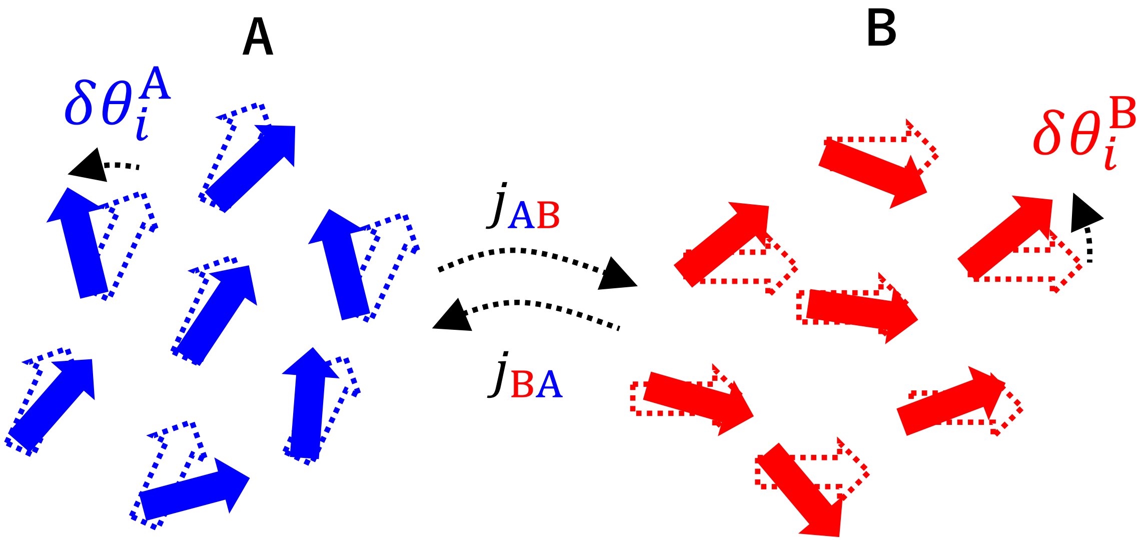
Order-by-disorder phenomena

$$\dot{\theta}_i^a = \sum_b \sum_{i=1}^{N_b} \frac{j_{ab}}{N_b} \sin(\theta_j^b - \theta_i^a) \; + \boldsymbol{\eta_i^a}$$

Noise

Probability distribution of $\delta\theta_i^a$ is $\Delta\phi$-dependent

"Accidental degeneracy" parameterized by $\Delta\phi = \phi_A - \phi_B$



Macroscopic spin (No noise)

$$\dot{\phi}_a = \sum_b j_{ab} \sin(\phi_b - \phi_a)$$

Macroscopic spin (With noise)

$$\dot{\phi}_a \approx \sum_b j_{ab}^{\star}(\Delta\phi) \sin(\phi_b - \phi_a)$$

$\Delta\phi$-dependent renormalized coupling

RH, PRX2024

Order-by-disorder phenomena

No noise

$$\Delta\dot{\phi} = -(j_{\mathrm{AB}} + j_{\mathrm{BA}}) \sin \Delta\phi = 0$$

Marginal when $j_{\mathrm{AB}} = -j_{\mathrm{BA}}$ $\Delta\phi(t) = \Delta\phi(0)$

With noise

$$\Delta\dot{\phi} = -\left(j^{\star}_{\mathrm{AB}}(\phi) + j^{\star}_{\mathrm{BA}}(\phi)\right) \sin \Delta\phi$$

$$\approx \frac{j_0 j_-^2 \sigma^2}{2} \frac{\cos \Delta\phi}{(j_0^2 - j_-^2 \cos^2 \Delta\phi)^2} \sin \Delta\phi$$

when $j_{AB} = -j_{BA} = j_-, j_{AA} = j_{BB} = j_0$

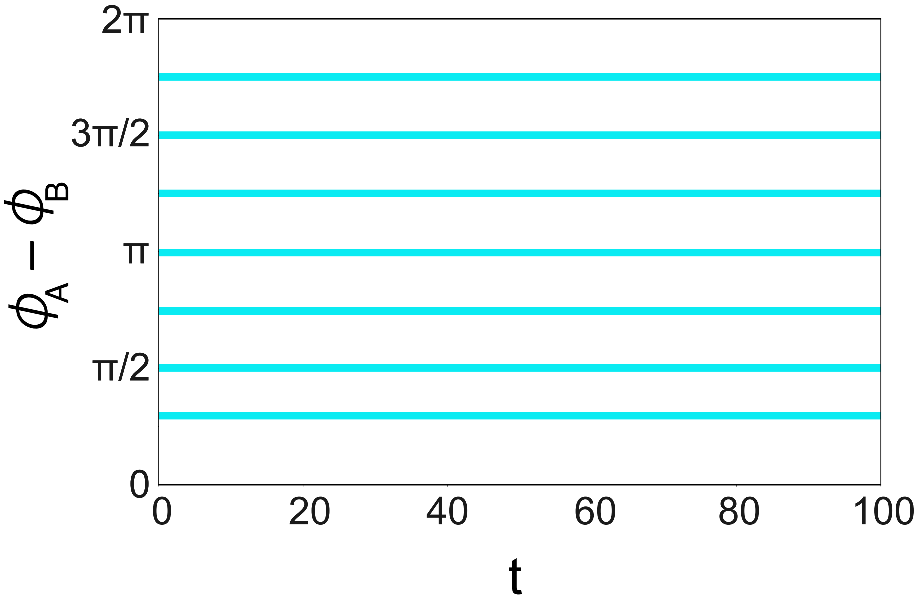
Fixed point at
$\Delta\phi_* = \pm\frac{\pi}{2}$!

"Entropic" force
favoring $\Delta\phi = \pm\frac{\pi}{2}$!

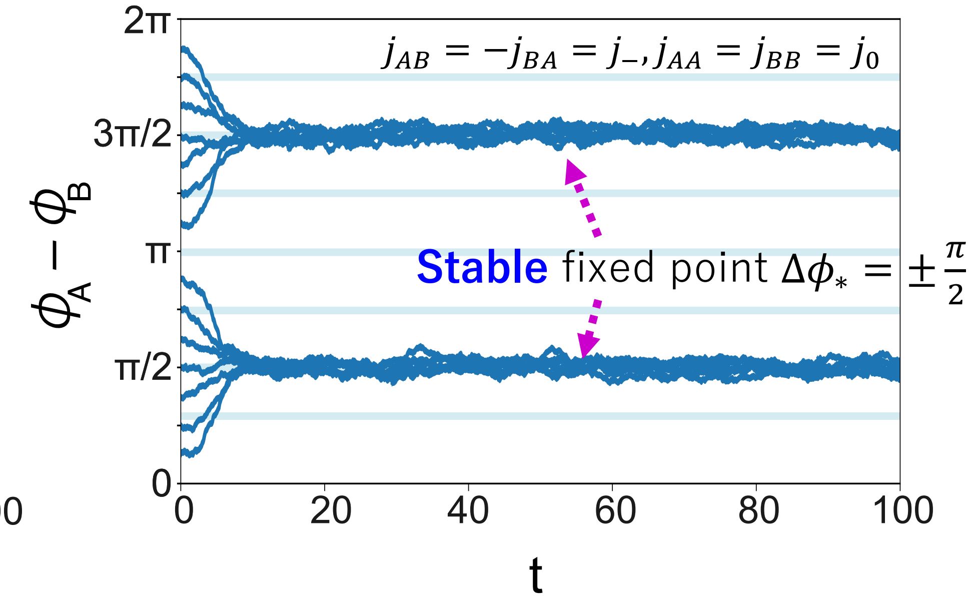
Order-by-disorder phenomena

$$j_{\mathrm{AB}} = -j_{\mathrm{BA}}$$

No noise



With noise



Order-by-disorder!

RH, PRX2024

Noise-induced non-reciprocal phase transition

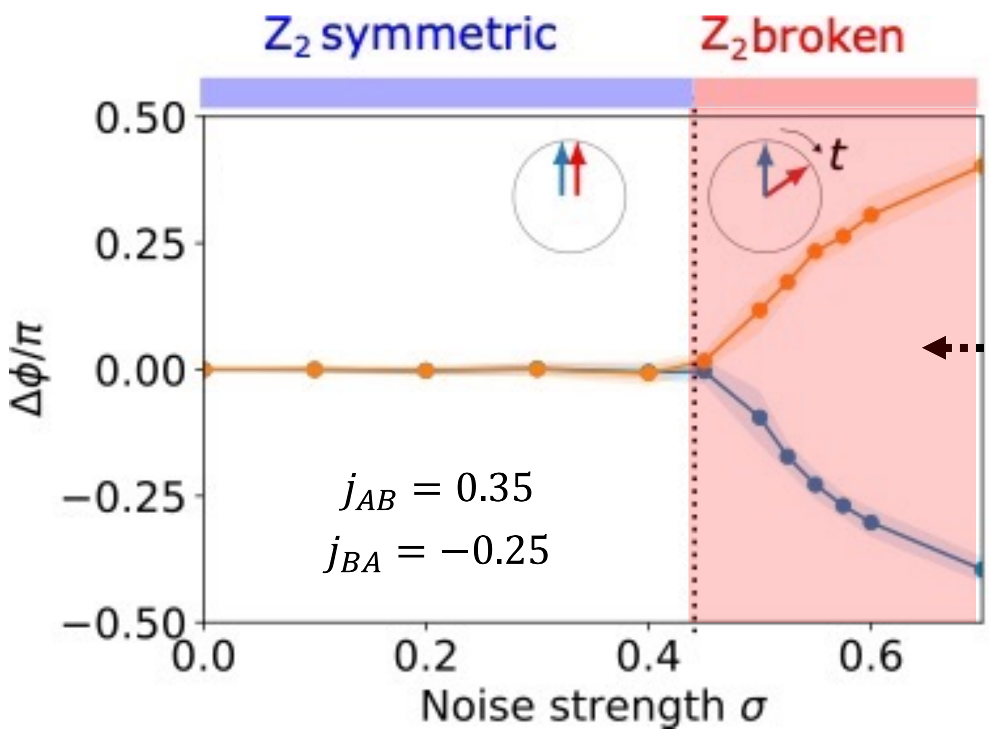
$$j_{\mathrm{AB}} > -j_{\mathrm{BA}}$$

$$\Delta\dot{\phi} = \left[-2j_+ + \frac{j_0 j_-^2 \sigma^2}{2} \frac{\cos\Delta\phi}{(j_0^2 - j_-^2\cos^2\Delta\phi)^2}\right]\sin\Delta\phi$$

$j_+ = j_{\mathrm{AB}} + j_{\mathrm{BA}} > 0$

"Energetics" favoring $\Delta\phi = 0$

"Entropic" force favoring $\Delta\phi = \pm\frac{\pi}{2}$



Noise **induced** symmetry breaking!

See also,
Fruchart*, RH*, Littlewood, Vitelli, Nature 2021

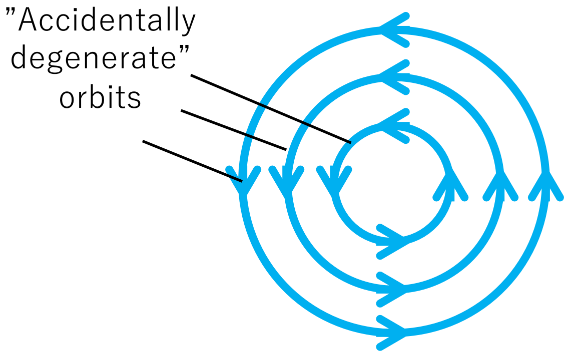
RH, PRX2024

$j_{AA} = j_{BB} = j_0$

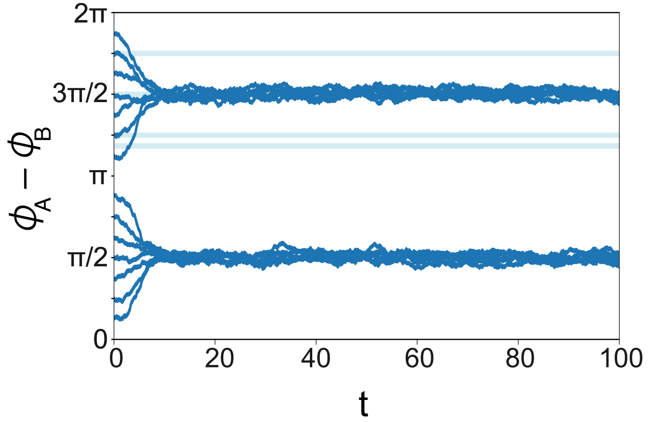
Summary

- Pointed out a direct analogy between geometrical and non-reciprocal frustration

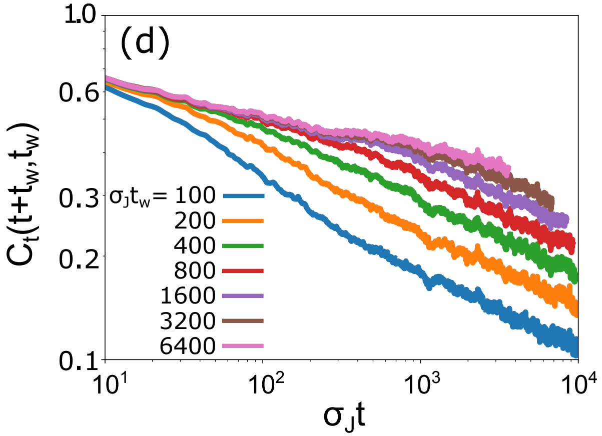
"Accidental degeneracy" of orbits



Order-by-disorder



Spin-glass-like state



RH, Phys. Rev. X **14**, 011029 (2024).
RH and Weis, in preparation

Derivation of renormalized coupling

Equation of motion

$$\dot\theta_i^a = -\sum_b \frac{j_{ab}}{N_b}\sum_{j=1}^{N_b}\sin\left(\theta_i^a - \theta_j^b\right) + \eta_i^a, \quad \longrightarrow \quad \dot\theta_i^a = -\sum_b j_{ab} r_b \sin(\theta_i^a - \phi_b) + \eta_i^a,$$

Order parameter $\psi_a = r_a e^{i\phi_a} = \frac{1}{N_a}\sum_i^{N_a} e^{i\theta_i^a}$

Order parameter dynamics in the absence of noise

$$\dot\phi_a = \sum_b j_{ab}\sin(\phi_b - \phi_a),$$

Marginal orbit (i.e. initial state dependent orbits) emerges when $j_{ab} = -j_{ba}$

Fluctuation dynamics $\delta\theta_i^a = \theta_i^a - \phi_a$

$$\delta\dot\theta_i^a \approx -\sum_b j_{ab}\cos(\phi_a(t) - \phi_b(t))\delta\theta_i^a + \eta_i^a.$$

Distribution function

$$\rho_i^a(t, \delta\theta_i^a; \phi(t)) = \frac{1}{\sqrt{\pi} w_a(t;\phi(t))} e^{-(\delta\theta_i^a)^2/w_a^2(t;\phi(t))}$$

with **configuration dependent** width $\quad w_a^2(t;\phi(t)) = 2\sigma\int_0^t d\tau e^{-2\int_\tau^t d\tau' \sum_b j_{ab}\cos\left(\phi_a(\tau') - \phi_b(\tau')\right)}$

Especially when $\Delta\phi_{ab} = \phi_a - \phi_b = \mathrm{const.}$, $\quad w_a^2(\phi) = \dfrac{\sigma}{\sum_b j_{ab}\cos\Delta\phi_{ab}}$

Derivation of renormalized coupling

Order parameter dynamics $\psi_a = r_a e^{i\phi_a}$

$$\dot{\psi}_a = (\dot{r}_a + r_a i\dot{\phi}_a)e^{i\phi_a} = \frac{i}{N_a}\sum_{i=1}^{N_a}\dot{\theta}_i^a e^{i\theta_i^a}$$

$$\begin{aligned}\dot{\phi}_a &= -\sum_b \frac{j_{ab}}{N_a}\sum_{i=1}^{N_a}\frac{r_b}{r_a}\sin(\theta_i^a - \phi_b)\cos(\theta_i^a - \phi_a) + \bar{\eta}_a \\ &= -\sum_b j_{ab}^\star(\phi(t))\sin(\phi_a - \phi_b) + \bar{\eta}_a\end{aligned}$$

with renormalized coupling $$j_{ab}^\star(\phi(t)) = j_{ab}\frac{r_b(\phi(t))}{r_a(\phi(t))}\left\langle \cos^2\delta\theta_i^a \right\rangle_{\phi(t)}$$

where $\langle\cdots\rangle_{\phi(t)} = \int d\delta\theta_i^a \rho_i^a\big(t, \delta\theta_i^a; \phi(t)\big)(\cdots)$

Distribution function

$$\rho_i^a(t, \delta\theta_i^a; \phi(t)) = \frac{1}{\sqrt{\pi}w_a(t;\phi(t))}e^{-(\delta\theta_i^a)^2/w_a^2(t;\phi(t))}$$

Here, self-averaging $\frac{1}{N_a}\sum_i^{N_a}(\cdots) = \langle\cdots\rangle_\phi$ is assumed.

Macroscopic noise strength

$$\langle\bar{\eta}_a(t)\bar{\eta}_b(t')\rangle \approx \frac{\sigma}{N_a}\delta_{ab}\delta(t-t') \quad \to 0\ (N_a \to \infty)$$

"Accidental degeneracy" of *orbits*

[Proof]

Continuity equation: $\frac{\partial \rho}{\partial t} = -\sum_i \frac{\partial(\rho \dot{\theta}_i)}{\partial \theta_i} = -\sum_i \left[\frac{\partial \rho}{\partial \theta_i}\dot{\theta}_i + \rho \frac{\partial \dot{\theta}_i}{\partial \theta_i}\right]$

$$\sum_i \frac{\partial \dot{\theta}_i}{\partial \theta_i} = \sum_{ij} [J_{ij} \cos(\theta_j - \theta_i)] = 0$$

$J_{ij} = -J_{ji}$

$$\dot{\theta}_\mathrm{i} = \sum_j J_{ij} \sin(\theta_j - \theta_i)$$

Therefore, $$\frac{\partial \rho}{\partial t} + \sum_i \frac{\partial \rho}{\partial \theta_i}\dot{\theta}_i = 0. \quad \blacksquare$$

※A similar theorem holds for non-reciprocally interacting particles and Heisenberg spins.

RH, PRX2024