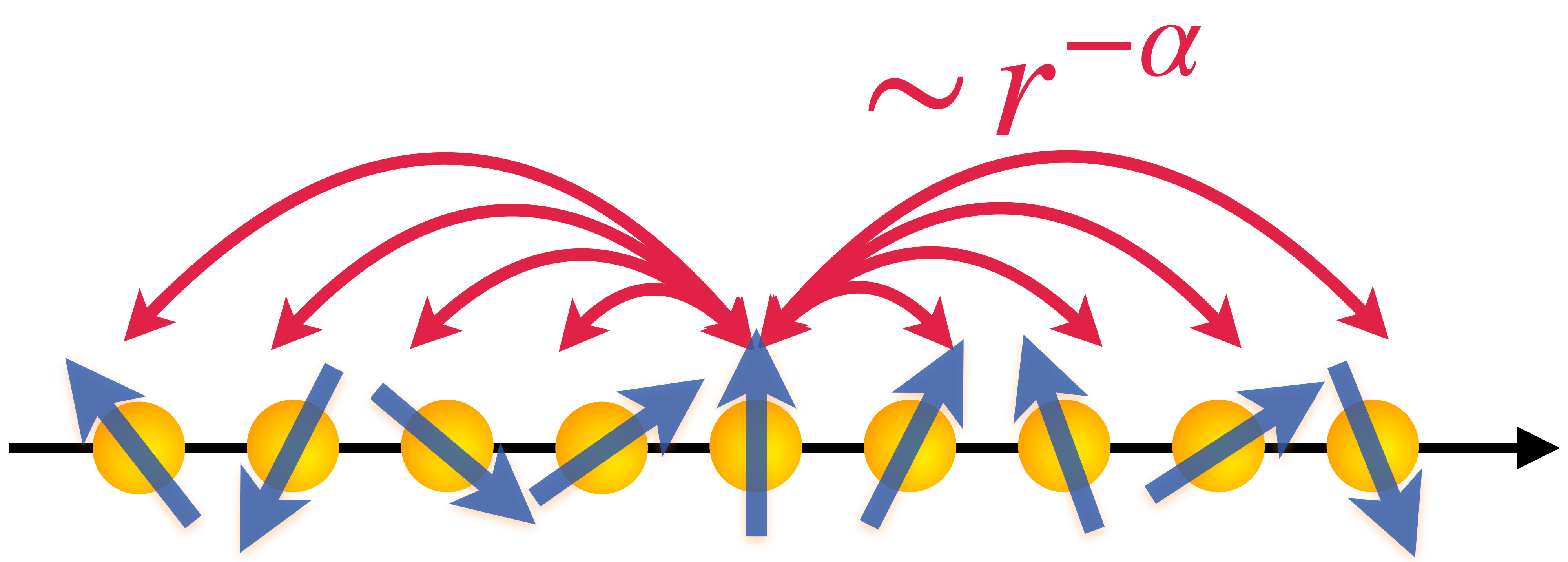
202506 KITP Workshop
“Hydrodynamics of low-dimensional interacting systems”



Energy Diffusion in the Long-range Interacting Spin Systems

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& MD

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arXiv:2502.10139

Outline

1. **Overview of Long-range Interacting (LRI) Systems**
2. **Energy Diffusion in the Long-range Interacting (LRI) Spin Systems**
   - Models & Dynamics : Transverse Ising・XYZ
   - Local Energy Current in Long-range Interacting Systems
   - Divergence of Thermal Conductivity (Green-Kubo formula)
   - Cumulant Power-law Clustering Theorem in the LRI Systems
   - Fluctuating Hydrodynamics for Anomalous Diffusion
   - The case of D (≥ 2) dimensions
3. **Conclusion**

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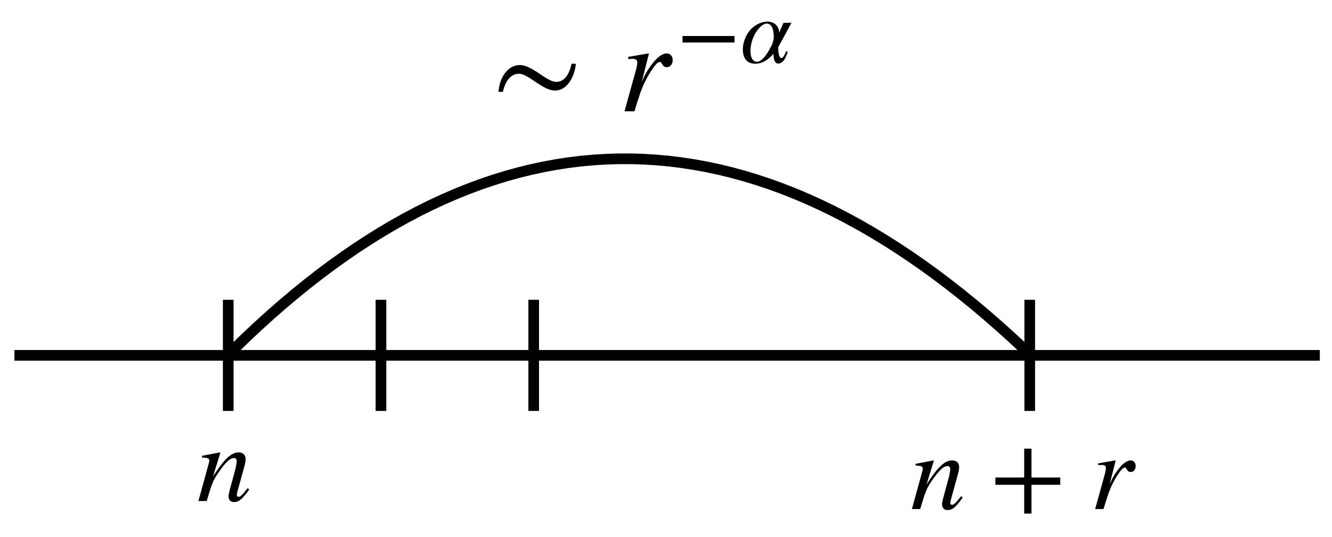
Long-range Interactions (LRI) are Ubiquitous in Nature

- Long-range Interaction : $V(r) \sim r^{-\alpha}$



e.g. $\alpha = 1$: Gravitational field, Plasma field

$\alpha = 3$: Magnetic dipole interaction

$\alpha = 6$: Lennard-Jones potential

Long-range Interactions (LRI) are Ubiquitous in Nature

- Recent experimental advances (Rydberg atom, Cold atomic gases, Ion trap etc)

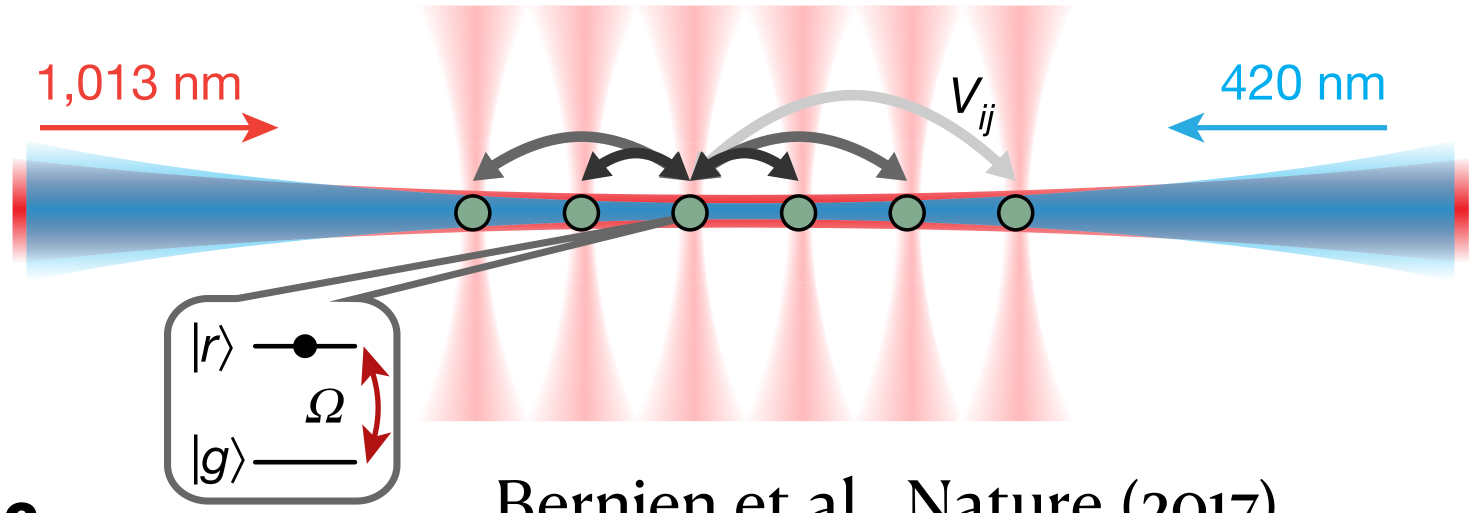
→Long-range interacting spin systems **with tunable power-law exponents α**

✓Rydberg atoms

Long-range Ising model

$$H = \sum_{\mathbf{r},\mathbf{r}'} \frac{1}{|\mathbf{r}-\mathbf{r}'|^{\alpha}} S^z_{\mathbf{r}} S^z_{\mathbf{r}'}$$

Aikawa et al., Phys. Rev. Lett. (2012)
Saffman et al., Rev. Mod. Phys. (2010)
Bendkowsky et al., Nature (2009)



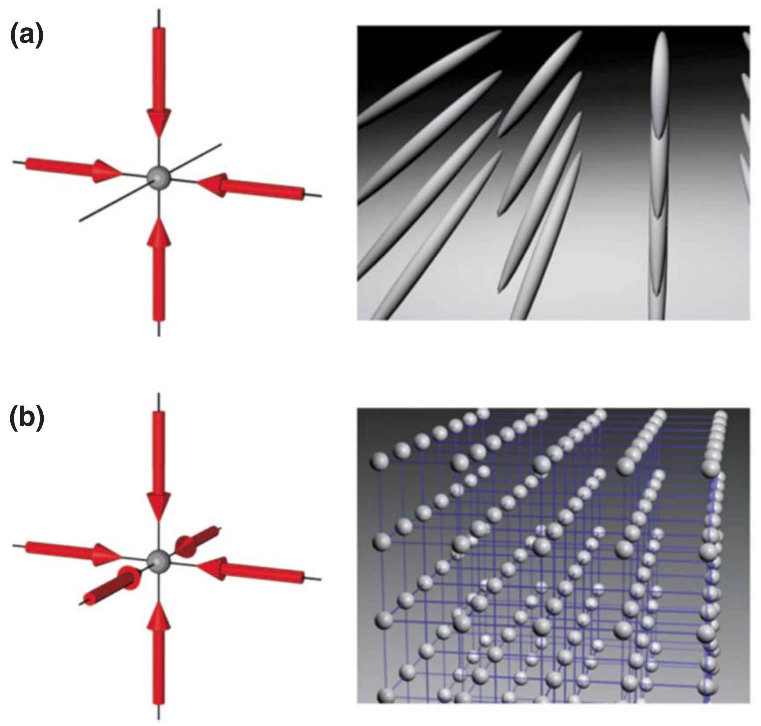
Bernien et al., Nature (2017)

$$\frac{\mathcal{H}}{\hbar} = \sum_i \frac{\Omega_i}{2}\sigma^i_x - \sum_i \Delta_i n_i + \sum_{i<j} V_{ij} n_i n_j$$

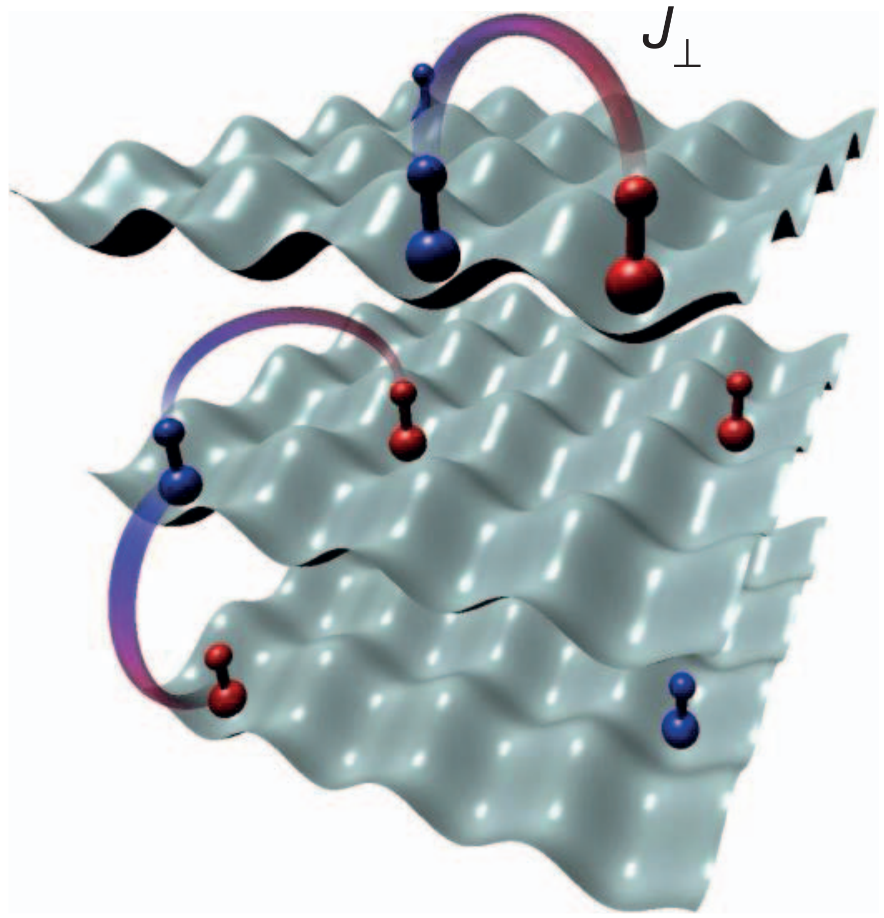
✓Cold atomic gases

Long-range Ising, XY model

Bloch et al., Nat. Phys. (2005)
Neyenhuis et al., Sci. Adv.(2017)

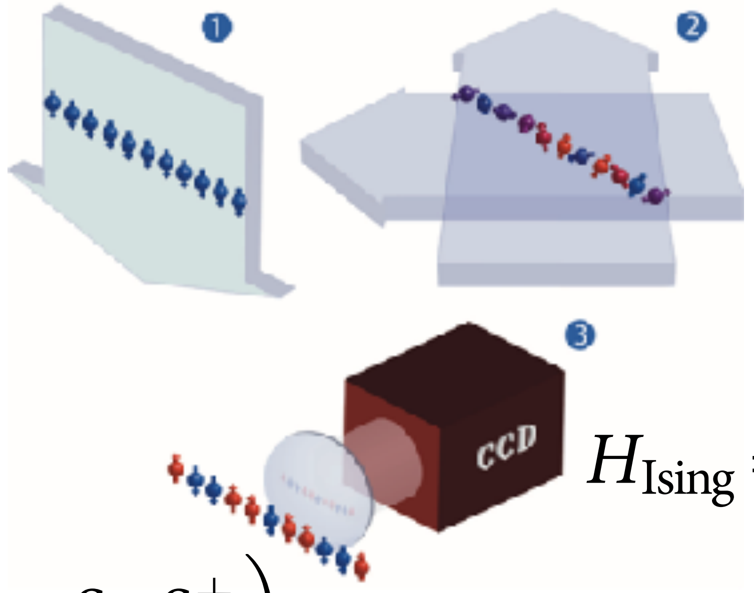


Yan et al., Nature (2013)



$$H = \frac{J_\perp}{2}\sum_{i>j} V_{\mathrm{dd}}(\boldsymbol{r}_i - \boldsymbol{r}_j)\left(S_i^+ S_j^- + S_i^- S_j^+\right)$$

Richerme et al., Nature (2014)



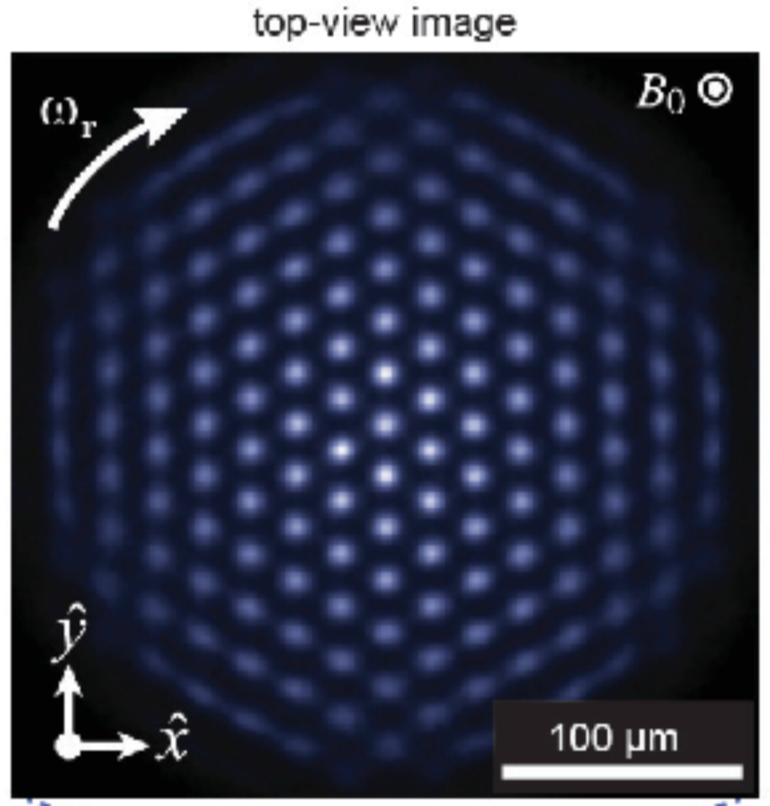
$$H_{\mathrm{Ising}} = \sum_{i<j} J_{i,j}\sigma_i^x\sigma_j^x$$

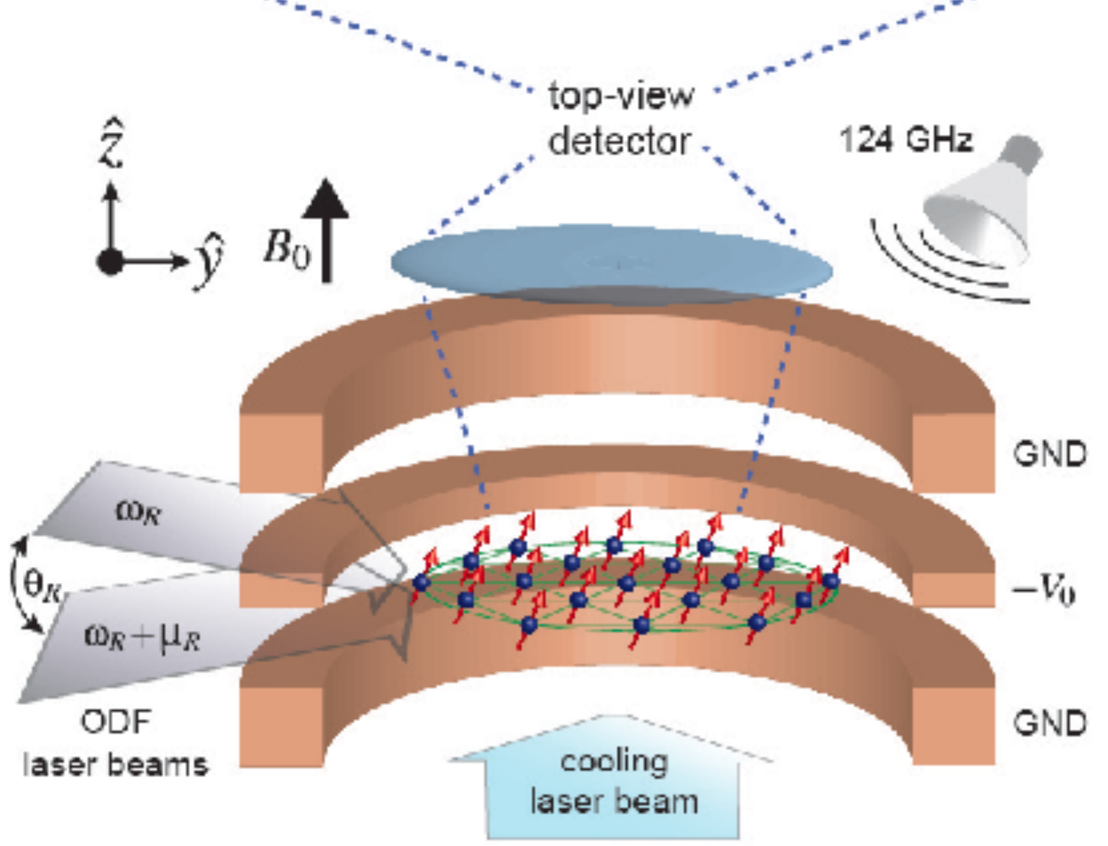
$$H_{XY} = \frac{1}{2}\sum_{i<j} J_{i,j}\left(\sigma_i^x\sigma_j^x + \sigma_i^z\sigma_j^z\right)$$

✓Ionic traps

Long-range Ising model

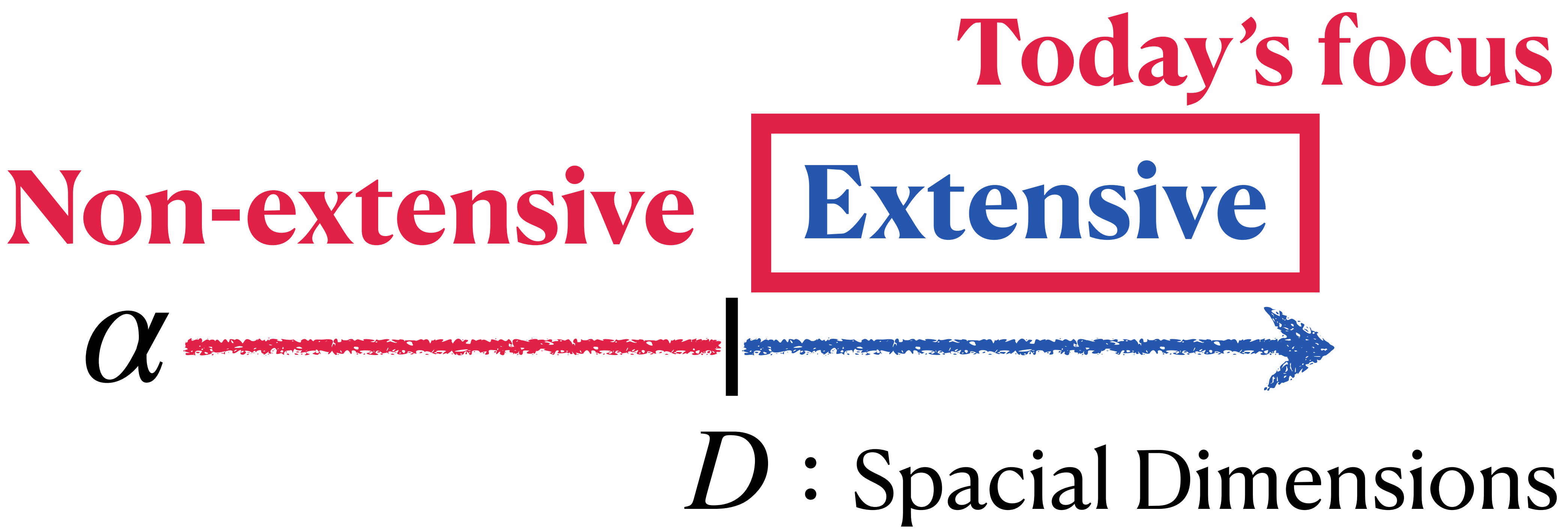
Britton et al., Nature (2012)

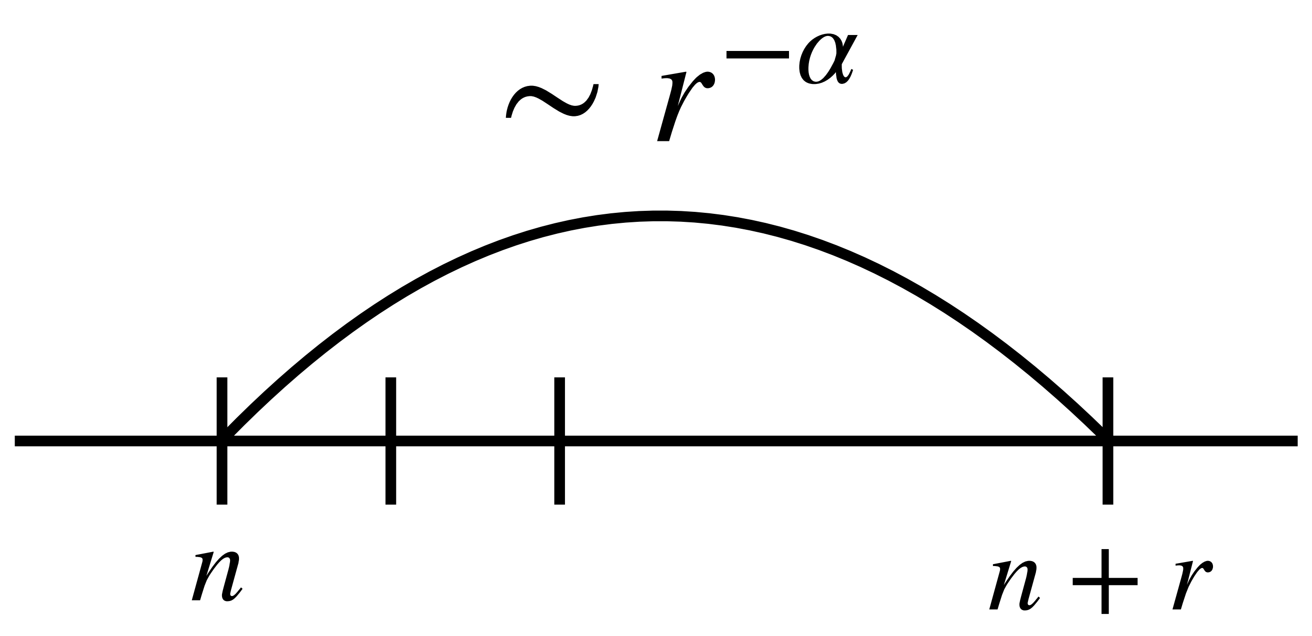




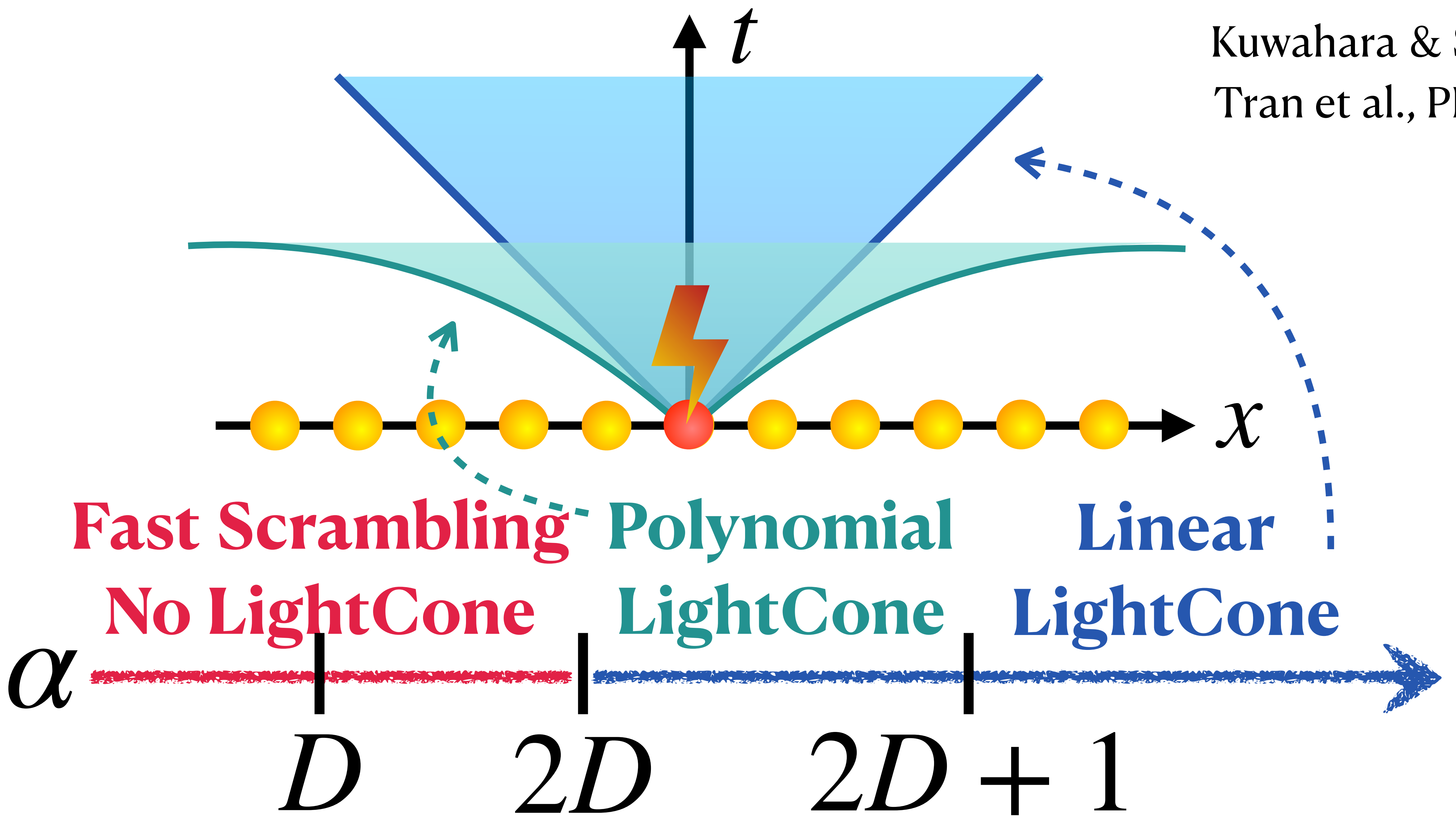
The effect of Long-range Interaction (LRI)

- **Change of Thermodynamics**





- **Information Propagation in LRI Systems**



Kuwahara & Saito PRX (2020)
Tran et al., PRX (2021)

The effect of Long-range Interaction (LRI)

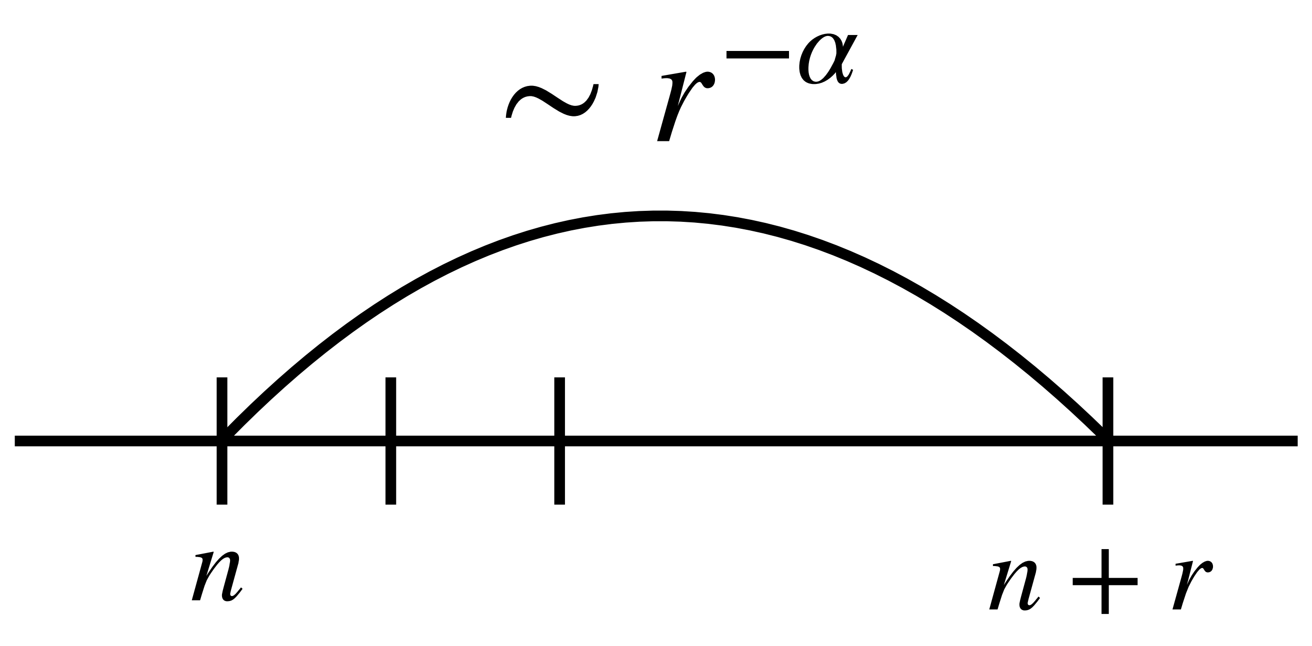
- **Change of Thermodynamics**

Today's focus

Non-extensive | **Extensive**

α ⟶

D : Spacial Dimensions



- **Energy Propagation in LRI systems**

✓**Spin systems**：Relevant for **Rydberg Atom, Cold Atom, Ion Trap etc.**

However, there are NO previous studies in spin systems, not even numerical studies.

→**We want to construct the Theory of Energy Diffusion in the LRI Spin Systems.**

α **Levy Diff.** | **Normal Diff.** ⟶

α_c : Universal ??

Q. What is α_c where a transition from Normal to Levy diffusion occurs ??

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3. **Conclusion**

Models : Typical Spin Systems

- Hamiltonian : 1-dim 2-local Classical LRI Spin Systems

$$H = \sum_{n=1}^{N} \epsilon_n$$

ϵ_n : Local energy at site n

N spins

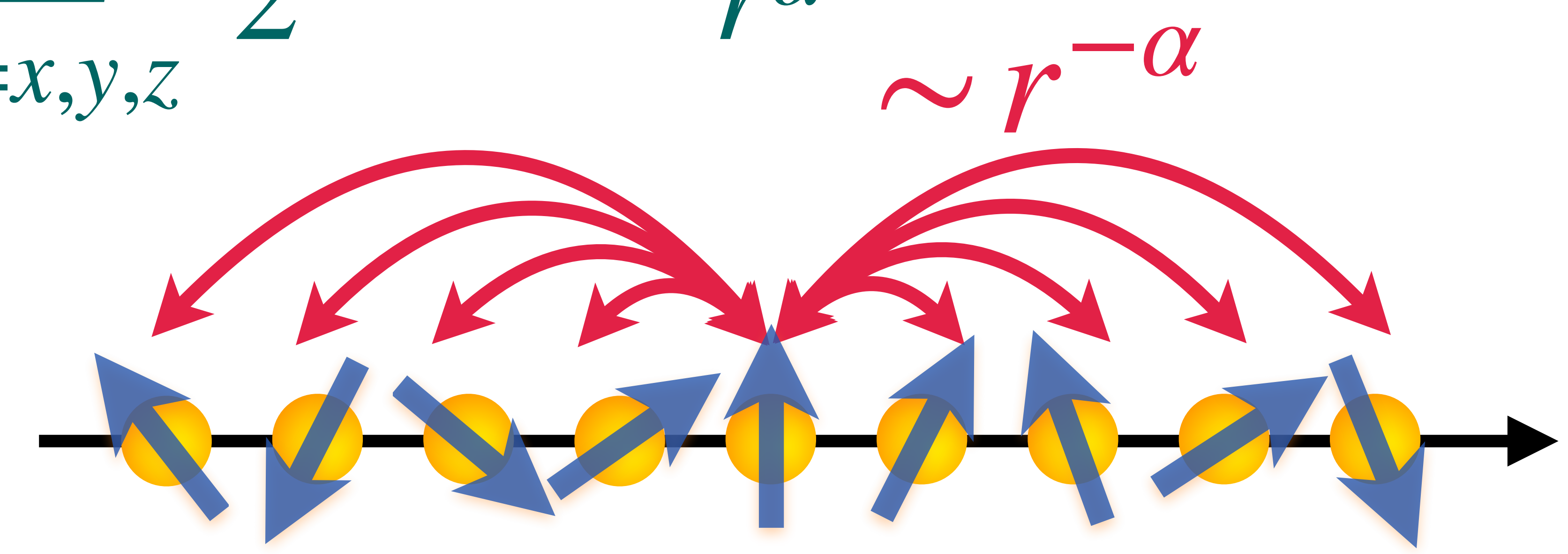
$n + N \equiv n$: P.B.C.

(a) LRI Transverse Ising Model

$$\epsilon_n = -\frac{J}{2}\sum_{r=1}^{N/2} \frac{S_n^z S_{n+r}^z + S_n^z S_{n-r}^z}{r^\alpha} - hS_n^x$$

(b) LRI XYZ Model (including XY)

$$\epsilon_n = -\sum_{r=1}^{N/2} \sum_{\sigma=x,y,z} \frac{J_\sigma}{2} \frac{S_n^\sigma S_{n+r}^\sigma + S_n^\sigma S_{n-r}^\sigma}{r^\alpha}$$



$\alpha > 1$: Extensivity holds

✓ Extensivity→ $\|\epsilon_n\| < \infty$: one site energy is well-defined

$\|\cdot\|$: operator norm (maximum value) $\left(\because \int_1^\infty dr r^{-\alpha} < \infty \right)$

✓ Typical models, and relevant for Rydberg atoms, cold atoms, & ionic trap.

Dynamics : Classical Spin Systems

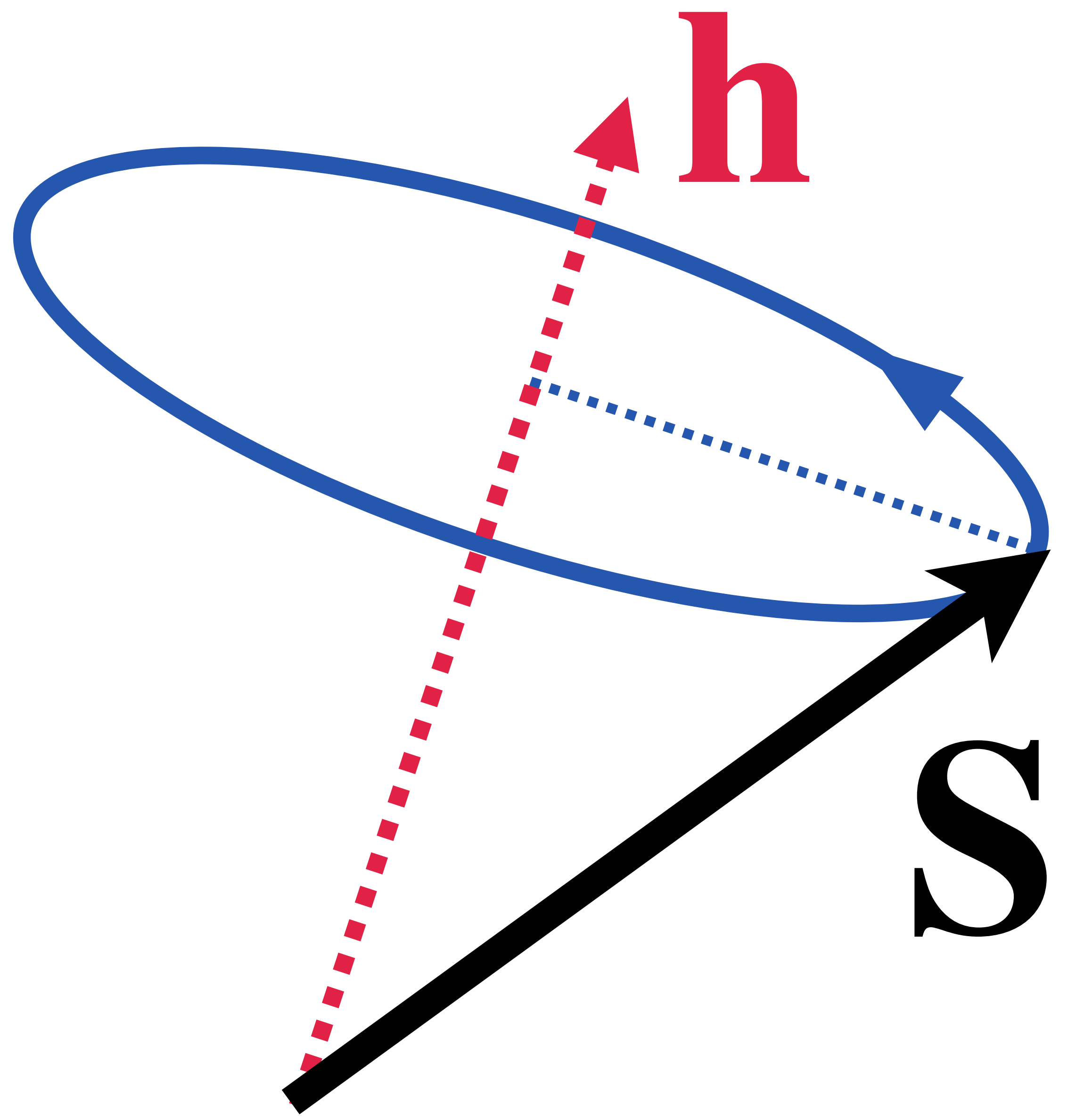
✓Single Spin : $H = -\mathbf{h}\cdot\mathbf{S}$

$$\partial_t \mathbf{S} = -\mathbf{S}\times\mathbf{h} = \frac{\delta H}{\delta \mathbf{S}}\times\mathbf{S}$$

For arbitrary function $A(\mathbf{S})$,

$$\partial_t A(\mathbf{S}) = \sum_a \frac{\delta A}{\delta S^a}\partial_t S^a = \sum_{abc}\frac{\delta A}{\delta S^a}\varepsilon^{abc}\frac{\delta H}{\delta S^b}S^c$$

ε^{abc} : Levi-Civita symbol



Torque dynamics
→Precession around the **h**-axis

✓Many-body Spin Systems : $H(\{\mathbf{S_i}\})$

$$\partial_t A(\{\mathbf{S_i}\}) = \{A, H\}$$

$$\{A, B\} := \sum_i \sum_{abc}\varepsilon^{abc}\frac{\partial A}{\partial S_i^a}\frac{\partial B}{\partial S_i^b}S_i^c$$: Spin Poisson bracket

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3. **Conclusion**

Local Energy Current in 1-Dim LRI Systems

- Local Energy on the site n $\quad H = \sum_n \epsilon_n$ satisfies. →Energy is locally conserved.

$$\epsilon_n = -\frac{J}{2}\sum_j \frac{S_n^z S_j^z}{d_{n,j}^\alpha} - hS_n^x \quad \text{: Transverse Ising}$$

$$\epsilon_n = -\sum_j \sum_{\sigma=x,y,z} \frac{1}{d_{n,j}^\alpha}\frac{J_\sigma}{2} S_n^\sigma S_j^\sigma \quad \text{: XYZ}$$

- Local Energy Current $\mathcal{J}_n^\epsilon$?? ❖ Criteria : **Continuity eq.** holds.

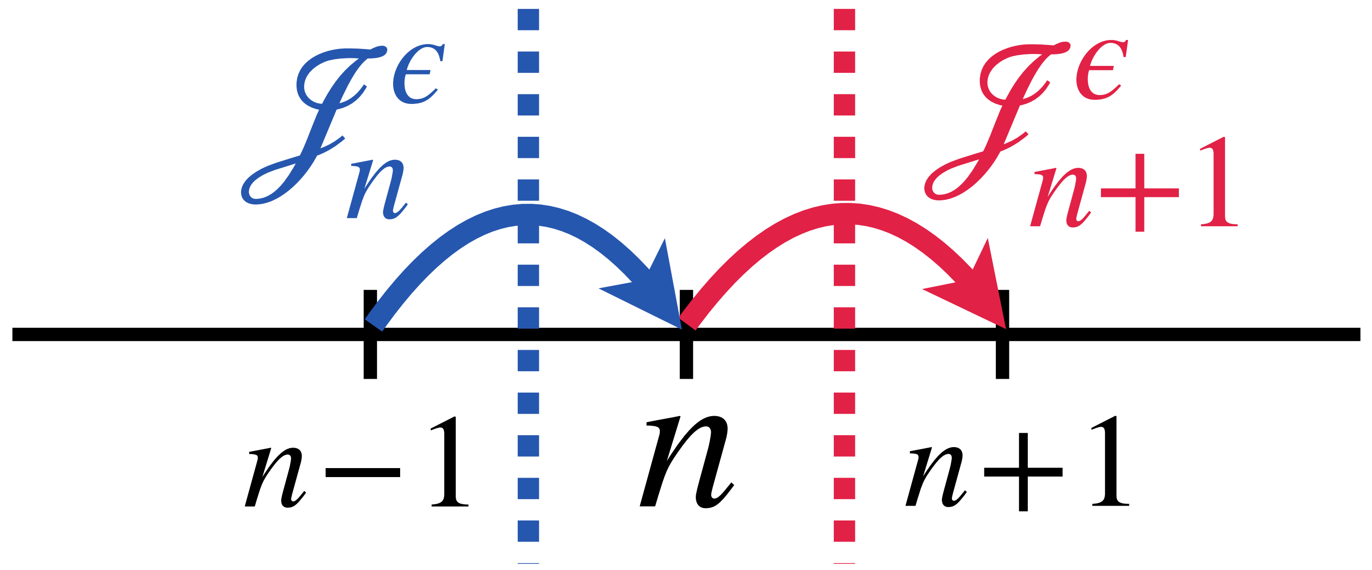
$$\partial_t \epsilon_n = -\mathcal{J}_{n+1}^\epsilon + \mathcal{J}_n^\epsilon$$

✓Nearest-neighbor interacting case can be straightforwardly derived

$$\partial_t \epsilon_n = \{\epsilon_n, H\} = \{\epsilon_n, \epsilon_{n+1}\} + \{\epsilon_n, \epsilon_{n-1}\}$$

$$= -\{\epsilon_{n+1}, \epsilon_n\} + \{\epsilon_n, \epsilon_{n-1}\}$$

$$= \mathcal{J}_{n+1}^\epsilon \qquad = \mathcal{J}_n^\epsilon$$

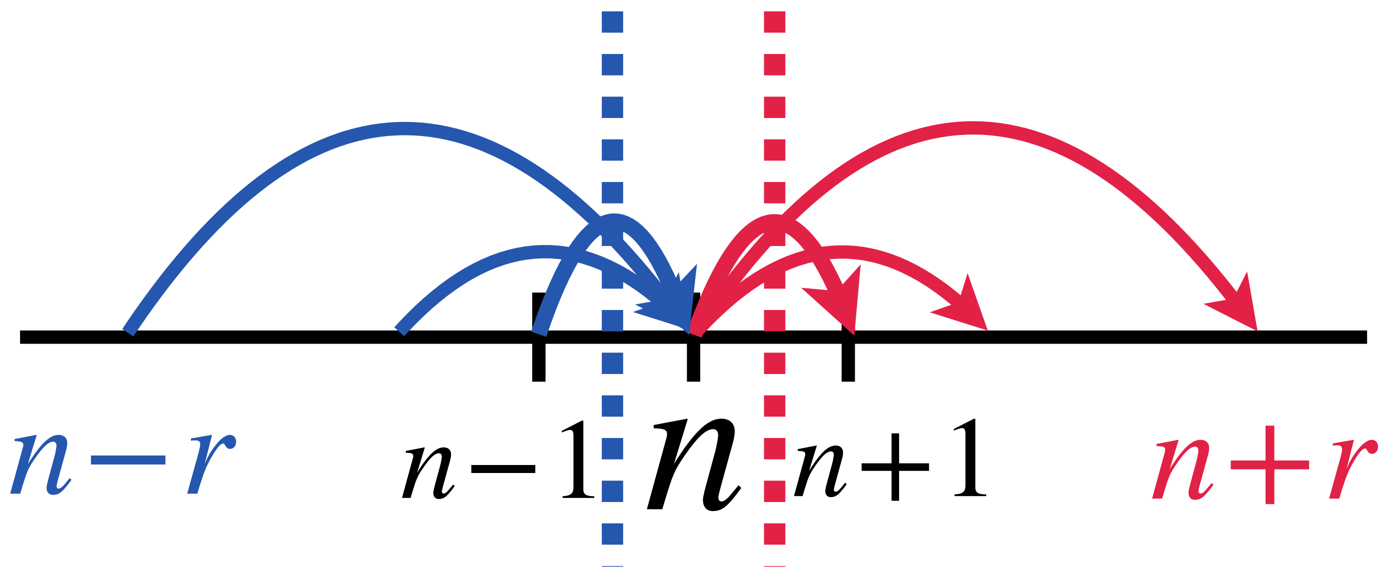


Local Energy Current in 1-Dim LRI Systems

✓ Continuity eq. for 1-dim long-range interacting systems HN & K.Saito, arXiv:2502.10139

$$\partial_t \epsilon_n = \{\epsilon_n, H\} = \sum_{m\neq n} \{\epsilon_n, \epsilon_m\} = \sum_{m\neq n} t_{n\leftarrow m} \qquad t_{n\leftarrow m} := \{\epsilon_n, \epsilon_m\}$$

$$= -\sum_{r=1}^{N/2} t_{n+r\leftarrow n} + \sum_{r=1}^{N/2} t_{n\leftarrow n-r}$$



~~$\mathcal{J}^{\epsilon}_{n+1}$?~~ ~~$\mathcal{J}^{\epsilon}_{n}$?~~

❖ Criteria : **Continuity eq.** holds.

$$\partial_t \epsilon_n = -\mathcal{J}^{\epsilon}_{n+1} + \mathcal{J}^{\epsilon}_{n}$$

This expression does **NOT** satisfy the criteria !!

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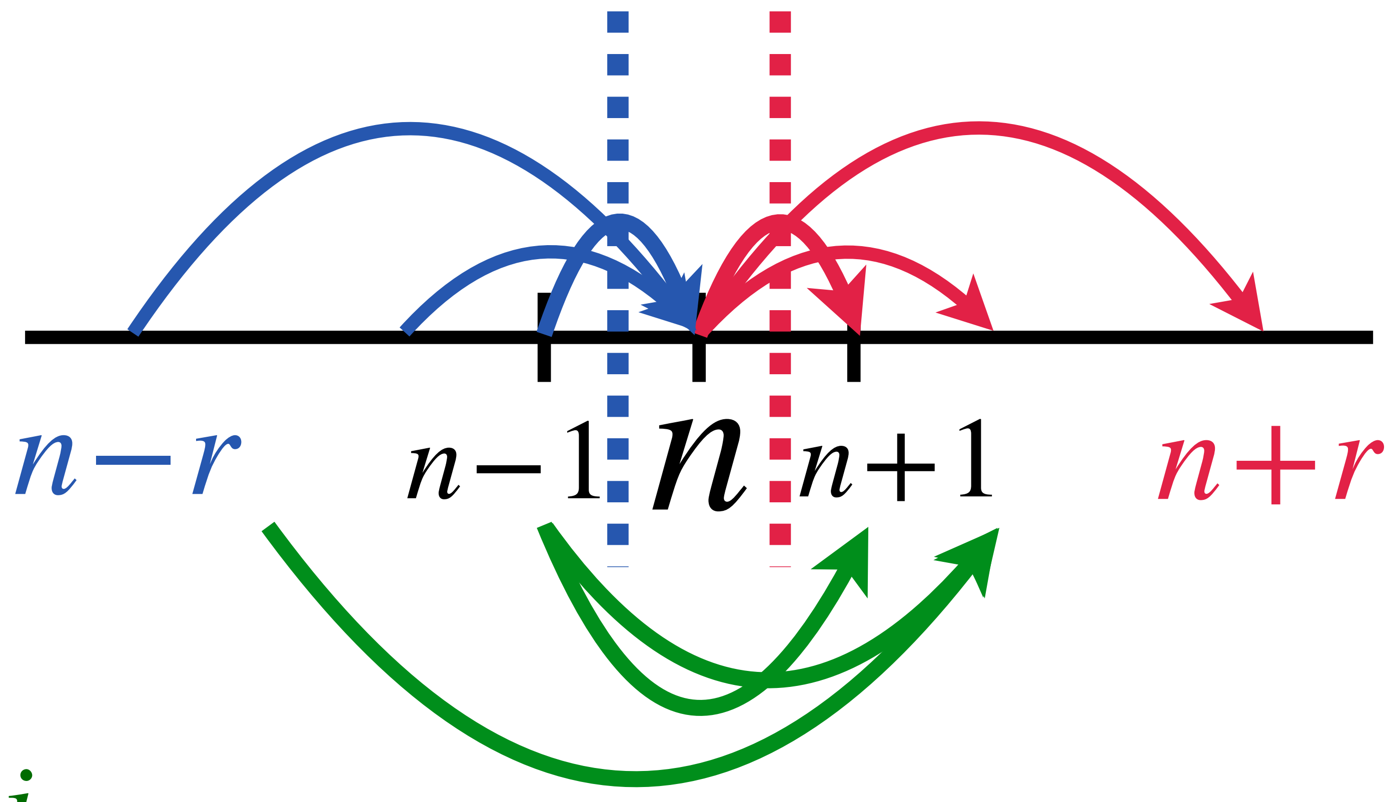
$$= -\sum_{r=1}^{N/2} t_{n+r\leftarrow n} + \sum_{r=1}^{N/2} t_{n\leftarrow n-r}$$

$$-\sum_{\substack{i\geq n+1 \\ j\leq n-1}} t_{i\leftarrow j} \qquad + \sum_{\substack{i\geq n+1 \\ j\leq n-1}} t_{i\leftarrow j}$$

$$= -\sum_{j<n+1\leq i} t_{i\leftarrow j} + \sum_{j<n\leq i} t_{i\leftarrow j}$$



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$$= -\sum_{r=1}^{N/2} t_{n+r\leftarrow n} + \sum_{r=1}^{N/2} t_{n\leftarrow n-r}$$

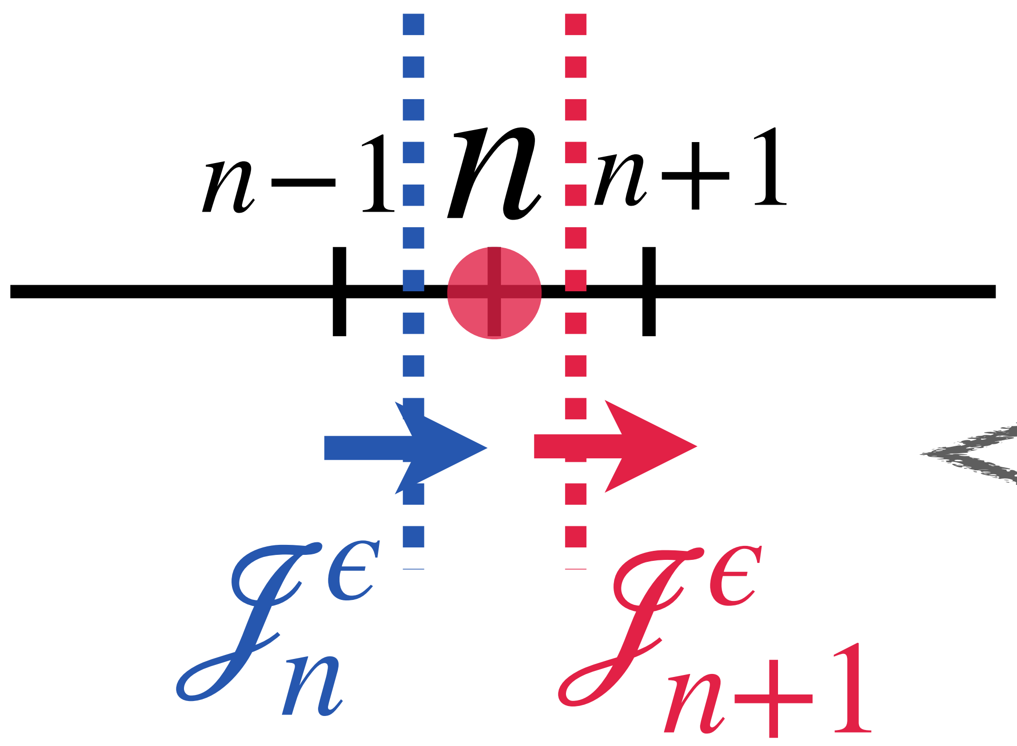
$$= -\sum_{j<n+1\leq i} t_{i\leftarrow j} + \sum_{j<n\leq i} t_{i\leftarrow j}$$

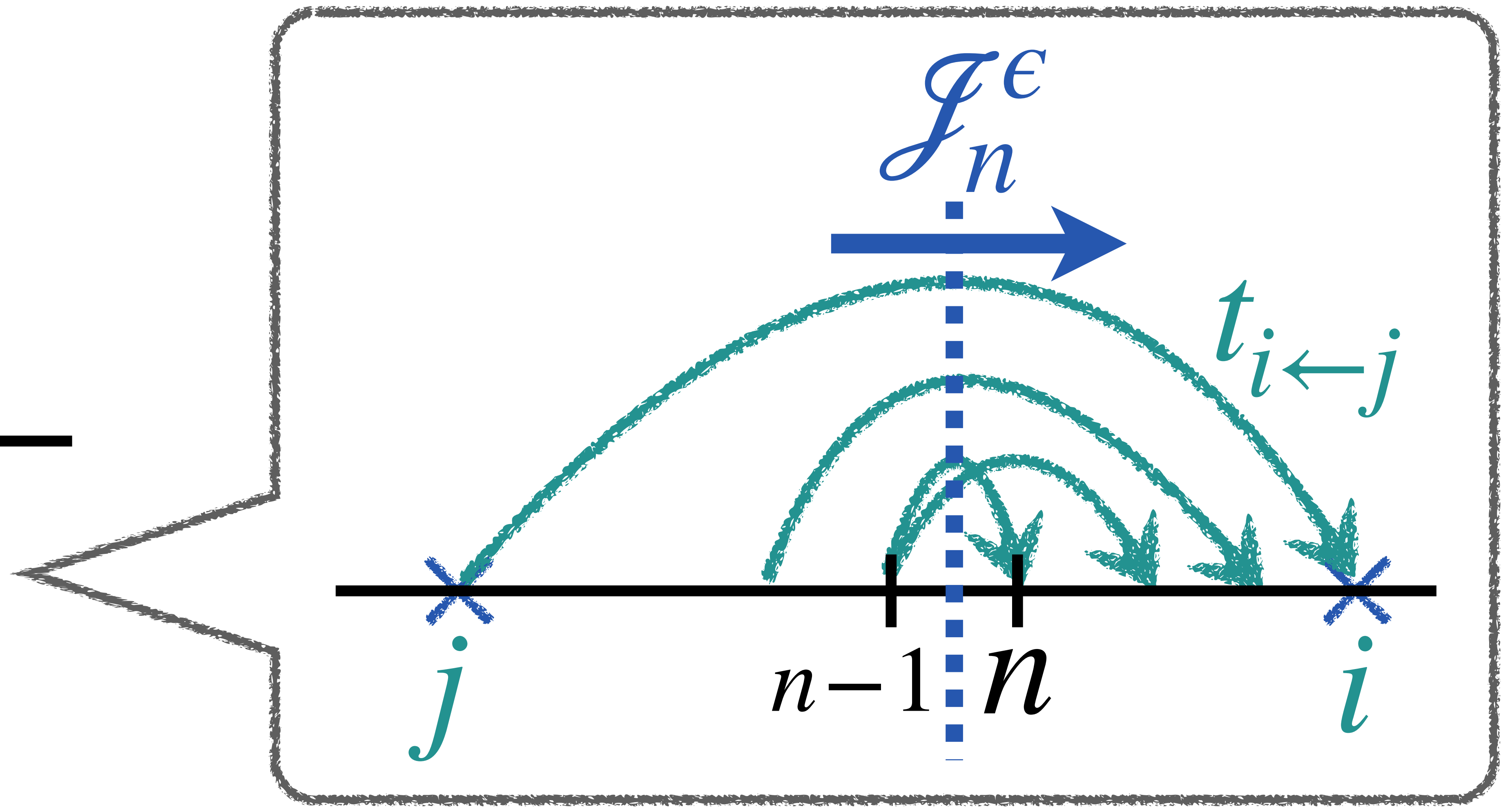
$$= \mathcal{J}^{\epsilon}_{n+1} \qquad = \mathcal{J}^{\epsilon}_{n}$$

❖ Criteria : **Continuity eq.** holds.

$$\partial_t \epsilon_n = -\mathcal{J}^{\epsilon}_{n+1} + \mathcal{J}^{\epsilon}_{n}$$

$$\mathcal{J}^{\epsilon}_{n} := \sum_{j<n\leq i} t_{i\leftarrow j}$$





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Energy Diffusion in the 1-Dim LRI Spin Systems：Outline

✓Normal Diffusion→**Thermal Conductivity (Green-Kubo formula)** is convergent.

$$\kappa_N = \frac{1}{k_B T^2}\int_0^\infty dt \sum_n \langle \mathcal{J}_n^\epsilon(t)\, \mathcal{J}_0^\epsilon \rangle_{\mathrm{eq}}$$

We focus on **The Amplitude of $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle_{\mathrm{eq}}$**

Thm.1：Cumulant Power-law Clustering

Consider k-local LRI systems on D-dim lattices.

For the regime $\alpha > D, T > T_c$,

$\langle \mathcal{O}_{X_1} \ldots \mathcal{O}_{X_n} \rangle_c \le$ const .

$$\sum_{i_1,\ldots,i_n \text{:connected}} \prod_{p=1}^{n-1} \|\mathcal{O}_{X_p}\| \frac{|X_{i_p}||X_{i_{p+1}}| e^{(|X_{i_p}|+|X_{i_{p+1}}|)/k}}{(d_{X_{i_p},X_{i_{p+1}}})^\alpha}$$

Thm. 2：Upper Bound on $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle_{\mathrm{eq}}$

For the prototypical 2-local LRI spin systems (Trans. Ising & XYZ),

$\langle \mathcal{J}_n^\epsilon\, \mathcal{J}_0^\epsilon \rangle_{\mathrm{eq}} < c' n^{2-2\alpha}$, $(1 < \alpha < 2)$

$\alpha > 3/2$ is Sufficient for **Normal Diffusion**

If $\alpha < 3/2$ ？：**Fluctuating Hydrodynamics**→**Levy Diffusion**

Energy Diffusion in the 1-Dim LRI Spin Systems：Outline

✓Normal Diffusion→**Thermal Conductivity (Green-Kubo formula)** is convergent.

$$\kappa_N = \frac{1}{k_{\mathrm{B}} T^2} \int_0^\infty dt \sum_n \langle \mathcal{J}_n^\epsilon(t)\, \mathcal{J}_0^\epsilon \rangle_{\mathrm{eq}}$$

We focus on **The Amplitude of** $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle_{\mathrm{eq}}$

Normal Diffusion and Green-Kubo formula

- If the energy diffusion is **normal**,

$$\partial_t \epsilon_n(t) = \mathscr{D}\,\nabla_n^2 \epsilon_n(t) \quad : \text{Diffusion eq.}$$

$$\mathscr{D} = \frac{\kappa}{c_V}, \quad \kappa \ : \text{Thermal conductivity} \quad c_V : \text{Specific Heat per unit volume}$$

- Green-Kubo formula for thermal conductivity

$$\kappa_N = \frac{1}{k_B T^2} \int_0^\infty dt\, C_N(t)\,, \quad C_N(t) = \sum_n \langle \mathscr{J}_n^\epsilon(t)\, \mathscr{J}_0^\epsilon \rangle_{\text{eq}}$$

- Normal diffusion → **Thermal conductivity converges to finite**.

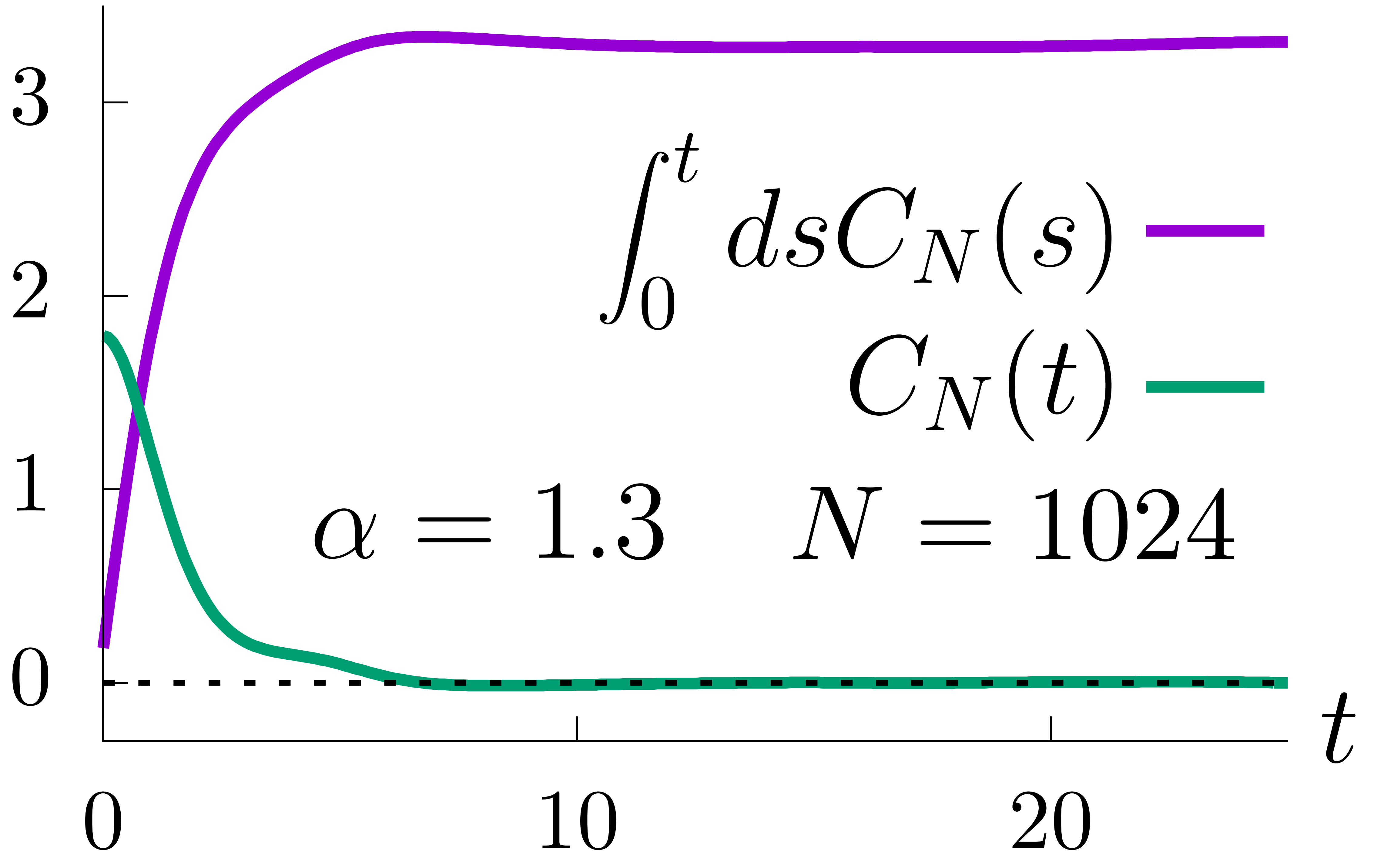
  - Remark : $c_V < \infty$ for $\alpha > 1$

Numerical Simulation : Total Current Correlation

- LRI Transverse Ising Model $H = \sum_n \epsilon_n$

$$\epsilon_n = -\frac{J}{2}\sum_r \frac{S_n^z S_{n+r}^z + S_n^z S_{n-r}^z}{r^\alpha} - hS_n^x$$

- Total current correlation : $C_N(t) = \sum_n \langle \mathcal{J}_n^\epsilon(t)\, \mathcal{J}_0^\epsilon \rangle_{\mathrm{eq}}$



✓ $C_N(t)$: **Rapid decay in time**

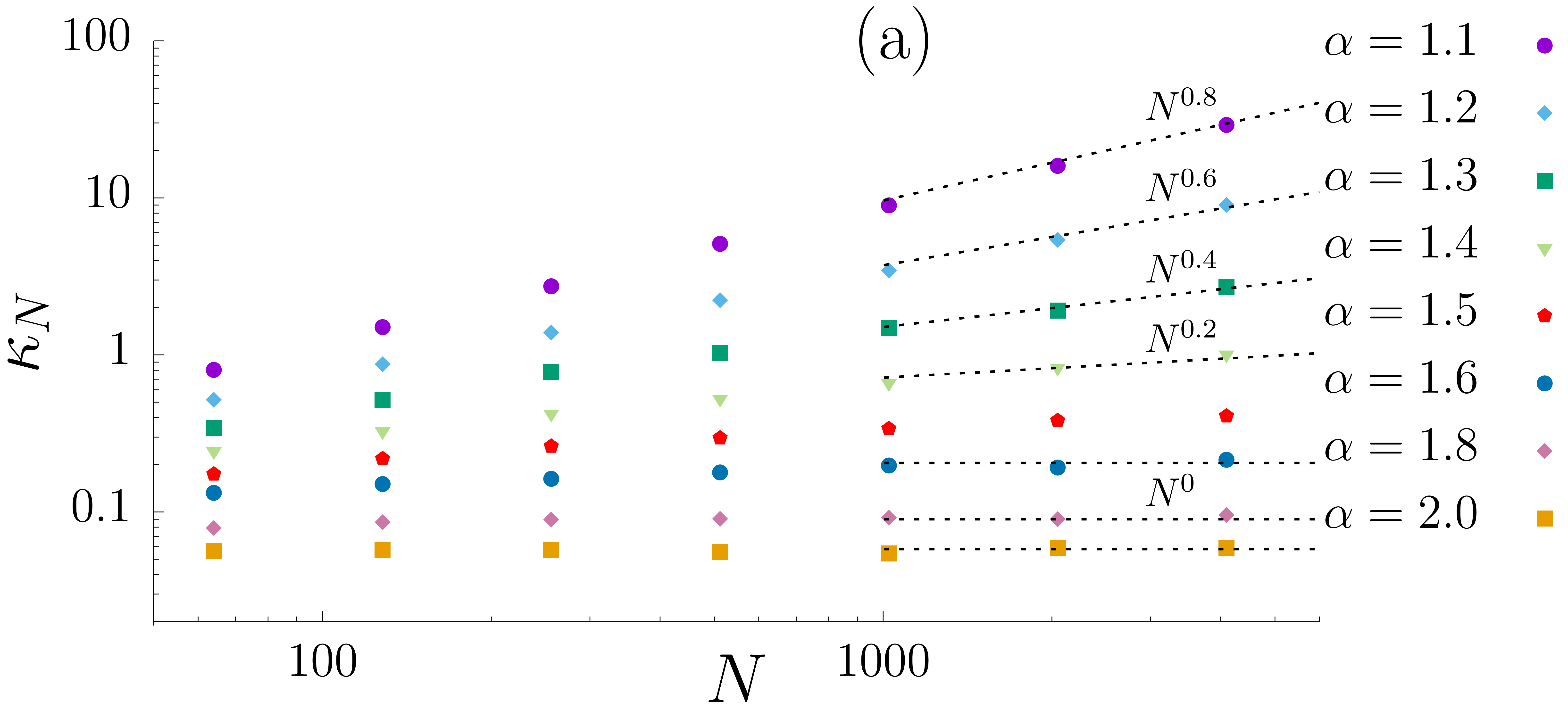
→ **No long-time tail** in $C_N(t)$

✓ Green-Kubo integral is convergent.

✓ Due to non-integrable systems & no continous translational symmetry.

Numerical Simulation : Thermal Conductivity

- Thermal conductivity : $\kappa_N = \frac{1}{k_B T^2} \int_0^\infty dt \, C_N(t)\,, \quad C_N(t) = \sum_n \langle \mathcal{J}_n^\epsilon(t)\, \mathcal{J}_0^\epsilon \rangle_{\rm eq}$



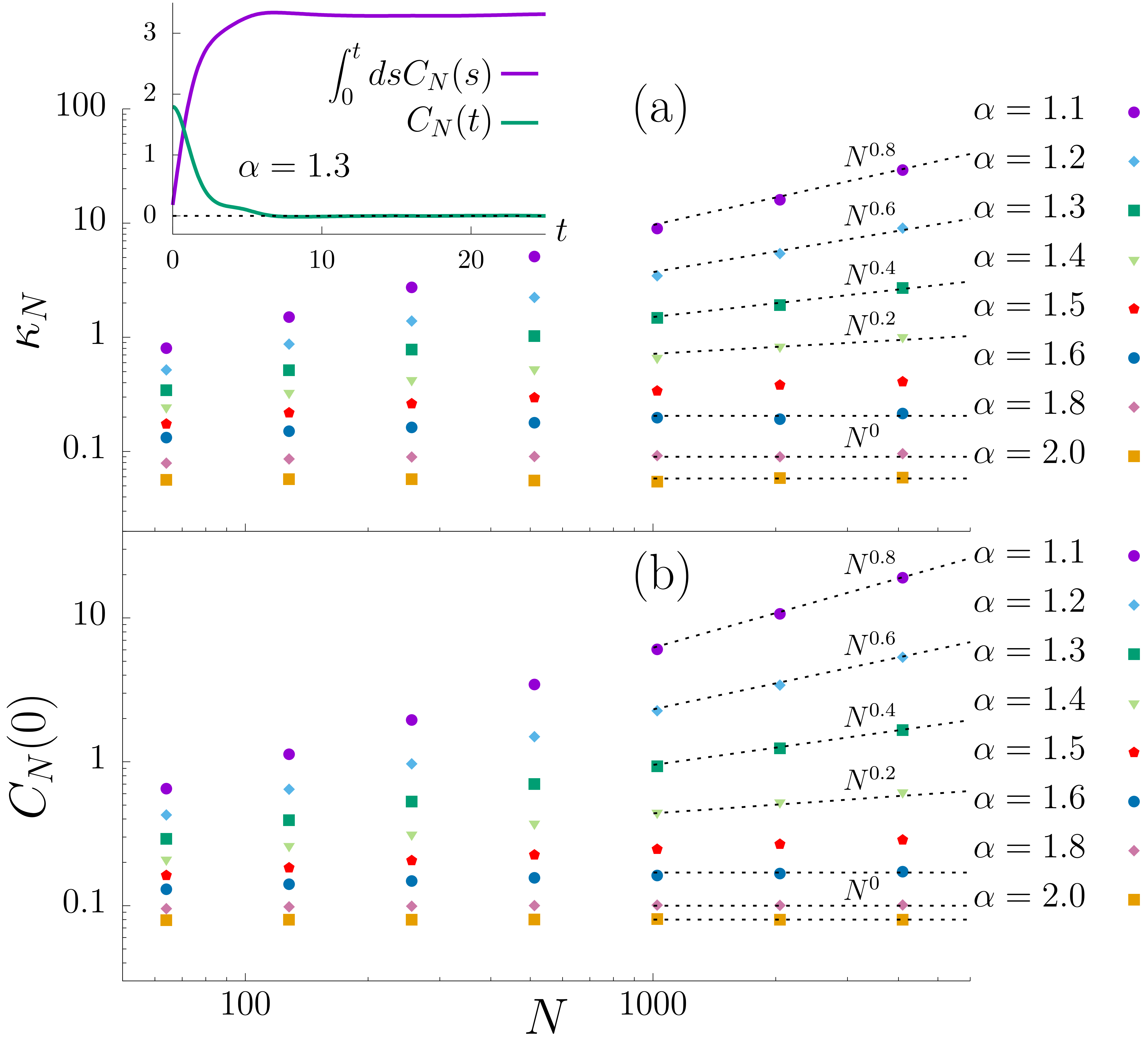
✓ Small α (< 1.5) seems to show the diverging thermal conductivity.

→What is the origin of the anomalous behavior ?

cf) There is NO long-time tail in $C_N(t)$.

Divergence of Thermal Conductivity (Green-Kubo formula)

We focus on **the equal-time current correlation** : $C_N(0) = \sum_n \langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle_{\rm eq}$



$$\kappa_N = \frac{1}{k_{\rm B} T^2} \int_0^\infty dt\, C_N(t) ,$$

$$C_N(t) = \sum_n \langle \mathcal{J}_n^\epsilon(t)\, \mathcal{J}_0^\epsilon \rangle_{\rm eq}$$

✓ Both are qualitatively same.

✓ **New mechanism :**

Anomalous enhancement of the amplitude of $C_N(0)$

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We focus on **The Amplitude of** $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle_{\mathrm{eq}}$

Thm.1：Cumulant Power-law Clustering

Consider k-local LRI systems on D-dim lattices.

For the regime $\alpha > D, T > T_c$,

$$\langle \mathcal{O}_{X_1} \dots \mathcal{O}_{X_n} \rangle_c \le \mathrm{const}\,.$$

$$\sum_{i_1,\dots,i_n\,:\mathrm{connected}} \prod_{p=1}^{n-1} \|\mathcal{O}_{X_p}\| \frac{|X_{i_p}||X_{i_{p+1}}| e^{(|X_{i_p}|+|X_{i_{p+1}}|)/k}}{(d_{X_{i_p},X_{i_{p+1}}})^\alpha}$$

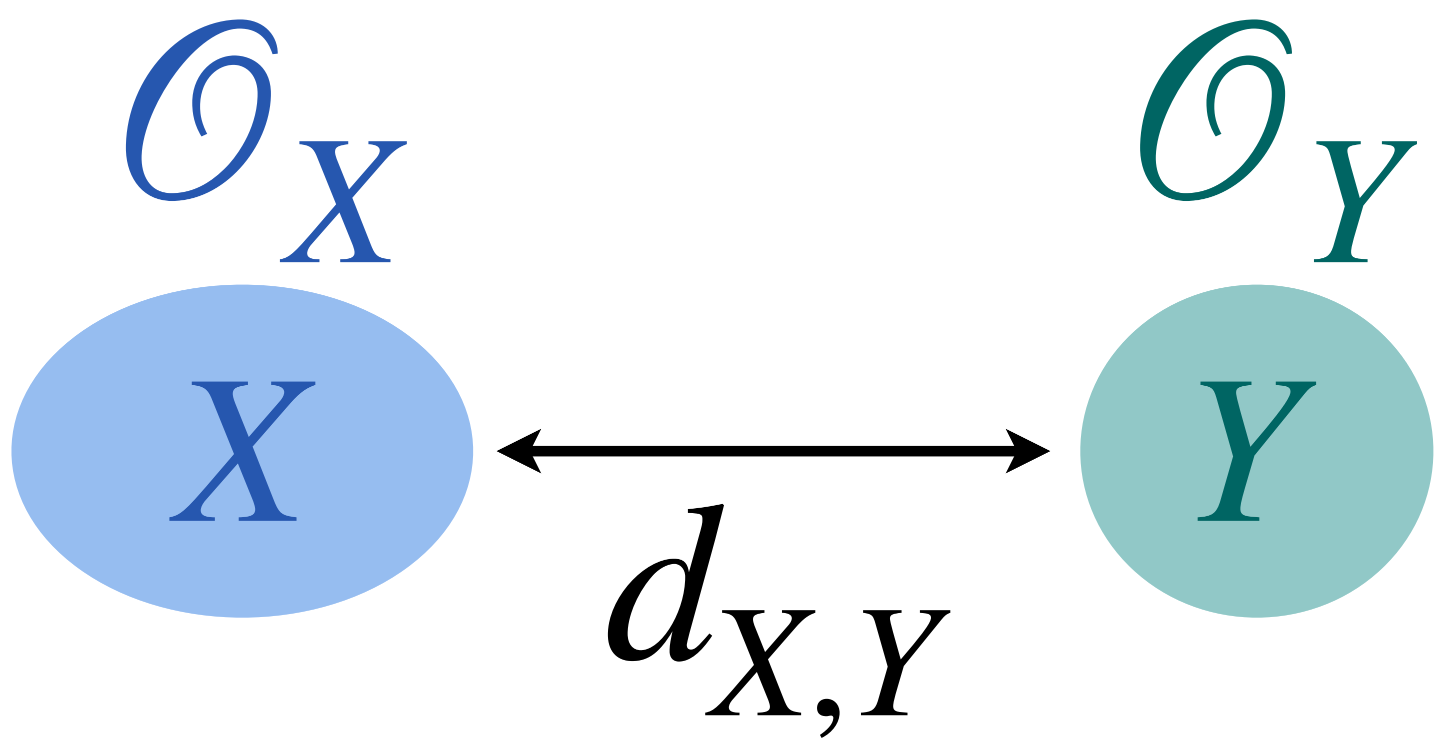
Clustering Theorem for Many-body Systems

✓ Clustering Theorem for Short-range Interacting Systems

Araki, Comm. Math. Phys. (1969)
Gross, Comm. Math. Phys. (1979)
Kliesh et al., Phys. Rev. X (2014)

$$\langle \delta\mathcal{O}_X \delta\mathcal{O}_Y \rangle_{\text{eq}} \le \text{const.} \cdot \|\mathcal{O}_X\| \cdot \|\mathcal{O}_Y\| e^{-c' d_{X,Y}}$$

$$\delta\mathcal{O}_X := \mathcal{O}_X - \langle \mathcal{O}_X \rangle_{\text{eq}}$$



✓ Clustering Theorem for Long-range Interacting Systems

Kim, Kuwahara, Saito
Phys. Rev. Lett. (2025)

$$\langle \delta\mathcal{O}_X \delta\mathcal{O}_Y \rangle_{\text{eq}} \le \text{const.} \cdot \|\mathcal{O}_X\| \cdot \|\mathcal{O}_Y\| \frac{|X||Y| e^{(|X|+|Y|)/k}}{d_{X,Y}^{\alpha}}$$

Proof: high-temperature cluster expansion technique

Cumulant Power-law Clustering Theorem for LRI Systems

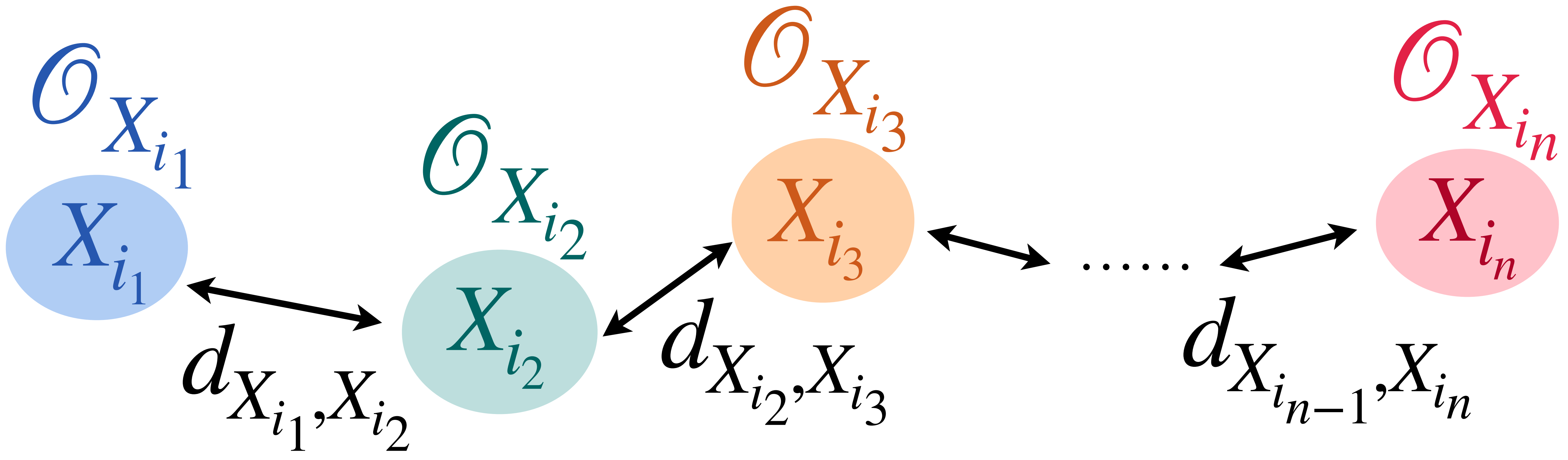
Thm.1 : Cumulant Power-law Clustering Theorem for LRI Systems HN & K.Saito, arXiv:2502.10139

Consider k-local LRI systems on D-dim lattices. For $\alpha > D, T > T_c$, the following inequality holds:

$$\langle \mathcal{O}_{X_1} \ldots \mathcal{O}_{X_n} \rangle_{\mathrm{c}} \leq \mathrm{const.} \sum_{\substack{i_1, \ldots, i_n \\ \text{:connected}}} \prod_{p=1}^{n-1} \| \mathcal{O}_{X_p} \| \frac{|X_{i_p}| |X_{i_{p+1}}| e^{(|X_{i_p}| + |X_{i_{p+1}}|)/k}}{(d_{X_{i_p}, X_{i_{p+1}}})^{\alpha}}$$

$$\langle \mathcal{O}_{X_1} \ldots \mathcal{O}_{X_n} \rangle_c := \left. \frac{\partial^n}{\partial \lambda_1 \ldots \partial \lambda_n} \right|_{\vec{\lambda}=0} \mu(\vec{\lambda})$$

$$\mu(\vec{\lambda}) = \ln \langle (\sum_i \lambda_i \mathcal{O}_{X_i}) \rangle$$



Proof: high-temperature cluster expansion technique

Cumulant Power-law Clustering Theorem for LRI Systems

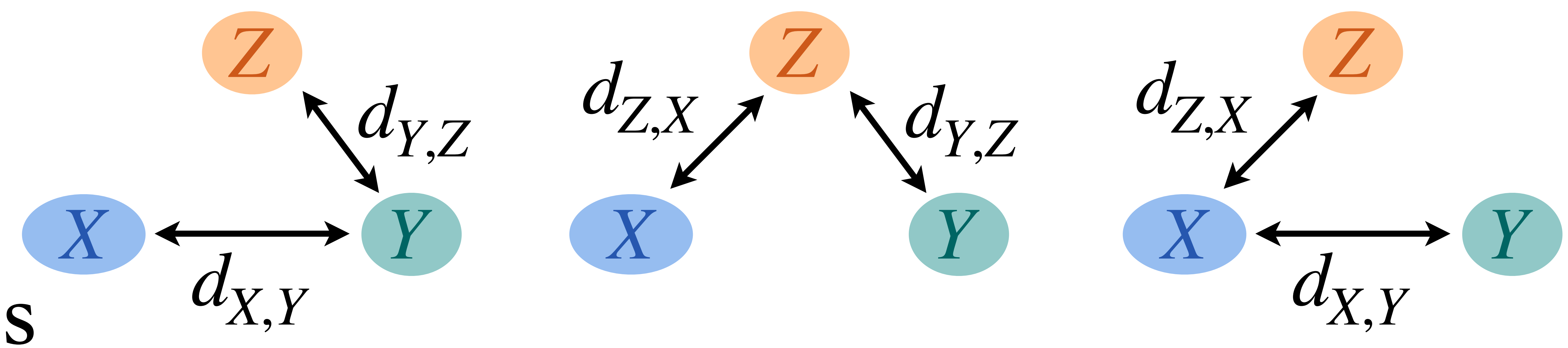
Thm.1：Cumulant Power-law Clustering Theorem for LRI Systems HN & K.Saito, arXiv:2502.10139

Consider k-local LRI systems on D-dim lattices. For $\alpha > D, T > T_c$, the following inequality holds:

$$\langle \mathcal{O}_{X_1} \ldots \mathcal{O}_{X_n} \rangle_{\mathrm{c}} \leq \mathrm{const.} \sum_{\substack{i_1, \ldots, i_n \\ \text{:connected}}} \prod_{p=1}^{n-1} \| \mathcal{O}_{X_p} \| \frac{|X_{i_p}||X_{i_{p+1}}| e^{(|X_{i_p}|+|X_{i_{p+1}}|)/k}}{(d_{X_{i_p}, X_{i_{p+1}}})^{\alpha}}$$

$$\langle \mathcal{O}_{X_1} \ldots \mathcal{O}_{X_n} \rangle_c := \frac{\partial^n}{\partial \lambda_1 \ldots \partial \lambda_n} \bigg|_{\vec{\lambda}=0} \mu(\vec{\lambda})$$

$$\mu(\vec{\lambda}) = \ln \langle (\sum_i \lambda_i \mathcal{O}_{X_i}) \rangle$$



e.g. case of 3rd order cumulants

$$\langle \delta\mathcal{O}_X \delta\mathcal{O}_Y \delta\mathcal{O}_Z \rangle_{\mathrm{eq}} \leq \frac{\mathcal{O}(1)}{(d_{X,Y})^{\alpha}(d_{Y,Z})^{\alpha}} + \frac{\mathcal{O}(1)}{(d_{Y,Z})^{\alpha}(d_{Z,X})^{\alpha}} + \frac{\mathcal{O}(1)}{(d_{Z,X})^{\alpha}(d_{X,Y})^{\alpha}}$$

Energy Diffusion in the 1-Dim LRI Spin Systems：Outline

✓Normal Diffusion→**Thermal Conductivity (Green-Kubo formula)** is convergent.

$$\kappa_N = \frac{1}{k_B T^2} \int_0^\infty dt \sum_n \langle \mathcal{J}_n^\epsilon(t)\, \mathcal{J}_0^\epsilon \rangle_{\mathrm{eq}}$$

We focus on **The Amplitude of** $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle_{\mathrm{eq}}$

Thm.1：Cumulant Power-law Clustering

Consider k-local LRI systems on D-dim lattices.

For the regime $\alpha > D, T > T_c$,

$\langle \mathcal{O}_{X_1} \dots \mathcal{O}_{X_n} \rangle_c \leq$ const .

$$\sum_{i_1,\dots,i_n} \prod_{p=1}^{n-1} \|\mathcal{O}_{X_p}\| \frac{|X_{i_p}||X_{i_{p+1}}| e^{(|X_{i_p}|+|X_{i_{p+1}}|)/k}}{(d_{X_{i_p},X_{i_{p+1}}})^\alpha}$$

:connected

Thm. 2：Upper Bound on $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle_{\mathrm{eq}}$

For the prototypical 2-local LRI spin systems (Trans. Ising & XYZ),

$\langle \mathcal{J}_n^\epsilon\, \mathcal{J}_0^\epsilon \rangle_{\mathrm{eq}} < c' n^{2-2\alpha}$, $\quad (1 < \alpha < 2)$

$\alpha > 3/2$ is Sufficient for **Normal Diffusion**

Upper Bound on Energy Current Correlation

Thm.2 : Upper Bound on Current Correlation for 1-dim LRI Spin Systems

HN & K.Saito, arXiv:2502.10139

For 1-dim prototypical 2-local LRI spin systems (Trans. Ising & XYZ), the equal-time equilibrium energy current correlation is upper-bound as

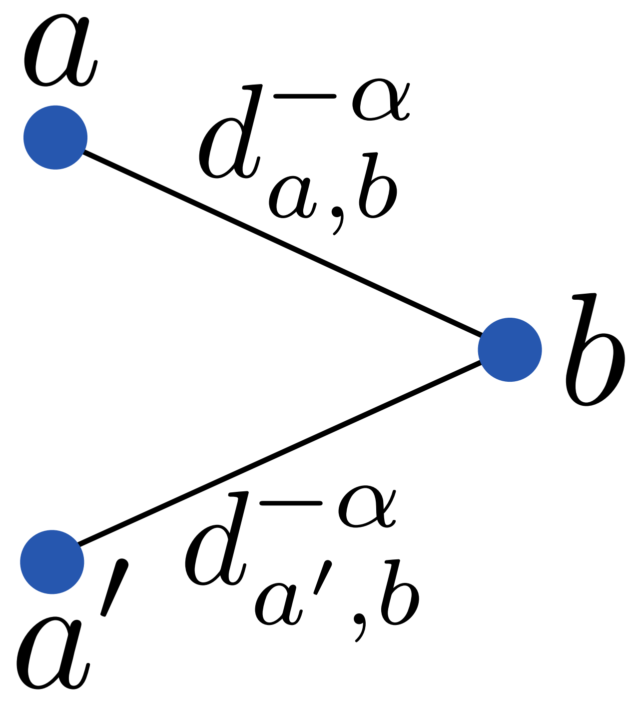
$$\langle \mathcal{J}_n^{\epsilon} \mathcal{J}_0^{\epsilon} \rangle_{\mathrm{eq}} < c' n^{2-2\alpha}, \quad (1 < \alpha < 2)$$

Sketch of Proof Symmetry of Hamiltonian + Cumulant Power-law Clustering Thm.

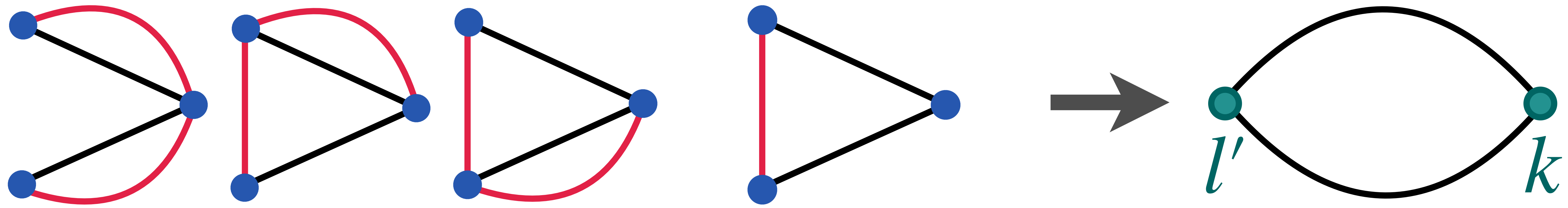
✓ Trans. Ising→ $\langle S_i^z \rangle_{\mathrm{eq}} = \langle S_i^y \rangle_{\mathrm{eq}} = 0, \quad \langle S_i^y S_j^y \mathcal{O} \rangle_{\mathrm{eq}} = \delta_{ij} \langle (S_i^y)^2 \mathcal{O} \rangle_{\mathrm{eq}}$

$$H = -J \sum_{i,j} \frac{S_i^z S_j^z}{d_{i,j}^{\alpha}} - h \sum_i S_i^x$$

$$\langle \mathcal{J}_n^{\epsilon} \mathcal{J}_0^{\epsilon} \rangle_{\mathrm{eq}} = \sum_{\substack{k \geq n > l \\ k' \geq 0 > l'}} \sum_{\substack{\{a,b\} = \{k,l\} \\ \{a',b'\} = \{k',l'\}}} \epsilon^{ab} \epsilon^{a'b'} \frac{J^2 h^2}{4} \delta_{b,b'} \frac{\langle S_a^x S_{a'}^x (S_b^y)^2 \rangle_{\mathrm{eq}}}{d_{a,b}^{\alpha} d_{a',b}^{\alpha}}$$



$$\langle S_a^x S_{a'}^x (S_b^y)^2 \rangle_{\mathrm{eq}} = \langle S_a^x S_{a'}^x (S_b^y)^2 \rangle_c + \langle S_a^x S_{a'}^x \rangle_c \langle (S_b^y)^2 \rangle_c$$



$$\sum_{k \geq n, l' < 0} d_{k,l'}^{-2\alpha} \leq \mathrm{const.} \cdot n^{2-2\alpha}$$

Upper Bound on Energy Current Correlation

Thm.2 : Upper Bound on Current Correlation for 1-dim LRI Spin Systems

HN & K.Saito, arXiv:2502.10139

For 1-dim prototypical 2-local LRI spin systems (Trans. Ising & XYZ), the equal-time equilibrium energy current correlation is upper-bound as

$$\langle \mathcal{J}_n^{\epsilon} \mathcal{J}_0^{\epsilon} \rangle_{\mathrm{eq}} < c' n^{2-2\alpha}, \quad (1 < \alpha < 2)$$

Sketch of Proof **Symmetry of Hamiltonian + Cumulant Power-law Clustering Thm.**

✓ Trans. Ising→ $\langle S_i^z \rangle_{\mathrm{eq}} = \langle S_i^y \rangle_{\mathrm{eq}} = 0, \quad \langle S_i^y S_j^y \mathcal{O} \rangle_{\mathrm{eq}} = \delta_{ij} \langle (S_i^y)^2 \mathcal{O} \rangle_{\mathrm{eq}}$

✓ XYZ→ $\langle S_i^\sigma \rangle_{\mathrm{eq.}} = 0, \quad \langle S_i^\sigma S_j^\tau \rangle_{\mathrm{eq.}} = \delta_{\sigma\tau} \langle S_i^\sigma S_j^\sigma \rangle_{\mathrm{eq.}} \quad (\sigma, \tau = x, y, z)$

Upper Bound on Energy Current Correlation

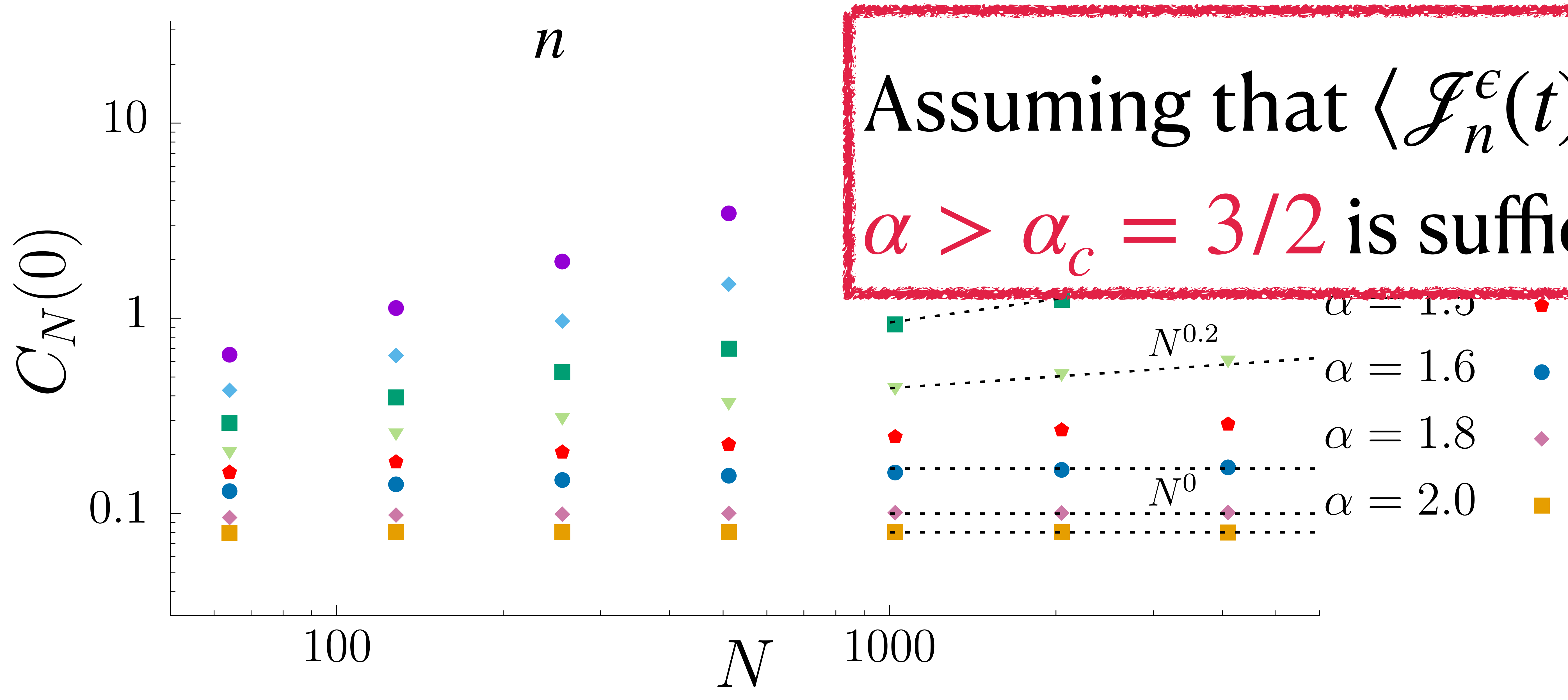
Thm.2 : Upper Bound on Current Correlation for 1-dim LRI Spin Systems

HN & K.Saito, arXiv:2502.10139

For 1-dim prototypical 2-local LRI spin systems (Trans. Ising & XYZ), the equal-time equilibrium energy current correlation is upper-bound as

$$\langle \mathcal{J}_n^{\epsilon} \mathcal{J}_0^{\epsilon} \rangle_{\mathrm{eq}} < c' n^{2-2\alpha}, \quad (1 < \alpha < 2)$$

$$\rightarrow \quad C_N(0) = \sum_n \langle \mathcal{J}_n^{\epsilon} \mathcal{J}_0^{\epsilon} \rangle_{\mathrm{eq}} < c'' N^{3-2\alpha} + \mathrm{const.} < \infty \quad (\alpha > 3/2)$$



Assuming that $\langle \mathcal{J}_n^{\epsilon}(t) \mathcal{J}_0^{\epsilon} \rangle$ is integrable,

$\alpha > \alpha_c = 3/2$ is sufficient for **Normal Diffusion.**

Energy Diffusion in the 1-Dim LRI Spin Systems：Outline

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$$\kappa_N = \frac{1}{k_B T^2}\int_0^\infty dt \sum_n \langle \mathcal{J}_n^\epsilon(t)\,\mathcal{J}_0^\epsilon\rangle_{\mathrm{eq}}$$

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Consider k-local LRI systems on D-dim lattices.

For the regime $\alpha > D, T > T_c$,

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$$\sum_{\substack{i_1,\dots,i_n\\ \text{:connected}}}\prod_{p=1}^{n-1}\|\mathcal{O}_{X_p}\|\frac{|X_{i_p}||X_{i_{p+1}}|e^{(|X_{i_p}|+|X_{i_{p+1}}|)/k}}{(d_{X_{i_p},X_{i_{p+1}}})^\alpha}$$

Thm. 2：Upper Bound on $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon\rangle_{\mathrm{eq}}$

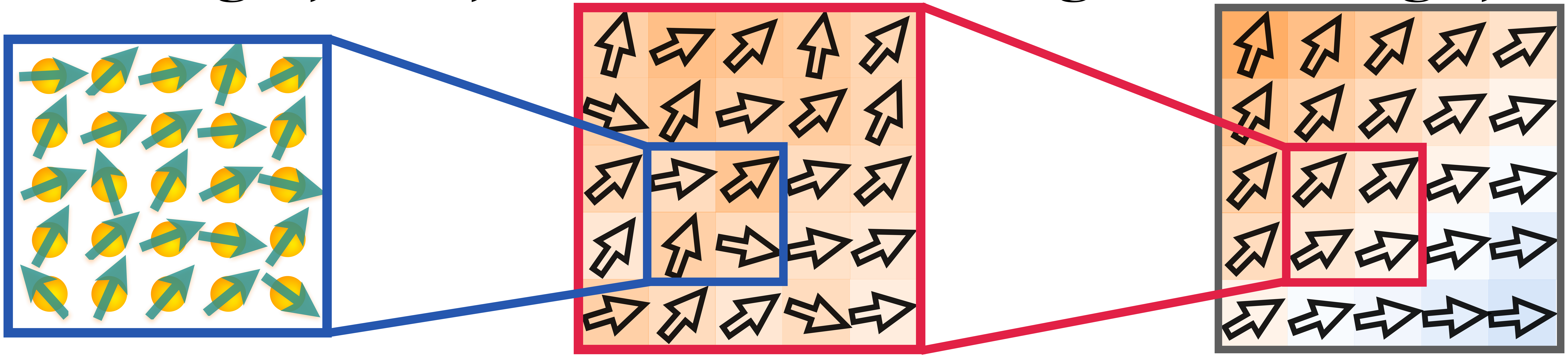
For the prototypical 2-local LRI spin systems (Trans. Ising & XYZ),

$\langle \mathcal{J}_n^\epsilon\,\mathcal{J}_0^\epsilon\rangle_{\mathrm{eq}} < c' n^{2-2\alpha}$, $(1 < \alpha < 2)$

$\alpha > 3/2$ is Sufficient for **Normal Diffusion**

If $\alpha < 3/2$ ？：**Fluctuating Hydrodynamics→Levy Diffusion**

Fluctuating Hydrodynamics for Short-range Interacting Systems



Micro
(Newton eq.)

MESO
(Fluctuating Hydrodynamics : FHD)

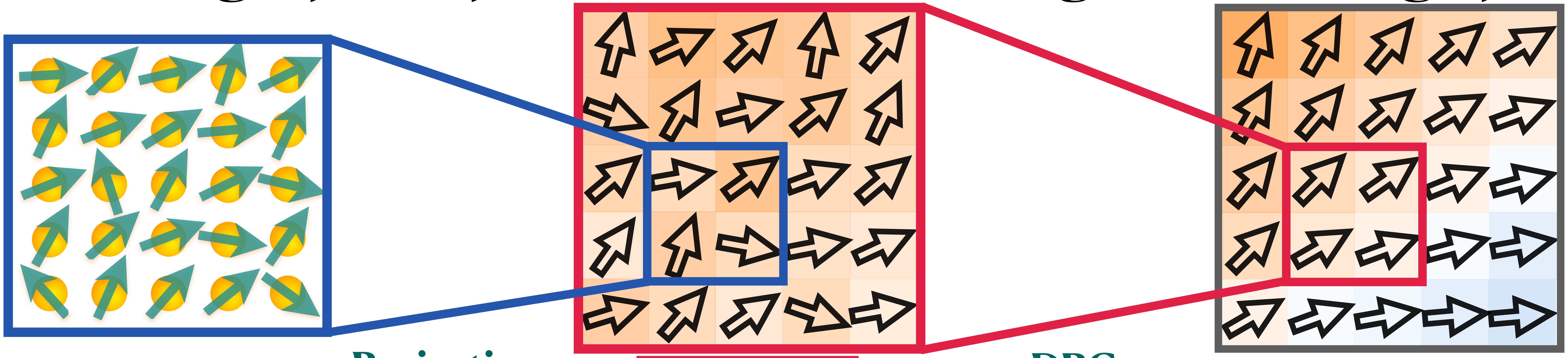
Macro
(Hydrodynamics)

✓Diverging viscosity in low-dim fluids
✓Diverging thermal conductivity in low-dim lattices
✓Long-range correlation in noneq. steady states

They cannot be described by macroscopic hydrodynamics.

→ Important is **The Fluctuation in Mesoscale.**

Fluctuating Hydrodynamics for Short-range Interacting Systems



Micro (Newton eq.) → **Projection** → **MESO (FHD)** → **DRG** → **Macro** (Hydrodynamics)

Projection: Zwanzig Phys. Rep. (1961), Saito et al., PRL (2021)

DRG: Forester, Nelson, Stephen Phys. Rep. (1977)

$$\partial_t u_a(x,t) = -\partial_x \left[J_a(\{u\}) - \sum_{a'} D_{a,a'} \partial_{x'} u_a(x,t) + \xi_{a,x}(t) \right]$$

Spohn, J. Stat. Phys. (2014)

- $\partial_t u_a(x,t)$: **Local conserved quantities**
- $J_a(\{u\})$: **Equilibrium current**
- $D_{a,a'}$: **Dissipation**
- $\xi_{a,x}(t)$: **Fluctuation**

✓FHD in LRI Spin Systems ??

✓Diverging thermal conductivity in $\alpha < 3/2$

$$\partial_t u_a(x,t) = -\partial_x \left[\sum_{a'} \int dx' \tilde{D}_{a,a'}(x-x') \partial_{x'} u_{a'}(x',t) + \xi_{a,x}(t) \right]$$

$\tilde{D}_{a,a'}(x-x')$: **Nonlocal Diffusion (Dissipation)**

HN & K.Saito, arXiv:2502.10139

Fluctuating Hydrodynamics of Energy for LRI Spin Systems

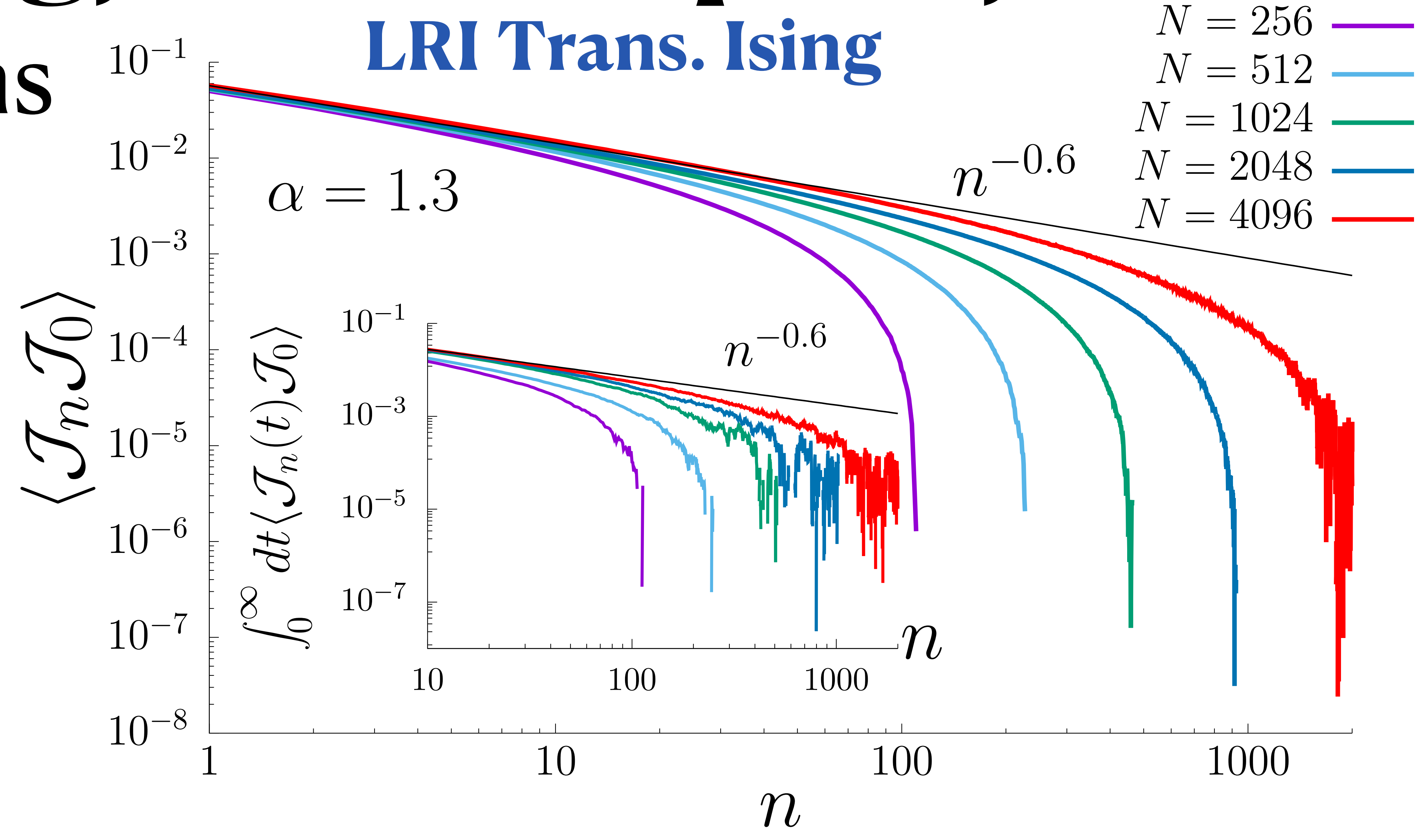
- Fluctuating Hydrodynamics for LRI Spin Systems

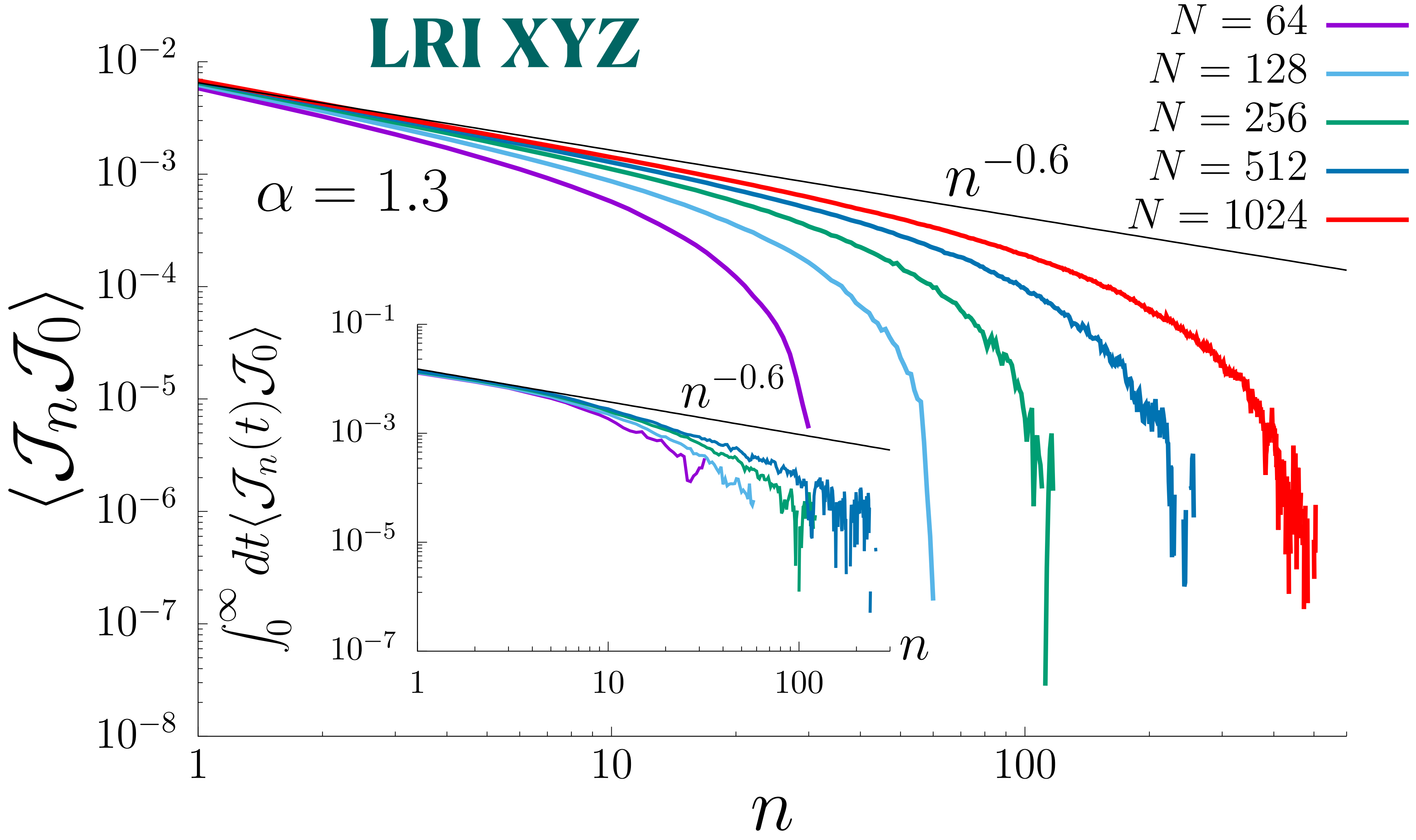
$$\partial_t \epsilon_n(t) = \nabla_n \left(\sum_m D_{n-m} \nabla_m \epsilon_m + \xi_n(t) \right)$$

$$\langle \xi_n(t) \xi_m(t') \rangle = 2D_{n-m} \delta(t - t')$$

$$D_n = T^{-2} c_V^{-1} \int_0^\infty dt \langle \mathcal{J}_n^\epsilon(t) \mathcal{J}_0^\epsilon \rangle$$

Thm.2 $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle_{\text{eq}} < c' n^{2-2\alpha}$





Fluctuating Hydrodynamics of Energy for LRI Spin Systems

- Fluctuating Hydrodynamics for LRI Spin Systems

$$\partial_t \epsilon_n(t) = \nabla_n \left(\sum_m D_{n-m} \nabla_m \epsilon_m + \xi_n(t) \right)$$

$$\langle \xi_n(t) \xi_m(t') \rangle = 2 D_{n-m} \delta(t-t')$$

$$D_n = T^{-2} c_V^{-1} \int_0^\infty dt \langle \mathcal{J}_n^\epsilon(t) \mathcal{J}_0^\epsilon \rangle \sim n^{2-2\alpha}$$

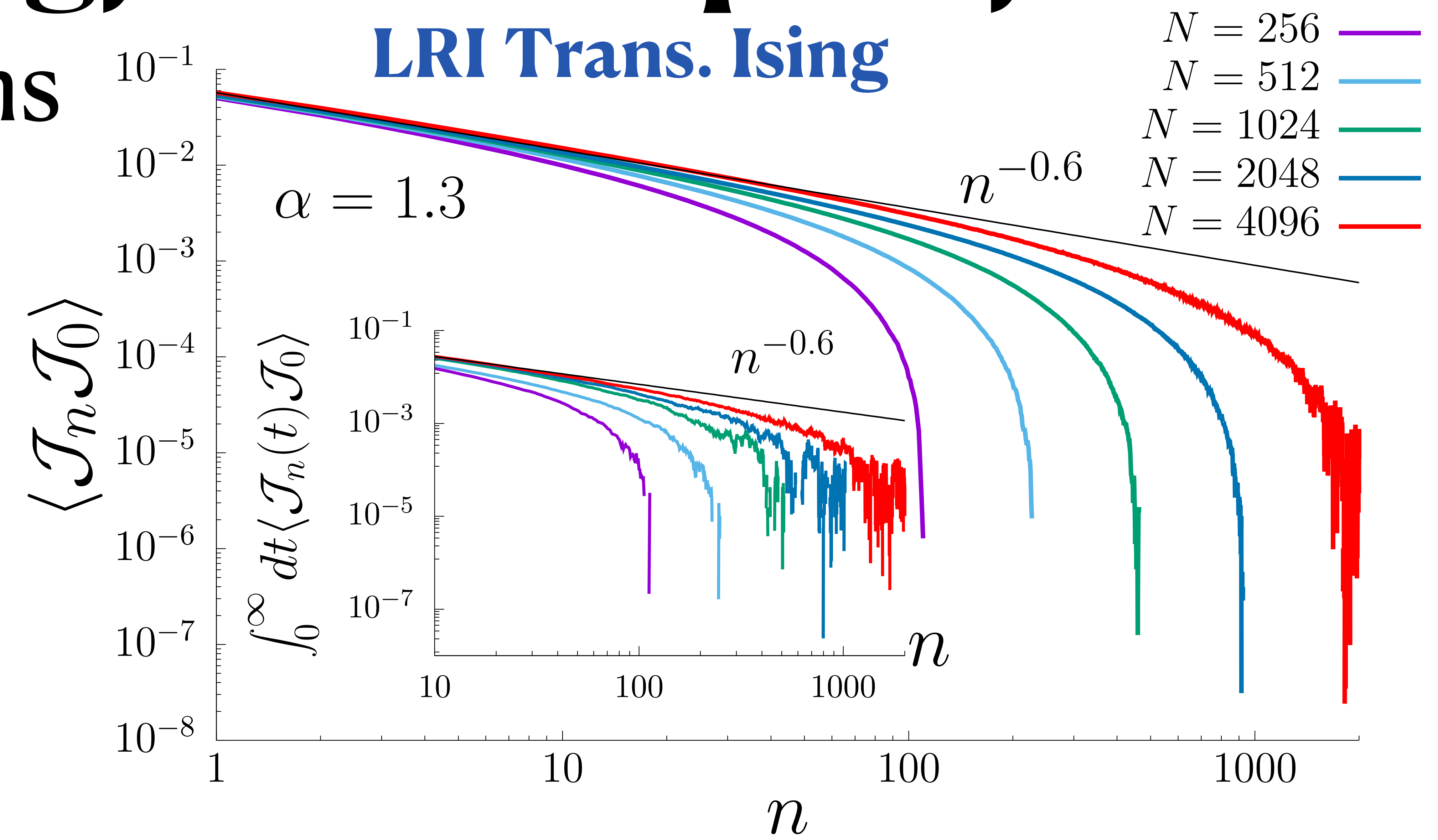
Thm.2 is optimal ! $\langle \mathcal{J}_n^\epsilon \mathcal{J}_0^\epsilon \rangle_{\mathrm{eq}} \sim n^{2-2\alpha}$

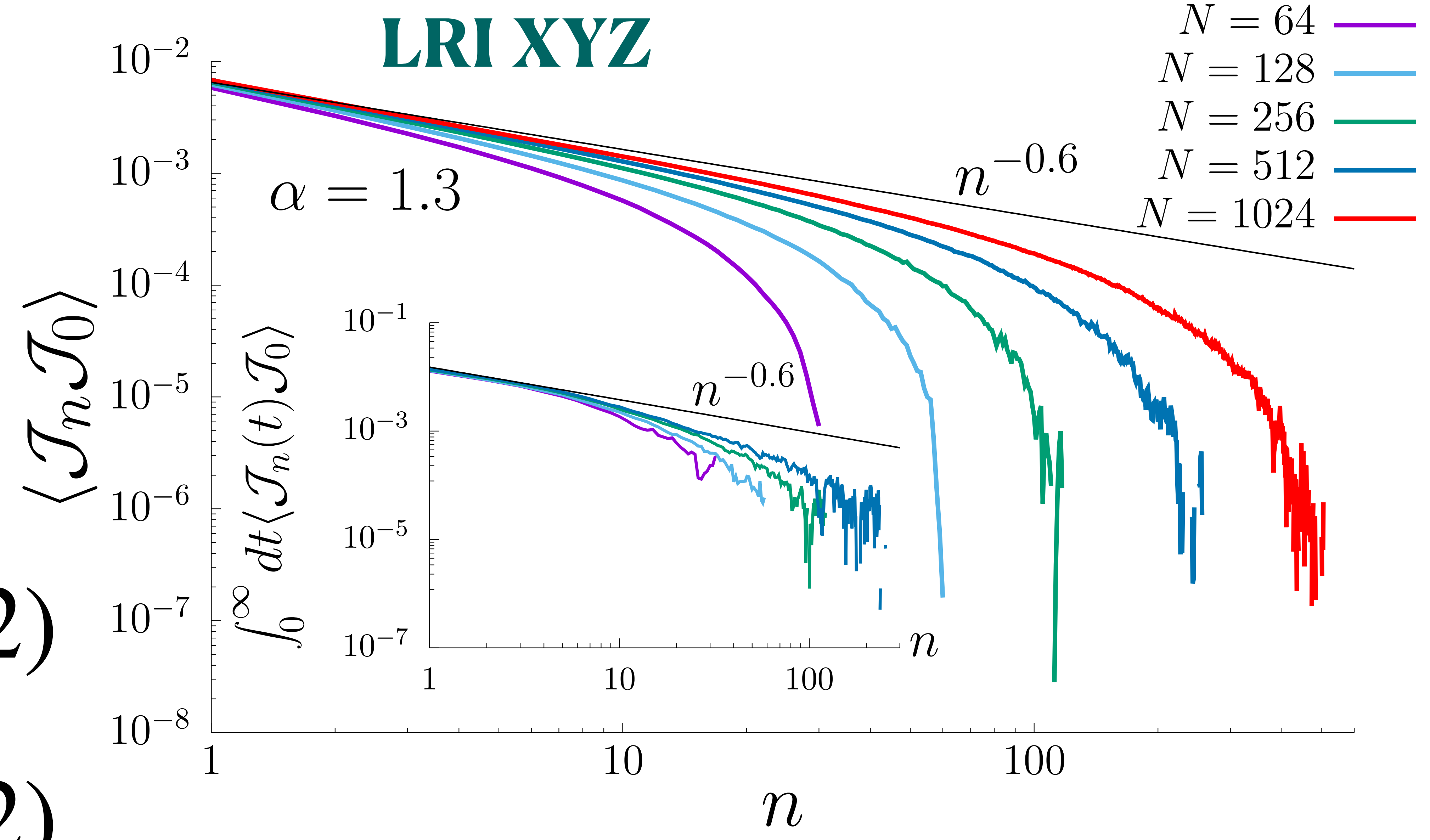
Fourier tr. $\langle \delta\tilde{\epsilon}(k,t) \delta\epsilon_0 \rangle_{\mathrm{eq}} \sim k^2 k^{2\alpha-3} \langle \delta\tilde{\epsilon}(k,t) \delta\epsilon_0 \rangle_{\mathrm{eq}}$

Levy Diffusion

$$\langle \delta\epsilon_n(t) \delta\epsilon_0(0) \rangle_{\mathrm{eq}} \sim \begin{cases} t^{-1/(2\alpha-1)} f\left(\frac{n}{t^{1/(2\alpha-1)}}\right) & (\alpha < 3/2) \\ t^{-1/2} f\left(\frac{n}{t^{1/2}}\right) & (\alpha \geq 3/2) \end{cases}$$

Normal Diffusion





Energy Diffusion in the 1-Dim LRI Spin Systems

✓Space-time energy correlation : $C(n,t) := \langle \delta\epsilon_n(t)\delta\epsilon_0 \rangle_{\mathrm{eq}}$

Outline

1. **Overview of Long-range Interacting (LRI) Systems**
2. **Energy Diffusion in the Long-range Interacting (LRI) Spin Systems**
    - Models & Dynamics : Transverse Ising・XYZ
    - Local Energy Current in Long-range Interacting Systems
    - Divergence of Thermal Conductivity (Green-Kubo formula)
    - Cumulant Power-law Clustering Theorem in the LRI Systems
    - Fluctuating Hydrodynamics for Anomalous Diffusion
    - **The case of $D\ (\geq 2)$ dimensions**
3. **Conclusion**

Green-Kubo Formula for D-dim LRI Systems

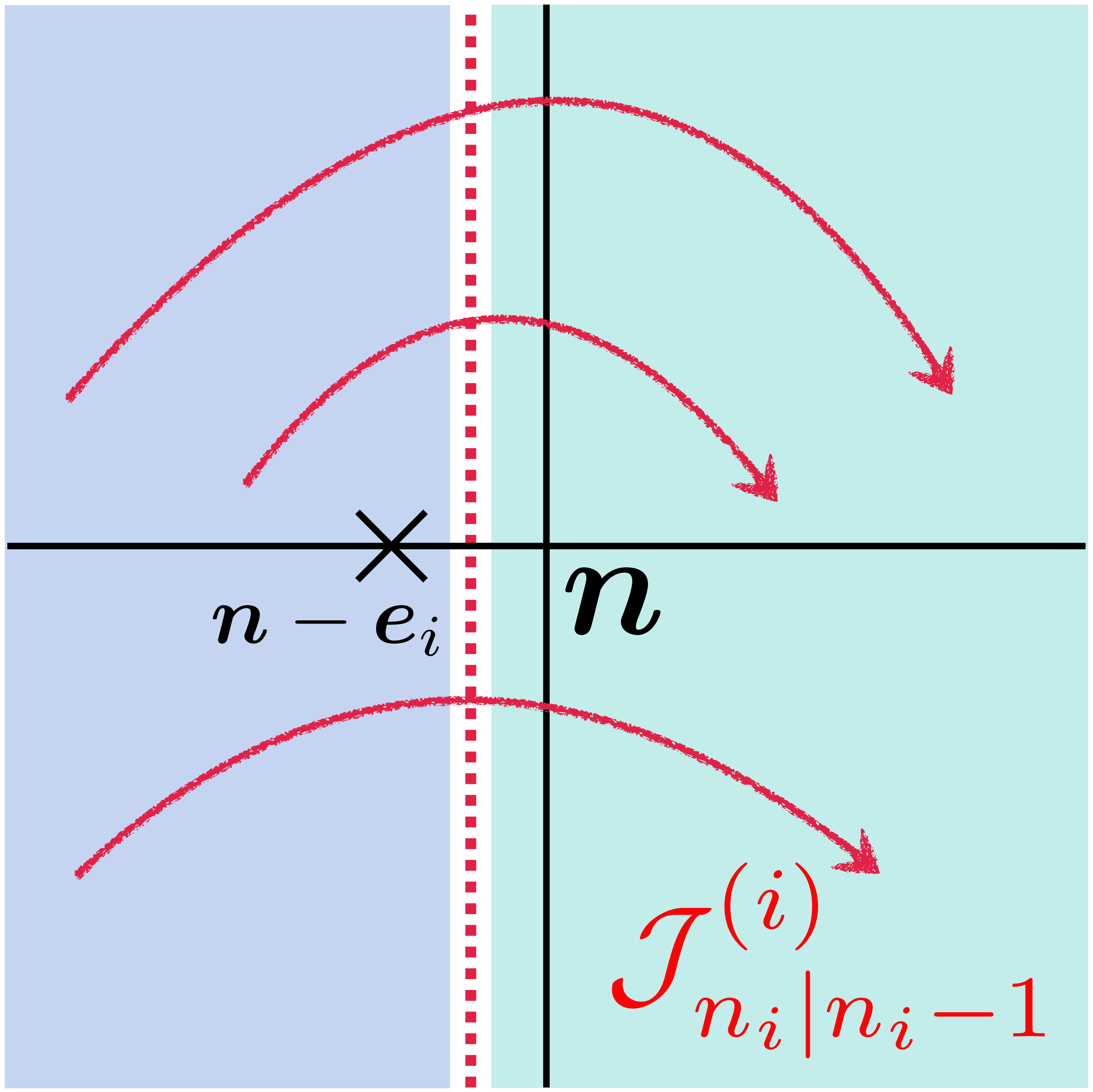
HN & K.Saito, arXiv:2502.10139

✓Green-Kubo formula : $\mathscr{D} = \dfrac{1}{c_{\mathrm{V}} k_{\mathrm{B}} T^2} \displaystyle\int_0^\infty dt C_N^{(D)}(t)$

$$C_N^{(D)}(t) = N^{1-D} \sum_{i=1}^{D} \sum_{n_i=1}^{N} \langle \mathscr{J}^{(i)}_{n_i|n_i-1}(t) \mathscr{J}^{(i)}_{0|-1} \rangle$$

$\mathscr{D}$: Diffusion constant

$$\mathscr{J}^{(i)}_{n_i|n_i-1} = \sum_{\substack{k:k_i \geq n_i \\ \ell:l_i < n_i}} t_{k \leftarrow \ell}$$



Thm.3 : Upper Bound on Current Correlation of D-dim LRI Spin Systems

HN & K.Saito, arXiv:2502.10139

For $D\,(\geq 2)$-dim prototypical 2-local LRI spin systems (Trans. Ising & XYZ) , assuming $\alpha > D$, equal-time total energy current correlation is finite : $C_N^{(D)}(0) < \infty$

Assuming that $C_N^{(D)}(t)$ is integrable, for $\alpha > D$, **Energy Diffusion is always Normal !!**

Conclusion

Energy Diffusion in the Arbitrary Dimensional LRI Spin Systems

HN & K.Saito, arXiv:2502.10139

- Energy diffusion in the prototypical 2-local classical LRI spin systems (Trans. Ising, XYZ model)
- **Mechanism : Anomalous enhancement of the equal-time current correlation**
- **Cumulant Power-law Clustering Theorem for LRI systems**
- 1-Dim : **Fluctuating Hydrodynamics**

1. $\alpha \geq \alpha_c = 3/2$: **Normal Diffusion**
2. $\alpha < \alpha_c = 3/2$: **Levy Diffusion** with exponent $2\alpha - 1$

- $D\,(\geq 2)$-Dim : For $\alpha > D$, **Normal Diffusion**

1-Dim	$\alpha<1$	$1<\alpha<3/2$	$3/2<\alpha$
	—	**Levy**	**Normal**
D-Dim	$\alpha<D$	$D<\alpha$	
$(D\geq 2)$	—	**Normal**	

Backup Slides

Quantum case

✓Thm.1 holds even for quantum case (for bounded systems).

✓Quantum Trans. Ising / XYZ model (& Non-integrable Fermionic systems)

：Thm. 2,3 also holds. We expect the same behavior (For 1-Dim, $\alpha_c = 3/2$).

However, the numerical confirmation of the rapid decay in time is demanding.

✓Non integrable Bosonic systems

：Unbound systems→Thm.1 cannot be directly applicable. Open question.

Proof : Cumulant Power-law Clustering Theorem for LRI Systems

✓ k-local LRI Hamiltonian :

$$H = \sum_{Z:|Z|\le k} h_Z \quad Z: \text{a subset of interacting sites} \quad J_{i,j} := \sum_{Z:Z\ni\{i,j\}} \|h_Z\| \le \frac{g}{(d_{i,j}+1)^\alpha} \quad : \text{LRI}$$

✓ Multi-phase spaces technique :

$\tilde{\rho} = \bigotimes_{i=1}^{N} \rho^{(i)}$: p.d.f in the multi-phase spaces

$O_X^{(i)}$: local physical quantity supported by the region X in the i-th space

$$O_X^{(i,j)} := O_X^{(i)} - O_X^{(j)}, \quad (i < j)$$

$$\tilde{O}^{(1,2,\ldots,n)}_{X_1,X_2,\ldots,X_n} := O_{X_1}^{(1)} \prod_{j=2}^{n} \left(\sum_{k=1}^{j-1} O_{X_j}^{(k,j)} \right) \quad : n\text{-th cumulant operator}$$

$$\langle O_{X_1} \ldots O_{X_n} \rangle_c = \mathrm{tr}\left(\tilde{\rho}\, \tilde{O}^{(1,\ldots,n)}_{X_1,\ldots,X_n} \right) \quad : n\text{-th cumulant}$$

Proof : Cumulant Power-law Clustering Theorem for LRI Systems

Lemma 1

If $Z_1, \dots, Z_m$ are NOT connected to $X_1, \dots, X_n$, $\mathrm{tr}\left(\tilde{O}_{Z_1} \dots \tilde{O}_{Z_m} \tilde{O}^{(1,\dots,n)}_{X_1,\dots,X_n}\right) = 0$

$\tilde{\rho} = \dfrac{e^{-\beta \tilde{H}}}{\mathcal{Z}^n}$: equilibrium p.d.f. in the multi-phase spaces $\quad \tilde{H} = \bigotimes_{i=1}^{n} H^{(i)}$

→We can take the cluster expansion using only connecting components

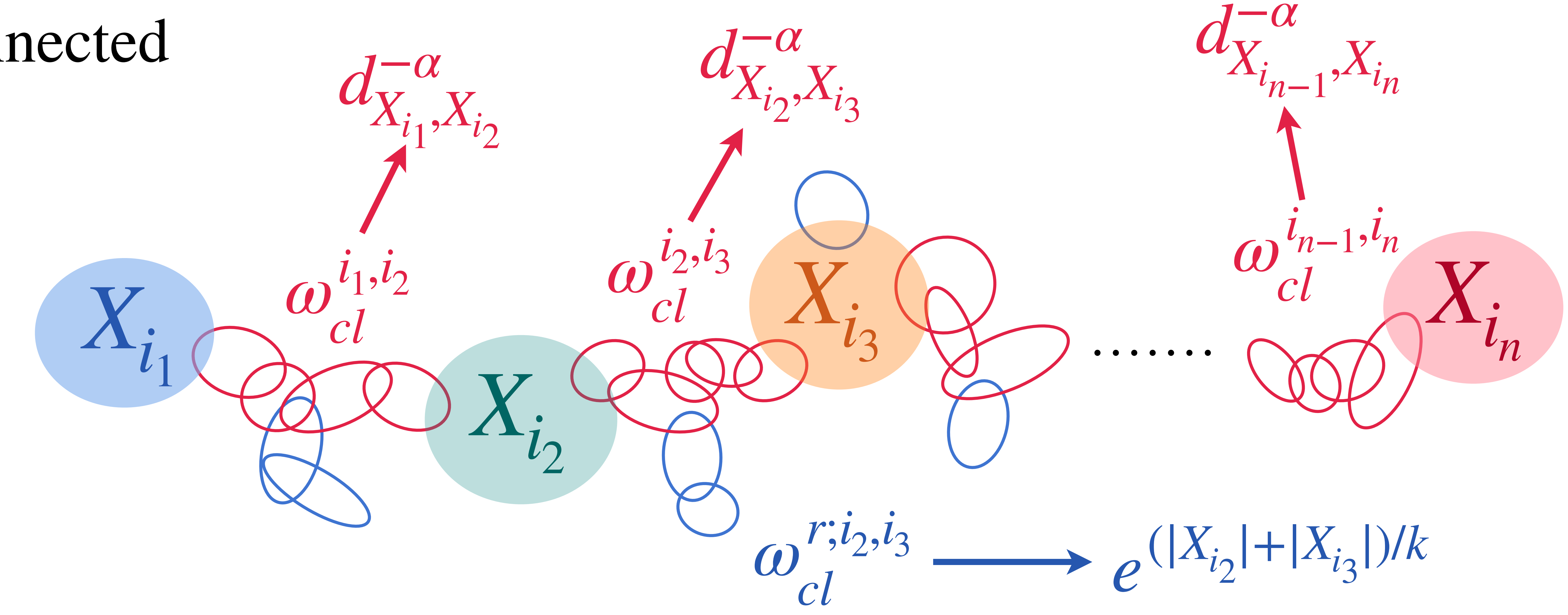
: By using Lemma 1, all disconnected paths vanish.

$$\langle O_{X_1} \dots O_{X_n} \rangle_c = \mathrm{tr}\left(\tilde{\rho} \tilde{O}^{(1,\dots,n)}_{X_1,\dots,X_n}\right) = \frac{1}{\mathcal{Z}^n} \mathrm{tr}\left(e^{-\beta \tilde{H}} \tilde{O}^{(1,\dots,n)}_{X_1,\dots,X_n}\right)$$

$$= \frac{1}{\mathcal{Z}^n} \sum_{m=0}^{\infty} \sum_{\substack{Z_1, \dots Z_m \\ \text{:connected}}} \frac{(-\beta)^m}{m!} \mathrm{tr}\left(\tilde{h}_{Z_1} \dots \tilde{h}_{Z_m} \tilde{O}^{(1,\dots,n)}_{X_1,\dots,X_n}\right)$$

Proof : Cumulant Power-law Clustering Theorem for LRI Systems

$$\langle O_{X_1}\dots O_{X_n}\rangle_c = \frac{1}{\mathcal{Z}^n}\sum_{m=0}^{\infty}\sum_{\substack{Z_1,\dots Z_m\\ \text{:connected}}}\frac{(-\beta)^m}{m!}\mathrm{tr}\left(\tilde{h}_{Z_1}\dots\tilde{h}_{Z_m}\tilde{O}^{(1,\dots,n)}_{X_1,\dots,X_n}\right)$$



$$\leq \text{const}\,.\sum_{\substack{i_1,\dots,i_n\\ \text{:connected}}}\prod_{p=1}^{n-1}\|\mathcal{O}_{X_p}\|\frac{|X_{i_p}||X_{i_{p+1}}|e^{(|X_{i_p}|+|X_{i_{p+1}}|)/k}}{(d_{X_{i_p},X_{i_{p+1}}})^{\alpha}}$$

Proof : Cumulant Power-law Clustering Theorem for LRI Systems

✓ Representation of n-th cumulant on multi-phase space

$$\langle O_{X_1} \ldots O_{X_n} \rangle_c = \mathrm{tr}\left(\tilde{\rho} \tilde{O}^{(1,\ldots,n)}_{X_1,\ldots,X_n} \right) \qquad \tilde{O}^{(1,\ldots,n)}_{X_1,\ldots,X_n} := O^{(1)}_{X_1} \prod_{j=2}^{n} \left(\sum_{k=1}^{j-1} O^{(k,j)}_{X_j} \right)$$

✓ Decomposition of n-th moment : $\langle O_{X_1} \ldots O_{X_n} \rangle = \langle O_{X_1} \ldots O_{X_n} \rangle_c + D_n$

D_n : All the summation of n-th representations decomposed into products of cumulant up to $(n-1)$-th order, involving $O_{X_1}, \ldots, O_{X_n}$

$$D_2 = \langle O_{X_1} \rangle_c \langle O_{X_2} \rangle_c$$

$$D_3 = \langle O_{X_1} O_{X_2} \rangle_c \langle O_{X_3} \rangle_c + \langle O_{X_2} O_{X_3} \rangle_c \langle O_{X_1} \rangle_c + \langle O_{X_3} O_{X_1} \rangle_c \langle O_{X_2} \rangle_c + \langle O_{X_1} \rangle_c \langle O_{X_2} \rangle_c \langle O_{X_3} \rangle_c$$

We show the following relation in the multi-phase space. $\langle O_{X_1} \ldots O_{X_n} \rangle = \mathrm{tr}\left(\tilde{\rho} \tilde{O}^{u}_{X_1,\ldots,X_n} \right)$

$$\tilde{O}^{(1,\ldots,n)}_{X_1,\ldots,X_n} = \tilde{O}^{u}_{X_1,\ldots,X_n} - \mathscr{D}_n \qquad \tilde{O}^{u}_{X_1,\ldots,X_n} := O^{(1)}_{X_1} O^{(1)}_{X_2} \prod_{j=3}^{n} \left(O^{(1)}_{X_j} + \sum_{k=2}^{j-1} O^{(k,j)}_{X_j} \right)$$

Proof : Cumulant Power-law Clustering Theorem for LRI Systems

✓ Goal of proof : $\tilde{O}^{(1,\ldots,n)}_{X_1,\ldots,X_n} = \tilde{O}^{u}_{X_1,\ldots,X_n} - \mathscr{D}_n$

✓ Induction : $n = 3$ case

$$\mathscr{D}_3 = \mathscr{D}_2(O^{(1,3)}_{X_3} + O^{(2,3)}_{X_3}) + (\tilde{O}^{(1,2)}_{X_1,X_2} + \mathscr{D}_2)O^{(3)}_{X_3}$$

O_{X_3} contributes to higher-order (≥ 2) cumulants

O_{X_3} contributes to 1st-order cumulants

$$\tilde{O}^{(1,2,3)}_{X_1,X_2,X_3} = O^{(1)}_{X_1}O^{(1,2)}_{X_2}(O^{(1,2)}_{X_3} + O^{(1,3)}_{X_3})$$

$$\tilde{O}^{u}_{X_1,X_2,X_3} = O^{(1)}_{X_1}O^{(1)}_{X_2}(O^{(1)}_{X_3} + O^{(2,3)}_{X_3})$$

$$\tilde{O}^{(1,2)}_{X_1,X_2} = O^{(1)}_{X_1}O^{(1,2)}_{X_2}$$

$$\tilde{O}^{u}_{X_1,X_2} = O^{(1)}_{X_1}O^{(1)}_{X_2} \qquad \mathscr{D}_2 = O^{(1)}_{X_1}O^{(2)}_{X_2}$$

$$\langle O_{X_1}\rangle_c\langle O_{X_2}O_{X_3}\rangle_c + \langle O_{X_2}\rangle_c\langle O_{X_1}O_{X_3}\rangle_c$$

$$\langle O_{X_1}O_{X_2}\rangle_c\langle O_{X_3}\rangle_c + \langle O_{X_1}\rangle_c\langle O_{X_2}\rangle_c\langle O_{X_3}\rangle_c$$

$$O^{(1)}_{X_1}O^{(2)}_{X_2}O^{(2,3)}_{X_3} + O^{(1)}_{X_2}O^{(2)}_{X_1}O^{(2,3)}_{X_3}$$

$$= O^{(1)}_{X_1}O^{(2)}_{X_2}(O^{(1,3)}_{X_3} + O^{(2,3)}_{X_3})$$

$$O^{(1)}_{X_1}O^{(1,2)}_{X_2}O^{(3)}_{X_3} + O^{(1)}_{X_1}O^{(2)}_{X_2}O^{(3)}_{X_3}$$

$$= (O^{(1)}_{X_1}O^{(1,2)}_{X_2} + O^{(1)}_{X_1}O^{(2)}_{X_2})O^{(3)}_{X_3}$$

Proof : Cumulant Power-law Clustering Theorem for LRI Systems

✓ Goal of proof : $\tilde{O}^{(1,\ldots,n)}_{X_1,\ldots,X_n} = \tilde{O}^{u}_{X_1,\ldots,X_n} - \mathscr{D}_n$

✓ Induction : $n = 3$ case

$$\mathscr{D}_3 = \mathscr{D}_2(O^{(1,3)}_{X_3} + O^{(2,3)}_{X_3}) + (\tilde{O}^{(1,2)}_{X_1,X_2} + \mathscr{D}_2)O^{(3)}_{X_3}$$

O_{X_3} contributes to higher-order (≥ 2) cumulants

O_{X_3} contributes to 1st-order cumulants

$$\tilde{O}^{(1,2,3)}_{X_1,X_2,X_3} = O^{(1)}_{X_1}O^{(1,2)}_{X_2}(O^{(1,2)}_{X_3} + O^{(1,3)}_{X_3})$$

$$\tilde{O}^{u}_{X_1,X_2,X_3} = O^{(1)}_{X_1}O^{(1)}_{X_2}(O^{(1)}_{X_3} + O^{(2,3)}_{X_3})$$

$$\tilde{O}^{(1,2)}_{X_1,X_2} = O^{(1)}_{X_1}O^{(1,2)}_{X_2}$$

$$\tilde{O}^{u}_{X_1,X_2} = O^{(1)}_{X_1}O^{(1)}_{X_2} \qquad \mathscr{D}_2 = O^{(1)}_{X_1}O^{(2)}_{X_2}$$

$$\begin{aligned}
\tilde{O}^{u}_{X_1,X_2,X_3} - \mathscr{D}_3 &= \tilde{O}^{u}_{X_1,X_2,X_3} - \mathscr{D}_2(O^{(1,3)}_{X_3} + O^{(2,3)}_{X_3}) - (\tilde{O}^{(1,2)}_{X_1,X_2} + \mathscr{D}_2)O^{(3)}_{X_3} \\
&= \tilde{O}^{u}_{X_1,X_2}(O^{(1)}_{X_3} + O^{(2,3)}_{X_3}) - \mathscr{D}_2(O^{(1,3)}_{X_3} + O^{(2,3)}_{X_3}) - \tilde{O}^{u}_{X_1,X_2}O^{(3)}_{X_3} \\
&= (\tilde{O}^{u}_{X_1,X_2} - \mathscr{D}_2)(O^{(1)}_{X_3} + O^{(2,3)}_{X_3}) \\
&= \tilde{O}^{(1,2)}_{X_1,X_2}(O^{(1)}_{X_3} + O^{(2,3)}_{X_3}) = \tilde{O}^{(1,2,3)}_{X_1,X_2,X_3}
\end{aligned}$$

Proof : Cumulant Power-law Clustering Theorem for LRI Systems

✓ Goal of proof : $\tilde{O}^{(1,\ldots,n)}_{X_1,\ldots,X_n} = \tilde{O}^{u}_{X_1,\ldots,X_n} - \mathscr{D}_n$

$$\tilde{O}^{(1,\ldots,n)}_{X_1,\ldots,X_n} := O^{(1)}_{X_1}\prod_{j=2}^{n}\left(\sum_{k=1}^{j-1} O^{(k,j)}_{X_j}\right)$$

✓ Induction : Assume n-case→proof $n+1$-case

$$\tilde{O}^{u}_{X_1,\ldots,X_n} := O^{(1)}_{X_1}O^{(1)}_{X_2}\prod_{j=3}^{n}\left(O^{(1)}_{X_j} + \sum_{k=2}^{j-1} O^{(k,j)}_{X_j}\right)$$

$$\tilde{O}^{u}_{X_1,\ldots,X_{n+1}} - \mathscr{D}_{n+1}$$

$$= \tilde{O}^{u}_{X_1,\ldots,X_{n+1}} - \mathscr{D}_n\sum_{l=1}^{n} O^{(l,n+1)}_{X_{n+1}} - (\tilde{O}^{(1,\ldots,n)}_{X_1,\ldots,X_n} + \mathscr{D}_n)O^{(n+1)}_{X_{n+1}}$$

$$= \tilde{O}^{u}_{X_1,\ldots,X_{n+1}} - \mathscr{D}_n\sum_{l=1}^{n} O^{(l,n+1)}_{X_{n+1}} - \tilde{O}^{u}_{X_1,\ldots,X_n}O^{(n+1)}_{X_{n+1}}$$

$$= (\tilde{O}^{u}_{X_1,\ldots,X_{n+1}} - \mathscr{D}_n)\sum_{l=1}^{n} O^{(l,n+1)}_{X_{n+1}}$$

$$= \tilde{O}^{(1,\ldots,n)}_{X_1,\ldots,X_n}\sum_{l=1}^{n} O^{(l,n+1)}_{X_{n+1}} = \tilde{O}^{(1,\ldots,n+1)}_{X_1,\ldots,X_{n+1}}$$

Lemma 2

$$\mathscr{D}_{n+1} = \mathscr{D}_n\sum_{l=1}^{n} O^{(l,n+1)}_{X_{n+1}} + (\tilde{O}^{(1,\ldots,n)}_{X_1,\ldots,X_n} + \mathscr{D}_n)O^{(n+1)}_{X_{n+1}}$$

Proof : Cumulant Power-law Clustering Theorem for LRI Systems

Lemma 2

$$\mathcal{D}_{n+1} = \mathcal{D}_n \sum_{l=1}^{n} O_{X_{n+1}}^{(l,n+1)} + (\tilde{O}_{X_1,\ldots,X_n}^{(1,\ldots,n)} + \mathcal{D}_n) O_{X_{n+1}}^{(n+1)}$$

$$\tilde{O}_{X_1,\ldots,X_n}^{(1,\ldots,n)} := O_{X_1}^{(1)} \prod_{j=2}^{n} \left(\sum_{k=1}^{j} O_{X_j}^{(k,j)} \right)$$

$O_{X_{n+1}}$ contributes to higher-order cumulants

$O_{X_{n+1}}$ contributes to 1st-order cumulants

Decomposition of $\mathcal{D}_n$ into $m\ (\leq n)$ products of cumulants

$$\tilde{O}_{X_{i(1)},\ldots,X_{i(l_1)}}^{(i(1),\ldots i(l_1))} \tilde{O}_{X_{i(l_1+1)},\ldots,X_{i(l_2)}}^{(i(l_1+1),\ldots,i(l_2))} \ldots \tilde{O}_{X_{i(l_{m-1}+1)},\ldots,X_{i(l_m)}}^{(i(l_{m-1}+1),\ldots,i(l_m))}$$

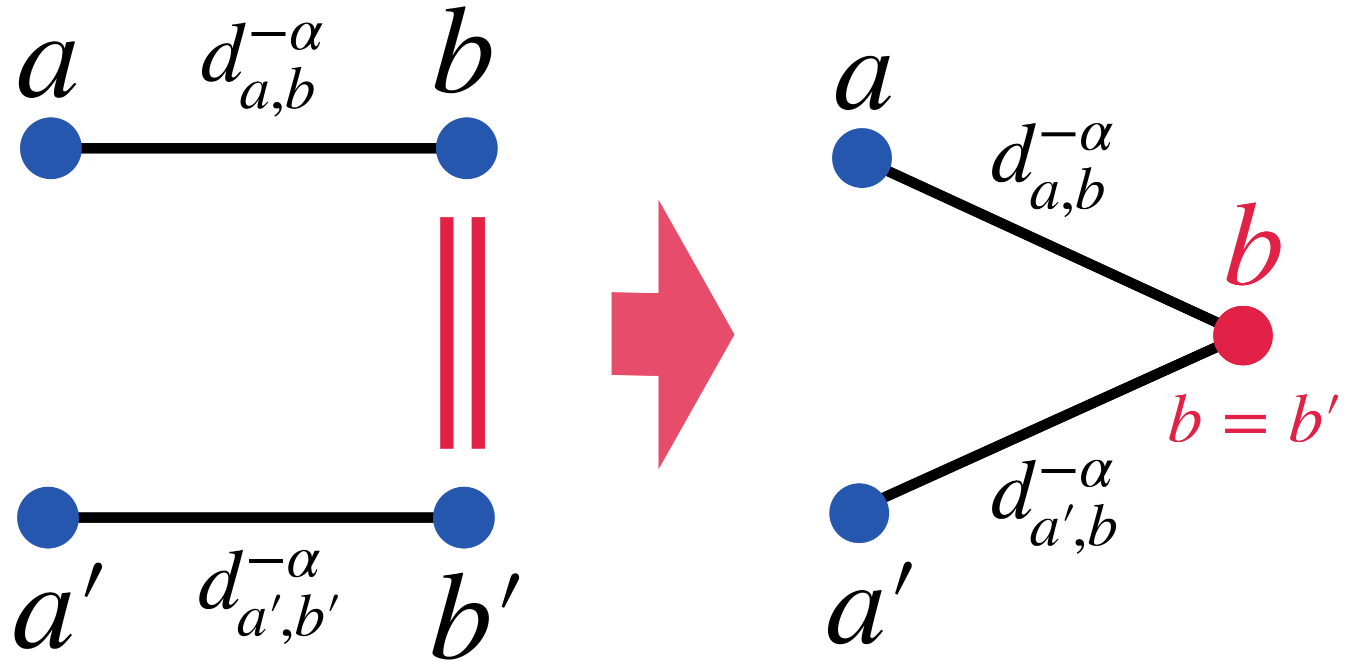
$$\tilde{O}_{X_{i(l_{p-1}+1)},\ldots,X_{i(l_p)}}^{(i(l_{p-1}+1),\ldots,i(l_p))} \left(O_{X_{n+1}}^{(i(l_{p-1}+1),n+1)} + \ldots + O_{X_{n+1}}^{(i(l_p),n+1)} \right) = \tilde{O}_{X_{i(l_{p-1}+1)},\ldots,X_{i(l_p)},X_{n+1}}^{(i(l_{p-1}+1),\ldots,i(l_p),n+1)}$$

Upper Bound on Energy Current Correlation：Trans. Ising Model

$$\langle \mathcal{J}_n \mathcal{J}_0 \rangle_{\text{eq.}} = \sum_{\substack{k \geq n > l \\ k' \geq 0 > l'}} \sum_{\substack{\{a,b\} = \{k,l\} \\ \{a',b'\} = \{k',l'\}}} \epsilon^{ab}\epsilon^{a'b'} \frac{J^2h^2}{4} \frac{\langle S_a^x S_b^y S_{a'}^x S_{b'}^y \rangle_{\text{eq.}}}{d_{a,b}^{\alpha} d_{a',b'}^{\alpha}}$$

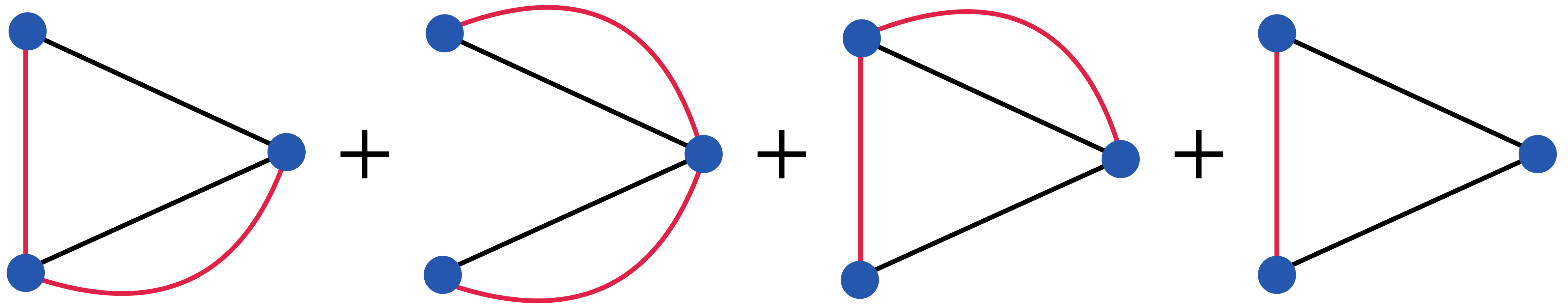
$$\langle S_i^y S_j^y \mathcal{O} \rangle_{\text{eq.}} = 0 \,, (i \neq j)$$

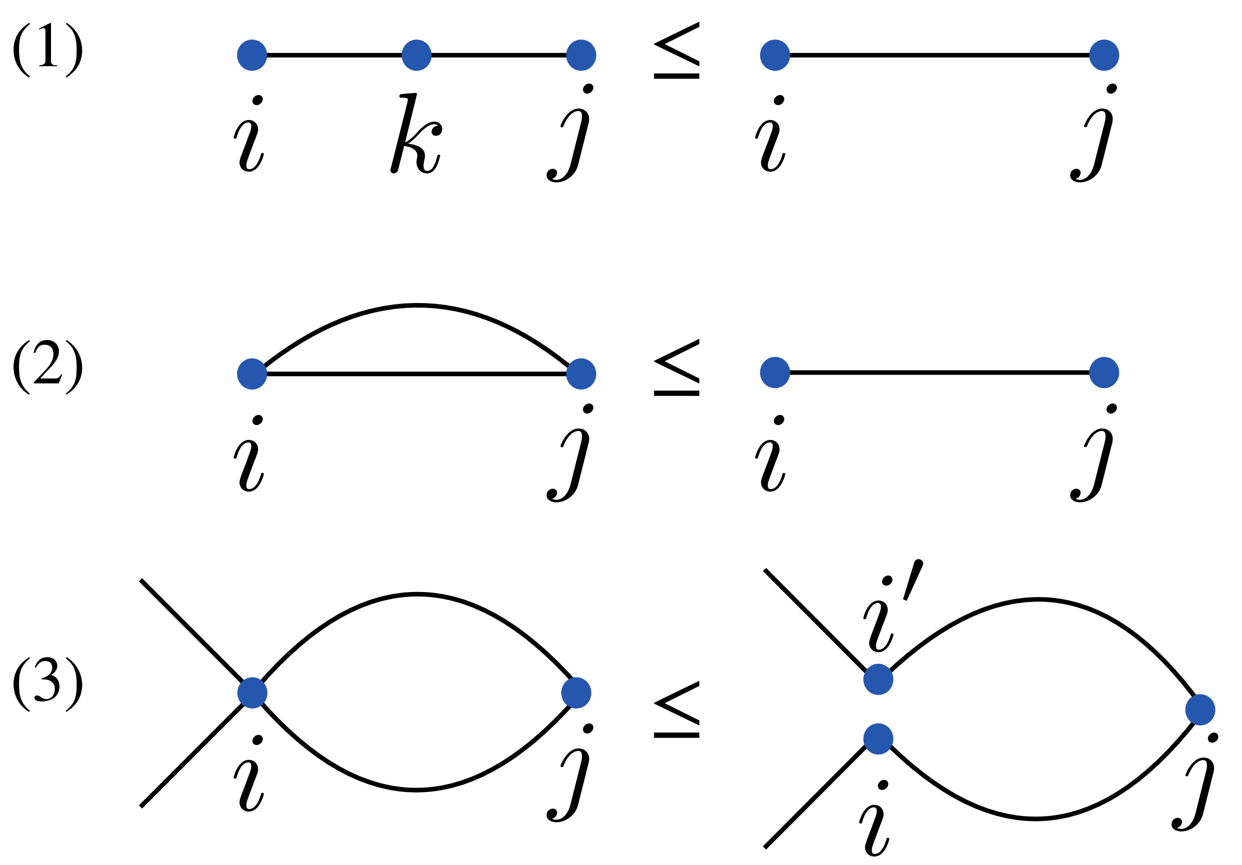
$$= \sum_{\substack{k \geq n > l \\ k' \geq 0 > l'}} \sum_{\substack{\{a,b\} = \{k,l\} \\ \{a',b'\} = \{k',l'\}}} \epsilon^{ab}\epsilon^{a'b'} \frac{J^2h^2}{4} \delta_{b,b'} \frac{\langle S_a^x S_{a'}^x (S_b^y)^2 \rangle_{\text{eq.}}}{d_{a,b}^{\alpha} d_{a',b}^{\alpha}}$$



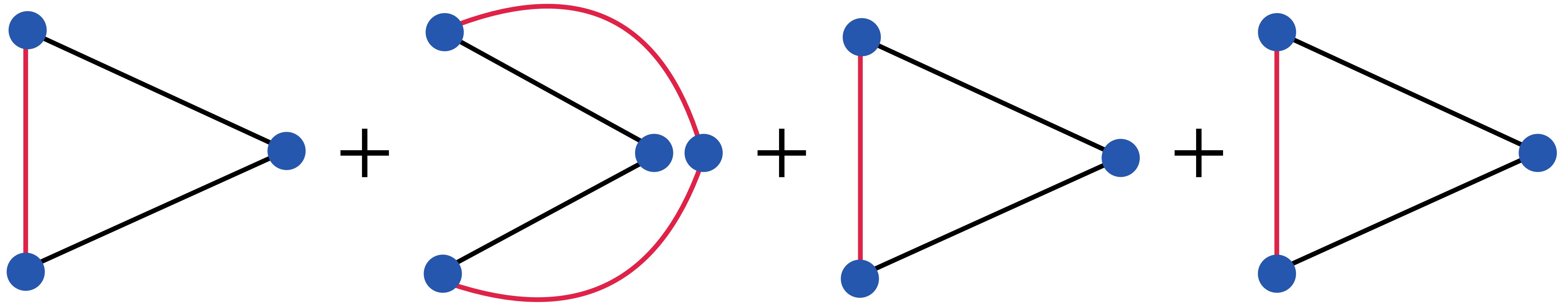
$$\langle S_a^x S_{a'}^x (S_b^y)^2 \rangle_{\text{eq.}} = \langle S_a^x S_{a'}^x (S_b^y)^2 \rangle_c + \langle S_a^x S_{a'}^x \rangle_c \langle (S_b^y)^2 \rangle_c$$

$\leq d_{a,a'}^{-\alpha} d_{a',b}^{-\alpha}$ $\leq d_{a,b}^{-\alpha} d_{b,a'}^{-\alpha}$ $\leq d_{a',a}^{-\alpha} d_{a,b}^{-\alpha}$ $\leq d_{a,a'}^{-\alpha}$

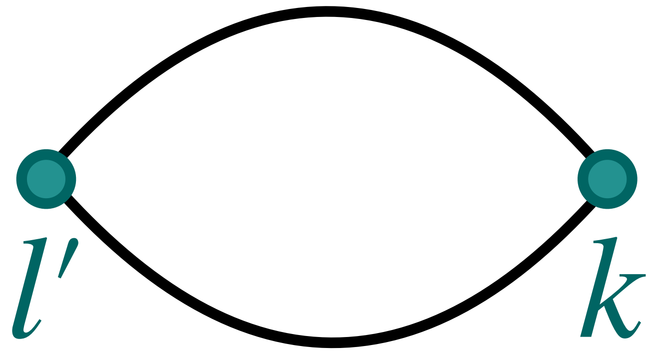




$\leq$



$\leq$

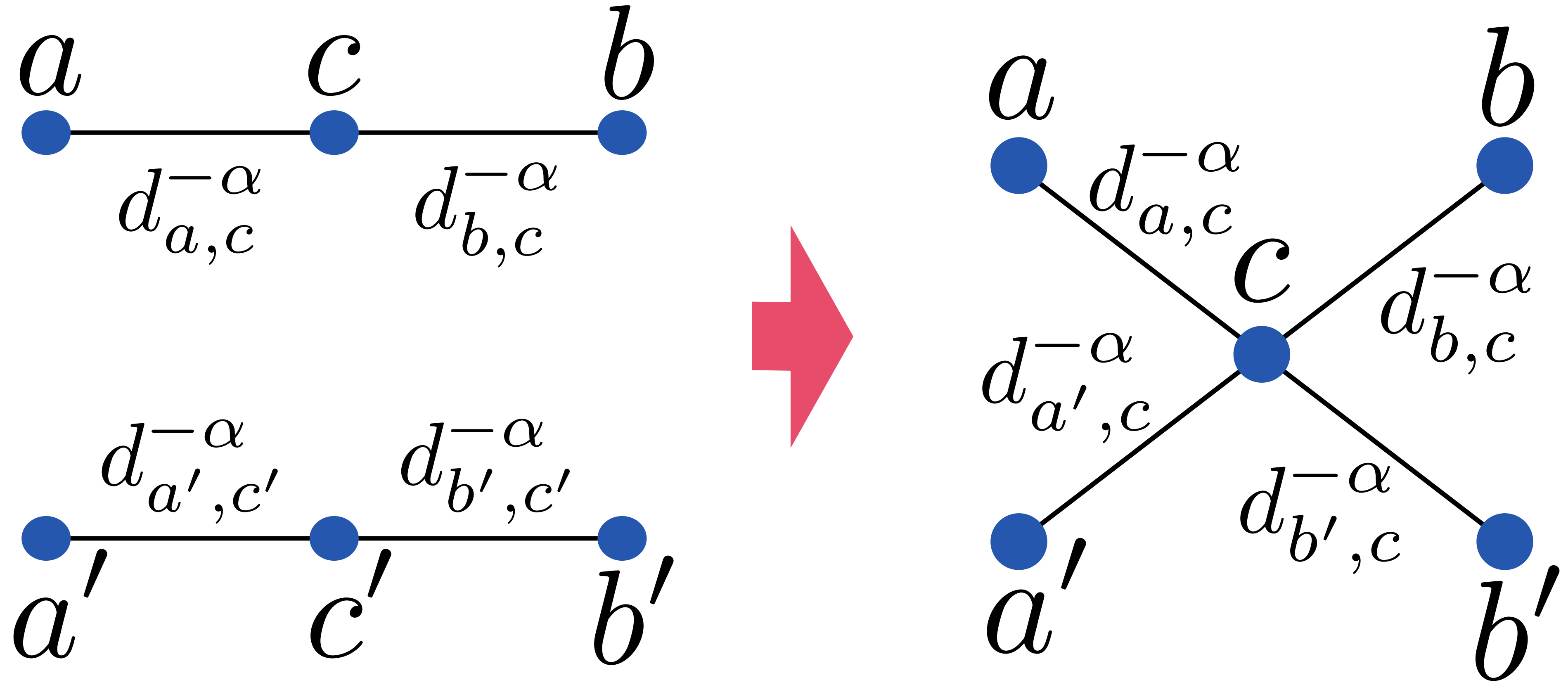


$$= \sum_{\substack{k \geq n \\ l' < 0}} d_{k,l'}^{-2\alpha} < \text{const.} \cdot n^{2-2\alpha}$$

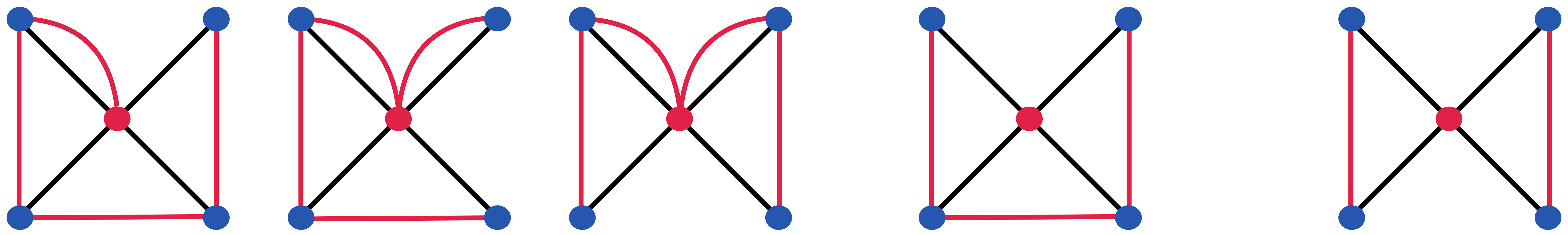
Upper Bound on Energy Current Correlation：XY model

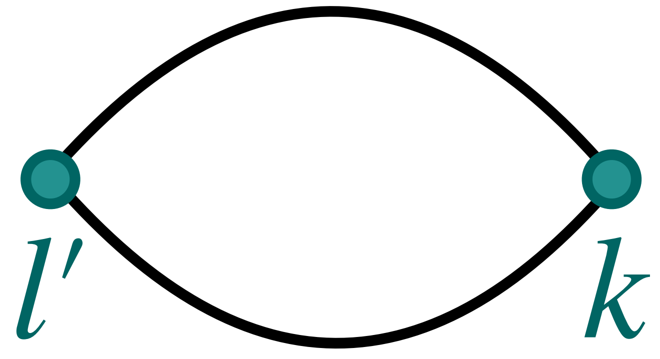
$$\langle \mathcal{J}_n \mathcal{J}_0 \rangle_{\text{eq.}} = \sum_{\substack{k \geq n > l \\ k' \geq 0 > l'}} \sum_{m,m'} \sum_{\substack{\{\sigma,\tau\}=\{x,y\} \\ \{\sigma',\tau'\}=\{x,y\}}} \sum_{\substack{\{a,b,c\}=\{k,l,m\} \\ \{a',b',c'\}=\{k',l',m'\}}} \epsilon^{\sigma\tau z}\epsilon^{\sigma'\tau' z}\epsilon^{abc}\epsilon^{a'b'c'} \frac{J_\sigma J_\tau J_{\sigma'} J_{\tau'}}{16} \frac{\langle S_a^\sigma S_b^\tau S_c^z S_{a'}^{\sigma'} S_{b'}^{\tau'} S_{c'}^z \rangle_{\text{eq.}}}{d_{a,c}^\alpha d_{b,c}^\alpha d_{a',c'}^\alpha d_{b',c'}^\alpha}$$

$$= \sum\sum\sum\sum \ldots \delta_{c,c'} \frac{\langle S_a^\sigma S_b^\tau S_{a'}^{\sigma'} S_{b'}^{\tau'} (S_c^v)^2 \rangle_{\text{eq.}}}{d_{a,c}^\alpha d_{b,c}^\alpha d_{a',c}^\alpha d_{b',c}^\alpha}$$



$$\langle S_a^\sigma S_b^\tau S_{a'}^{\sigma'} S_{b'}^{\tau'} (S_c^z)^2 \rangle_{\text{eq.}} = \langle SSSS(S^z)^2 \rangle_c + \sum \langle SSSS \rangle_c \langle (S^z)^2 \rangle_c + \sum \langle SS \rangle_c \langle SS \rangle_c \langle (S^z)^2 \rangle_c$$

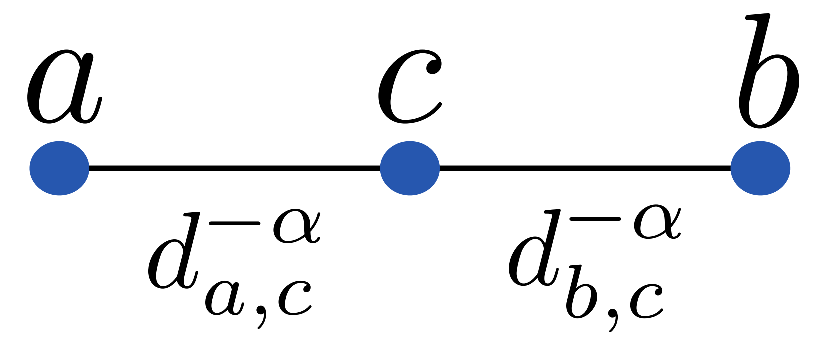




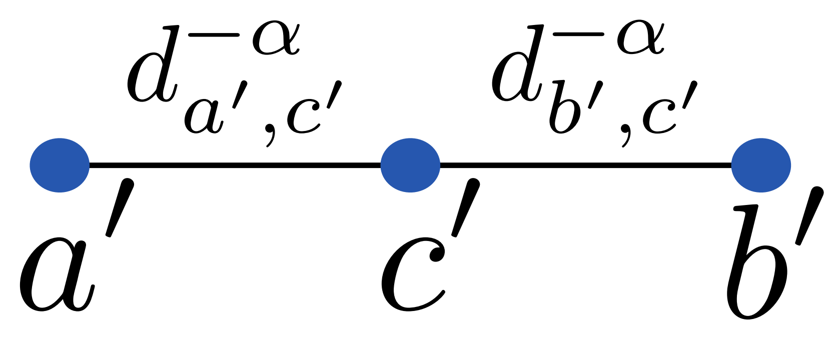
$$\sum_{\substack{k \geq n \\ l' < 0}} d_{k,l'}^{-\alpha} < \text{const.} \cdot n^{2-2\alpha}$$

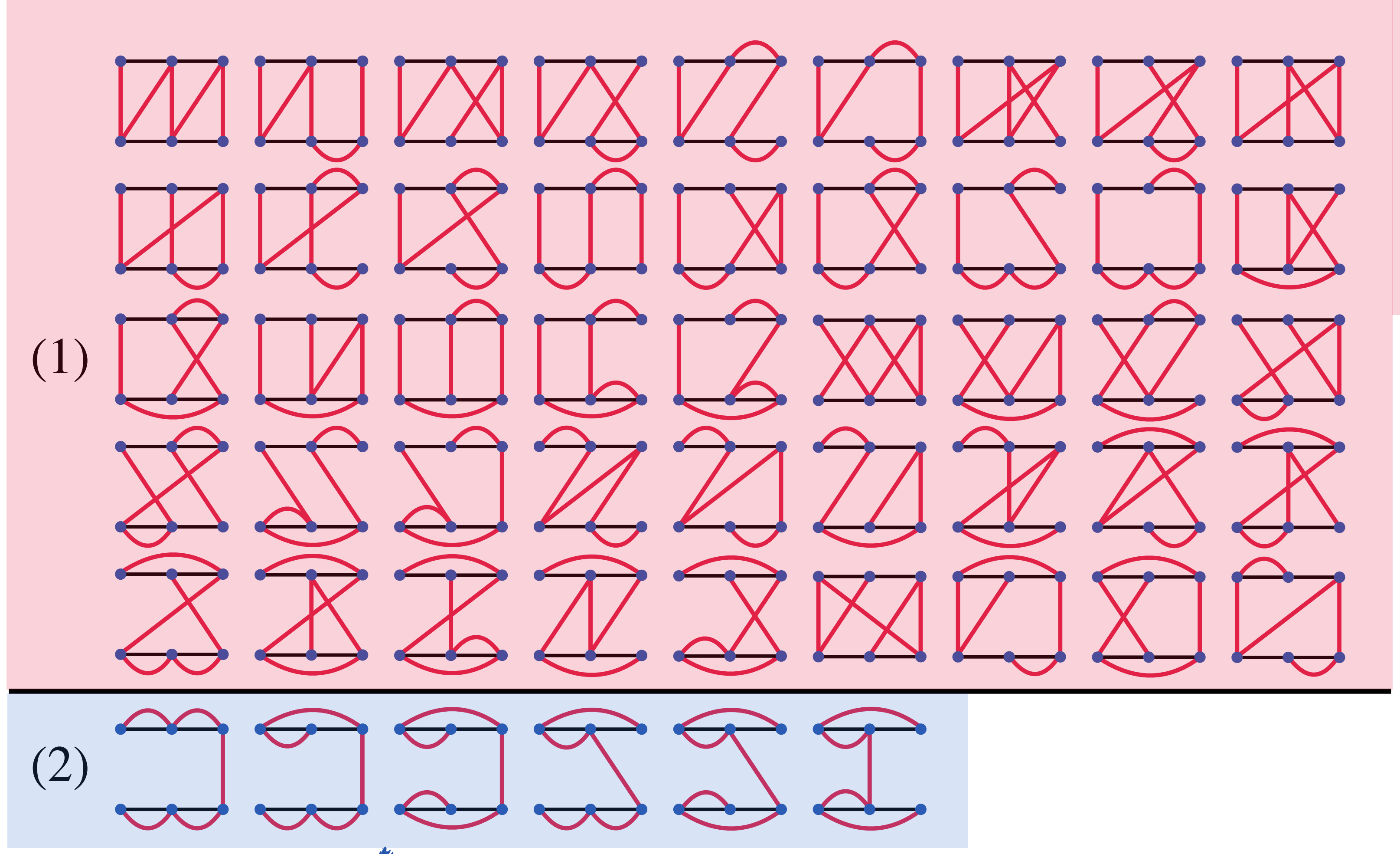
Upper Bound on Energy Current Correlation：XYZ model

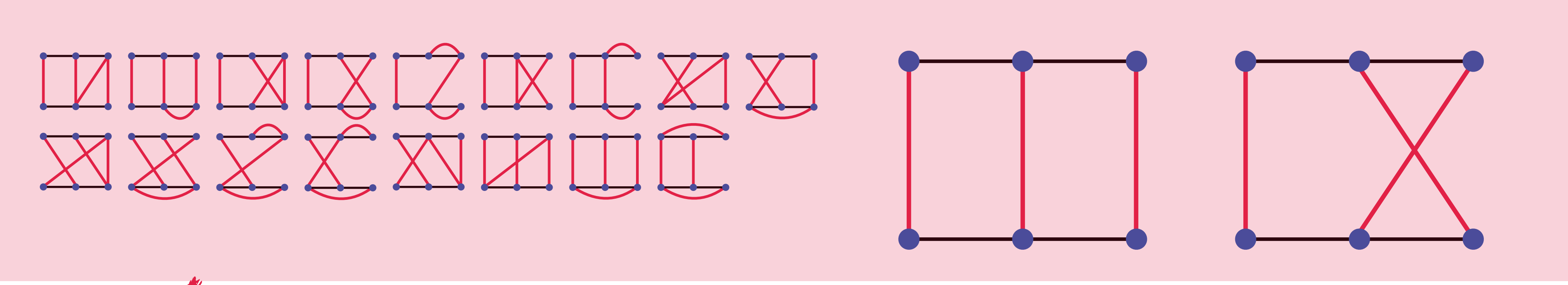
$$\langle \mathscr{J}_n \mathscr{J}_0 \rangle_{\text{eq.}} = \sum_{\substack{k \geq n > l \\ k' \geq 0 > l'}} \sum_{m,m'} \sum_{\substack{\{\sigma,\tau,\upsilon\}=\{x,y,z\} \\ \{\sigma',\tau',\upsilon'\}=\{x,y,z\}}} \sum_{\substack{\{a,b,c\}=\{k,l,m\} \\ \{a',b',c'\}=\{k',l',m'\}}} \epsilon^{\sigma\tau\upsilon}\epsilon^{\sigma'\tau'\upsilon'}\epsilon^{abc}\epsilon^{a'b'c'} \frac{J_\sigma J_\tau J_{\sigma'} J_{\tau'}}{16} \frac{\langle S_a^\sigma S_b^\tau S_c^\upsilon S_{a'}^{\sigma'} S_{b'}^{\tau'} S_{c'}^{\upsilon'} \rangle_{\text{eq.}}}{d_{a,c}^\alpha d_{b,c}^\alpha d_{a',c'}^\alpha d_{b',c'}^\alpha}$$

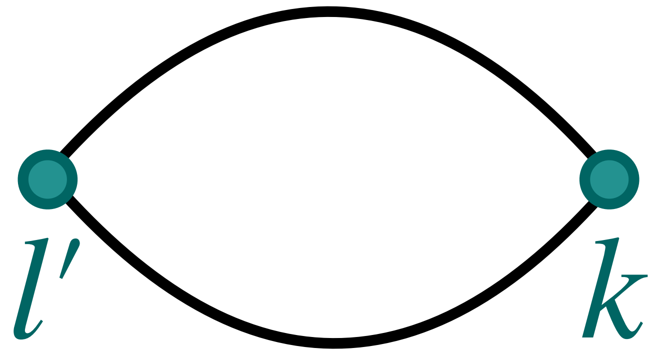


$$\langle S_a^\sigma S_b^\tau S_c^\upsilon S_{a'}^{\sigma'} S_{b'}^{\tau'} S_{c'}^{\upsilon'} \rangle_{\text{eq.}} = \langle SSSSSS \rangle_c + \sum \langle SS \rangle_c \langle SSSS \rangle_c + \sum \langle SS \rangle_c \langle SS \rangle_c \langle SS \rangle_c$$





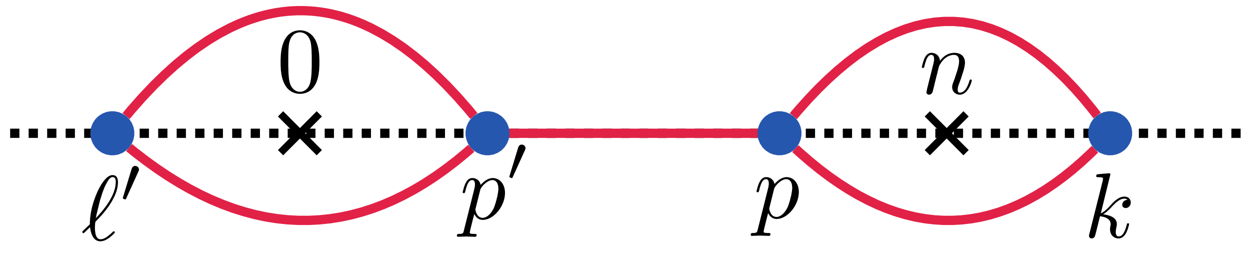




$$\sum_{\substack{k \geq n \\ l' < 0}} d_{k,l'}^{-\alpha} \leq \text{const.} \cdot n^{2-2\alpha}$$

DOMINANT !!

for $\alpha < 2$



$$\leq \text{const.} \cdot n^{-\alpha}$$

Derivation of Fluctuating Hydrodynamics in LRI Spin Systems

① Coarse-graining

HN & K.Saito, arXiv:2502.10139
K.Saito et al., Phys. Rev. Lett. 2021

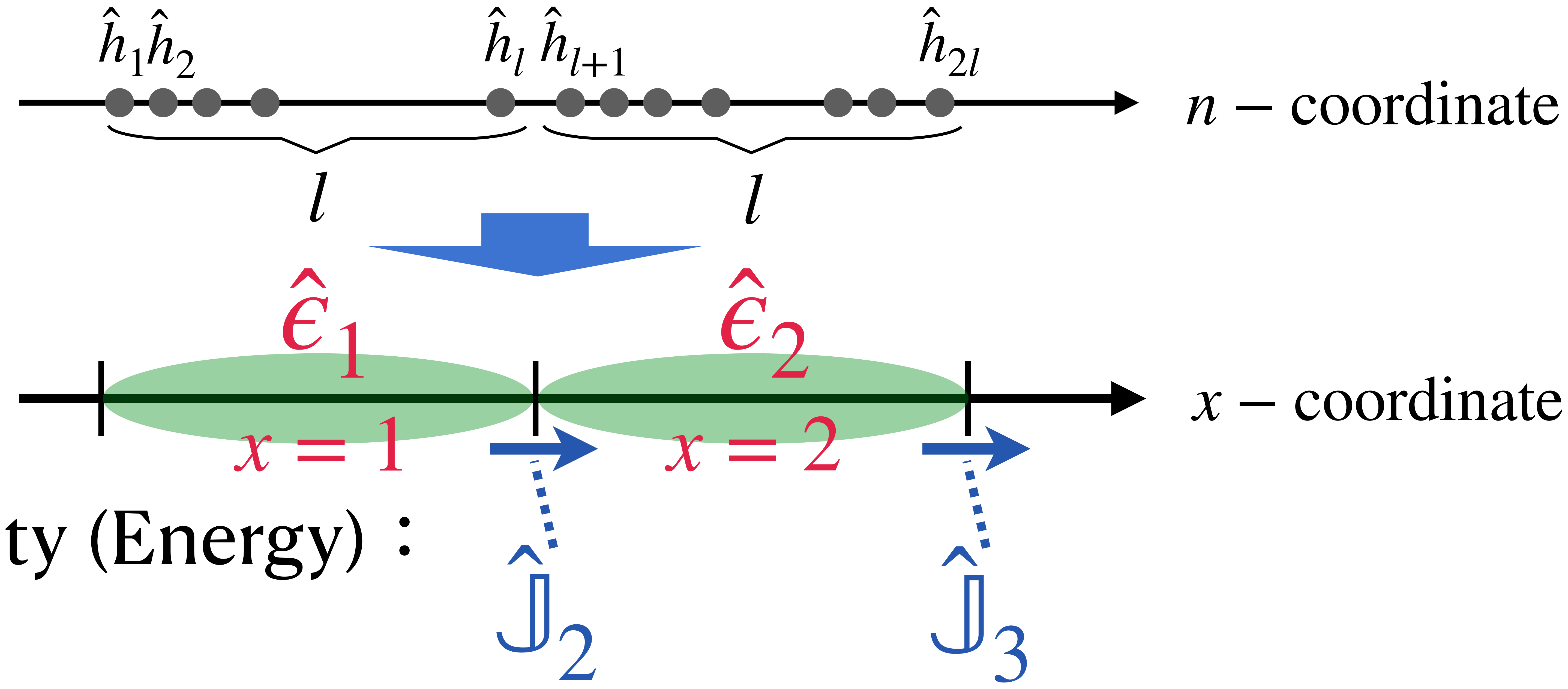
Microscopic Phase Space：

$$\Gamma := (S_1^x, S_1^y, S_1^z, \ldots, S_N^x, S_N^y, S_N^z)$$

Conserved Quantities (Micro)：

$\hat{h}_n$ only Local Energy

Continuity Eq.： $\partial_t \hat{h}_n = -\partial_n \hat{J}_n$



Coarse-grained Conserved Quantity (Energy)：

$$\hat{\epsilon}_x := \frac{1}{l} \sum_{n=(x-1)l+1}^{xl} \hat{h}_n$$

Continuity Eq. ： $\partial_t \hat{\epsilon}_x = -\partial_x \hat{\mathbb{J}}_x, \quad \hat{\mathbb{J}}_x = \hat{J}_{(x-1)l+1}$

Derivation of Fluctuating Hydrodynamics in LRI Spin Systems

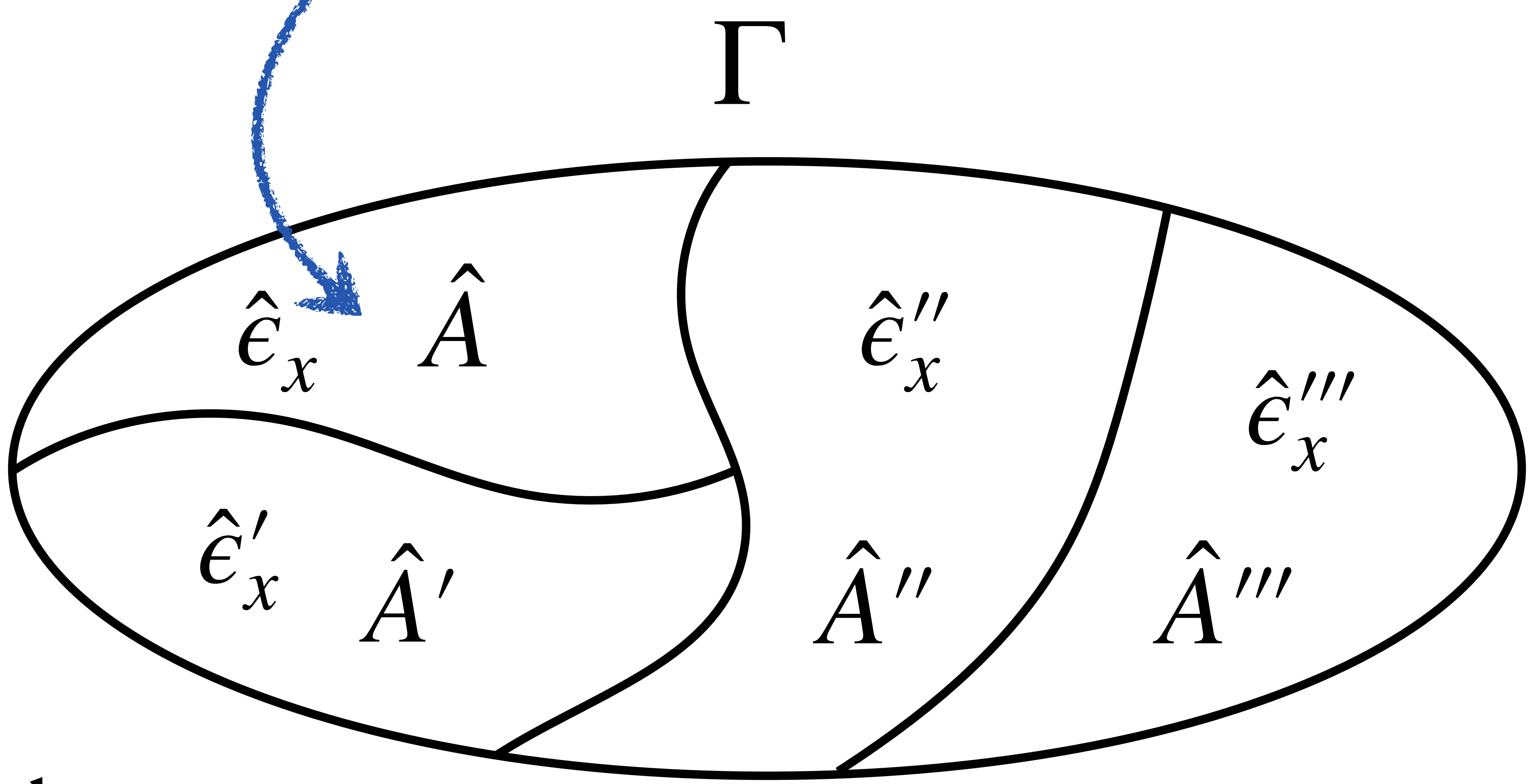
② Projection

Zwanzig, Phys. Rev. 1961

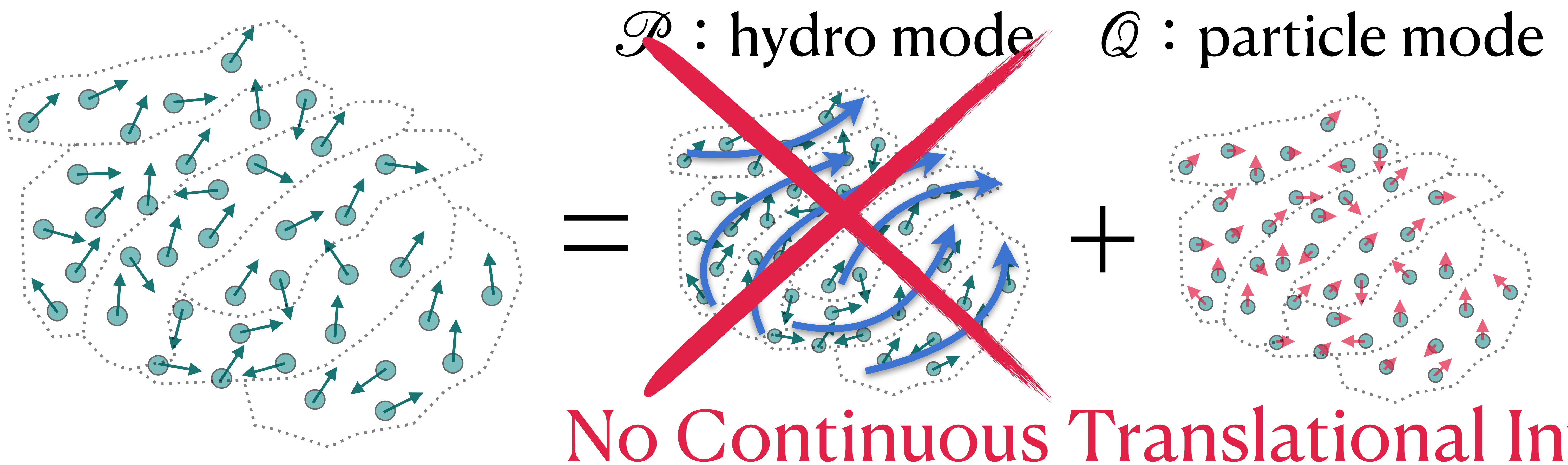
$$(\mathscr{P}\hat{A})[\Gamma] := \Omega^{-1}(\Gamma)\int d\Gamma' \hat{A}(\Gamma')\prod_x \delta(\hat{\epsilon}_x(\Gamma') - \hat{\epsilon}_x(\Gamma)),$$

$$\Omega(\Gamma) = \int d\Gamma' \prod_x \delta(\hat{\epsilon}_x(\Gamma') - \hat{\epsilon}_x(\Gamma))$$

if $\hat{\epsilon}_x(\Gamma) = \hat{\epsilon}_x(\Gamma')$,
then $(\mathscr{P}\hat{A})(\Gamma) = (\mathscr{P}\hat{A})(\Gamma')$



By projection operator $\mathscr{P}$, any function $\hat{A}$ is redefined in terms of coarse-grained quantities.



No Continuous Translational Invariance

→ **Hydro Mode $\mathscr{P}$ does NOT Exist !!**

Derivation of Fluctuating Hydrodynamics in LRI Spin Systems

③ Derivation of **Fokker-Planck eq.** for Coarse-grained p.d.f.

- p.d.f for Coarse-grained Conserved Quantities (Coarse-grained Energy)

$$f(\epsilon, t) := \int d\Gamma \hat{\rho}(\Gamma, t) \prod_x \delta(\hat{\epsilon}_x(\Gamma) - \epsilon_x)\,, \quad \partial_t \hat{\rho}(\Gamma, t) = \{H, \hat{\rho}(\Gamma, t)\}$$

- Markovian Approximation→Derivation of Fokker-Planck eq.

$$\partial_t f(\epsilon, t) = \sum_{x,x'} \frac{\delta}{\delta \epsilon_x} \Omega(\epsilon) \left(\partial_x \partial_{x'} K(x, x') \frac{\delta}{\delta \epsilon_x} \left(\frac{f(\epsilon, t)}{\Omega(\epsilon)} \right) \right),$$

$$K(x, x') = \int_0^\infty ds \int d\Gamma\, \Omega^{-1}(\epsilon) \prod_z \delta(\hat{\epsilon}_z(\Gamma) - \epsilon_z) \left[\hat{\mathbb{J}}_x(\Gamma) e^{s\mathcal{Q}\mathbb{L}} \hat{\mathbb{J}}_{x'}(\Gamma) \right]$$

$$\sim \int_0^\infty ds \langle \mathbb{J}_x\, e^{s\mathbb{L}} \mathbb{J}_{x'} \rangle_{\text{local Gibbs}} \sim \int_0^\infty ds \langle \mathbb{J}_x(0) \mathbb{J}_{x'}(s) \rangle_{\text{eq}}$$

Derivation of Fluctuating Hydrodynamics in LRI Spin Systems

$$\partial_t f(\epsilon, t) = \int d\Gamma \hat{\rho}(\Gamma, t) \sum_x \partial_x \hat{\mathbb{J}}_x(\Gamma) \frac{\delta}{\delta \epsilon_x} \prod_x \delta(\hat{\epsilon}_x(\Gamma) - \epsilon_x)$$

$$= \int d\Gamma \left[\mathscr{P}\hat{\rho}(\Gamma, t) + \mathscr{Q}\hat{\rho}(\Gamma, t)\right] \sum_x \partial_x \hat{\mathbb{J}}_x(\Gamma) \frac{\delta}{\delta \epsilon_x} \prod_x \delta(\hat{\epsilon}_x(\Gamma) - \epsilon_x)$$

$$= \int d\Gamma \mathscr{Q}\hat{\rho}(\Gamma, t) \sum_x \partial_x \hat{\mathbb{J}}_x(\Gamma) \frac{\delta}{\delta \epsilon_x} \prod_x \delta(\hat{\epsilon}_x(\Gamma) - \epsilon_x)$$

$$= \sum_{x,x'} \frac{\delta}{\delta \epsilon_x} \int_{-\infty}^{t} ds \int \mathscr{D}\epsilon' \Omega(\epsilon)(\partial_x \partial_{x'} K(x, x'; t-s)) \frac{\delta}{\delta \epsilon'_{x'}} \left(\frac{f(\epsilon', s)}{\Omega(\epsilon')} \right)$$

$$K(x, x'; t-s) = \int d\Gamma\, \Omega^{-1}(\epsilon) \prod_z \delta(\hat{\epsilon}_z(\Gamma) - \epsilon_z) \left[\hat{\mathbb{J}}_x(\Gamma) \left(e^{(t-s)\mathscr{Q}\mathbb{L}} \prod_y \delta(\hat{\epsilon}_y(\Gamma) - \epsilon'_y) \mathbb{J}_{x'}(\Gamma) \right)\right]$$

$$\sim \int d\Gamma\, \Omega^{-1}(\epsilon) \prod_z \delta(\hat{\epsilon}_z(\Gamma) - \epsilon_z) \left[\hat{\mathbb{J}}_x(\Gamma)(e^{(t-s)\mathscr{Q}\mathbb{L}} \mathbb{J}_{x'}(\Gamma))\right] \prod_y \delta(\epsilon_y - \epsilon'_y)$$

Derivation of Fluctuating Hydrodynamics in LRI Spin Systems

④ Derivation of Fluctuating Hydrodynamic Eq. (Langevin Eq.)

Corresponding Langevin eq.

$$\partial_t \epsilon_x(t) = -\partial_x \Big[\sum_{x'} K(x,x') \frac{\partial}{\partial x'} \frac{\delta \bar{S}}{\delta \epsilon_{x'}} + \xi_x(t) \Big]$$

$$= -\partial_x \Big[\sum_{x'} K(x,x') \frac{\partial}{\partial x'} \beta_{x'} + \xi_x(t) \Big]$$

$$= -\partial_x \Big[\sum_{x',x''} K(x,x') \frac{\partial}{\partial x'} \Lambda(x',x'') \epsilon_{x''} + \xi_x(t) \Big]$$

$$= -\partial_x \Big[\sum_{x'} D(x,x') \epsilon_{x'}(t) + \xi_x(t) \Big]$$

$$\langle\langle \xi_x(t) \xi_{x'}(t') \rangle\rangle = 2K(x,x')\delta(t-t'),$$

$$\bar{S} = \ln \Omega(\epsilon)$$

$$\Lambda(x,x') = \left(\frac{\partial \epsilon_{x'}}{\partial \beta_x} \right)^{-1} \Bigg|_{\mathrm{eq}} = \left(\langle \delta\epsilon_x \delta\epsilon_{x'} \rangle_{\mathrm{eq}} \right)^{-1}$$

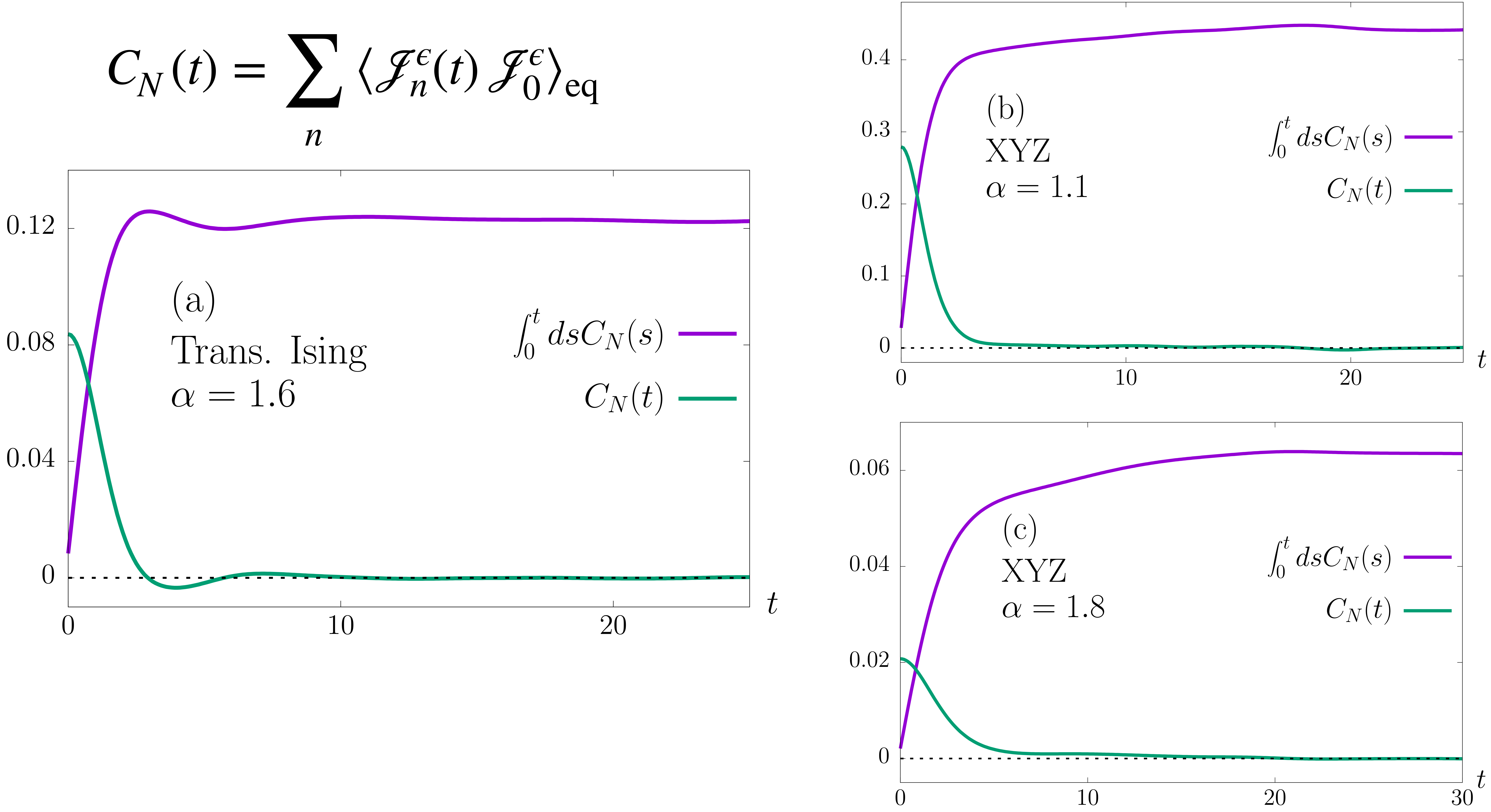
$$\sim (c_V k_B T^2)^{-1} \delta_{x,x'}$$

$$D(x,x') = \sum_{x''} K(x,x'') \Lambda(x'',x')$$

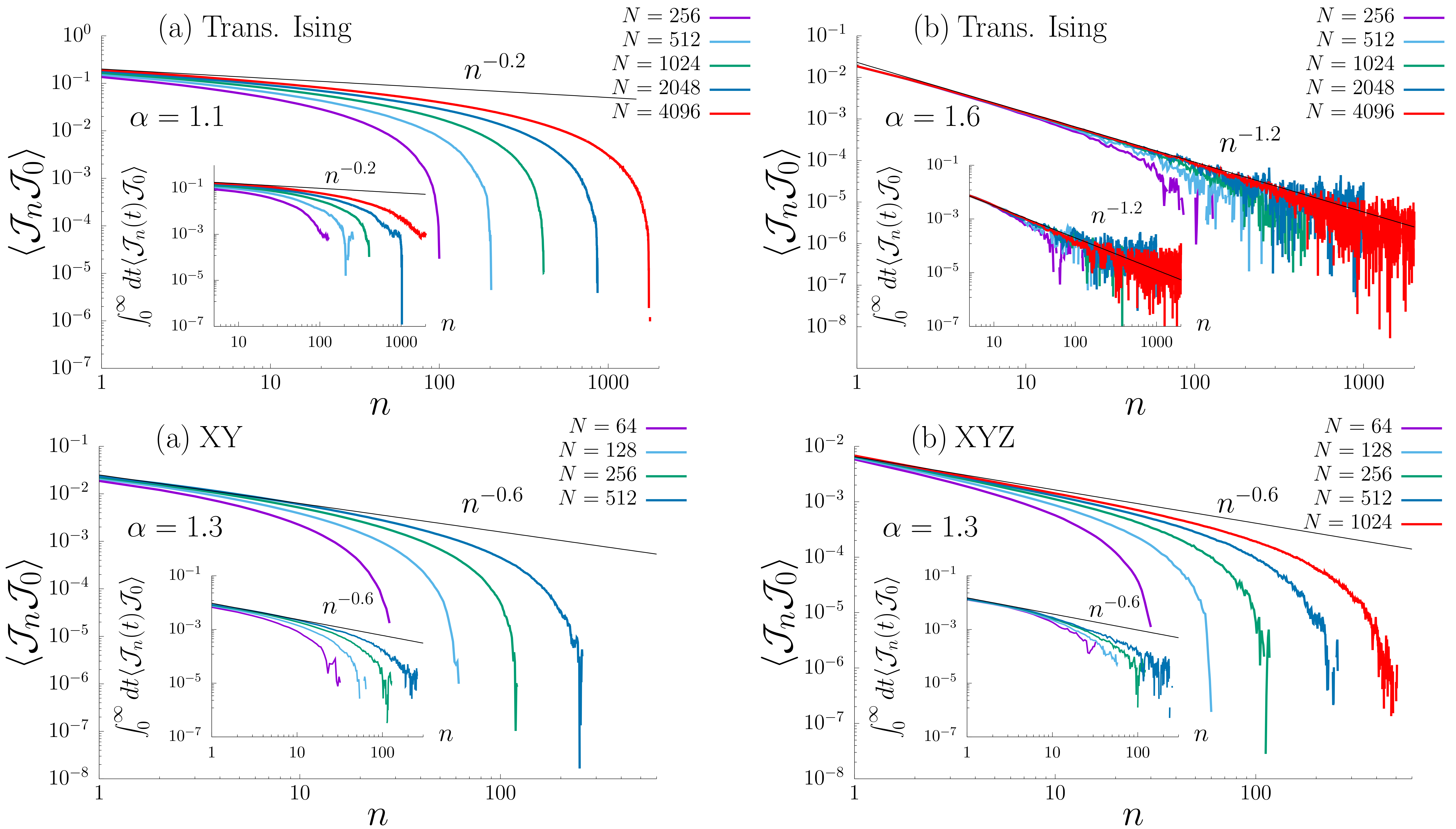
$$= K(x,x')/(c_V k_B T^2)$$

Rapid Decay of Total Current Correlation : Appendix

$$C_N(t) = \sum_n \langle \mathcal{J}_n^\epsilon(t)\, \mathcal{J}_0^\epsilon \rangle_{\text{eq}}$$



Optimality of Upper Bound on Current Correlation : Appendix

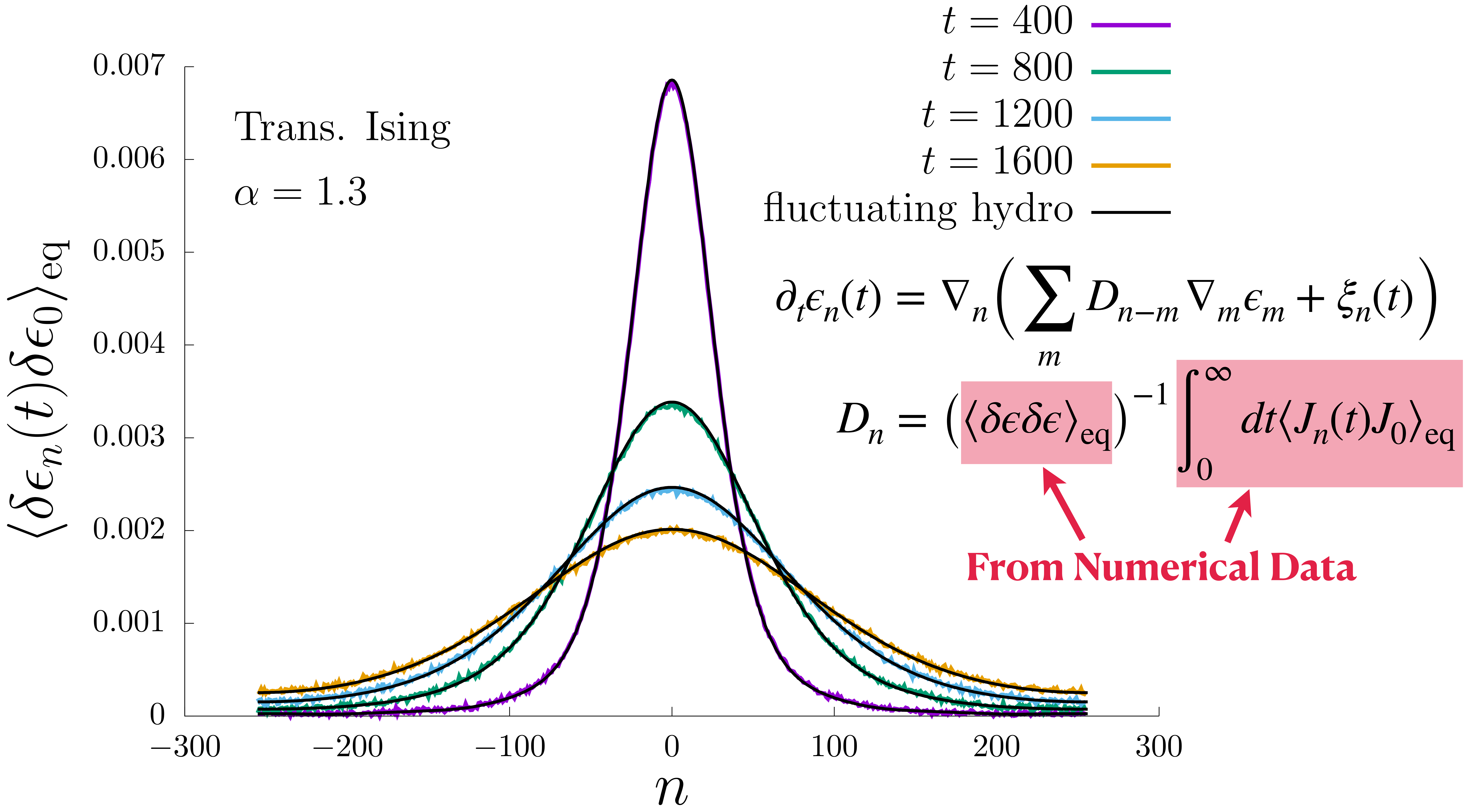


Energy Diffusion in LRI Spin Systems：Appendix

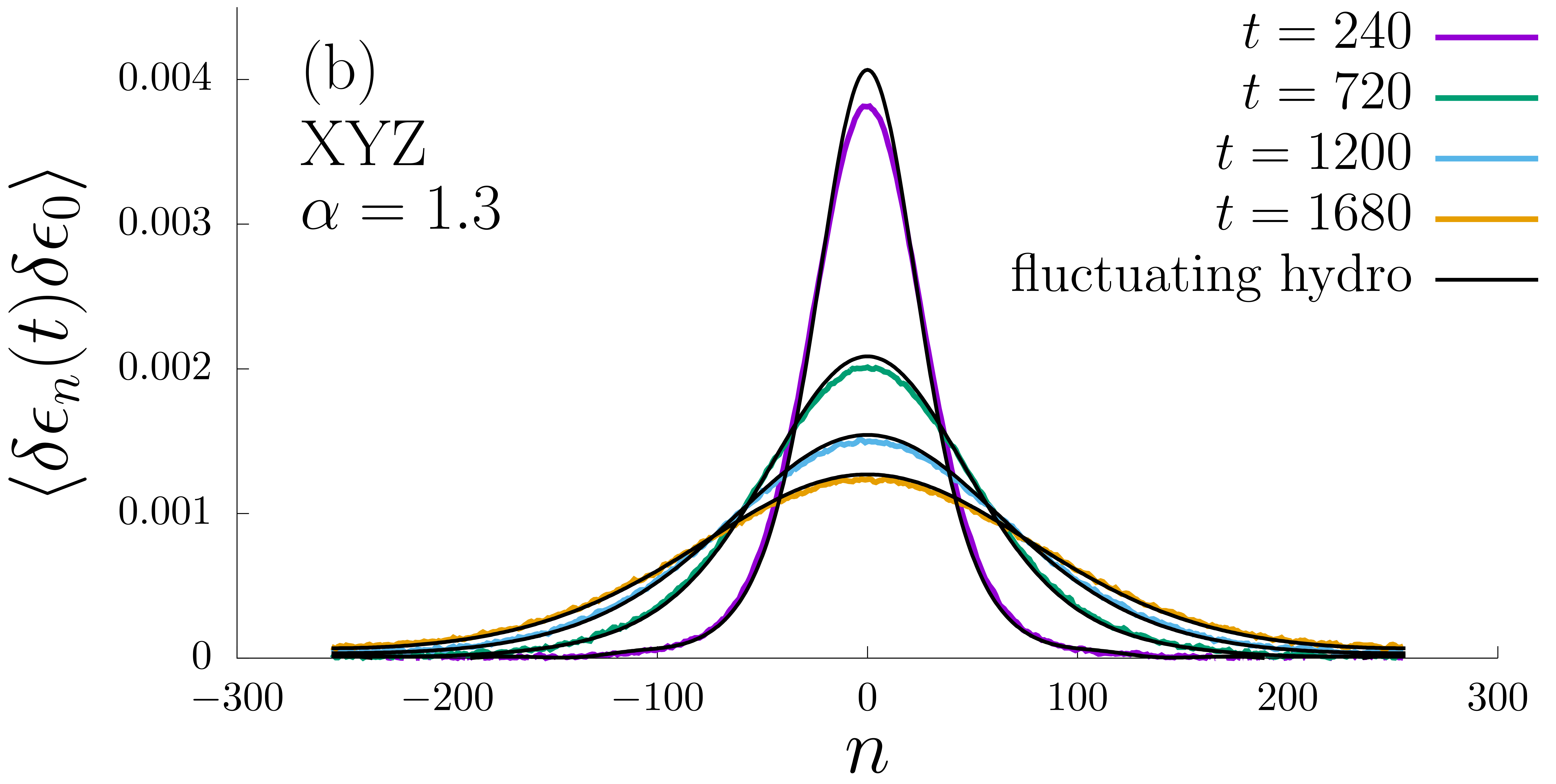
✓Space-time Energy Correlation：$C(n,t) := \langle \delta\epsilon_n(t)\delta\epsilon_0 \rangle_{\mathrm{eq}}$

Energy Diffusion in LRI XY models : Appendix

Validity of Fluctuating Hydrodynamics in LRI Spin Systems



Validity of Fluctuating Hydrodynamics in LRI Spin Systems



Derivation of Continuity Eq. for D-dim LRI Systems

✓ Continuity Eq. for $D\,(\geq 2)$-dim LRI Systems

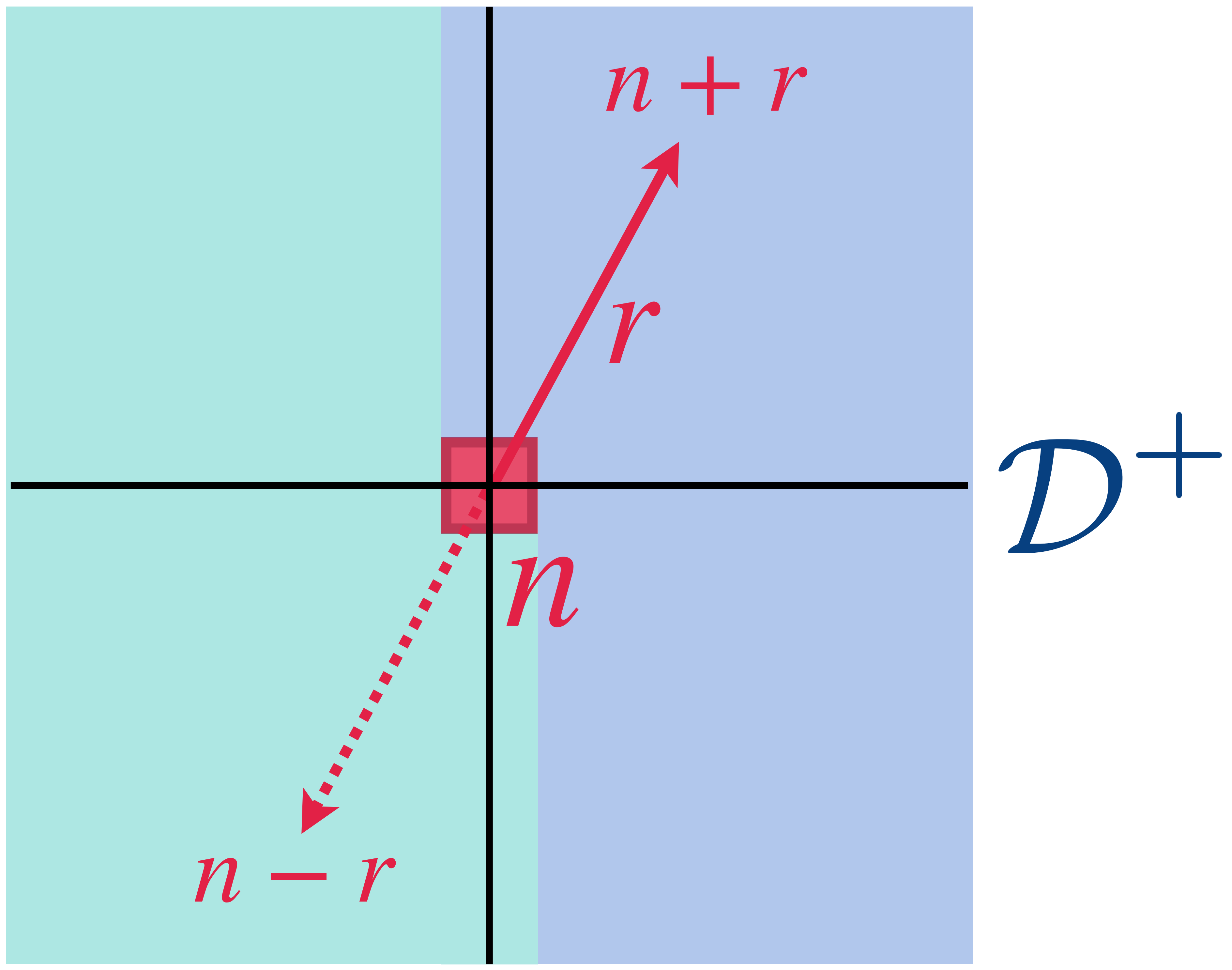
$$\partial_t \epsilon_n = \{\epsilon_n, H\} = \sum_{m \neq n} \{\epsilon_n, \epsilon_m\} = \sum_{m \neq n} t_{n \leftarrow m} \qquad t_{n \leftarrow m} := \{\epsilon_n, \epsilon_m\}$$

$$= \sum_{r \in \mathscr{D}^+} t_{n \leftarrow n+r} - \sum_{r \in \mathscr{D}^+} t_{n-r \leftarrow n}$$

$$= - \sum_{r \in \mathscr{D}^+} \nabla_r \mathscr{J}^{\epsilon}_{n:r}$$

$$\mathscr{J}^{\epsilon}_{n:r} := - t_{n \leftarrow n+r}$$

$$\nabla_r A_{n:r} := A_{n:r} - A_{n-r:r}$$



Derivation of Continuity Eq. for D-dim LRI Systems

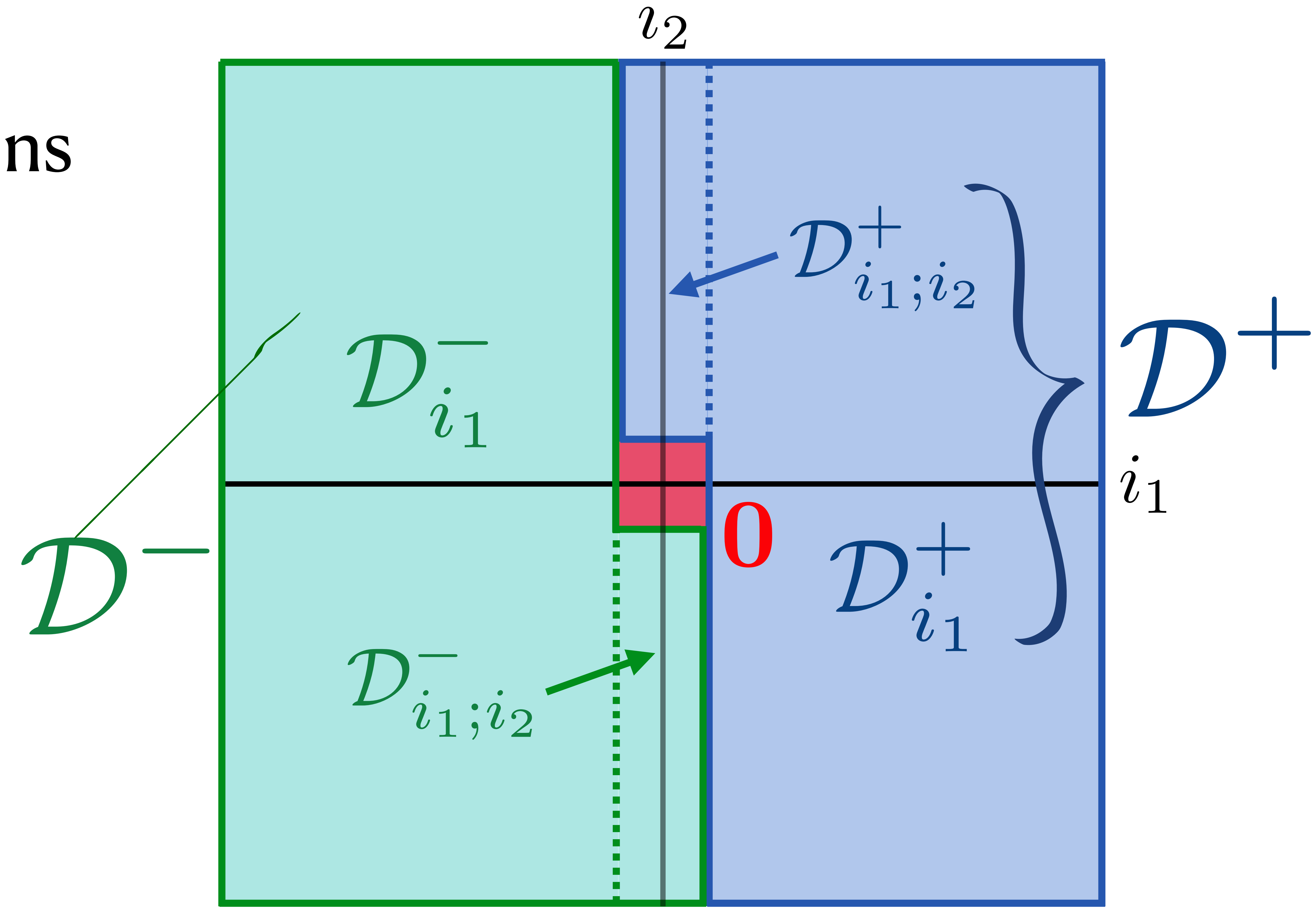
✓ Continuity Eq. for $D\,(\geq 2)$-dim LRI Systems

$$\partial_t \epsilon_n = -\sum_{r\in\mathscr{D}^+} \nabla_r \mathscr{J}^{\epsilon}_{n:r} \qquad \mathscr{J}^{\epsilon}_{n:r} := -\,t_{n\leftarrow n+r} \qquad t_{n\leftarrow m} := \{\epsilon_n, \epsilon_m\}$$

$$\nabla_r A_{n:r} := A_{n:r} - A_{n-r:r}$$

✓ Constructive definition of $\mathscr{D}^+$

= one of the two point-symmetric divisions of all lattices excluding the origin



Derivation of Green-Kubo Formula for D-dim LRI Systems

✓ Green-Kubo Formula for $D\ (\geq 2)$-dim LRI Systems

$S(t)$: Mean Square Displacement

$$S(t) = \left\langle \sum_{i=1}^{D} n_i^2(t) \right\rangle = \sum_{n} \sum_{i=1}^{D} n_i^2 P(n,t) = \frac{1}{k_\mathrm{B} T^2 c_\mathrm{V}} \sigma(t) \sim 2\mathscr{D} t \quad \mathscr{D} : \text{Diff. const.}$$

$$P(n,t) = \mathcal{N}^{-1} \langle \delta\epsilon_n(t) \delta\epsilon_0(0) \rangle \,,\ \mathcal{N} := \sum_n \langle \delta\epsilon_n \delta\epsilon_0 \rangle$$

$$\sigma(t) = \sum_{i=1}^{D} \sum_{n} n_i^2 \langle \delta\epsilon_n(t) \delta\epsilon_0(0) \rangle$$

Here, we use the Continuity Eq.

$$\partial_t \epsilon_n = - \sum_{r \in \mathscr{D}^+} \nabla_r \mathscr{J}^{\epsilon}_{n:r}$$

Derivation of Green-Kubo Formula for D-dim LRI Systems

✓ Green-Kubo Formula for $D\,(\geq 2)$-dim LRI Systems

$$\sigma(t) = \sum_{i=1}^{D}\sum_{n} n_i^2 \langle \delta\epsilon_n(t)\delta\epsilon_0(0)\rangle = \sum_{i=1}^{D}\sum_{n} n_i^2 \langle -\left(\int_0^t ds \sum_{r\in\mathscr{D}^+} \nabla_r \mathscr{J}_{n:r}(s)\right)\delta\epsilon_0(0)\rangle$$

$$= \sum_{i=1}^{D}\sum_{n}\sum_{r\in\mathscr{D}^+}(2r_i n_i + r_i^2)\int_0^t ds \langle \mathscr{J}_{n:r}(s)\delta\epsilon_0(0)\rangle = \sum_{i=1}^{D}\sum_{n}\sum_{r\in\mathscr{D}^+}(2r_i n_i + r_i^2)\int_0^t ds \langle \mathscr{J}_{n:r}(0)\delta\epsilon_0(-s)\rangle$$

$$= \sum_{i=1}^{D}\sum_{n}\sum_{r\in\mathscr{D}^+}(2r_i n_i + r_i^2)\int_0^t ds \langle \mathscr{J}_{n:r}(0)\left(-\sum_{r'\in\mathscr{D}^+}\int_0^s ds' \nabla_{r'}\mathscr{J}_{0:r'}(-s')\right)\rangle$$

$$= \sum_{i=1}^{D}\sum_{n}\sum_{r,r'\in\mathscr{D}^+}(2r_i n_i + r_i^2)\int_0^t ds \langle -(\mathscr{J}_{n:r}(0) - \mathscr{J}_{n-r':r}(0))\int_0^s ds' \mathscr{J}_{0:r'}(-s')\rangle$$

$$= \sum_{i=1}^{D}\sum_{n}\sum_{r,r'\in\mathscr{D}^+} 2r_i r'_i \int_0^t ds \int_0^s ds' \langle \mathscr{J}_{n:r}(0)\mathscr{J}_{0:r'}(-s')\rangle = \sum_{i=1}^{D}\sum_{n}\sum_{r,r'\in\mathscr{D}^+} 2r_i r'_i \int_0^t ds \int_0^s ds' \langle \mathscr{J}_{n:r}(s')\mathscr{J}_{0:r'}(0)\rangle$$

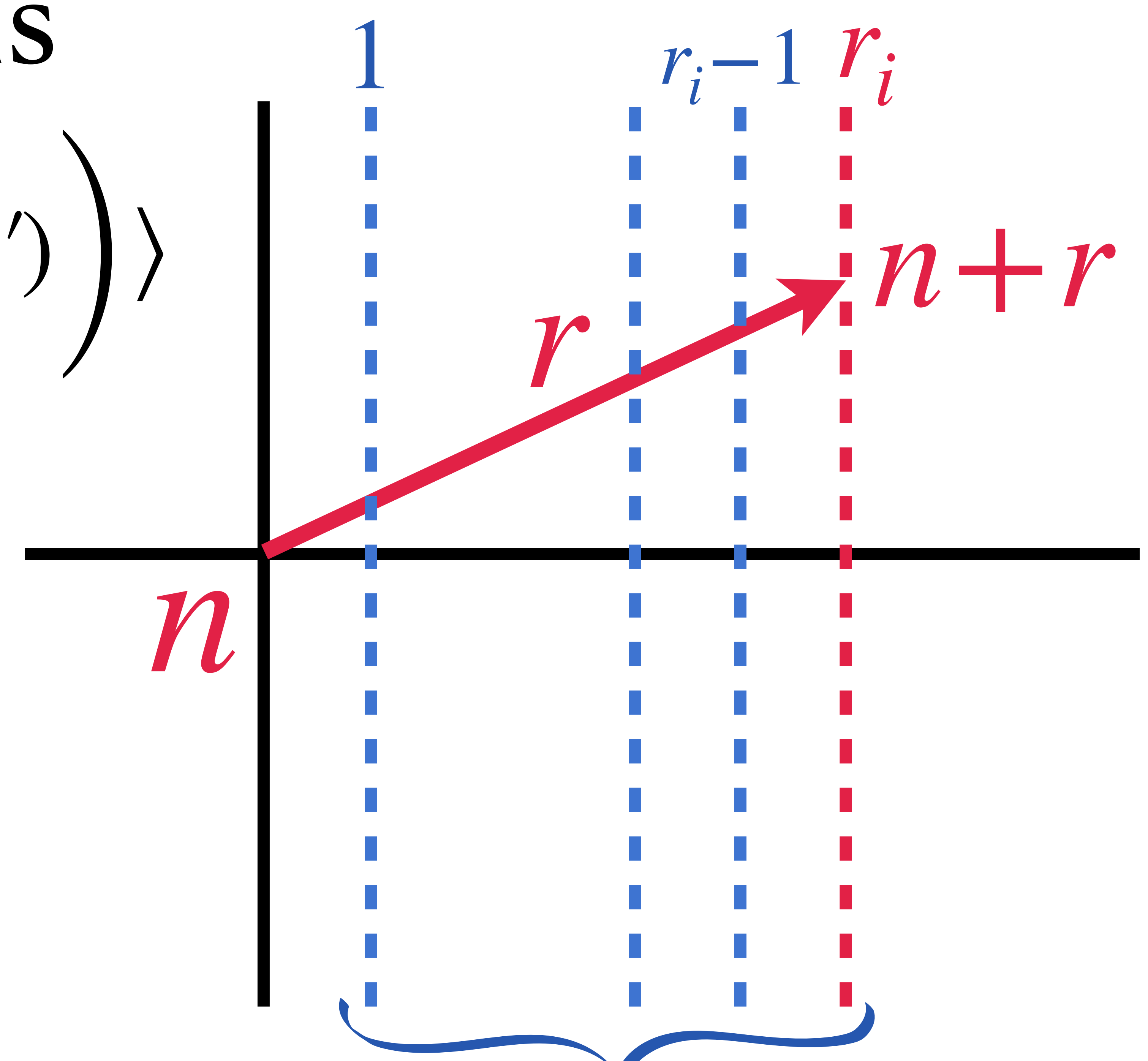
$$= \sum_{i=1}^{D}\frac{1}{N^D}\langle\left(\int_0^t ds \sum_{n}\sum_{r\in\mathscr{D}_i^+} r_i \mathscr{J}_{n:r}(s)\right)\left(\int_0^t ds' \sum_{n'}\sum_{r'\in\mathscr{D}^+} r'_i \mathscr{J}_{n':r'}(s')\right)\rangle$$

Derivation of Green-Kubo Formula for D-dim LRI Systems

✓ Green-Kubo Formula for $D\,(\geq 2)$-dim LRI Systems

$$\sigma(t) = \sum_{i=1}^{D} \frac{1}{N^D} \left\langle \left(\int_0^t ds \sum_{n} \sum_{r \in \mathscr{D}_i^+} r_i \mathscr{J}_{n:r}(s) \right) \left(\int_0^t ds' \sum_{n'} \sum_{r' \in \mathscr{D}^+} r'_i \mathscr{J}_{n':r'}(s') \right) \right\rangle$$

$$= \sum_{i=1}^{D} \frac{1}{N^D} \left\langle \left(\int_0^t ds \sum_{n_i} \mathscr{J}^{(i)}_{n_i|n_i-1}(s) \right) \left(\int_0^t ds' \sum_{n'_i} \mathscr{J}^{(i)}_{n'_i|n'_i-1}(s') \right) \right\rangle$$

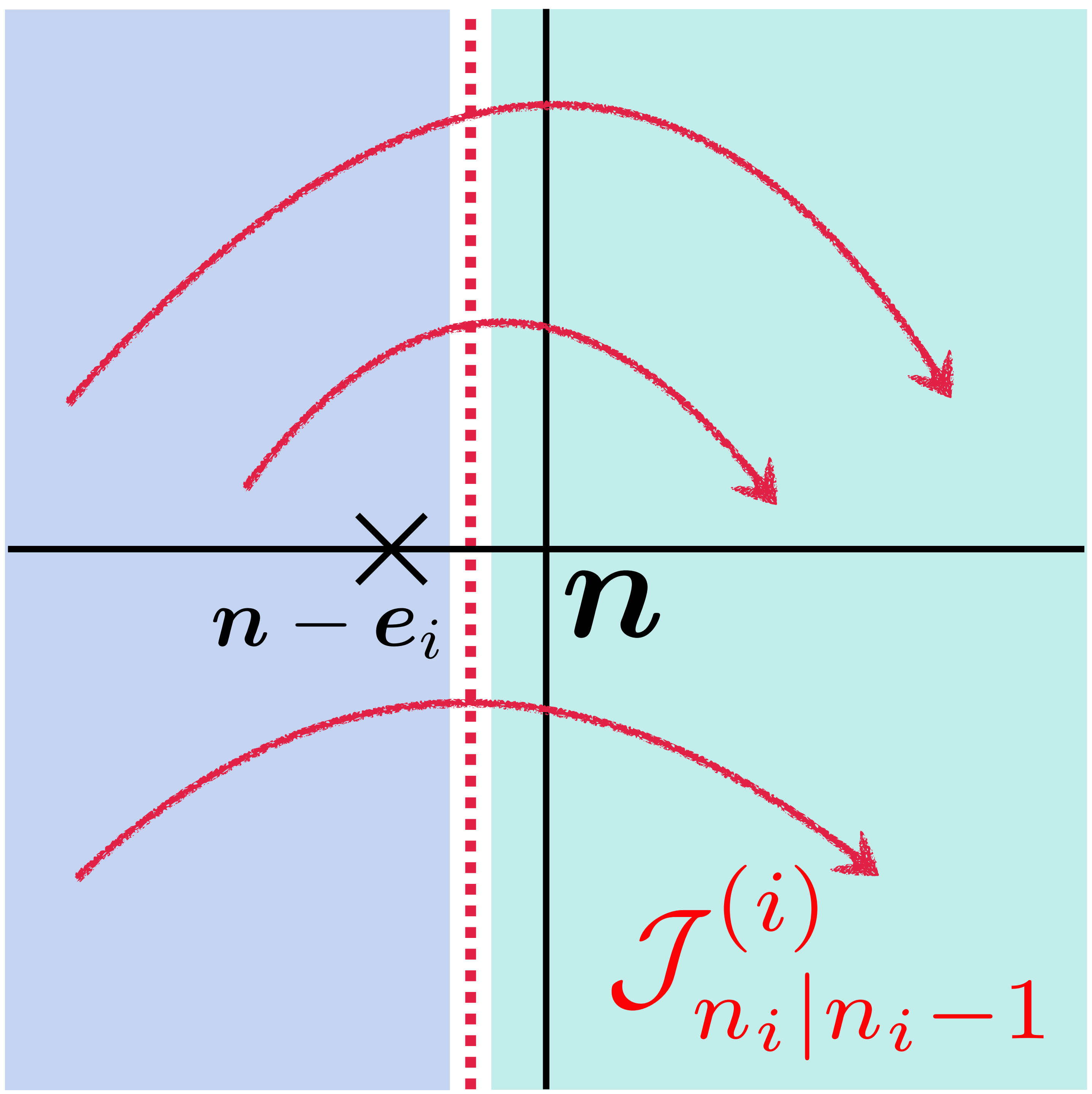


We count $\mathscr{J}_{n:r} = t_{n \leftarrow n+r}$, r_i times ($=$the number of layers that cross the boundary in the i-th direction)

$$\mathscr{J}^{(i)}_{n_i|n_i-1} = \sum_{\substack{k:k_i \geq n_i \\ \ell:l_i < n_i}} t_{k \leftarrow \ell}$$

$$\mathscr{D} = \frac{N^{1-D}}{c_V k_B T^2} \int_0^\infty dt$$

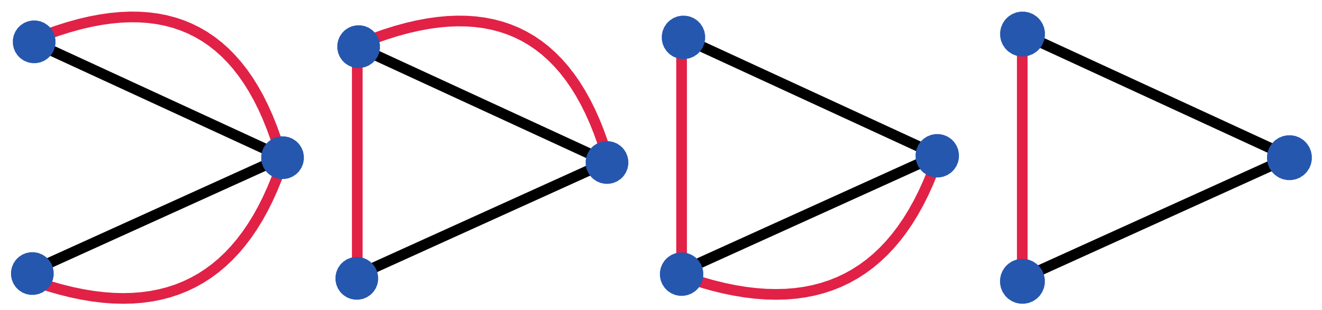
$$\sum_{i=1}^{D} \sum_{n_i=1}^{N} \langle \mathscr{J}^{(i)}_{n_i|n_i-1}(t) \mathscr{J}^{(i)}_{0|-1} \rangle$$



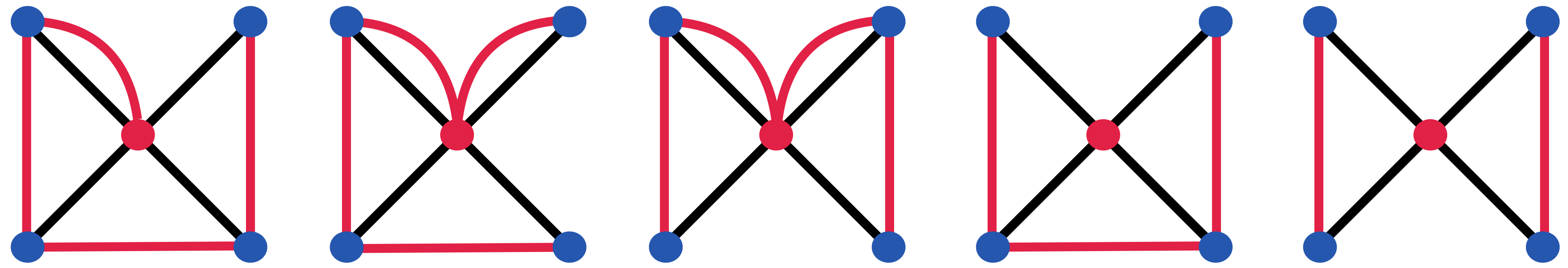
Upper Bound on Total Current Correlation：D-dim Trans. Ising・XY

$$C_N^{(D)}(0) = N^{1-D} \sum_{i=1}^{D} \sum_{n_i=1}^{N} \langle \mathcal{J}^{(i)}_{n_i|n_i-1} \mathcal{J}^{(i)}_{0|-1} \rangle_{\mathrm{eq}}$$

Trans. Ising

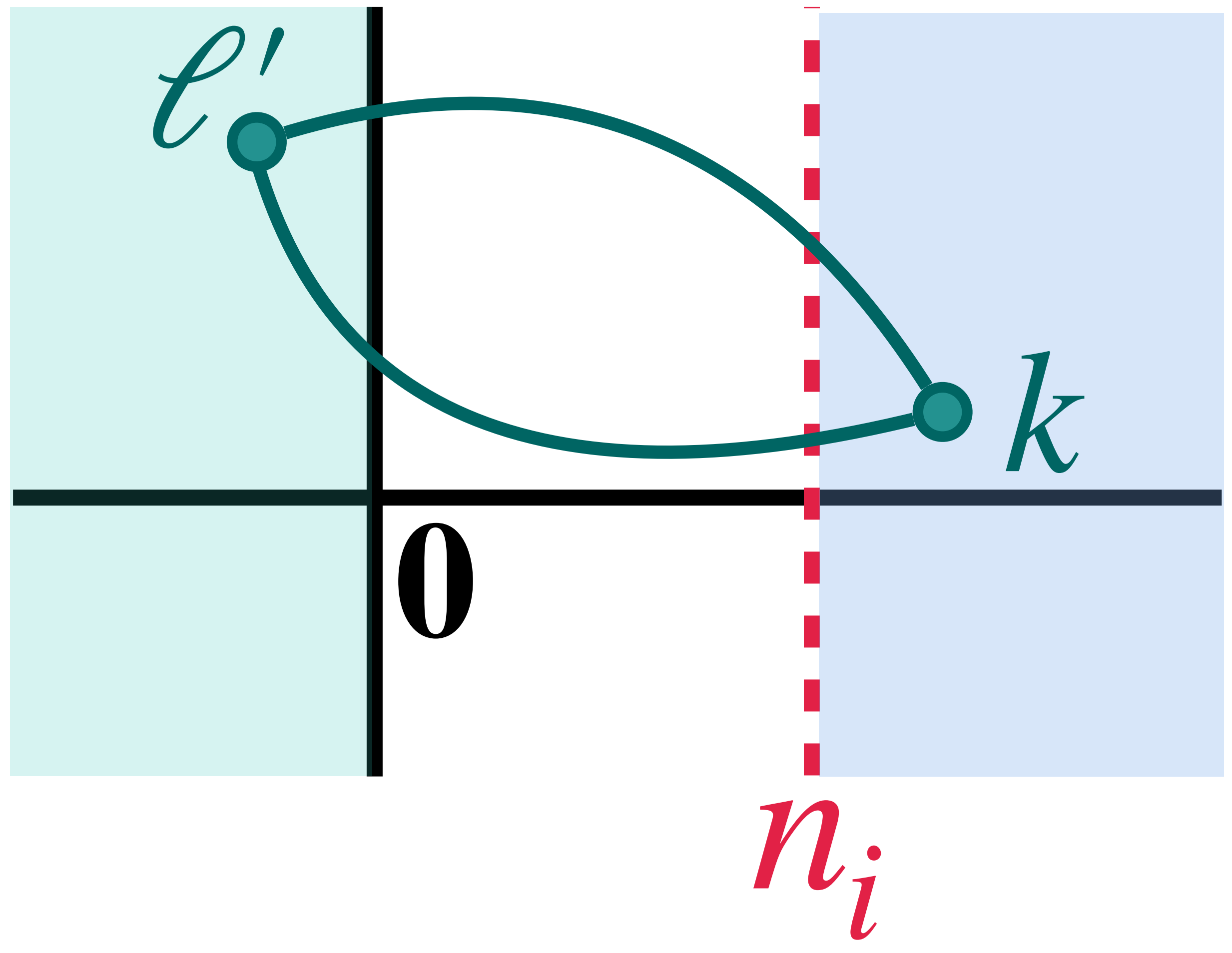


XY



$$\langle \mathcal{J}^{(i)}_{n_i|n_i-1} \mathcal{J}^{(i)}_{0|-1} \rangle \leq \mathcal{O}(1) \sum_{\substack{k:k_i \geq n_i \\ \ell':\ell'_i < 0}} \frac{1}{d_{k,\ell'}^{2\alpha}}$$

$$\leq \mathcal{O}(1) N^{D-1} n_i^{D+1-2\alpha}$$



$$C_N^{(D)}(0) \leq \mathcal{O}(1) N^{D+2-2\alpha} + \mathrm{const.} < \infty \quad (\alpha > D/2 + 1)$$

Upper Bound on Total Current Correlation：D-dim Trans. Ising・XY

$$\sum_{\substack{k:k_i \geq n_i \\ \ell':\ell'_i < 0}} \frac{1}{d_{k,\ell'}^{2\alpha}} \leq \int_{\substack{x_i \geq n_i \\ y_i < 0}} dx_i dy_i \prod_{k \neq i} \int_{-N/2}^{N/2} dx_k dy_k \left\{ (x_i - y_i)^2 + \sum_{j \neq i} (x_j - y_j)^2 \right\}^{-2\alpha/2}$$

$$\leq \Gamma_{D-1} \int_{-N/2}^{N/2} d^{D-1}u \int_{\substack{x_i \geq n_i \\ y_i < 0}} dx_i dy_i \int_0^\infty d\xi \xi^{D-2} \{ (x_i - y_i)^2 + \xi^2 \}^{-2\alpha/2}$$

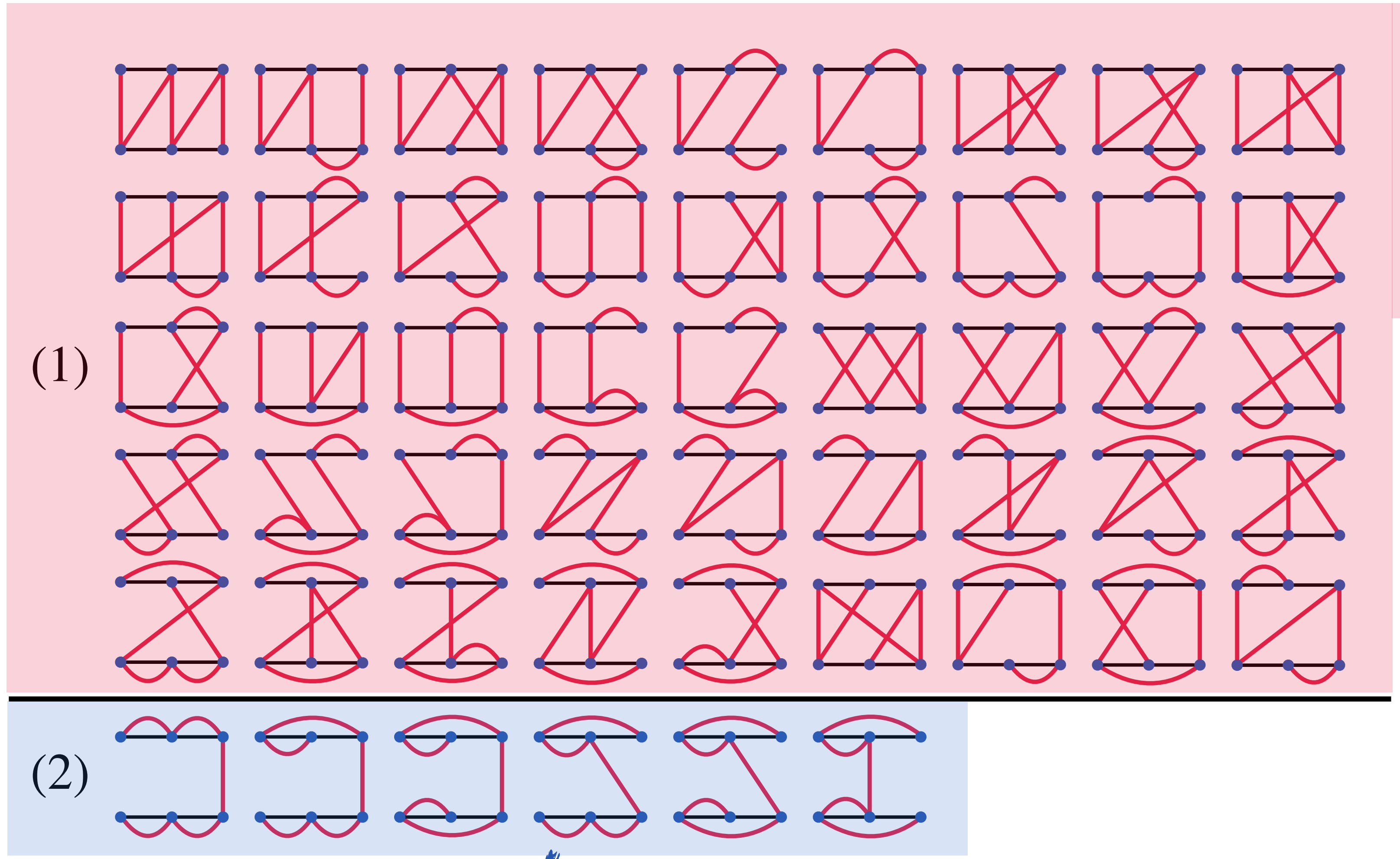
$$\leq \frac{\Gamma_{D-1} B(\frac{D-1}{2}, \alpha - \frac{D-1}{2})}{2} N^{D-1} \int_0^{N/2} dxdy (n_i + x + y)^{D-1-2\alpha}$$

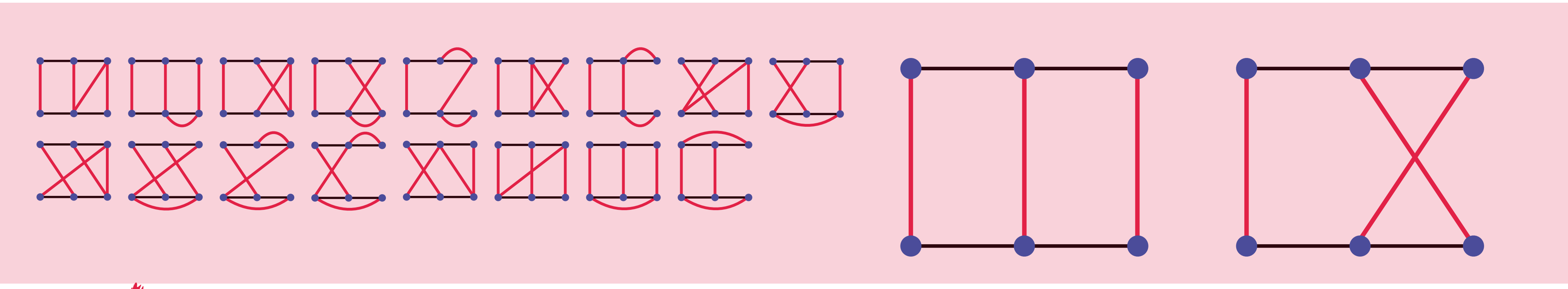
$$\leq \frac{\Gamma_{D-1} B(\frac{D-1}{2}, \alpha - \frac{D-1}{2})}{2(D-2\alpha)(D+1-2\alpha)} N^{D-1} n_i^{D+1-2\alpha}$$

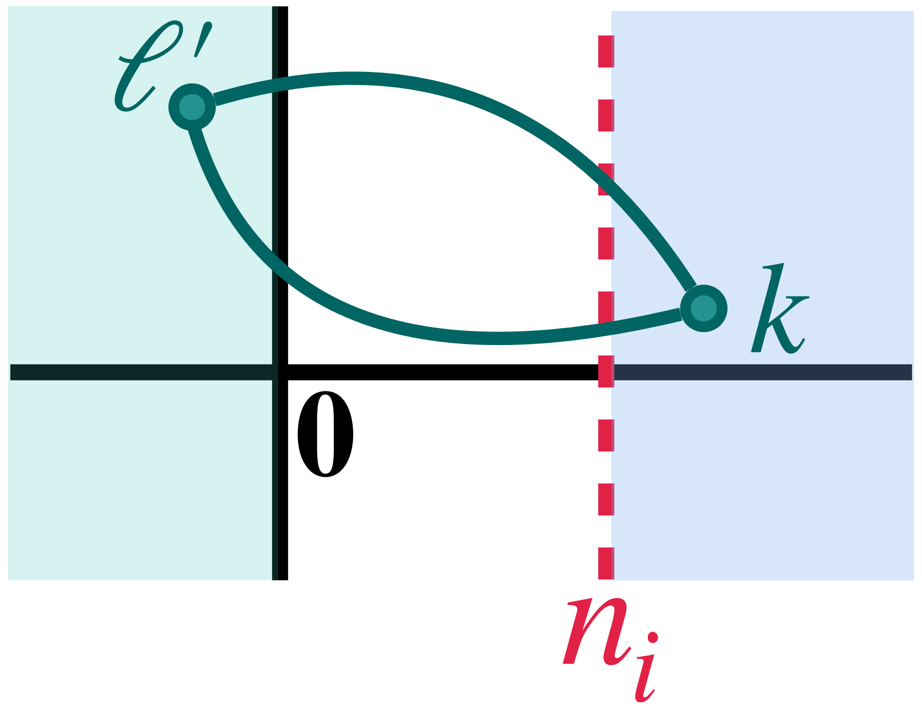
Upper Bound on Total Current Correlation：D-dim XYZ

$$C_N^{(D)}(0) = N^{1-D} \sum_{i=1}^{D} \sum_{n_i=1}^{N} \langle \mathcal{J}^{(i)}_{n_i|n_i-1} \mathcal{J}^{(i)}_{0|-1} \rangle_{\mathrm{eq}}$$

$$\langle S_a^\sigma S_b^\tau S_c^\upsilon S_{a'}^{\sigma'} S_{b'}^{\tau'} S_{c'}^{\upsilon'} \rangle_{\mathrm{eq.}} = \langle SSSSSS \rangle_c + \sum \langle SS \rangle_c \langle SSSS \rangle_c + \sum \langle SS \rangle_c \langle SS \rangle_c \langle SS \rangle_c$$





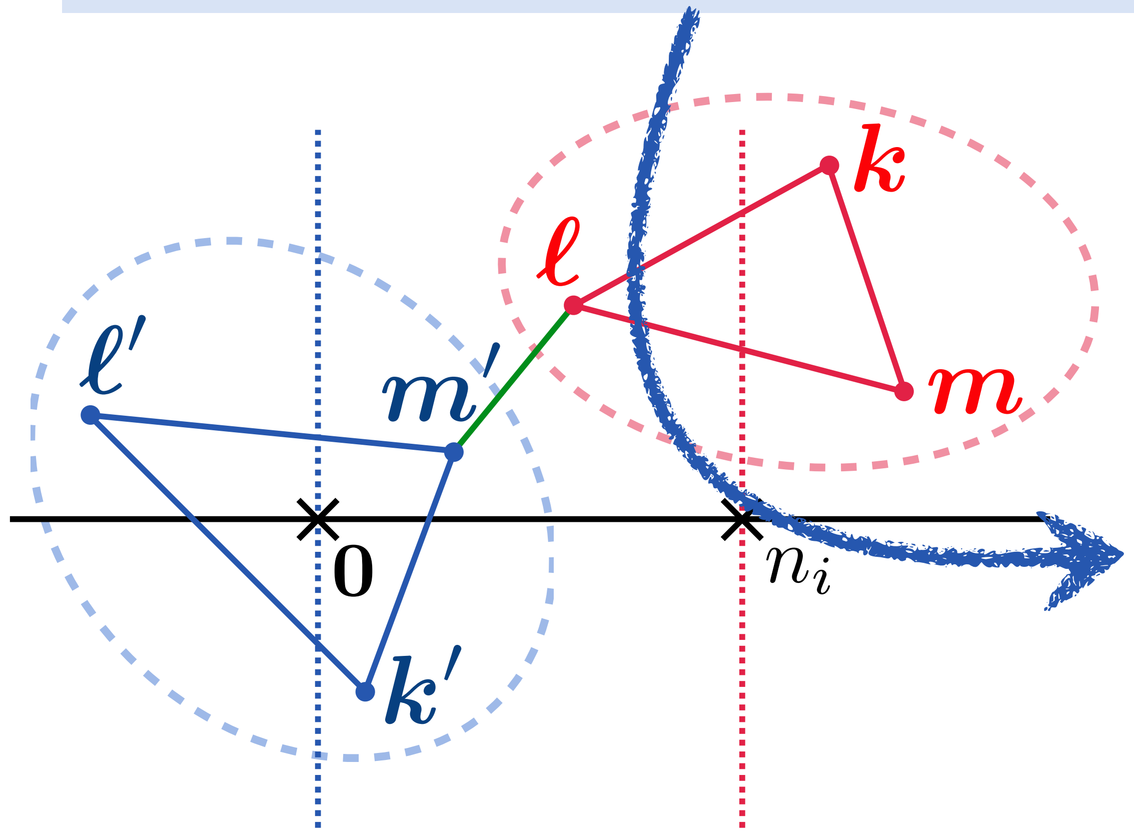


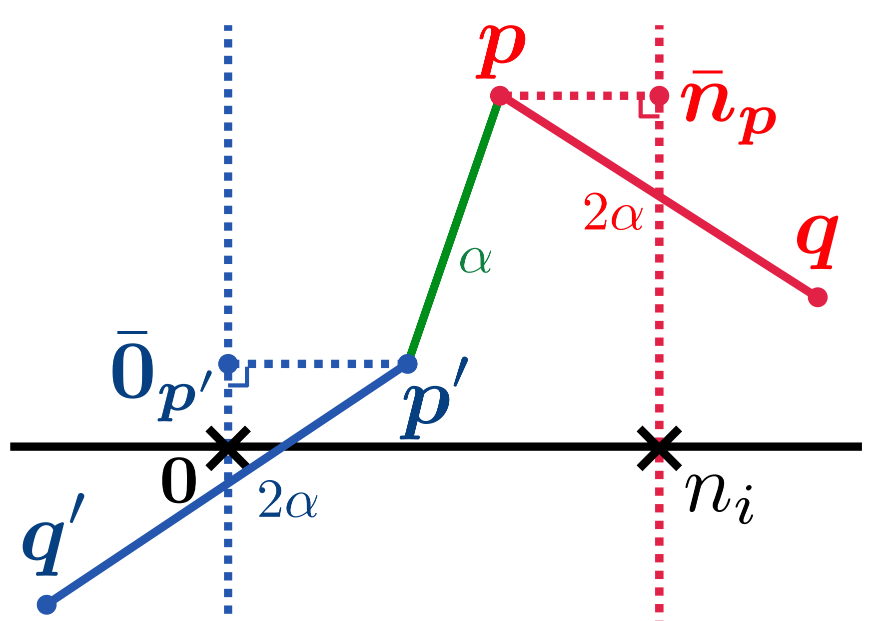
$$\langle \mathcal{J}^{(i)}_{n_i|n_i-1} \mathcal{J}^{(i)}_{0|-1} \rangle \leq \mathcal{O}(1) \sum_{\substack{k:k_i \geq n_i \\ \ell':\ell'_i < 0}} \frac{1}{d_{k,\ell'}^{2\alpha}}$$

$$\leq \mathcal{O}(1) \boxed{N^{D-1} n_i^{D+1-2\alpha}}$$

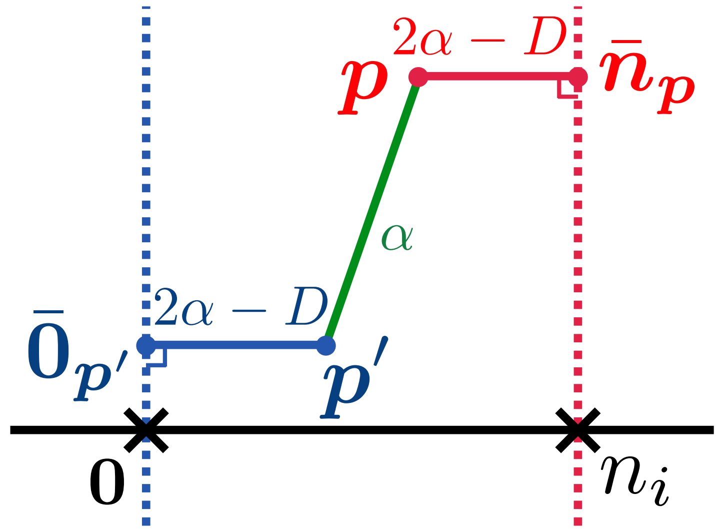
DOMINANT !!

for $D \geq 2$

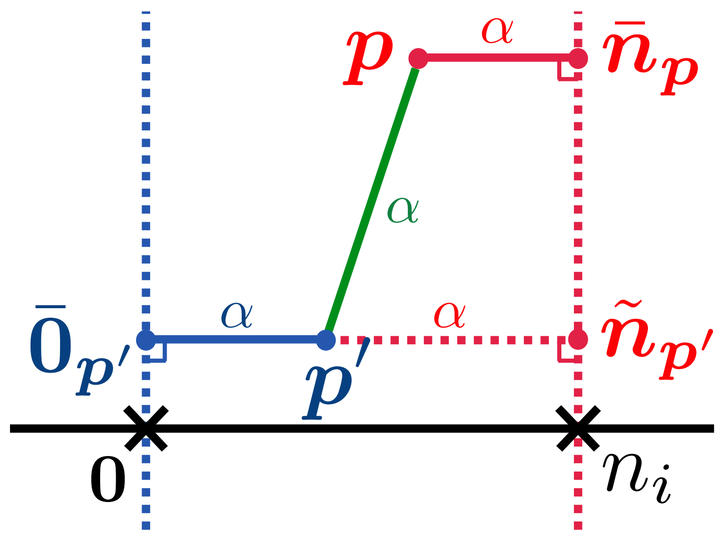




$\leq$



$\leq$



$\leq \mathcal{O}(1)\, n_i^{-\alpha}$