The triple-α reaction at low temperatures by an exact three-body model

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The direct triple-α process is discussed by a non-adiabatic Faddeev HHR* expansion method. The smoothly varying cross sections in photo-disintegration of $^{12}C(2^+_1 \to 0^+)$ are obtained at off-resonant energies, and the values of HHR* are much smaller than those of the adiabatic models for 0.15 < E < 0.35 MeV. The resultant reaction rates have a strong temperature dependence, as well as NACRE, and they are expressed in analytic forms. From the comparison between the calculations, the current evaluated rates are found to be reduced by about 10^{-4} at T_9 = 0.05, because of an accurate description of ^{8}Be break-up. The present rates do not have a component of the non-resonant sequential process between α+^{8}Be. Therefore, this component could be eliminated by hand, to update the rates in NACRE & REACLIB. Using the new rates, the direct process is exemplified to be important for helium burning at T_9= 0.01 in accreting white dwarfs, and the resultant ignition density is found to be insensitive to temperatures in 0.01 < T_9 < 0.05, from the reduction of rates at T_9= 0.05.

1.1 Triple-α reaction

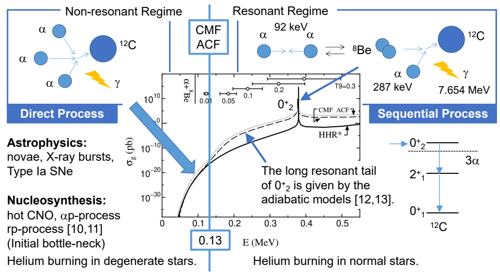
- **Triple-α reaction** plays an important role in nucleosynthesis heavier than ^{12}C, because no stable nuclei exist in mass number A=5 and A=8 [1,2].
- This reaction, followed by $^{12}C(\alpha,\gamma)^{16}O$ [3], controls C/O ratio at the end of helium burning phase in stars, and it affects up to the nucleosynthesis in extreme conditions, *e.g.* supernova explosion.
- In contrast to $^{12}C(\alpha,\gamma)^{16}O$, triple-α reaction is currently well-understood through the experimental studies of 0^+_2 (E_R = 0.379 MeV) in ^{12}C (*e.g.* [4]).
- *i.e.*, the reaction rates have been determined relatively well with the **sequential process** via the narrow resonances: α+α→^{8}Be, α+^{8}Be→^{12}C.
- Pioneering works: **CF88**[5], [6,7]; Exp. update: NACRE [8]

1.4 Present Report

- Show the calculated photo-disintegration of ^{12}C, using a non-adiabatic Faddeev HHR* expansion method, and discuss 3α reaction rates.
- Discuss the difference between the non-adiabatic and adiabatic models.
- The rates are given in an analytic expression.
- Explain that the evaluated rates could be reduced at T_9= 0.05.
- Example of the new rates: Ignition critical density of helium burning in accreting white dwarfs.

1.2 Direct triple-α process

- Apart from the sequential process, triple-α reaction from 3α continuum states is referred to as **direct process** [9].



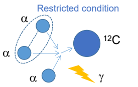
Is this boundary realistic?

1.3 Theoretical studies of triple-α reaction

- The direct process is generally expected to be *very slow*, because three α-particles almost simultaneously collide and fuse into a ^{12}C nucleus.
- Thus, this process is neglected or is treated in some approximations.

- **NACRE:** ^{8}Be is assumed to be bound as a particle, and the rates have been estimated by an improved model based on the pioneering works.

- To study the triple-α reaction dynamically, **Adiabatic approximation** of 3α continuum states has been adopted in *e.g.* [14], and recently in
  - **CMF:** Coulomb Modified Faddeev method [12],
  - **ACF:** Adiabatic Channel Function expansion method [13].



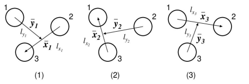
- CMF & ACF may have achieved the successful progress quantitatively.
- However, the precise description at off-resonant energies still seems to remain in an open question.

- **Non-adiabatic approach** of 3α has also been performed rigorously in
  - **HHR:** Hyper-spherical Harmonics & *R*-matrix expansion [15-17],
  - **HHR*:** Improvement in the asymptotic region, and *S*-factors [9].

2.1 Faddeev HH expansion

- Three-body Schrödinger equation (Faddeev equation):

$$\begin{cases} (\hat{T}_{(1)} + \hat{V}_{23} + \hat{V}_{3\alpha} - E)\Psi_{(1)} = -\hat{V}_{23}\Psi_{(2)} - \hat{V}_{23}\Psi_{(3)} \\ (\hat{T}_{(2)} + \hat{V}_{31} + \hat{V}_{3\alpha} - E)\Psi_{(2)} = -\hat{V}_{31}\Psi_{(3)} - \hat{V}_{31}\Psi_{(1)} \\ (\hat{T}_{(3)} + \hat{V}_{12} + \hat{V}_{3\alpha} - E)\Psi_{(3)} = -\hat{V}_{12}\Psi_{(1)} - \hat{V}_{12}\Psi_{(2)} \end{cases}$$



Three identical sets of equations are found, because of the symmetric 3α system.

Faddeev eq. is reduced as

$$(\hat{T}_{(3)} + \hat{V} - E)\Psi_{(3)} = 0$$

$$\hat{V} = \sum_{i<j}^{3} \hat{V}_{ij} + \hat{V}_{3\alpha}$$

3α interaction; α+α interaction; CDb int. [9]

Hyper-spherical coordinates are

$$\rho^2 = x_i^2 + y_i^2$$

$$\theta_i = \arctan\left(\frac{x_i}{y_i}\right) \quad \Omega_5^{(3)} \equiv (\theta_3, \hat{\boldsymbol{x}}_3, \hat{\boldsymbol{y}}_3)$$

$$\Psi_{(3)} = \rho^{-5/2} \sum_\gamma \chi^l_\gamma(\rho)\, \Phi^\gamma_{lm}(\Omega_5^{(3)})$$

- Coupled-channel (CC) equations with hyper-radial wavefunctions:

$$[T_\gamma + U^l_{\gamma\gamma}(\rho) - \epsilon]\, \chi^l_\gamma(\rho) = -\sum_{\gamma' \neq \gamma} U^l_{\gamma'\gamma}(\rho)\chi^l_{\gamma'}(\rho)$$

$$T_\gamma = \frac{d^2}{d\rho^2} - \frac{(K+3/2)(K+5/2)}{\rho^2}, \quad U^l_{\gamma'\gamma}(\rho) = -\frac{2m_N}{\hbar^2}V^l_{\gamma'\gamma}(\rho), \quad \epsilon = -\frac{2m_N}{\hbar^2}E.$$

CC equations for inelastic scattering *e.g.* [18,19] (if *L*= *K*+3/2).

Hyper-radial wavefunctions; Hyper-harmonic functions

The hyper-angle part is solved with Jacobi polynomials.

2.2 *R*-matrix expansion

CC equations, $(\boldsymbol{T}+\boldsymbol{U})\boldsymbol{X} = \epsilon\boldsymbol{X}$, are solved by matrix diagonalization.

Eigenstates:

$$\chi^l_{\gamma i}(\rho) = \sum_n c^n_{\gamma i}\varphi^K_n(\rho)$$

$$E(l_i^+) = -\frac{\hbar^2}{2m_N}\epsilon_i$$

- *R*-matrix expansion:

Continuum states with scattering boundary condition are expanded by the resultant eigenfunctions.

$$\tilde{\gamma}_{\alpha i} = \sqrt{\frac{\hbar^2}{2m_N a_c}}\chi^l_{\alpha i}(a_c)$$

Reduced width amplitudes

$$\chi^{l\,in}_{\alpha\beta}(k,\rho) = \sum_i A_{i\beta}(k)\chi^l_{\alpha i}(\rho) \quad \text{---- Eq.(A)}$$

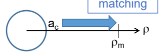
$$R^l_{\alpha\beta}(E,a_c) = \sum_i \frac{\tilde{\gamma}_{\alpha i}\tilde{\gamma}_{\beta i}}{E(l_i^+) - E}$$

R-matrix

a_c : channel radius

To include the long-range Coulomb couplings, CC equations in the external region are solved numerically from ρ = a_c to ρ = ρ_m, $\chi^{l\,in}_{\alpha\beta}(k,a_c) \longrightarrow \chi^{l\,ext}_{\alpha\beta}(k,\rho)$.

Asymptotic matching



$$\tilde{\chi}^l_{\gamma\gamma_0}(k,\rho) = \sum_{\gamma'} C_{\gamma'\gamma_0}(k)\chi^{l\,ext}_{\gamma\gamma'}(k,\rho)$$
$$\to \frac{i}{2}\left[I^{(\gamma_0)}_{\gamma,K+3/2}(\eta_\gamma;k\rho_m) - \sum_{\gamma'} S^l_{\gamma'\gamma_0}(E)O^{(\gamma')}_{\gamma,K+3/2}(\eta_\gamma;k\rho_m)\right]$$

S-matrix; Coupled-Coulomb waves [19]

$$\tilde{\chi}^{l\,in}_{\gamma\gamma_0}(k,\rho) = \sum_{\gamma'} C_{\gamma'\gamma_0}(k)\chi^{l\,in}_{\gamma\gamma'}(k,\rho) \leftarrow \text{Eq.(A)}$$
$$= \sum_i D_{i\gamma_0}(k)\chi^l_{\gamma i}(\rho)$$

Continuum states

Cross sections between 2^+_1 and 0^+ continuum states are calculated from these wavefunctions. Quadruple precision is required to execute stable calculations.

2.3 Analytic form of reaction rates

- Photo-disintegration cross sections of $^{12}C(2^+_1 \to 0^+)$:

$$\sigma_g(E) \propto \sum_{m'mq} \int \left|\langle \Psi_{f,2m'} | \mathcal{M}^E_{2q} | \Psi^{(+)}_{0m} \rangle\right|^2 d\Omega^k_5$$

0+ continuum states; 2^+_1 bound state

- Three-body *S*-factors:

$$S_{3\alpha}(E) \equiv E_g^2\sigma_g(E)\exp\left(\frac{2\pi\eta_0}{\sqrt{E}} + aE\right) = s_0(1 + s_1E + s_2E^2)$$

The coefficients are obtained from the calculated cross sections of HHR*. $s_0,\ s_1,\ s_2,\ \eta_0,\ a$

- Non-resonant reaction rates (Direct process):

$$\langle R_{3\alpha}\rangle_{NR} = N_A^2\frac{45\pi\sqrt{\pi}}{2m_N^3}\frac{\hbar^3}{c^2}\frac{(\pi\eta_0)^{1/3}}{(k_BT)^{13/6}(1+ak_BT)^{5/6}}\tilde{S}(T_a)\exp\left[-\frac{3(\pi\eta_0)^{2/3}}{T_a^{1/3}}\right]$$

- Resonant reaction rates (Sequential process):

$\sigma_g(E)$; $\Gamma_\gamma(0^+_2)\ \ E(0^+_2)$

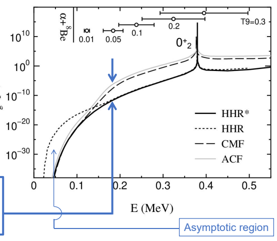
$$\langle R_{3\alpha}\rangle_R = \frac{9\sqrt{3}\pi^3}{4}\frac{N_A^2\hbar^5}{m_N^3}\frac{\Gamma_\gamma(0^+_2)}{(k_BT)^3}\exp\left(-\frac{E(0^+_2)}{k_BT}\right)$$

- Reaclib format [20]:

$$\langle R_{3\alpha}\rangle = \sum_i \exp\left(a_{i0} + a_{i1}/T_9 + a_{i2}/T_9^{1/3} + a_{i3}T_9^{1/3} + a_{i4}T_9 + a_{i5}T_9^{5/3} + a_{i6}ln(T_9)\right)$$

3.1 Photo-disintegration of ^{12}C

- In **HHR***, the smoothly varying cross sections are obtained at off-resonant energies. **No kink at 0.13 MeV.**
- The result of HHR* is much smaller than that of CMF & ACF for 0.15 < E < 0.35 MeV.
- α+^{8}Be feature of 0^+_2 is deduced without any assumption.

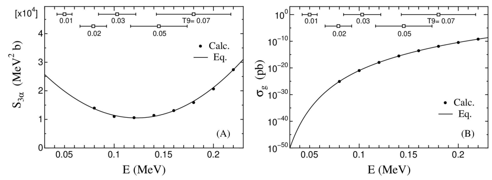


Most of the differences seem to stem from the assumed internal adiabatic feature.

- CMF has been developed in the three-body bound states and p+d reactions.
- The internal motion of ^{8}Be in break-up channels is described as a certain approximation within a certain range.

- For ACF, the adiabatic potential is characterized by α+^{8}Be at short radii and by 3α at large radii, and their σ_g for E > 0.13 MeV has been interpreted as a part of the 0^+_2 resonance.

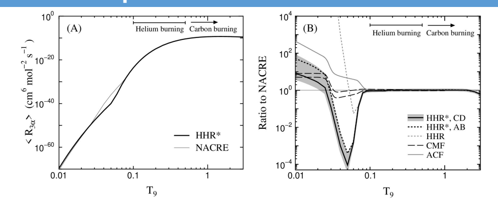
3.2 S-factors and σ_g



- The calculated σ_g for photo-disintegration of ^{12}C are expressed with the 3-body S-factors.
  - Compared with σ_g, the energy dependence of $S_{3\alpha}$ is quite weak.
  - The reaction rates below T_9 = 0.02 are generated from the extrapolated $S_{3\alpha}$, assuming that $S_{3\alpha}$ has weak variation to E.

	s_0 (MeV^2 b)	s_1 (MeV^{-1})	s_2 (MeV^{-2})	η_0 (MeV$^{1/2}$)	a (MeV^{-1})	$E(0^+_2)$ (MeV)	$\Gamma_\gamma(0^+_2)$ (meV)
CDb	3.717×10^4	−11.71	47.79	4.158	5.0	0.3796	3.9

3.3 Comparison between reaction rates



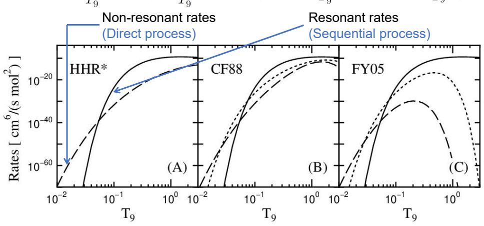
- The derived rates are consistent with NACRE for 0.08 < T_9 < 3.
- In contrast, the reaction rates are reduced by 10^{-4} at T_9 = 0.05.
  - This is caused by the accurate description of ^{8}Be break-up.
  - σ_g of HHR* is reduced from that of ACF and CMF at **E = 0.18 MeV**.
  - The difference in σ_g for E > 0.2 MeV cannot be found in the rates.
- **Astrophysical impact** of direct process is small, because the difference is found before helium burning temperatures in normal stars.
- The impact might be found in studies of accreting degenerate stars.

3.4 Translation into REACLIB

- A simple expression of the triple-α reaction rates:

$$\langle R_{3\alpha}\rangle = \frac{11.45}{T_9^{13/6}}\exp\left(-\frac{37.219}{T_9^{1/3}} - 1.47T_9\right) + \frac{2.966\times10^{-8}}{T_9^3}\exp\left(-\frac{4.4053}{T_9}\right)$$

Non-resonant rates (Direct process) — Resonant rates (Sequential process)



- CF88 [5] & FY05 [21] in Reaclib [20] have three components.
- HHR* does not have a component of non-resonant sequential process between α+^{8}Be (dotted curve).
- This means that the statistically generated ^{8}Be is broken-up immediately by the third α-particle before its lifetime at T_9 ≈ 0.05.
- To update the rates in [8,20], this contribution could be eliminated by hand.

4. Example of new rates

- Ignition critical density ρ_0 of helium burning in accreting white dwarfs
  - Energy generation rates of the triple-α reaction:

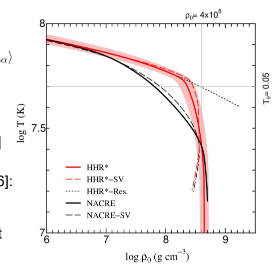
$$\epsilon_{3\alpha} = N_A\frac{Q}{6}\left(\frac{X_{He}}{4}\right)^3\rho_0^2 f_{sc}\langle R_{3\alpha}\rangle$$

Q : Q-value of 3α reaction
X_{He} : Helium mass fraction
f_{sc} : Electron screening

  - Salpeter & Van Horn [22]
  - N. Itoh et al. [23]
- Nuclear heating time scale [6]:

$$\tau_{He} = \frac{C_pT}{\epsilon_{3\alpha}} = 10^6 \text{ yr}$$

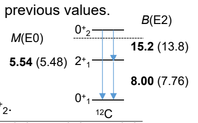
C_p : Specific heat at constant pressure



- The direct process plays an important role in helium burning of accreting white dwarfs at T =10^7 K, which seems to be consistent with [6,7].
- The resulting density is found to become insensitive to temperatures in 0.01 < T_9 < 0.05, because of the reduction of rates at T_9= 0.05.

5. Calculated lowest three states in ^{12}C

- Root-mean-square radius of 0^+_2, 2^+_1, 0^+_1
  - HHR* appears to be consistent with the previous values.
  - $R_{rms}(0^+_2)$ is longer than $R_{rms}(0^+_1)$.
  - $R_{rms}(2^+_1)$ is similar to $R_{rms}(0^+_1)$.
- Reduced transition probability
  - HHR* gives a better reproduction of *B*(E2), *M*(E0), and the decay width of 0^+_2.



		HHR*	HHR [15]	CMF [12]	ACF [13]	Exp. [24]
$E(0^+_2)$	(MeV)	0.3796	0.380	0.378	0.379	0.379
$E(2^+_1)$	(MeV)	−2.836	−2.875	−2.83	−2.836	−2.835
$E(0^+_1)$	(MeV)	−8.005		−7.789	−9.242	−7.275
$R_{rms}(0^+_2)$	(fm)	3.41		3.43	4.00	
$R_{rms}(2^+_1)$	(fm)	2.39	2.459		2.40	
$R_{rms}(0^+_1)$	(fm)	2.36			2.30	2.35–2.48
$B(E2;0^+_2\to2^+_1)$	(e^2fm^4)	15.2		8.7	34.6	13.8
$B(E2;2^+_1\to0^+_1)$	(e^2fm^4)	8.00			12.4	7.76
$M(E0;0^+_2\to0^+_1)$	(e fm^2)	5.54			6.44	5.48
$\Gamma(0^+_2)$	(eV)	7.1		6.9	15.8	9.3

6. Summary

- In this poster, I have discussed **the direct triple-α process** by using the non-adiabatic Faddeev **HHR*** expansion method.
- I have illustrated that the photo-disintegration cross sections of $^{12}C(2^+_1 \to 0^+)$ calculated by HHR* are much smaller than those by CMF & ACF for 0.15 < E < 0.35 MeV. **The boundary between the resonant and non-resonant regimes does not seem to be realistic.**
- The resultant rates have the strong temperature dependence at T_9 = 0.1, as well as NACRE, and they are expressed in **the analytic forms**.
- From the comparison between the calculations, I have found that the current standard rates can be reduced by about **10^{-4} at T_9 = 0.05**, because of the accurate description of ^{8}Be break-up. The present rates do not have the component of the non-resonant sequential process between α+^{8}Be. I, therefore, have found that **this component could be eliminated by hand**, to update the rates in NACRE & REACLIB.
- Using the new rates, I have also exemplified that the direct process is important for helium burning at T_9= 0.01 in accreting white dwarfs. From the reduction of rates at T_9= 0.05, the resulting ignition density has been found to be **insensitive to temperatures in 0.01 < T_9 < 0.05.**

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