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# Non-reciprocal frustration physics *Non-equilibrium physics Many-body (equilibrium) physics*

### Ryo Hanai

*Yukawa Institute for Theoretical Physics (YITP), Kyoto University* 



R. Hanai, Phys. Rev. X **14**, 011029 (2024).





Ground state:  $x = \{x_i\}$  that minimizes the energy E

Reciprocal coupling  $J_{ii} = J_{ii}$ 



### Non-reciprocally interacting systems



http://animal.memozee.com/vie w.php?tid=3&did=3513



e.g., A. Pluchino, et al., Inter. J. Mod. Phys. C 16, 515 (2005).





### Collective phenomena in non-reciprocal many-hody systems a 2 h 5 h 26 h 35 h 38 h anmon in Sarah A.M. Loose, 1986, 2006, 2007, 2008, 2008, 2008, 2008, 2008, 2008, 2008, 2008, 2008, 2008, 2008, 2008, 20<br>Personali A.M. Martynec <sup>2</sup>*ICTP – International Centre for Theoretical Physics, Strada Costiera, 11, 34151 Trieste, Italy* <sup>3</sup>*Technische Universit¨at Berlin, Straße des 17. Juni 135, 10623 Berlin, Germany* (Dated: June 23, 2022)  $W$  study a two-dimensional, non-two-dimensional, where each spin interaction interaction interactions on  $\mathbb{R}^n$ nearest neighbours in a certain angle around its current orientation, in analogy to a vision cone of a vision c

**a b**

Biodiversity in ecosystems  $\frac{S}{2}$ 



B. Kerr, et al., Nature 2002



*A* × area

ou<br>ah

**C.** Scheibn exceptional points and soft crystals and soft crystals in soft crystals in soft crystals in soft crystals in soft

Loos, Klapp, and Martynec, arXiv:2206.105 $19$ 

PHYSICAL REVIEW E<br>VOI  $\cdots$ 

No stress

contract the contract of the c

detectable by a non-zero entropy production rate.

0 Time (s) 2,000

**Fig. 1 | Developing starfish embryos self-organize into living control crystals. a, Time sequence of still images showing crystal assembly and dissolution assembly and dissolution and dissolution** 

0 h 12 h 26 h 38 h 44 h

### Spatially uniform traveling cellular patterns a

*y*

*K*o > 0

 $\left( a\right)$ 

Video2). Scale bar, 2 mm. **d**, Spinning embryos (yellow arrows) in the crystal

Odd elasticity of the finite vision cones.

d also in active systems. Using the system of the Monte-Carlo simulations we demonstrate  $\mathsf{Long\text{-}range\,}$ 

order in 2D

the emergence of a long-range ordered phase. A necessary ingredient is a configuration-dependent is a configuration-

 $\mathbf{r}$  in a flock reaction in a flock reaction in a flock reaction in  $\mathbf{r}$ 

**288** We report on a study of asymultipute in the concentration of active matterns in the concentration of  $\frac{1}{2}$ . interface in the experimental system<br>pattern appears via a supercritical pa symmetry and begins to drift with co<br>the drifting pattern as a function of t increases with the square root of the asymmetry. This behavior is in acco however, with the predictions of a n spatial modes with wave numbers  $q$ 

PACS number(s):  $47.54.+r$ ,  $47.20.K$ 

### in particular, concerning the emergence of phases and **I. INTRODUCTION**

dynamical systems [1]. Typically, an initiall<br> **EX** CON uniform system develops such a pattern, de stresses. The appropriate external incring. All example interactions were develops such a<br>
periment to be discussed in this paper is show a one-dimensional wave vector, we uncontracted a cone of the cone **b**, Based of the odd models in the original rate of the original state of the original state of the original state of the state of Stationary, one-dimensional patterns occudynamical systems [1]. Typically, an initially a one-dimensional wave vector, when it is di appropriate external forcing. An example fro  $\gamma_{\rm 0u, Baf}$  being the by changing as experimentally  $\gamma_{\rm 0u, Baf}$ 

اء<br>ا  $\text{FIG. 5.}$   $\text{Pab}$  and  $\text{Qb}$   $\text{Qb}$  and  $\text{Qb}$   $\text{Qb}$   $\text{Qb}$   $\text$  $v_o = 139.4 \text{ mm/s}$  transiation in time. It is periodic in space to the appear to the finite length of the experimental apparature translation in time. It is periodic in space (net  $(h)$ ) even are the space in space (net in  $\frac{1}{2}$  is increased as shown in Eig. 1(a). This pattern has certa ിG. 5. Pabana, Agհանա⊪©anaվe∳oխաαolestani  $(b)$   $(f)$  asymme is invariant under translation in the direction **Saha, Agudo-Canadejo, Golesting Canadeg a text of the consider a two-dimensions** 

volume 49, nubredhichi, et al., Phys. Rev. 195 2020

Department of Physics, Memorial Univers cialized to the from slower-rotating embryos34 manifests and slower-rotating embryos34 manifests and slower-rotating embryos34 manifests and slower- $\Gamma$ itions (Fig. 2c). To contributions (Fig. 2c). To contributions (Fig. 2c). To contributions (Fig. 2c). To concern (Fig. 2c). To consider (Fig. 2c). To consider (Fig. 2c). To consider (Fig. 2c). To consider (Fig. 2c). T  $\sigma$   $\mu$   $\sigma$ Department of Physics, Memorial Universed transference is a processed to the contract of the c Aligned Chiral

PHYSICAL REVIEW E  $W^{\bullet}$  $\sum_{i=1}^{\infty}$  Each spin interaction interacts only with the spin interaction interaction in  $\mathcal{L}$ 

VOLUME 49. NUMBER 1

**JANUA** 

**a**

 ${\bf Spatially}$  uniform traveling cellular patterns at a driven interface

**NATURE PHYSICS** ARTICLES ARE PHYSICS ARTICLES ARE PHYSICS ARTICLES ARTICLES ARE PHYSICS ARE PHYSICS ARE PHYSICS ARE PHYSICS ARE PHYSICS A

nada A1.<br>. n in<br><sub>ind,</sub>  $L^1$  is  $\mathbf{F} = \mathbf{F} \times \mathbf{F}$  and  $\mathbf{F} = \mathbf{F} \times \mathbf{F}$  number of total number of total number of  $\mathbf{F}$ coupled and solution of Physics, Memorial University of Newfoundland, St. John's, Newfoundland, Canony Canon Probability distributions *P*() in the LRO phase at *T* = 0*.*15.

interface in the experimental system known as the printer's instability. We find that the traveling pattern appears via a supercritical parity-breaking transition, at which the pattern loses its reflection<br>symmetry and begins to drift with constant speed. From measurements of the degree of asymmetry of<br>the drifting patte increases with the square root of the control parameter, and that the drift velocity is linear in the asymmetry. This behavior is in accord with recent theoretical predictions. Our results do not agree, asymmetry. I als behavior is in accord with recent theoretical predictions. Our results do not agree<br>
1. INTRODUCTION<br>
however, with the predictions of a model of the parity-breaking transition involving the coupling o

### I. INTRODUCTION propagates with constant velocity.

ciently far out of equilibrium by the applica Stationary, one-dimensional patterns occur in many which is initially straight. As the interface is a proportie external forcing. An example from the interface is a proportie of the interface is the interface is a proportie of the interface is a proportie of the interface coupled the two independent sheart of finders of the two indees of an oil-air inter<br>experiment show which is initially straight. As the interface is driver<br>which is initially straight. As the interface is driver permients to be ussues used in this paper is shown.<br>This figure shows video images of an oil-air and interface is the interface the appropriate external forcing. A<br>which is initially straight. As the interface is appropria dynamical systems [1]. Typically, an initially spatially uniform system develops such a pattern, described by a one-dimensional wave vector, when it is driven sufficiently far out of equilibrium by the application of an appropriate external forcing. An example from the ex- $20$  gyptment to be discussed in this paper is shown in Fig. 1.<br> $20$  and figure shows video images of an oil-air interface, which is initially straight. As the interface is driven out  $\frac{ce}{ce}$ spins **TV**<sub>C</sub>  $\mathbf{V} = \mathbf{S}$ ,  $\mathbf$ translation in time. It is periodic in space (neglecting

shown to be one of ten possible generic secondary<br>bilities of stationary one-dimensional patterns let and Iooss [3]. Parity-breaking bifurcations 1(b) is an example of a parity-breaking bifurcation<br>a bifurcation was postulated by Coullet et al. [ odi<br>Latio, as:<br>John by<br>dobs o the traple of<br>mple of was pos<br>o be dis<br>ionary<br>[3]. Pa<br>e subjee<br>2,4–19].<br>paritional post<br>of the states<br>is facter.<br>[20–3 explanation for phenomena observed in other exp tal systems, to be discussed below. Parity-breal tionary, symmetric<br>asymmetric pattern cently been the subject of a substantial amount<br>retical work [2,4-19]. Experimentally, both loca The transition from the stationary, symmetric<br>of Fig. 1(a) to the traveling, asymmetric pattern

### **SPATIALLY UNIFORM TRAVELING CELLULAR I** Department of Physics, Memorial University of a study of asymm

face at the front of the apparatus was monitored with a charge-coupled-device (CCD) video camera and monitor, and data were recorded on a VCR or stored on a personal computer using a video frame grabber. Images of the interface presented in this paper have been contrast enhanced, but are otherwise unprocessed. For quantitative analysis of the interface shape, the interface height as a function of  $x$  was extracted from video images by having the computer trace along the path of darkest pixels from a given starting point.

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In the experiments reported here, the minimum width of the gap between the cylinders was 0.5 mm, set with the micrometer screws. The stability of the stationary fingering pattern observed when only one cylinder rotated was very sensitive to the parallelism of the cylinder axes; this fact was used to optimize the cylinder alignment. From the way in which the stationary pattern appeared at its

 $\alpha$  and  $\alpha$ spatial modes with wave numbers  $q$  and  $zq$ .  $PACS$  number(8):  $41.54. +1, 41.20.$ Ky, 06.10.5w

## Non-reciprocal interaction ≈Frustration physics



### Non-reciprocal friendship ≈Frustration physics Like **A B**

Dislike



### Non-reciprocal friendship: source of frustration



# Geometrical frustration

### **Geometrically frustrated system**

A system that cannot satisfy all the constitutes' "desire" to minimize all interaction energies.



No configuration can make all spins happy

### $\blacksquare$ **Properties of a Classical Spin Liquid: The Heisenberg Pyrochlore Antiferromagnet** R. Moessner and J. T. Chalker phenomena in geometrical frustrated systems Accidental degeneracy induces exot

> Order-by-disorder





Willain, et al., J. Physique (1980)



 $\triangleright$  Spin ice

(Image from https://scglass.uchicago.edu/

▶ Quantum/Classical spin liquid *q* ≠ 3 and *q* ≠ 4, respectively [14]. An antiferromag-





R. Mossener and J. T. Chalker, PRL 1998 T. Imai and Y. Lee, Physics Today 2016 .<br>I



(Image from Wikipedia)

Villain, et al., J. Physique (1980)

**Accidental degeneracy:** Not protected by symmetry nor topology



### to be a consequence of their structures, with magnetic ions arbu dioordo tet i IV (IISOIC) Compounds in this class include SrCr8Ga4O<sup>19</sup> (SCGO) namics) decays in time *t* as k**S***i*s0d ? **S***i*s*t*dl ≠ exps2*cTt*d, where *c* is a constant. This behavior is in striking contrast to that of the kagomé Heisenberg antiferromagnet, previously the best-studied example of geometric frustration, in namics) decays in time *t* as k**S***i*s0d ? **S***i*s*t*dl ≠ exps2*cTt*d, where *c* is a constant. This behavior is in striking contrast to that the karakter and the karakter and the kange of the kange of the kange of the kange of the ka Order by disorder phenomena

is reduced towards *TF* [1]. An important step towards

sites of a kagomé lattice, and fluoride, and fluoride (Example) **the limit**  $\sigma$   $\sigma$   $\sigma$  $t_{\text{ref}}$  is in Fig. 1. Magnetic correlations in Fig. 1. Magnetic correlations in Fig. 1. Magnetic correlations in  $\mathcal{L}$ erromagnetic XY spins of [2,3,5,6] and muon spin relaxation [4,7] measurements, the limit *T* ! 0 [12]. Additionally, we find, in agreement Moessner and Chalker, PRL1998, PRB1998 perimentally in most pyrochlore antiferromagnets [13] is a pyrochlore la Antiferromagnetic XY spins on a pyrochlore lattice  $(LXdII]$ , the freezing transition observed exabsent from a We take, as a general starting point, *n*-component clas-



Disordered state

sharing units: the kagomé and pyrochlore lattices have lattices have  $\bigcap_{n=1}^{\infty}$ 

sharing units: the kagomé and pyrochlore lattices have

## Geometrical vs Non-reciprocal frustration

Accidental degeneracy of ground state



Geometrical frustration Mon-reciprocal frustration



Energy cannot be defined ...

May not even converge to a static state...

## Geometrical vs Non-reciprocal frustration

### Accidental degeneracy of ground state



### Geometrical frustration Mon-reciprocal frustration

"Accidental degeneracy" of orbits



Ø *Dynamical counterpart* of order-bydisorder and spin glass(-like) state occurs!

This talk: order by disorder

### Dissipative XY spin dynamics

$$
\dot{\theta}_i = \sum_j J_{ij} \sin(\theta_j - \theta_i)
$$

Here, couplings are non-reciprocal in general  $J_{ij} \neq J_{ji}$ 

## Dissipative XY spin dynamics

Reciprocal case  $(J_{ij} = J_{ji})$ : Energy minimization problem

$$
\dot{\theta}_i = -\frac{\partial E(\theta)}{\partial \theta_i} \quad \text{with} \quad E(\theta) = -\sum_{i,j} J_{ij} \cos(\theta_j - \theta_i)
$$

Potential with geometrically frustration



Accidentally degenerate ground states



 $\triangleright$  Orbits



# "Accidental degeneracy" of *orbits*

 $\triangleright$  Anti-symmetric case  $(J_{ij} = -J_{ji})$ 



Conservation of phase volume = **Non-dissipative** dynamics



Marginal





Common feature between geometrically frustrated and non-reciprocal system: **Marginal orbits!** 

### Order-by-disorder phenomena in geometrically frustrated systems

**► Zero temperature**  $\rightarrow$  **Finite temperature** 



 $\dot{\theta} = -\frac{\partial F}{\partial \theta}$  $\partial \theta$  $=\frac{1}{2}$  $\overline{\partial L}$  +  $T$  $\partial S$  $\partial \theta$ Entropic force **Entropic force** selects the orbit! = order by disorder Marginal

### Order-by-disorder phenomena in geometrically frustrated systems

**► Zero temperature**  $\rightarrow$  **Finite temperature** 

### Q: Does the same physics occur in non-reciprocally interacting systems?



Dynamics  $(T > 0)$ :

 $\dot{\theta} = -\frac{\partial F}{\partial \theta}$  $\partial \theta$  $=\frac{1}{2}$  $\frac{1}{\partial \theta} + T$  $\partial S$  $\partial \theta$ Entropic force

**Entropic force** selects the orbit! = order by disorder



$$
\dot{\theta}_i^a = \sum_b \sum_{i=1}^{N_b} \frac{j_{ab}}{N_b} \sin(\theta_j^b - \theta_i^a)
$$
  
A  
"Accidental  
degeneracy"  
parameterized by  

$$
\Delta \phi = \phi_A - \phi_B
$$
  

$$
j_{AB}
$$
  

$$
j_{BB}
$$
  

$$
j_{BB}
$$

Macroscopic spin (No noise)

$$
\dot{\phi}_a = \sum_b j_{ab} \sin(\phi_b - \phi_a)
$$



Macroscopic spin (No noise)

$$
\dot{\phi}_a = \sum_b j_{ab} \sin(\phi_b - \phi_a)
$$



 $\Delta\phi$ -dependent renormalized coupling





### Order-by-disorder phenomena  $j_{AB} = -j_{BA}$

No noise With noise

(2)





### Summary  $\mathbf{I}$   $\mathbf{$

• Pointed out a direct analogy between geometrical and non-reciprocal frustration



RH, Phys. Rev. X **14,** 01102<mark>9</mark> (**2**024). (2) RH and Weis, in preparation  $\frac{1}{2}$ 6×10<sup>3</sup>

Jt

### Derivation of renormalized coupling pled *Delivation* for reflormance A*,* B*,* C*, ...,* following the Langevin equation,  $\alpha$ alized coupling  $\ldots$  contract coupled nature  $\sigma$ but ferromagnetically within the same community,  $\mathbf{r}$ *<i>D***crivation of Ferrolli**

Equation of motion *b j*

$$
\dot{\theta}_i^a = -\sum_b \frac{j_{ab}}{N_b} \sum_{j=1}^{N_b} \sin\left(\theta_i^a - \theta_j^b\right) + \eta_i^a, \quad \dots \quad \downarrow \qquad \dot{\theta}_i^a = -\sum_b j_{ab} r_b \sin\left(\theta_i^a - \phi_b\right) + \eta_i^a,
$$
\nOrder parameter  $\psi_a = r_a e^{i\phi_a} = \frac{1}{N_a} \sum_i^{N_a} e^{i\theta_i^a}$ 

dex next part of dimension in the eperopee of point <u>aer parameter uynar</u> <u>Order parameter dynamics in the absence of noise</u>  $T$ <u>r paramete</u> r*<sup>i</sup> · r*˙*<sup>i</sup>* = ⇢ *r a meter dynamics in the absence of noise* 

$$
\dot{\phi}_a = \sum_b j_{ab} \sin(\phi_b - \phi_a), \ \ \blacktriangleleft
$$

 $\sum_{i=1}^{n}$  waighted of protection of state dependent of pris<sub>i</sub> emerges  $t \mapsto t$  is governed by the renormalized couplings,  $t \mapsto t$ community coupling is taken to be geometrically/non $j_{ab}$   $\sin(\phi_b - \phi_a)$ ,  $\leftarrow$  Marginal orbit (i.e. initial state dependent orbits) emerges can take di↵erent orbits (*t*)=(A(*t*)*,* B(*t*)*, ...*) depend- $\mu = -J_{ba}$ 

$$
\frac{\text{Fluctuation dynamics}}{\delta \theta_i^a} = \theta_i^a - \phi_a \qquad \delta \dot{\theta}_i^a \approx -\sum_b j_{ab} \cos(\phi_a(t) - \phi_b(t)) \delta \theta_i^a + \eta_i^a.
$$

community control is taken to be generally non-trivial control in the geometrical control of  $\alpha$ Distribution function *ij* (*|r<sup>i</sup> r<sup>j</sup> |*)  $\blacksquare$ Distribution function *ij* (*|r<sup>i</sup> r<sup>j</sup> |*)*.*

$$
\rho_i^a(t,\delta\theta_i^a;\phi(t))=\frac{1}{\sqrt{\pi}w_a(t;\phi(t))}e^{-(\delta\theta_i^a)^2/w_a^2(t;\phi(t))}
$$

with **configuration dependent** width  $w_a^2(t; \phi(t)) = 2\sigma \int_0^t a^2(t; \phi(t)) dt$ with **configuration dependent** width This proves the desired Liouville-type theorem

with **configuration dependent** width 
$$
w_a^2(t;\phi(t)) = 2\sigma \int_0^t d\tau e^{-2\int_\tau^t d\tau' \sum_b j_{ab} \cos\left(\phi_a(\tau') - \phi_b(\tau')\right)}
$$

To proceed, we consider the dynamics of fluctuations ✓*<sup>a</sup> <sup>i</sup>* = ✓*<sup>a</sup> <sup>i</sup> <sup>a</sup>* caused by noise. Assuming weak noise ⇢*a <sup>i</sup>* (✓*<sup>a</sup>*  $\Delta \phi_{ab}$ Especially when  $\Delta\phi_{ab} = \phi_a - \phi_a$  $J_0$ <br> *i*  $w_a^2(\phi) = \frac{\sigma}{\sigma}$ *w*2 *<sup>a</sup>*(*t*; (*t*)) = 2  $\mathbf{r}$   $\mathbf{r}$  $w_a(\psi) = \frac{\sum_b j_{ab} \cos \Delta}{2}$  $\overline{f}$  +  $\$  $\mathcal{L}^2$  ipecially when  $\Delta \phi_{ab} = \phi_a - \phi_b = \text{const.}$ ,  $w_a^2(\phi) = \frac{\sigma}{\sigma_a}$  $\sum_b$  *Jab*  $\cos \Delta \phi_{ab}$ Especially when  $\Delta\phi_{ab} = \phi_a - \phi_b = \text{const.}$ ,  $w_a^2(\phi) = \frac{\sigma}{\sum_i q_i \sigma^2}$  $\sum_b j_{ab}$  cos Δ $\phi_{ab}$ 

### Derivation of renormalized coupling  $\mathcal{L}$  provide here the analysis of the analysis of the analysis of the analysis of  $\mathcal{L}$ tivation of refluctuatized by  $\mathbb{R}^n$ Derivation metrically and non-reciprocally frustrated systems. For that are a $\mathcal{A}$  are above fluctuations induced by the above fluctuations induced by  $\mathcal{A}$ *a*  $\alpha$  *derivation of renormal N<sup>a</sup>*  $\overline{a}$

by-disorder phenomena (OBDP) occurring in both geometrically and non-reciprocally frustrated systems. For noise. From Order parameter dynamics  $\psi_a = r_a e^{i \phi_a}$ pled XY-model grouped into a few communities *a* = pled XY-model grouped into a few communities *a* = A*,* B*,* C*, ...,* following the Langevin equation,

$$
\dot{\psi}_a = (\dot{r}_a + r_a i \dot{\phi}_a) e^{i\phi_a} = \frac{i}{N_a} \sum_{i=1}^{N_a} \dot{\theta}_i^a e^{i\theta_i^a}
$$

$$
\dot{\phi}_a = -\sum_b \frac{j_{ab}}{N_a} \sum_{i=1}^{N_a} \frac{r_b}{r_a} \sin(\theta_i^a - \phi_b) \cos(\theta_i^a - \phi_a) + \bar{\eta}_a
$$
\n
$$
= -\sum_b j_{ab}^{\star}(\phi(t)) \sin(\phi_a - \phi_b) + \bar{\eta}_a
$$

with renorm = *<sup>i</sup> ,* (B2)  $\frac{1}{\phi}$   $j_{ab}^*(\phi(t)) = j_{ab} \frac{r_b(\phi(t))}{r_b(\phi(t))} \left\langle \cos^2 \delta \theta_i^a \right\rangle_{\phi(t)}$ *rb*((*t*))  $r_b(\phi(t))$  $r_a(\phi(t))$  $\left\langle \cos^2\delta\theta^a_i \right\rangle_{\phi(t)}$ with renormalized coupling X <sup>r</sup>*<sup>i</sup> · <sup>f</sup> <sup>a</sup>*  $r_a(\phi(t))$  $qa \setminus$  $\int i / \phi(t)$ 

where  $\langle \cdots \rangle_d$ where  $\langle \cdots \rangle_{\phi(t)} = j \omega \sigma_i \rho_i \rho_i$ *ab*((*t*)) = *jab*  $\delta\theta_i^a \rho_i^a(t, \delta\theta_i^a; \phi(t))$  (...) P*<sup>N</sup><sup>a</sup>*  $\frac{1}{2}$ *ij* <sup>r</sup>*<sup>j</sup> · <sup>f</sup> <sup>a</sup>* where  $\langle \cdots \rangle_{\phi(t)} = \int d\delta \theta_i^a \rho_i^a(t, \delta \theta_i^a; \phi(t)) (\cdots)$ 

 $\mathbf{r}$  is governed by the renormalized couplings,  $\mathbf{r}$  $\rho_i^a(t, \partial \theta_i^a; \phi(t)) = -\n\begin{cases}\n\frac{\partial^a}{\partial t^a(t, \partial \theta_i^a; \phi(t))}{\partial t^a(t, \partial \theta_i^a; \phi(t))} = -\n\end{cases}$ positive that the positive term is a crucial positive term in the electron control of the e $\Gamma$  $\rho_i^a(t, \delta \theta_i^a; \phi(t)) = \frac{1}{\sqrt{\pi}w_a(t; \phi(t))} e^{-(\delta \theta_i^a)^2/w_a^2(t; \phi(t))}$ Distribution function

Here, selt the maging  $\frac{1}{n}\sum_{i=1}^{N_a}$  (...)  $\equiv$  (...), is assumed.  $N_a$   $\Delta i$  ( ) – 1  $\beta$  is assumed Here, self-averaging  $\frac{1}{N} \sum_{i=1}^{N_a} (\cdots) = \langle \cdots \rangle_a$  is assumed.  $g$ ing  $\frac{1}{N_a} \sum_i a(\cdots) = \langle \cdots \rangle_{\phi}$  is assumed. ing on their in the self-ave ity. We will show below that this "accidental degeneracy"  $t = \frac{1}{2} \sum_{i=1}^{N} x_i$  $\log \frac{1}{N_a} \sum_i u(\cdots) = \langle \cdots \rangle_\phi$  is assumed. Here, self-averaging  $\frac{1}{N_a}\sum_i^{N_a}(\cdots) = \langle \cdots \rangle_\phi$  is assumed.

 $M$ acroscopic noise strength *t* 0 ), and <sup>h</sup>*h*(✓*<sup>a</sup>* Macroscopic noise strength sumed that the system self-averages, i.e., <sup>h</sup>*h*(✓*<sup>a</sup>* To proceed, we consider the dynamics of fluctuations ⇢*a <sup>i</sup>* (✓*<sup>a</sup>*  $\mathbf{r}$ initial condition in the absence of stochastic-of stochastic-of stochastic-of-stochastic-of-stochastic-of-stochastic-of-stochastic-of-stochastic-of-stochastic-of-stochastic-of-stochastic-of-stochastic-of-stochasti iviacioscopi  $\sim t$   $\sim$ 

*<sup>i</sup> ,* (B1)

bio noise strength

\n
$$
\langle \bar{\eta}_a(t)\bar{\eta}_b(t')\rangle \approx \frac{\sigma}{N_a} \delta_{ab} \delta(t-t') \longrightarrow 0 \left(N_a \to \infty\right)
$$

## "Accidental degeneracy" of *orbits*

### **[Proof]**

Continuity equation: 
$$
\frac{\partial \rho}{\partial t} = -\sum_{i} \frac{\partial (\rho \dot{\theta}_{i})}{\partial \theta_{i}} = -\sum_{i} \left[ \frac{\partial \rho}{\partial \theta_{i}} \dot{\theta}_{i} + \rho \right]
$$

$$
\sum_{i} \frac{\partial \dot{\theta}_{i}}{\partial \theta_{i}} = \sum_{ij} [J_{ij} \cos(\theta_{j} - \theta_{i})] = 0
$$
  

$$
\dot{\theta}_{i} = \sum_{j} J_{ij} \sin(\theta_{j} - \theta_{i})
$$
  
Therefore, 
$$
\frac{\partial \rho}{\partial t} + \sum_{i} \frac{\partial \rho}{\partial \theta_{i}} \dot{\theta}_{i} = 0.
$$

※A similar theorem holds for non-reciprocally interacting particles and Heisenberg spins.