# GPE Calculations for Superfluid Neutron Quantum Vortices and Superconducting Proton Fluxtubes in Neutron Stars

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Nucleosynthesis and Evolution of Neutron Stars

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- Introduction of Neutron Star, Pulsar and Glitch
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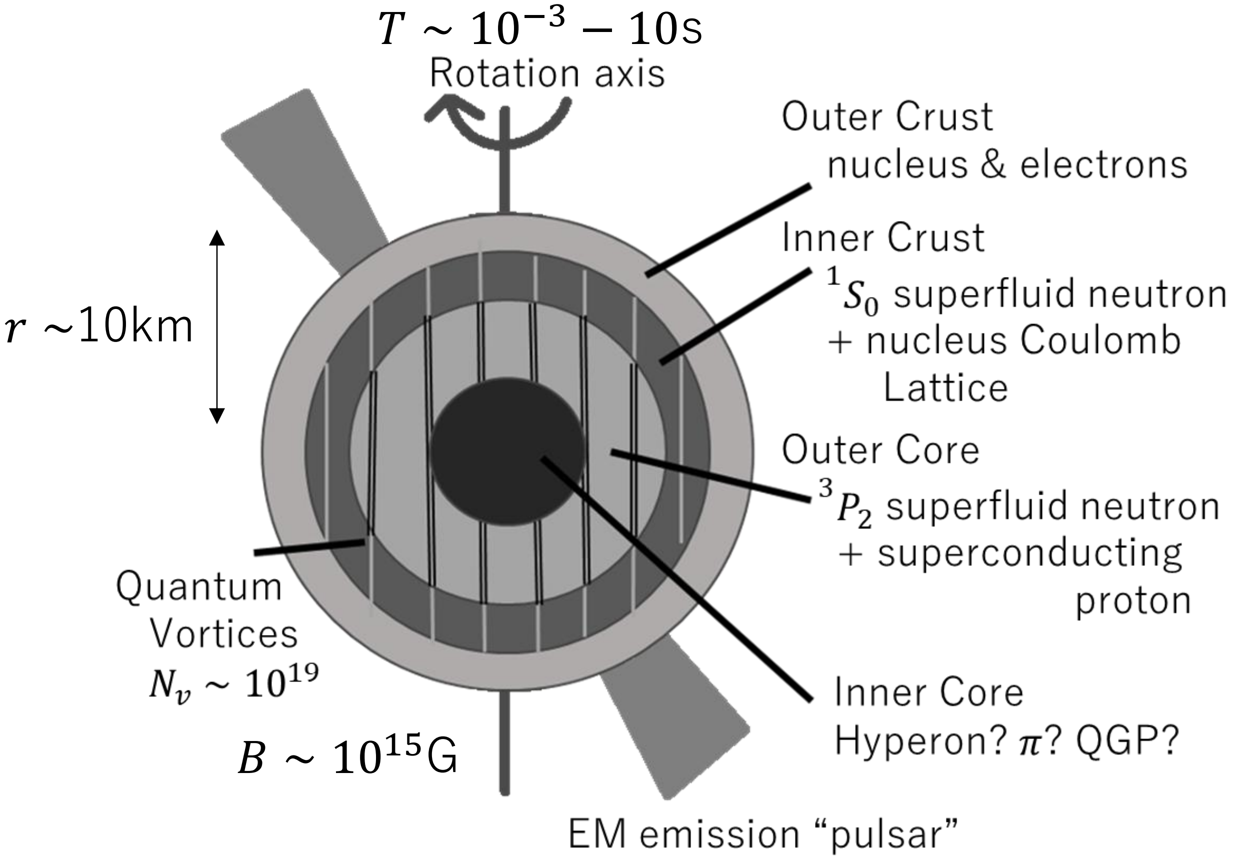
  - Magnetic field effect
  - Proton tube effect
- Future Work

# Neutron Star and its Structure

- Topic:

  Neutron Star's Pulsar Glitch

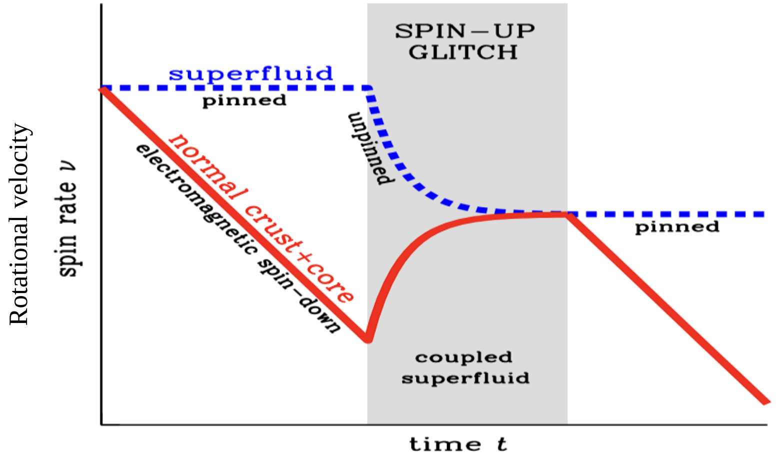
- Strong EM radiation from the magnetic poles
  → Pulse-like signal is observed
  →"***Pulsar***"



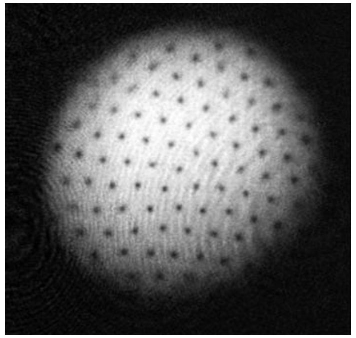

Neutron Star Structure

# Superfluidity and Pulsar Glitch

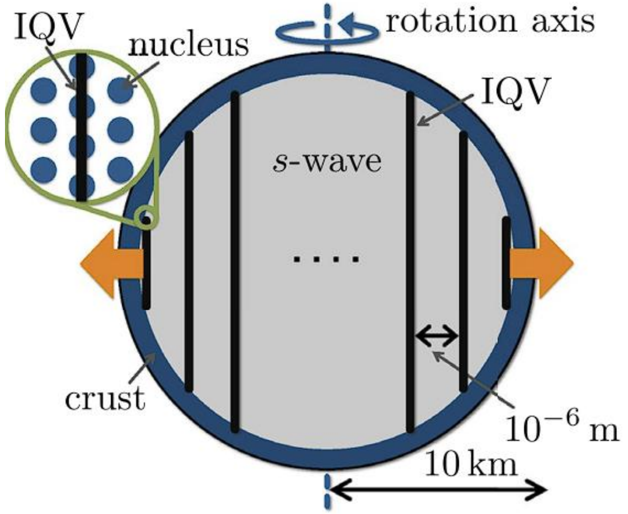
- Some pulsars shows sudden changes of the rotation period →“*Glitch*”
- Quantized Vortices of superfluid can act as a trigger of Glitch.




Glitch[1]



$^4$He quantum vortex[2]




Vortex avalanche model[3]
(Image by [4])

[1] Paul S. Ray, et al., ApJ, **879**(2):130, jul 2019.
[2] W. Ketterle, MIT Physics Annual. (2001)
[3] Anderson, Philip W. and Naoyuki Itoh, Nature **256** (1975): 25-27., K. S. Cheng, et al., ApJ **330**:835 (1988)
[4] G. Marmorini, S. Yasui, M. Nitta, Scientific Reports, **14**:7857 (2024)

# Many Types of Glitch Pattern

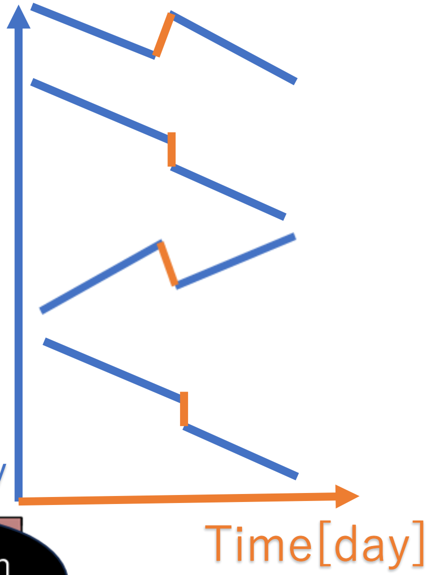
- There are different types of glitches!
- Neutron Stars have strong magnetic field!

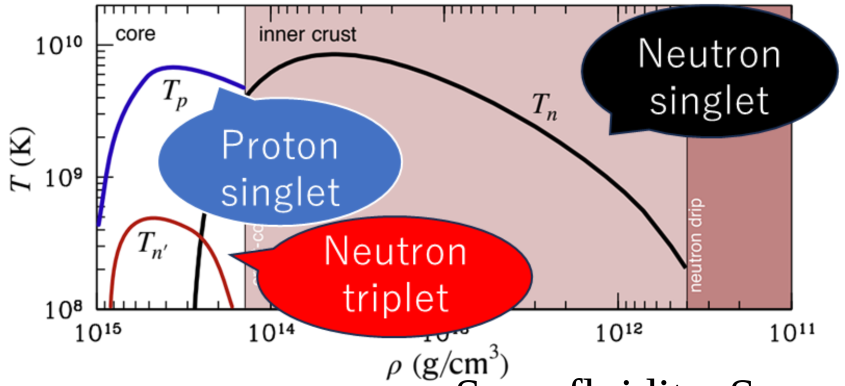
- In the Inner Core region:
  Magnetic Flux Tube (Proton Superconductor)
  + Quantum Vortex (Neutron Superfluid)

→To investigate these topics, we analyze the structure of $^3P_2$ Vortices and the effect of interaction between magnetic field and proton.

Typical Observed Glitch Pattern

| Name | Types | Glitch Pattern |
|---|---|---|
| Vela pulsar [1] | Rotation-powered pulsar | |
| IE 2259 + 586 [2] | Magnetar | |
| NGC300 ULC-1[3] | Accretion-powered pulsar | |
| PSR B0540-69[4] | Rotation-powered pulsar | |




Superfluidity, Superconductivity critical temperatures depends on the density[5]

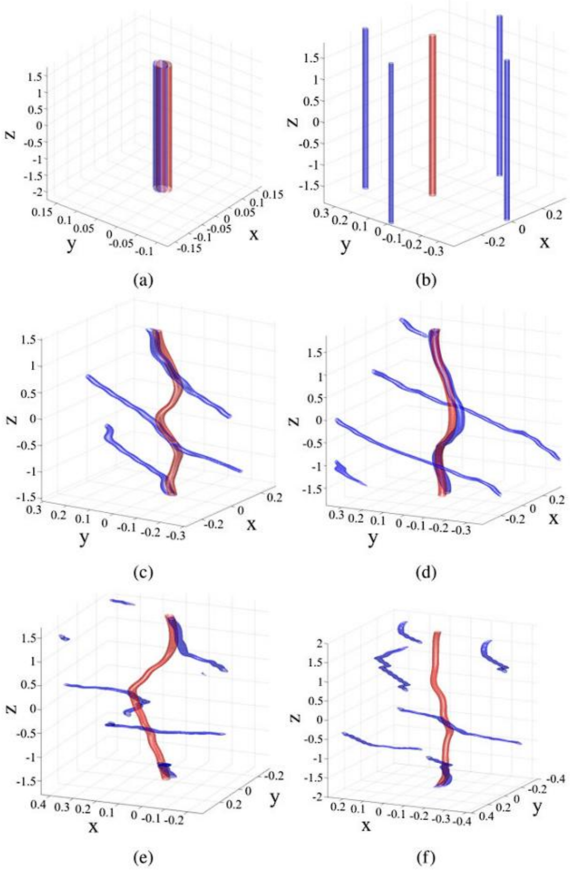
[1]VAUGHAN A. LARGE, M. and B. MILLS. Nature, **220**:340-341, 1968.,K. S. Cheng, et al. ApJ, **330**:835, July 1988.
[2] V. M. Kaspi, et al. ApJ, **588**(2):L93, apr 2003., R. Archibald, V. Kaspi, C. Y. Ng, et al. Nature, **497**:591?593, 2013., George Younes, et al. ApJL, **896**(2):L42, jun 2020.
[3] Paul S. Ray, et al., ApJ, **879**(2):130, jul 2019.
[4] Youli Tuo, et al., ApJL, **967**(1):L13, may 2024.
[5] Wynn C. G. Ho, Kostas Glampedakis, Nils Andersson, MNRAS, Volume **422**, Issue 3, May 2012, Pages 2632–2641

# Previous Works

[1] K. H. Thong , A. Melatos and L. V. Drummond, MNRAS **521**, 5724-5737 (2023)
[2] L. V. Drummond and A. Melatos, MNRAS **475**, 1, 910-920(2018)

- $^1S_0$ superfluidity neutron + $^1S_0$ superconductivity proton is already done [1,2].
- TODO:
  $^3P_2$ superfluidity neutron + $^1S_0$ superconductivity proton

| | Proton | Neutron | Magnetic Field |
|---|---|---|---|
| Previous Work[1,2] | $^1S_0$ | $^1S_0$ | Fixed |
| Today's talk | $^1S_0$ | $^3P_2$ | Fixed |



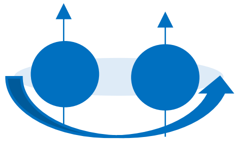
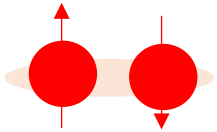

①Simulate the interaction between neutron's $^1S_0$ vortex and proton's flux tube.[1]

Neutron vortex

Proton flux tube

# Methods: GPE+GLE for the Outer Core of Neutron Stars

- BEC neutron and proton are treated as bosonic cooper pair.
- Using GPE/GLE for describe order parameter of neutron/proton.
- Components:

| | phase | Components of order parameter, Cooper pair | charge | Equation |
|---|---|---|---|---|
| neutron | $^3P_2$ superfluidity | $J = -2, -1, 0, 1, 2$  $\Delta L > 0$ | 0 | Gross-Pitaevskii equation (GPE) (weak interaction, bosonic model) |
| proton | $^1S_0$ superconductivity | $J = 0$  $\Delta L = 0$ | $2e$ | Ginzburg-Landau equation (GLE) |

# Formalisms: GPE for ${}^3P_2$ Superfluid Neutrons

- ${}^3P_2$ superfluidity : using Gross-Pitaevskii equation (GPE)[1]

$c_0, c_1, c_2$ are parameter, we set $c_0 > 0, c_1 > 0, c_2 < 0$.

$$E[\Psi] \equiv \langle \hat{H} \rangle_0 = \int d\mathbf{r} \left\{ \sum_{m=-2}^{2} \psi_m^* \left[ -\frac{\hbar^2 \nabla^2}{2M} + U_{\mathrm{trap}}(\mathbf{r}) - pm + qm^2 \right] \psi_m + \frac{c_0}{2} n^2 + \frac{c_1}{2} |\mathbf{F}|^2 + \frac{c_2}{2} |A_{00}|^2 \right\}$$

$$A_{00}(\mathbf{r}) \equiv \langle \hat{A}_{00}(\mathbf{r}) \rangle_0 = \frac{1}{\sqrt{5}} [2\psi_2(\mathbf{r})\psi_{-2}(\mathbf{r}) - 2\psi_1(\mathbf{r})\psi_{-1}(\mathbf{r}) + \psi_0^2(\mathbf{r})]$$

$$f_x = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} & 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad f_y = i \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -\sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} & 0 & -\sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}} & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$F_\nu(\mathbf{r}) \equiv \langle \hat{F}_\nu(\mathbf{r}) \rangle_0 = \sum_{m,m'=-2}^{2} \psi_m^*(\mathbf{r}) (f_\nu)_{mm'} \psi_{m'}(\mathbf{r}) \quad (\nu = x, y, z)$$

$$n = \sum_{m=-2}^{2} |\psi_m|^2$$

$$f_z = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

$p$: linear Zeeman term, $p = -g\mu_B B$, $q$: quadratic Zeeman term

[1] M. Ueda, M. Koashi, Phys. Rev. A **65** (2002) 063602

# Formalisms: GLE for $^1S_0$ Superconducting Protons

- $^1S_0$ superconductivity : using Ginzburg-Landau equation (GLE) [1]

$$\langle E \rangle = \int dr^3 \left( -\frac{\hbar^2}{2M} |(\nabla - iA)\phi|^2 + U_{\text{trap}}|\phi|^2 + \alpha|\phi|^2 + \frac{\beta}{2}|\phi|^4 \right)$$

$A$:vector potential, $\alpha < 0, \beta > 0$.

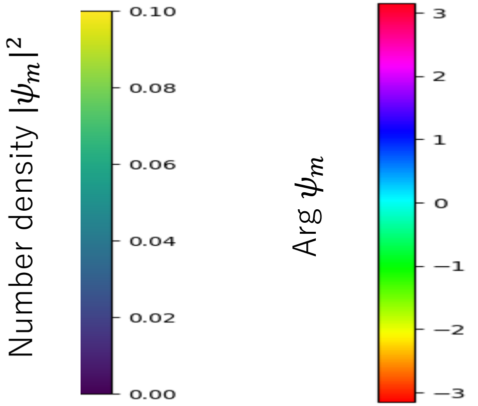
- Interaction $n$ & $p$ Term [2]:

$$H_{int} = \int dr^3 \left( \sum_{m=-2}^{2} \eta|\psi_m|^2|\phi|^2 + \sum_{m=-2}^{2} \xi J_m \cdot J_p \right)$$

Now we set $\eta$, and neglect the current term ($\xi = 0$).

[1] Schmid A. Physik der kondensierten Materie, **5**, 302 (1966), Kopnin N. B., Journal of Low Temperature Physics, **129**, 219 (2002). Tiknham M., Kasamatsu K., Ueda M., Phys. Rev. A, **65,** 023603 (2002), Ebisawa H., Fukuyama H., Progress of Theoretical Physics, **46,** 1042 (1971)
[2] K. H. Thong , A. Melatos and L. V. Drummond, MNRAS **521**, 5724-5737 (2023)

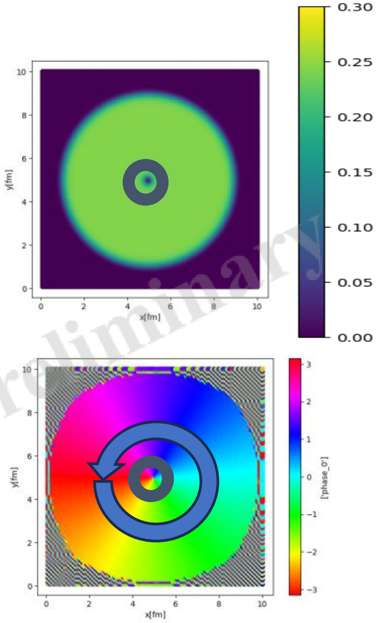
# Vortex Shape




- $^1S_0$

$|\psi|^2$

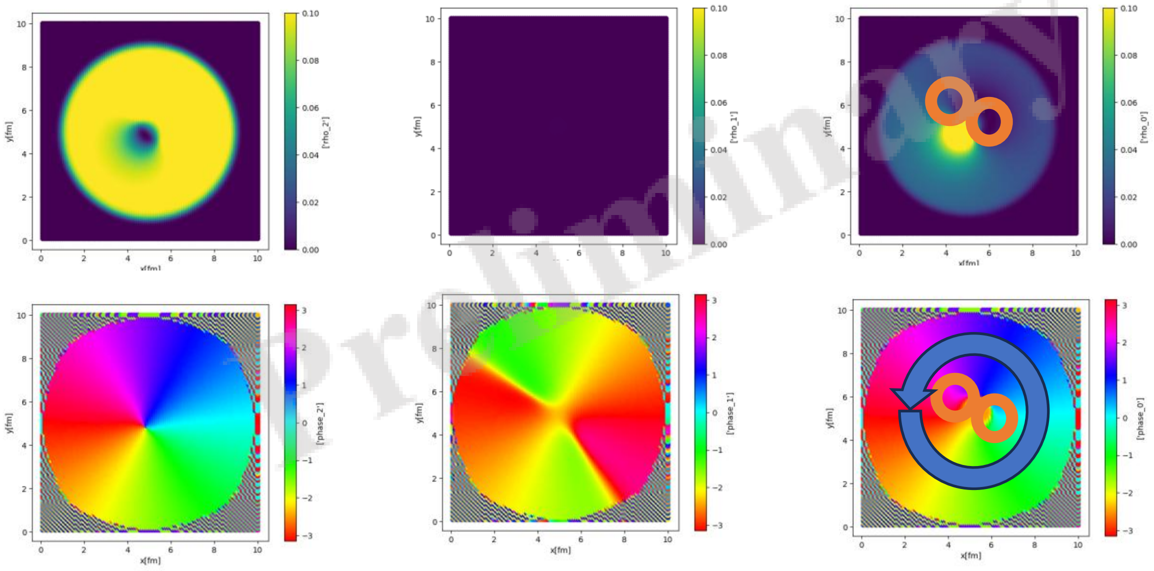
Args($\psi$)




- $^3P_2$

$|\psi_m|^2$

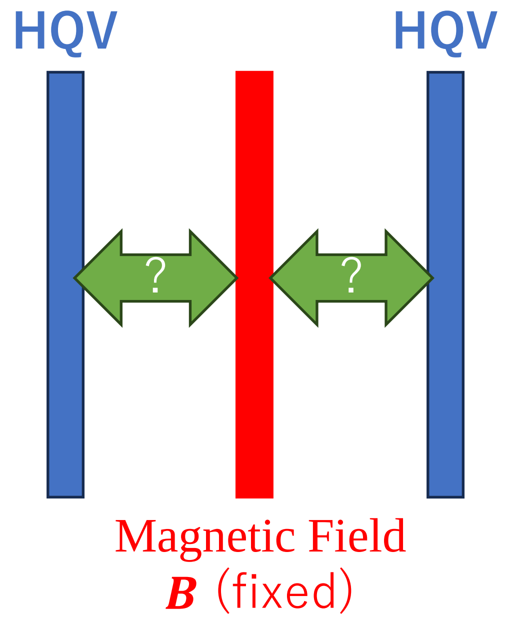
Args($\psi_m$)




$\psi_2$ $\psi_1$ $\psi_0$

- $^1S_0$ superfluidity make Integer-Quantized Vortices (IQV)
- $^3P_2$ superfluidity make Half-Quantized Vortices (HQV).
  - Create two vortices with angular momentum $\hbar$

# Magnetic Field vs Vortex in Outer Core




Magnetic Flux of Flux tube:

$$\Phi_0 = \frac{\pi \hbar c}{2e} \approx B r_{FT}^2 \pi$$

$$p = g_n \mu_N B$$

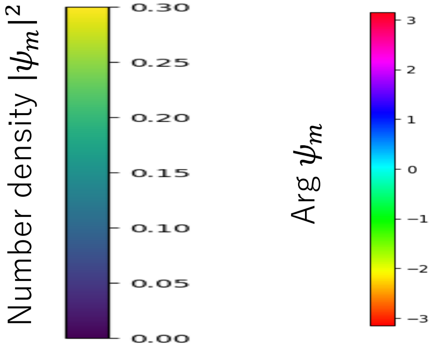
$$= \frac{79.0\mathrm{fm}^2 \pi}{\pi r_{FT}^2 [\mathrm{fm}^2]} \mathrm{MeV}$$

$r_{\mathrm{FT}}$ is the radius of flux tube.

- Interaction between Magnetic Field and Neutron Vortices
  
  Set and fix tube-like magnetic flux with ${}^3P_2$ quantum vortices.
  
  →Analyze the shape of vortices of spinor superfluid.
  
  ※ Here, we neglect p-n interaction ($\eta = 0$).
- We put the constant magnetic flux tube:

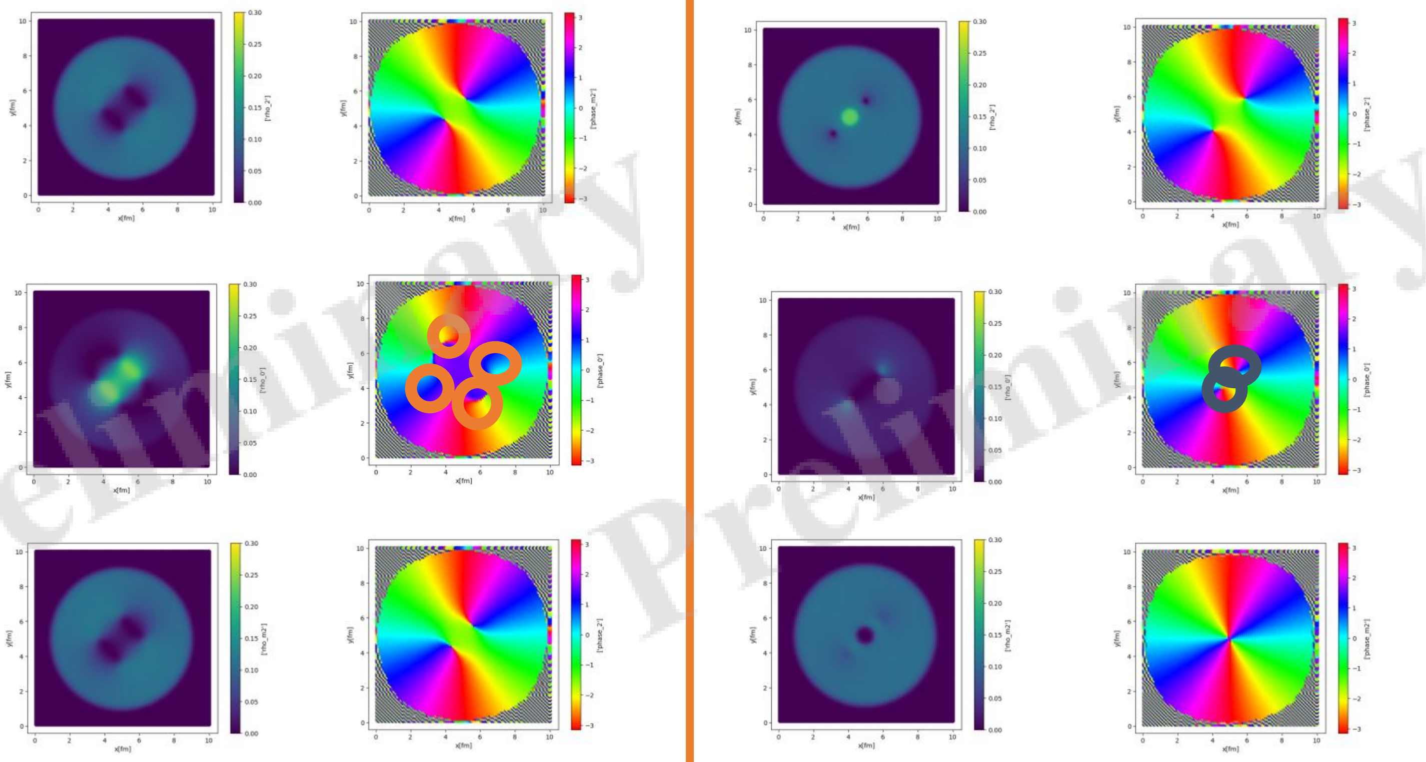
$$p(r) = \begin{cases} 0 \quad \mathrm{MeV} & r > r_{\mathrm{FT}} \\ \frac{79.0\mathrm{fm}^2}{r_{FT}^2} \quad \mathrm{MeV} & r < r_{\mathrm{FT}} \end{cases}$$

# Results① Magnetic Field Effect




- $N_v = 2$
- When tube-like magnetic field applied, spin polarized in tube area.
- The vortex shape of $m = 0$ component is changed

| | $B = 0$ | $p = 3.2 \times 10^2$MeV, $r_{\mathrm{FT}} = 0.5$fm |
|---|---|---|
| $m = 2$ $\psi_2$ | | |
| $m = 0$ $\psi_0$ | | |
| $m = -2$ $\psi_{-2}$ | | |



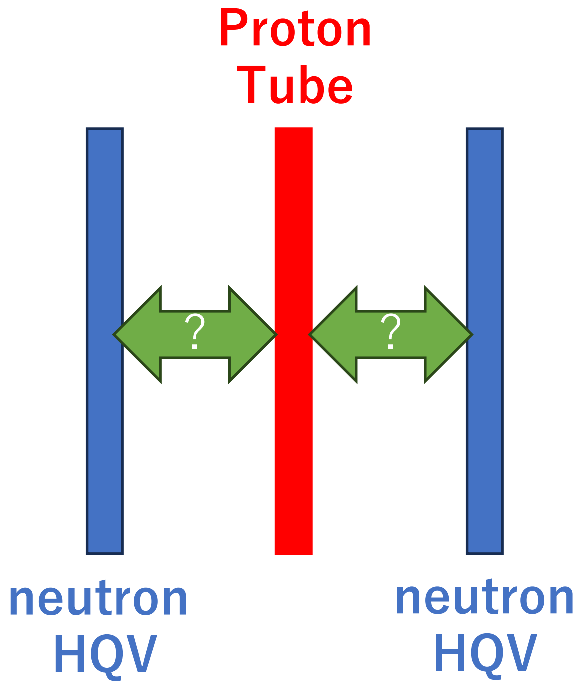

$|\psi_m|^2$ **4 HQVs** $\mathrm{Arg}(\psi_m)$

**2 IQV**

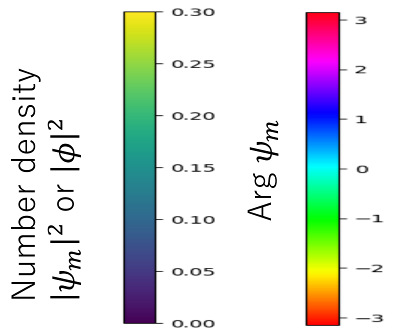
# Proton and Neutron Interaction

- Proton-Neutron Interaction Effect:
  - For simplicity:
    - The density of proton and neutron are same.
    - $-\alpha = \beta = c_0$.
    - No magnetic effect directly ($p = q = 0$).
  - $\eta$ is set as $-1.0 \times \beta$ or $-10 \times \beta$, and $\xi = 0$.
  - Proton flux tube & Neutron vortices are set as parallel

- Here, we show the case of 2 Vortices (4 HQV) and 1 Proton Tube.



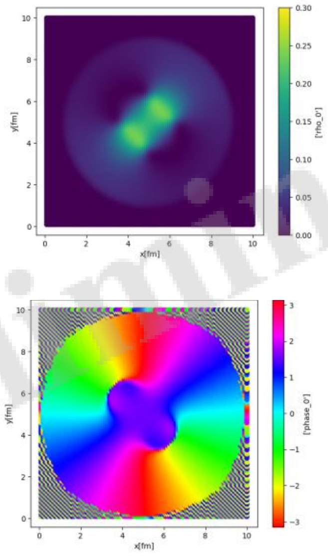
# Results② Proton and Neutron Interaction



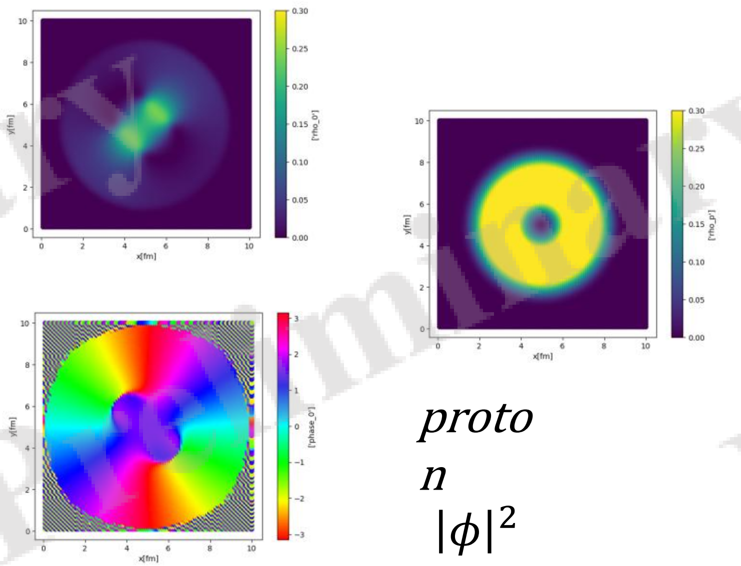

- Neutron and Proton interaction: $N_v = 2, N_{FT} = 1$
- It seems that there are no effect on the vortex shape.

$|\psi_0|^2$

$\mathrm{Arg}(\psi_0)$

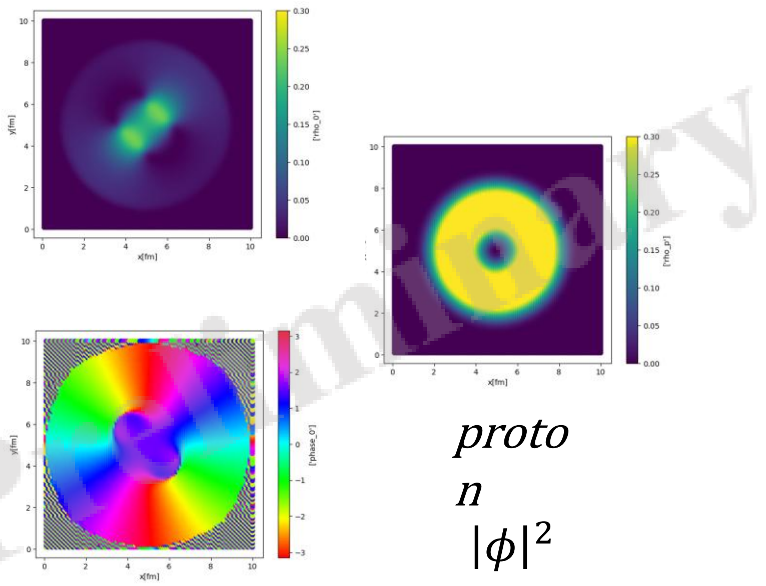


No Interaction
**4 HQVs**



*proton* $|\phi|^2$

$\eta = -1.0 \times c_0$
**4 HQVs**



*proton* $|\phi|^2$

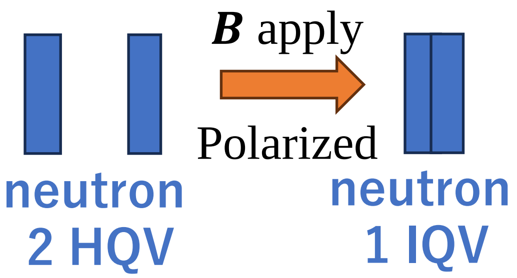
$\eta = -10 \times c_0$
**4 HQVs**

# Discussion

1. Magnetic Field Effect:
   - Strong magnetic fields cause local spin polarization.
   - Local spin polarization can change vortex shapes.

     → Magnetic Field can destroy nature as $^3P_2$ Vortex.

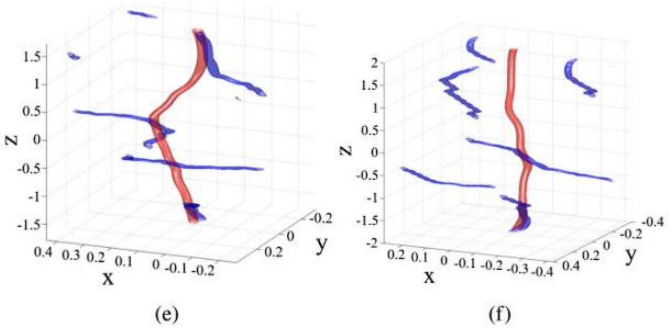
   ※This effect is not seen in $^1S_0$ Vortex.




2. Proton-Neutron interaction:
   - If the vortex and flux tube are parallel, the effect is barely noticeable.

# Future Works

- More realistic setup : density ratio, phenomenological parameters, etc.
- Treat all interaction in one calculation.
- Solving magnetic field self consistently
  - Zeeman term can affect on vortex shape and components density ratio.
  - Feedback of neutron → magnetic field.
    Can spin ≠ 0 neutron affect on magnetic field?
- Flux Tube ∦ vortices:
  - Vortices may be attracted by flux tube[1].




$B$ ∦ vortex simulation
( $^1S_0$neutron & $^1S_0$proton) [1]

[1] K. H. Thong , A. Melatos and L. V. Drummond, MNRAS **521**, 5724-5737 (2023)

# Summary

- Analise Interaction between $^3P_2$ neutron vortex and proton flux tube
  - Vortex vs. Magnetic field
    - Spin polarize → change vortex shape!
  - Vortex vs. Proton
    - small effect.
- Future Work:
  - More realistic setup, treat all interaction
  - Feedback effect: Proton/Neutron → Magnetic field

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# Thank you for your attention!

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