# The Kardar-Parisi-Zhang universality class for driven interfaces and... integrable spin chains?

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Talk plan

First half: mini review of the Kardar-Parisi-Zhang class

Second half: mysterious link to integrable spin chains

Questions welcome during the lecture!

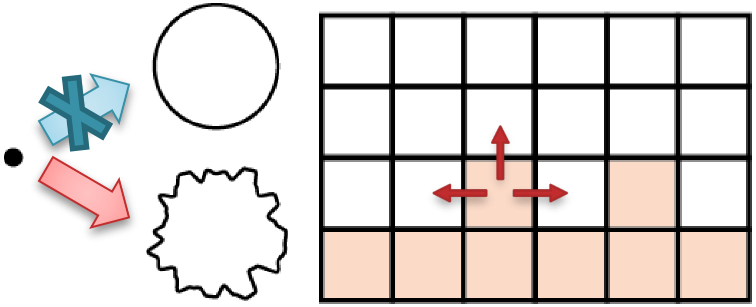
# Random Growth in Real World

YouTube [original by Calahorra Mountain Club]

@Calahorra, Spain

Poplar fluff on fire

Fire front is rough

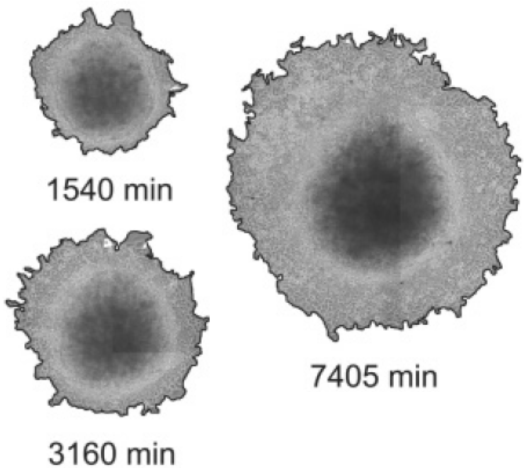


Random growth

$\Rightarrow$ rough interfaces

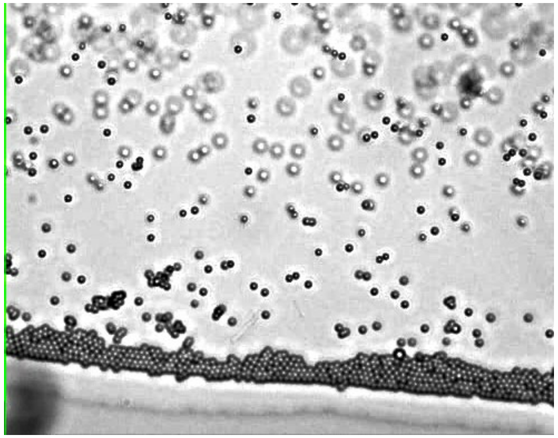
# Various Examples

proliferating cancer cells




[Huergo *et al.* PRE 2012]

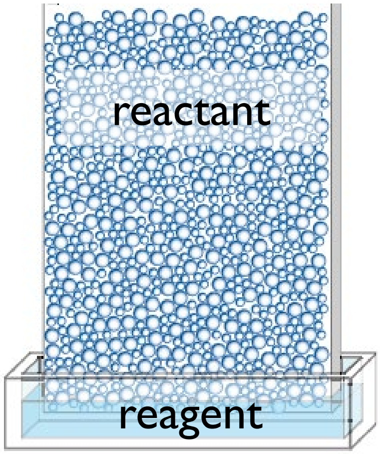
particle deposition

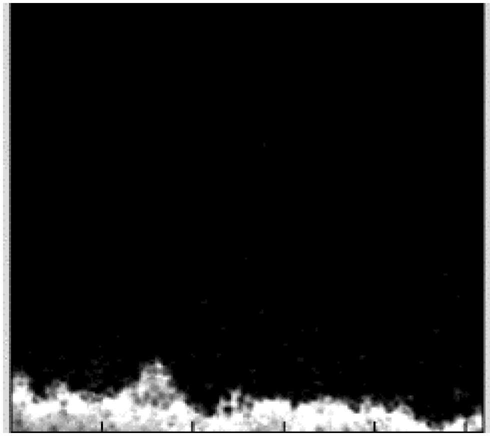


[Yunker *et al.* Nature 2011]

chemical reaction

(in disordered medium)



[Atis *et al.*, PRL 2013]

Microscopically, totally different processes.

Nevertheless, macroscopically,

similar rough interfaces are developed.

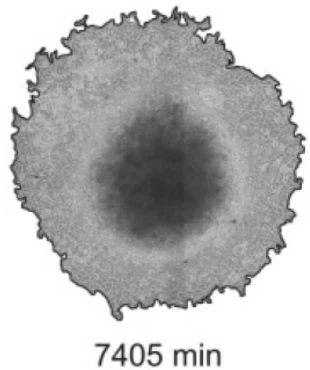
# Questions

circular

fire front

[YouTube]

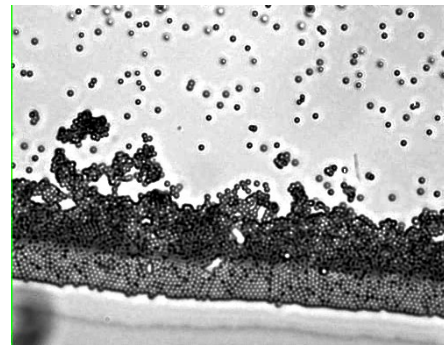
cancer cell growth




[Huergo *et al.* 2012]

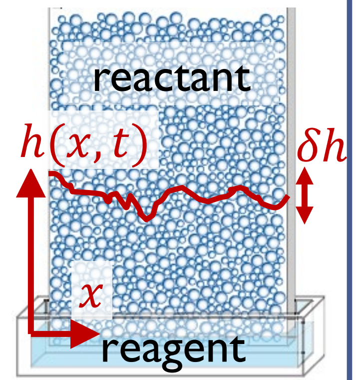
flat

particle deposition



[Yunker *et al.* 2011]

chem' reaction




[Atis *et al.* 2013]

**Q1: Any physics in common?** ⇨ Yes!

- Kardar-Parisi-Zhang (KPZ) universality class [KPZ, PRL 56, 889 (1986)]

$$\boxed{\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \sqrt{D}(\text{noise})}$$

for 1D ⇨ fluctuation

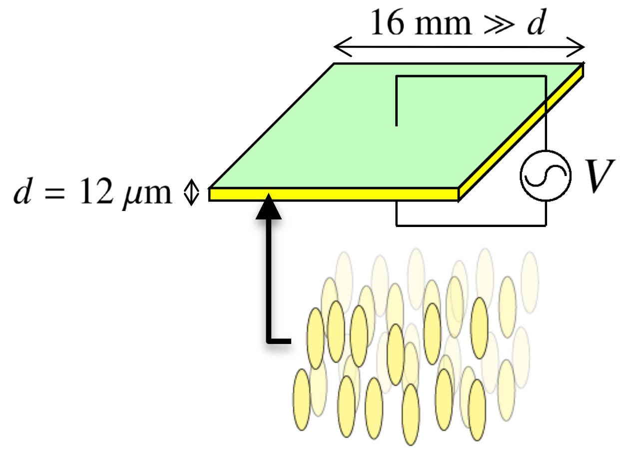
$$\boxed{\begin{array}{l} h(x,t) = v_\infty t + \delta h(x,t), \\ \delta h \sim t^{1/3}, \text{(corr. len.)} \sim t^{2/3} \end{array}}$$

- 1D models solved exactly! (despite non-eq many-body)
  Exact & nontrivial statistical properties. How universal? Integrability needed?

**Q2: Does the shape matter?**

[review: KaT, Physica A 504, 77 (2018)]

# Experiment



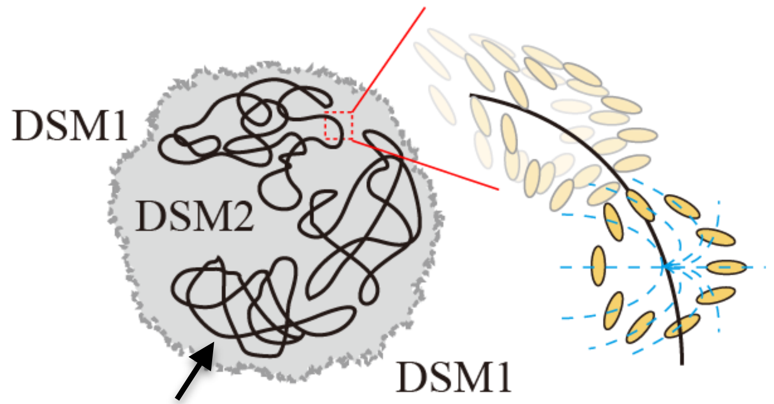

liquid crystal MBBA

Convection of nematic liquid crystal
driven by electric field

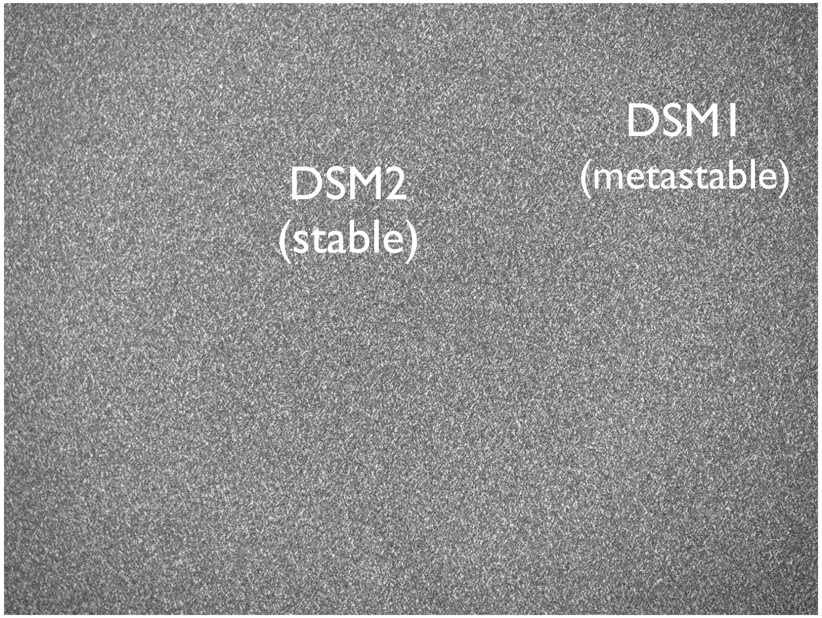
Two turbulent states at high enough $V$

Metastable: DSM1 = defect-less turbulence

Stable: DSM2 = defect-filled turbulence




Topological defect lines
in nematic director field



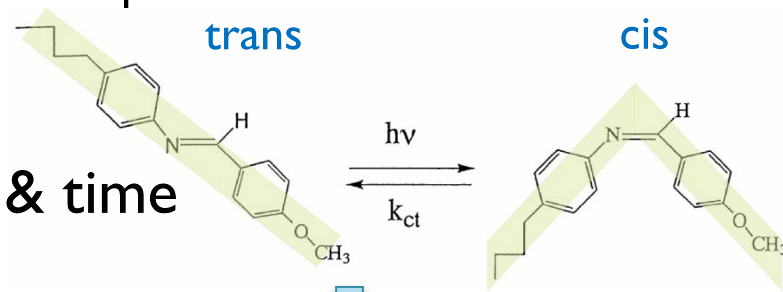

35V, 250Hz Speed x2, — 200 μm
(homeotropic alignment)

Growing DSM2 interfaces!

[review: KaT, Physica A 504, 77 (2018)]

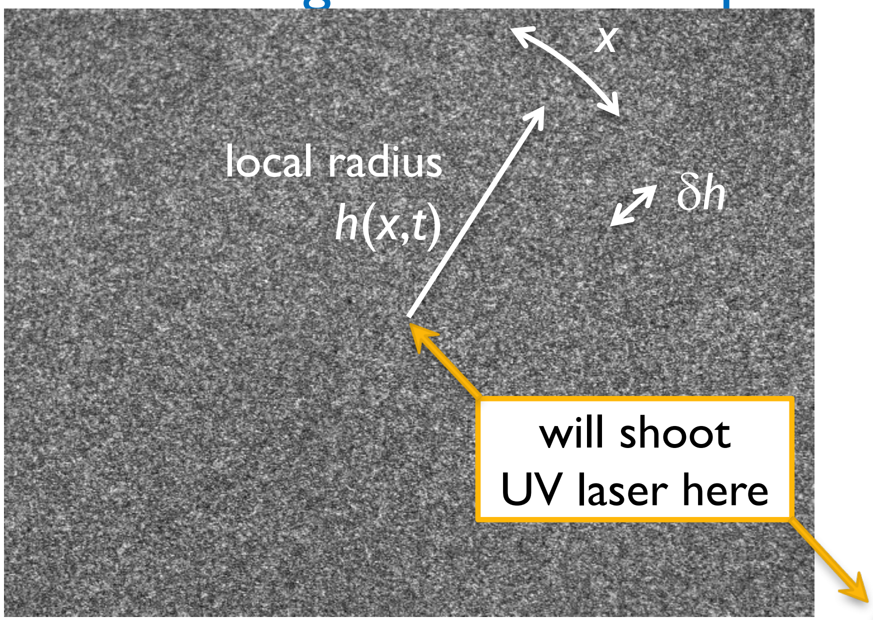
# Trigger the Growth

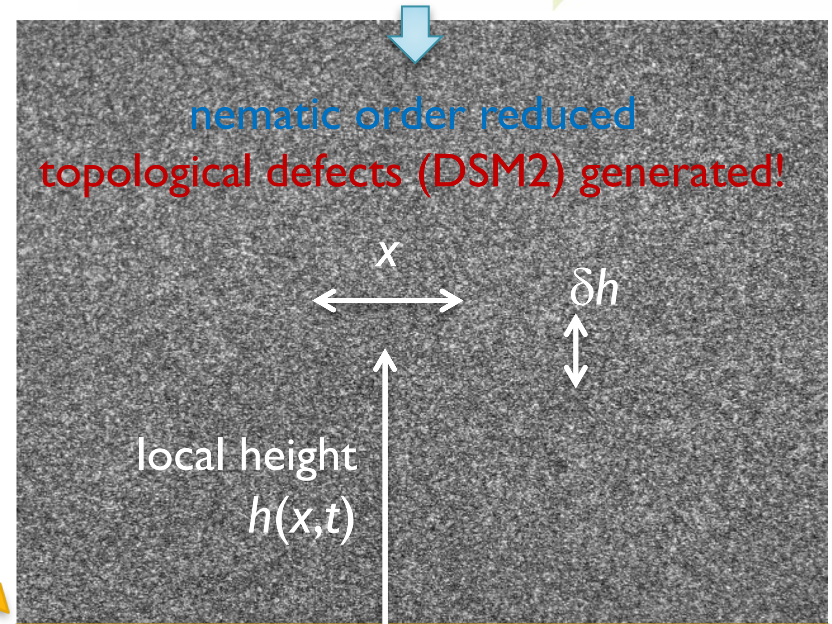
photo-isomerization of MBBA




Used UV laser to generate DSM2

- Growth starts at target position & time
- Can design the initial shape!



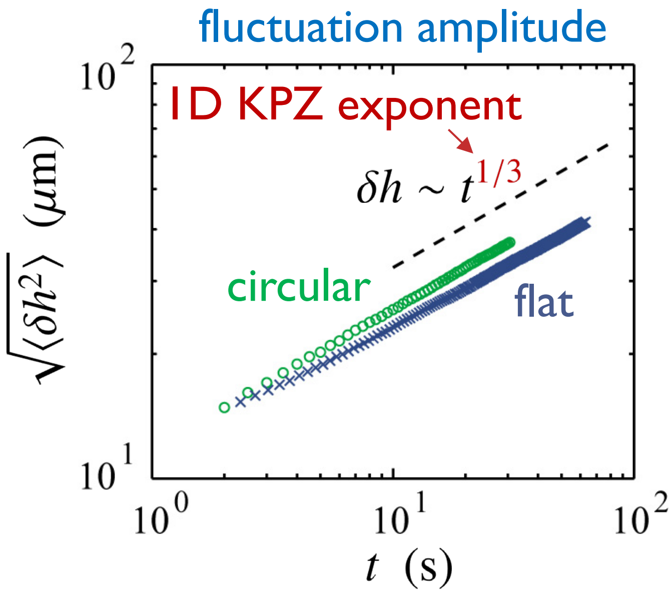




26V, 250Hz Speed x5, — 200μm

We generated both circular and flat interfaces (~1000 times)
and studied interface fluctuations

[KaT & Sano, J. Stat. Phys. 147, 853 (2012)
review: KaT, Physica A 504, 77 (2018)]

# Exponent & Distribution



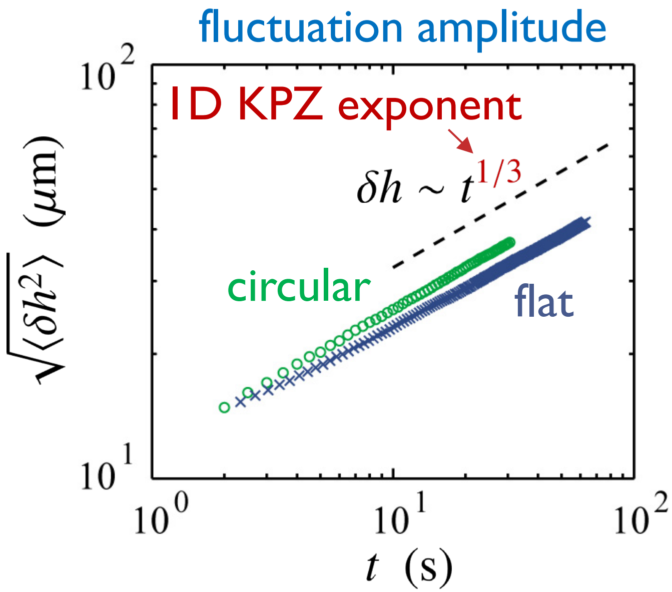

$$\boxed{h \simeq v_\infty t + (\Gamma t)^{1/3} \chi}$$

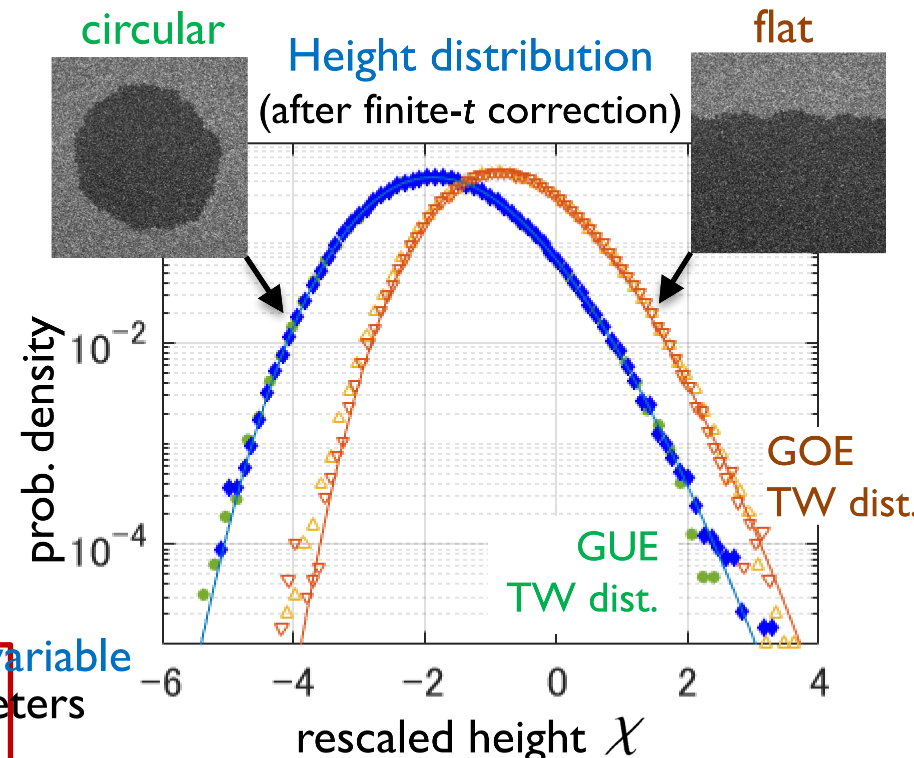
$\chi$ : rescaled variable
$v_\infty, \Gamma$ : parameters

- Both circular & flat cases show the same KPZ exponent.
  (∴ KPZ class)
- Exact studies predict far beyond, e.g., distribution. Does it agree?

[KaT & Sano, J. Stat. Phys. 147, 853 (2012); review: KaT, Physica A 504, 77 (2018)]

# Exponent & Distribution




$h \simeq v_\infty t + (\Gamma t)^{1/3}\chi$

$\chi$ : rescaled variable

$v_\infty, \Gamma$ : parameters

circular : $h \simeq v_\infty t + (\Gamma t)^{1/3}\chi_{\mathrm{GUE}}$

flat : $h \simeq v_\infty t + (\Gamma t)^{1/3}\chi_{\mathrm{GOE}}$

- Distribution agrees too!
  circular/flat ⇒ GUE/GOE Tracy-Widom distribution (to define)
- Universal, yet geometry-dependent! (even in $t \to \infty$)
  "KPZ class splits into different universality subclasses"

# Tracy-Widom Distribution

= distribution of the largest eigenvalue of Gaussian random matrices

e.g.) Gaussian Unitary Ensemble (GUE)

complex
Hermitian matrix $A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1N} \\ A_{21} & A_{22} & \dots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \dots & A_{NN} \end{pmatrix}$

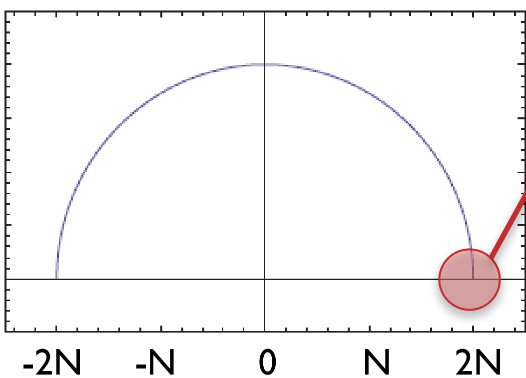
Gaussian
mean 0 variance $N/2$

$$A_{ij} = \bar{A}_{ji} = a_{ij} + i b_{ij}$$

$$A_{ii} = a_{ii}$$

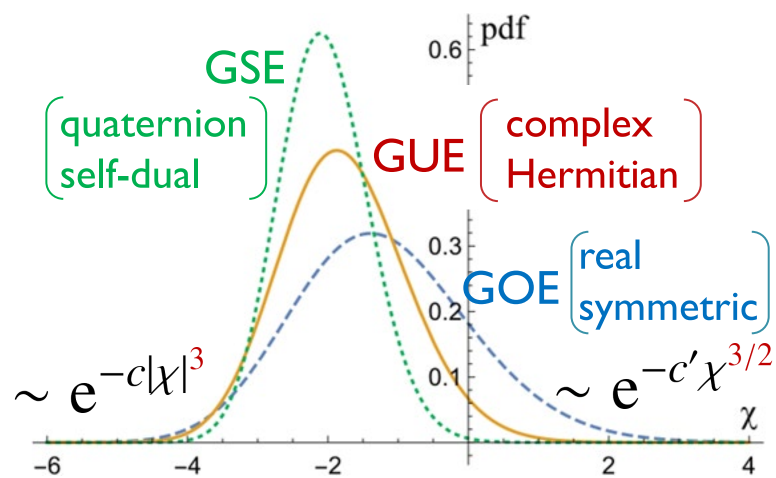
mean 0 variance $N$

distribution of all $N$ eigenvalues
(Wigner's semicircle law)




$$\boxed{\lambda_{\max} \simeq 2N + N^{1/3}\chi_{\mathrm{GUE}}}$$

$\mathrm{pdf}(\chi_{\mathrm{GUE}}) \equiv$
GUE Tracy-Widom dist.



[review: KaT, Physica A 504, 77 (2018)]

# How Come?

1D KPZ equation (coefficients fixed, w/o loss of generality)

$$\boxed{\frac{\partial h}{\partial t} = \frac{1}{2}\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\left(\frac{\partial h}{\partial x}\right)^2 + \eta}$$

$\eta(x,t)$: white Gaussian noise

Cole-Hopf transformation $\boxed{Z(x,t) = \mathrm{e}^{h(x,t)}}$

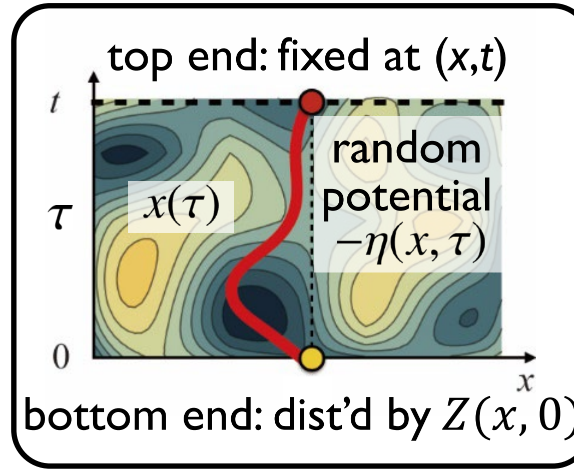
stochastic heat equation

$$\boxed{\frac{\partial Z}{\partial t} = \frac{1}{2}\frac{\partial^2 Z}{\partial x^2} + \eta Z}$$

path integral (Feynman-Kac formula)

directed polymer (DP)'s partition function

$$\boxed{Z(x,t) = \int \mathcal{D}x(\tau)\ \exp\left\{-\int \mathrm{d}\tau \left[\frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 - \eta(x,\tau)\right]\right\}}$$




$$h(x,t) = \log Z(x,t)$$

height = -(DP free energy)

polymer elasticity random potential

for $N$th-order moment $\Psi(\vec{x},t) \equiv \langle Z(x_1,t)\cdots Z(x_N,t)\rangle$

$N$-body bosons (attractive Lieb-Liniger model, quantum integrable)

$$\boxed{\frac{\partial \Psi}{\partial t} = -H_{\mathrm{LL}}\Psi, \quad H_{\mathrm{LL}} \equiv -\frac{1}{2}\sum_n \frac{\partial^2}{\partial {x_n}^2} - \frac{1}{2}\sum_{n\neq m}\delta(x_n - x_m)}$$

Bethe ansatz → GUE TW

[Calabrese et al., Dotsenko, 2010]

(there's also a rigorous approach)

[review: KaT, Physica A 504, 77 (2018)]

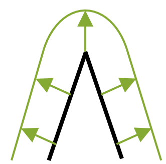
# How Come?

- Recap: $\boxed{\text{interface } h(x,t)} \Leftrightarrow \boxed{\text{polymer part. func. } Z(x,t) = e^h} \Leftrightarrow \boxed{\text{bosons}}$
- Geometry dependence? ➜ Consider the initial conditions

## circular case

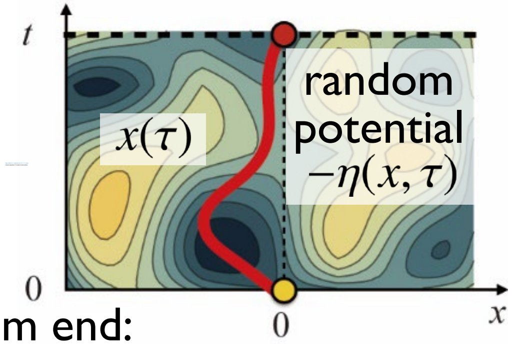
$$h(x,0) = -\kappa|x|$$

$$Z(x,0) = e^{h(x,0)} \xrightarrow{\kappa\to\infty} \delta(x)$$



### polymer picture

top end: fixed at a point



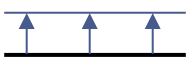

bottom end:
dist'd by $Z(x,0) = \delta(0) \to$ fixed at (0,0)

circular = "point-to-point problem"

## flat case

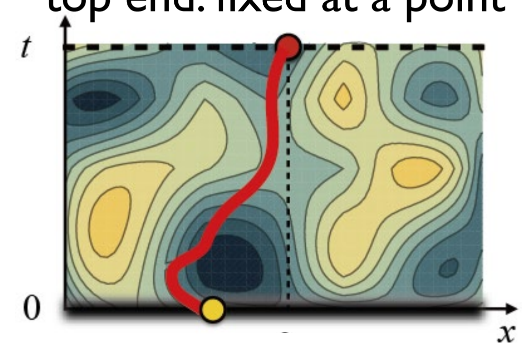
$$h(x,0) = 0$$

$$Z(x,0) = e^{h(x,0)} = \text{const}$$



### polymer picture

top end: fixed at a point

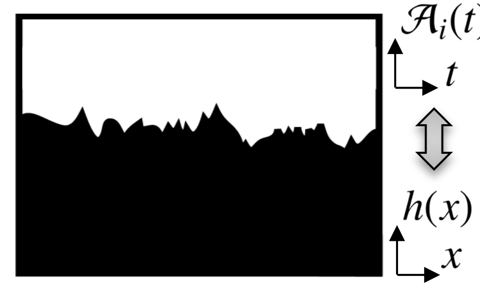



bottom end: uniformly distributed

flat = "line-to-point problem"

**Different boundary ⇒ different statistics** (distribution, correlation, ...)

# Spatial Correlation




Outcome from exact studies:

$$\langle h(x_1,t)\cdots h(x_n,t)\rangle \xrightarrow{\text{rescaled, } t\to\infty} \langle \mathcal{A}_i(x_1^{\mathrm{res}})\cdots \mathcal{A}_i(x_n^{\mathrm{res}})\rangle$$

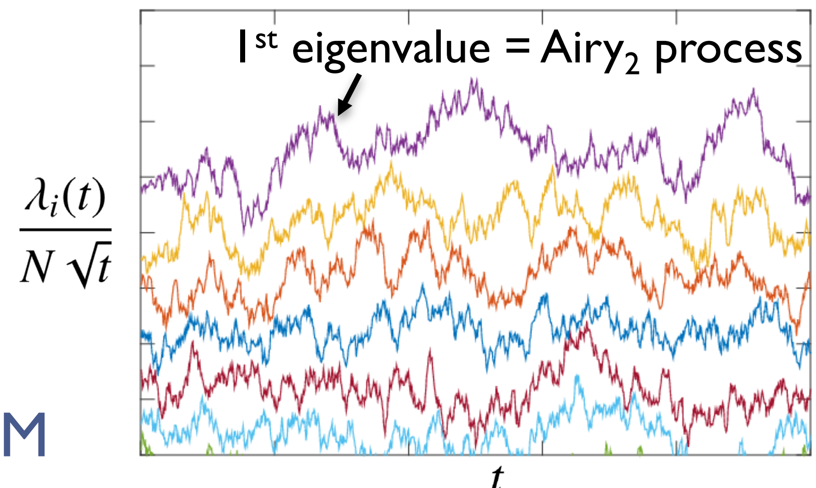
with $\mathcal{A}_i = \begin{cases} \mathcal{A}_2 \text{ (Airy}_2\text{ process) for circular case} \\ \mathcal{A}_1 \text{ (Airy}_1\text{ process) for flat case} \end{cases}$

analytic formulae known [Prahofer & Spohn JSP 2002; Sasamoto JPA 2005]

- Airy$_2$ process = Dyson Brownian motion of GUE matrices

$$A(t) = \begin{pmatrix} A_{11}(t) & \cdots & A_{1N}(t) \\ \vdots & \ddots & \vdots \\ A_{N1}(t) & \cdots & A_{NN}(t) \end{pmatrix}$$

eigenvalues: $\lambda_i(t)$




- Airy$_1$ process ≠ GOE Dyson BM
  (KPZ-random matrix relation is only partial)

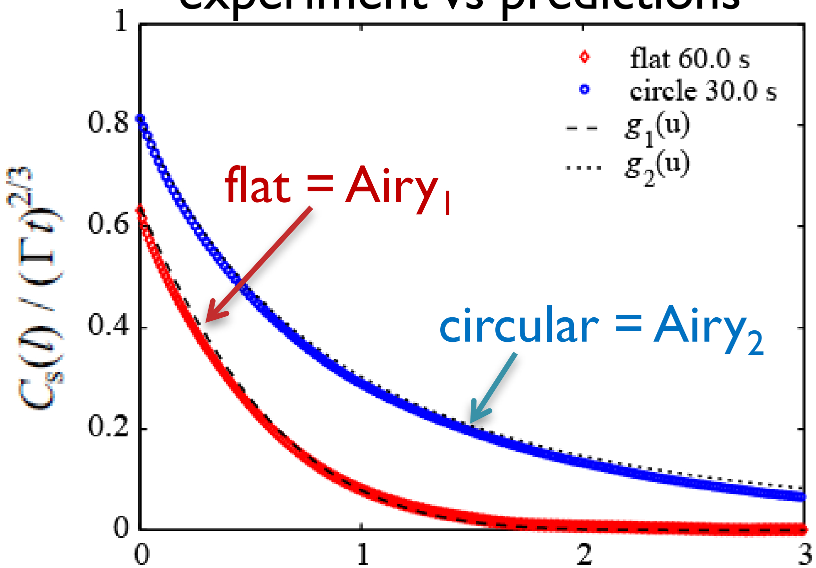
[review: KaT, Physica A 504, 77 (2018)]

# Spatial Correlation Observed in Experiment

$$\boxed{C_s(\ell,t) \equiv \langle \delta h(x+\ell,t)\delta h(x,t)\rangle} \text{ with } \delta h(x,t) \equiv h - \langle h\rangle$$

$$\simeq (\Gamma t)^{2/3} g_i\left(\frac{\ell}{\xi(t)}\right) \text{ with } g_i \equiv \langle \delta\mathcal{A}_i(u_0+u)\delta\mathcal{A}_i(u_0)\rangle$$

experiment vs predictions




rescaled length $\ell/\xi(t)$

*Qualitatively* different decay

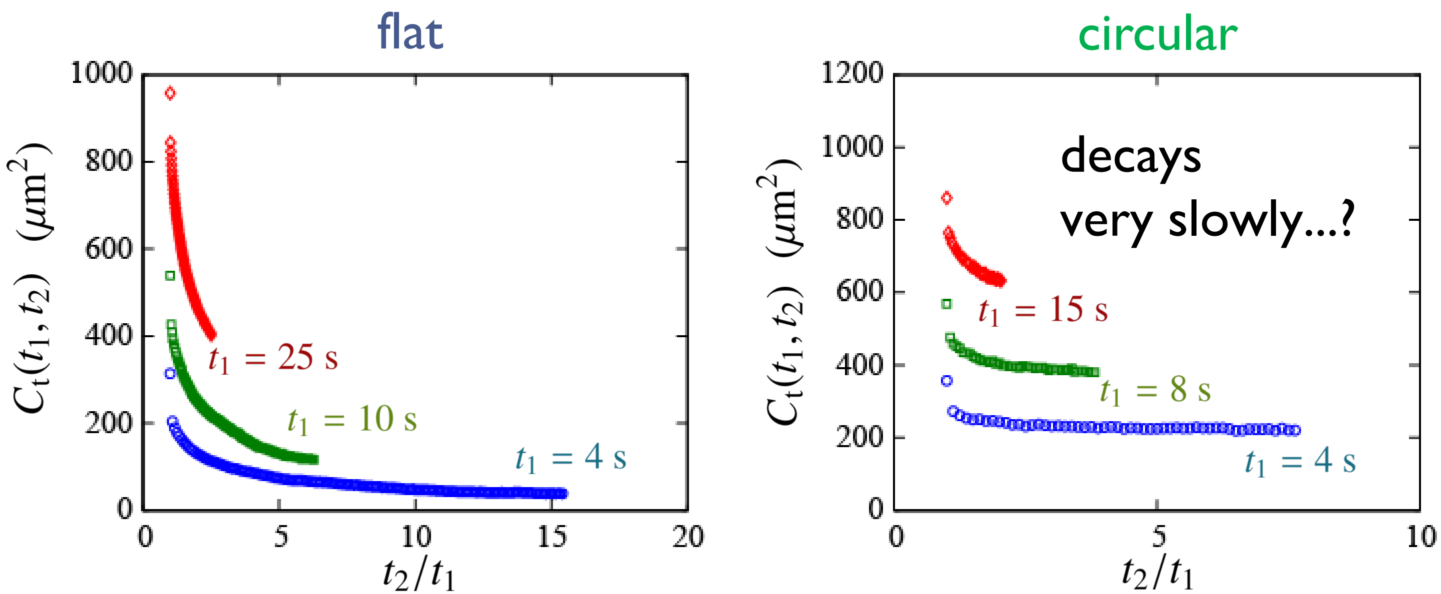
- Circular: $g_2(u) \sim u^{-2}$
- Flat: $g_1(u) \sim \exp(-\frac{1}{3}u^3)$

[review: KaT, Physica A 504, 77 (2018)]

# From Exp't to Theory & Math

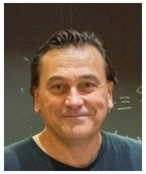
Time correlation $\boxed{C_{\mathrm{t}}(t_1, t_2) \equiv \langle \delta h(x, t_1) \delta h(x, t_2) \rangle}$ $\delta h(x,t) \equiv h(x,t) - \langle h(x,t) \rangle$

(theoretically difficult & little was understood)



[review: KaT, Physica A 504, 77 (2018)]

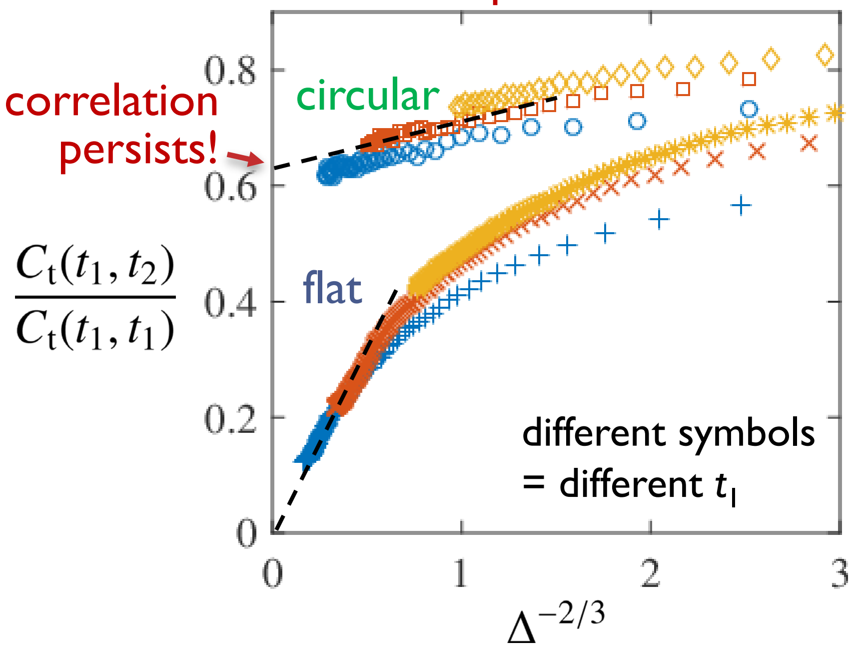
# From Exp't to Theory & Math



De Nardis (left)
Le Doussal (right)

Time corr. $\boxed{C_t(t_1, t_2) \equiv \langle \delta h(x, t_1) \delta h(x, t_2) \rangle}$

experiment



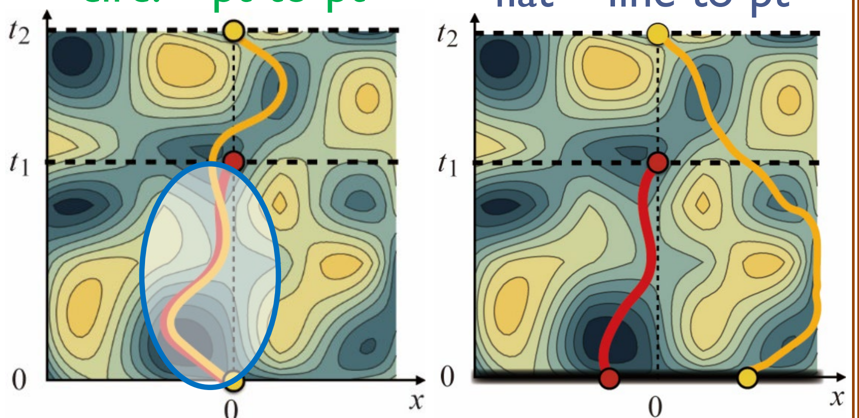

dimensionless time increment $\Delta \equiv \frac{t_2 - t_1}{t_1}$

theory

2-time problem = 2-polymer problem

circ. = pt-to-pt

flat = line-to-pt




polymers overlap → correlation persists!

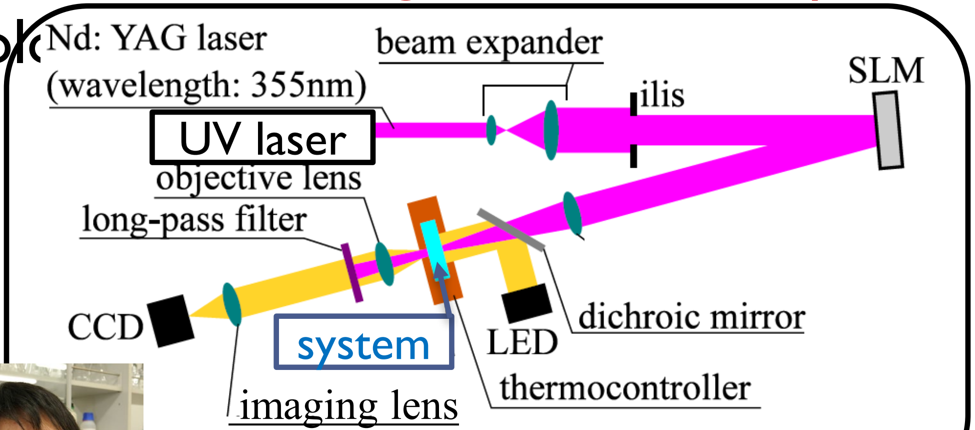
Approximate solution obtained by mapping to bosons.

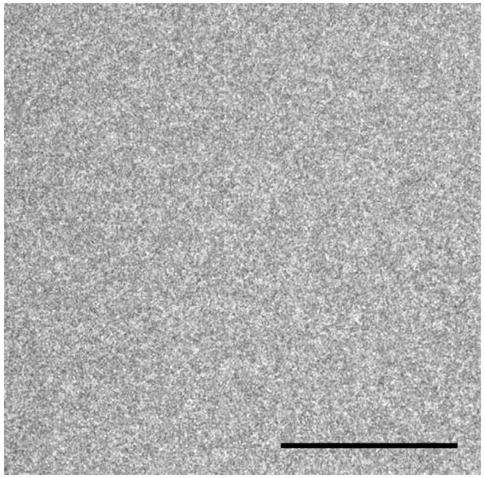
Later, solved mathematically & became a theorem.

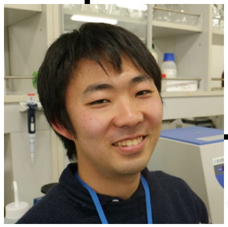
[Johansson, Probab. Theory Relat. Fields 2019]

# Exploring Various Geometries

We can now design the initial shape arbitrarily by laser hol

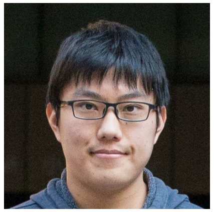


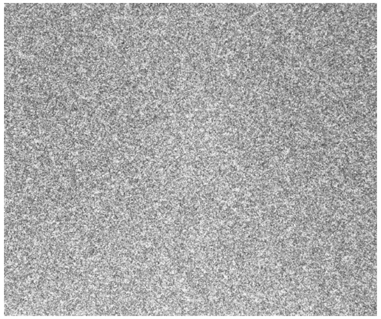





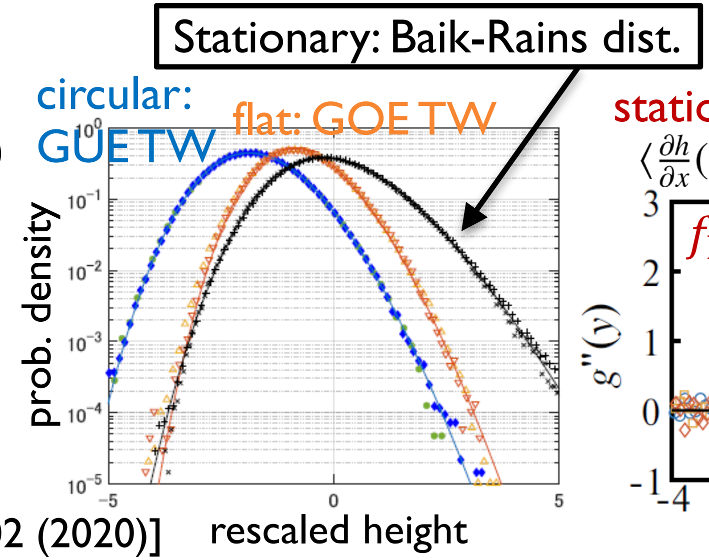
[Fukai & KaT, PRL 119, 030602 (2017)]

realized KPZ stationary state
(by generating, instead of waiting)

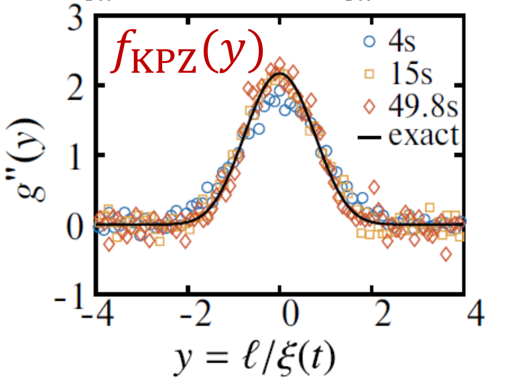




[Iwatsuka, Fukai, KaT, PRL 124, 250602 (2020)]



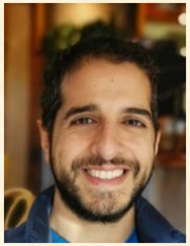

stationary 2pt corr. func.
$\langle \frac{\partial h}{\partial x}(x+\ell, t+t_0)\frac{\partial h}{\partial x}(x,t_0)\rangle$



# The Kardar-Parisi-Zhang universality class for driven interfaces and... <u>integrable spin chains?</u>

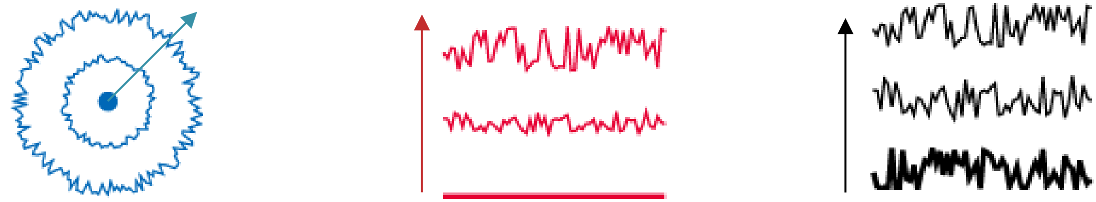
<u>Research part is based on collaboration with</u>

Kazuaki Takasan (Univ. Tokyo)
Ofer Busani (Univ. Edinburgh)
Patrik L. Ferrari (Bonn Univ.)
Romain Vasseur (Univ. Geneva)
Jacopo De Nardis (CY Cergy Paris Univ.)

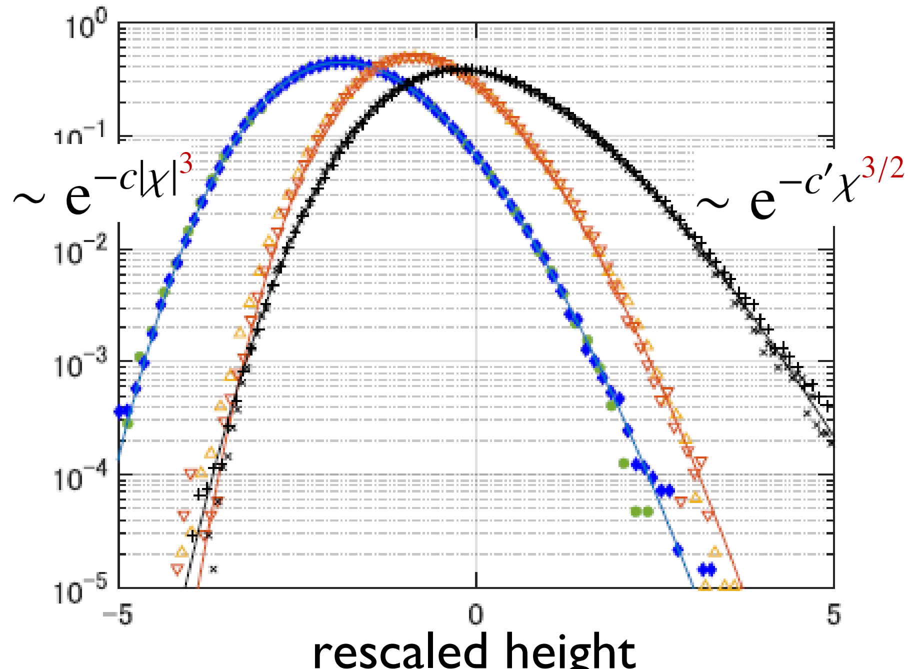




K. A. Takeuchi *et al.*, "Partial Yet Definite Emergence of the Kardar-Parisi-Zhang Class in Isotropic Spin Chains", Phys. Rev. Lett. **134**, 097104 (2025) [review: Takeuchi, Physica A 504, 77 (2018)]

# Reminder: KPZ Universality Subclasses



| | Circular | Flat | Stationary |
|---|---|---|---|
| exponents | height fluctuation $\delta h \sim t^{1/3}$, correlation length $\xi(t) \sim t^{2/3}$ | | |
| height distribution | GUE Tracy-Widom | GOE Tracy-Widom | **Baik-Rains** |

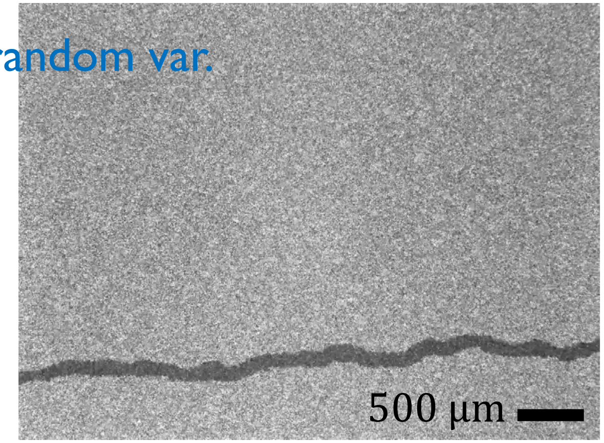



All these are asymmetric distributions.

[review: Takeuchi, Physica A 504, 77 (2018)]

# KPZ Stationary Subclass


liquid crystal experiment

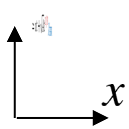



speed x5

[Iwatsuka, Fukai, Takeuchi, PRL 2020]

Note) stationary interface of 1D KPZ eq. = Brownian motion if $x$ is regarded as time

$$h(x,t)$$

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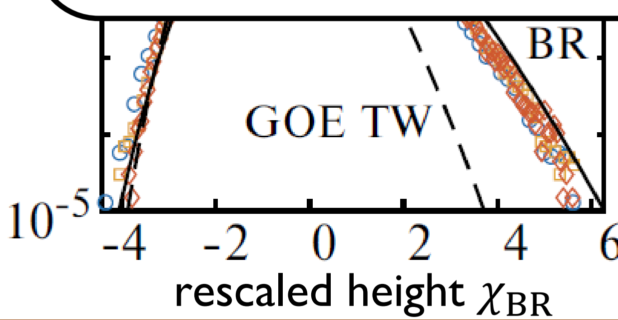
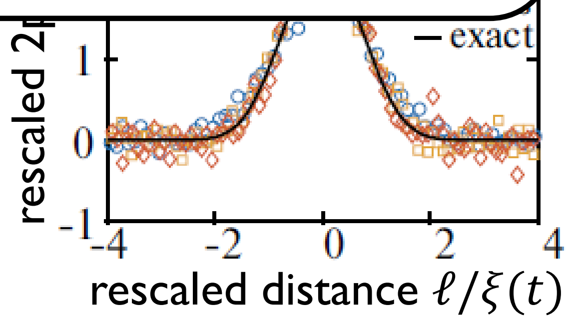



… and more known from exact studies

# Relation to Hydrodynamics?

KPZ's prehistory is in hydrodynamics [Forster, Nelson, Stephen, PRA 1977]
(before KPZ PRL 1986)

KPZ equation $\boxed{\frac{\partial}{\partial t} h(\vec{x}, t) = \nu \nabla^2 h + \frac{\lambda}{2} \left( \vec{\nabla} h \right)^2 + \eta(\vec{x}, t)}$

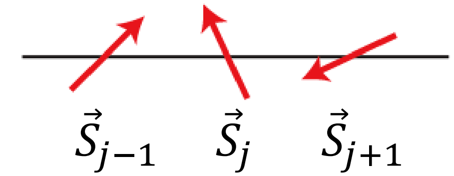
Take the gradient & define $\vec{v}(\vec{x}, t) \equiv -\lambda \vec{\nabla} h$

$\rightarrow \boxed{\frac{\partial \vec{v}}{\partial t} + \left( \vec{v} \cdot \vec{\nabla} \right) \vec{v} = \nu \nabla^2 \vec{v} - \lambda \vec{\nabla} \eta}$ : noisy Burgers equation
(toy model for fluid & shock waves)

Remarks

- KPZ's RG done & exponents obtained already in 1977.
- KPZ may describe
  some class of nonlinear fluctuating hydrodynamics
  (e.g., anharmonic chains; see Spohn's lecture notes arXiv:1505.05987)

e.g.) 1D Heisenberg model




# Now… Integrable Systems

Infinitely many conserved charges

→ Infinitely many hydrodynamic equations?

$$\frac{\partial q_1}{\partial t} + \frac{\partial j_1}{\partial x}(q_1, q_2, \cdots) = 0$$
$$\frac{\partial q_2}{\partial t} + \frac{\partial j_2}{\partial x}(q_1, q_2, \cdots) = 0$$
$$\vdots$$

A better approach

- Diagonalize all conserved charges
  → Eigenstates $|\theta_1, \cdots, \theta_N\rangle$ ($\theta_i$: quasimomentum, $N$: # of particles)
- $N \to \infty$ limit → quasimomentum density $\rho(\theta)$
- Generalized hydrodynamics (GHD)

$$\boxed{\frac{\partial}{\partial t}\rho(\theta; x, t) = -\frac{\partial}{\partial x}\left[v^{\mathrm{eff}}(\theta; x, t)\rho(\theta; x, t)\right]}$$

[Castro-Alvaredo et al., PRX 2016; Bertini et al., PRL 2016]

with an explicit formula for $v^{\mathrm{eff}}(\theta; x, t)$

→ Powerful! Drude weight can be directly evaluated.

# Transport of Integrable Systems

- AC conductivity (of $i$th charge): $\sigma_i(\omega) = \underline{D_i}\delta(\omega) + \underbrace{\sigma_i^{\mathrm{reg}}(\omega)}_{\text{normal (diffusive)}}$

- Drude weight: $D_i = \lim_{t\to\infty}\lim_{L\to\infty}\frac{1}{L}\langle \underbrace{J_i(t)}_{=\int j_i dx\text{: total current}} J_i(0)\rangle$

  = ballistic contribution

- Formula by Doyon & Spohn $\boxed{D_i = \lim_{L\to\infty}\sum_{j,k}\frac{\langle J_i Q_j\rangle\langle Q_k J_i\rangle}{\langle Q_j Q_k\rangle}}$

  $Q_i = \int q_i dx$ total charge

  [SciPost Phys 2017]

  - $D_i > 0$ if $J_i$ overlaps with some charges (usually the case) ➜ ballistic transport in most integrable systems
  - $D_i = 0$ if $J_i$ doesn't overlap with any charge ➜ diffusive

- Heisenberg XXZ chain $\boxed{H = J\sum_{j=1}^{N}\left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta\, S_j^z S_{j+1}^z\right)}$ (integrable for all $\Delta$)

  - No overlap $\langle J_{S^z} Q_j\rangle = 0$ for $\Delta > 1$ ➜ $D_{S^z} = 0$, diffusive
  - Ballistic for $\Delta < 1$
  - $\Delta = 1$: critical. GHD ➜ $\sigma_{S^z}(\omega) \sim \omega^{-1/3}$, $\xi(t) \sim t^{2/3}$: superdiffusive

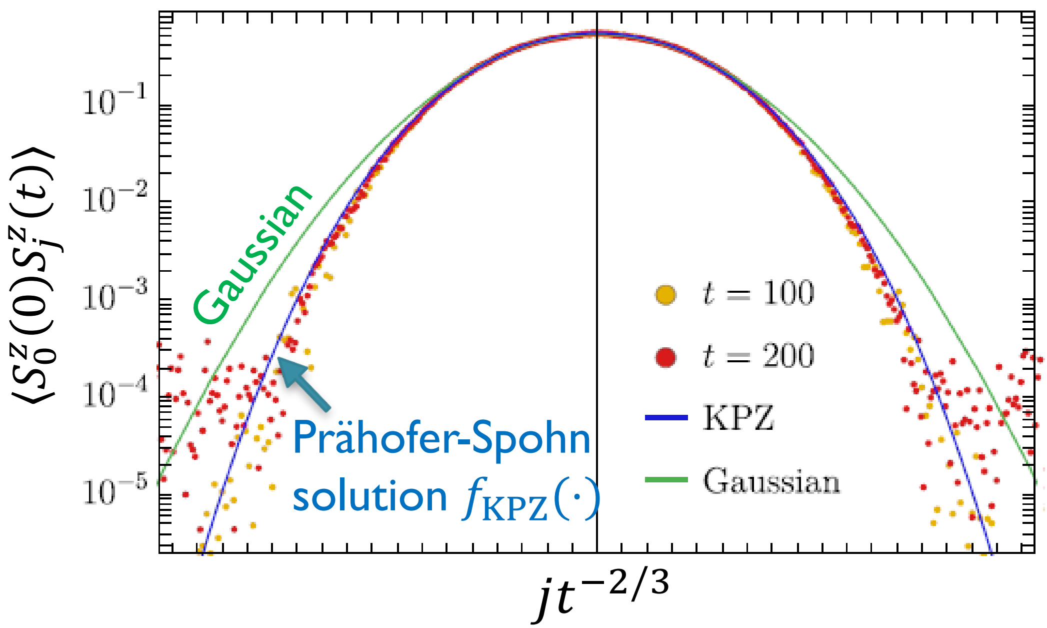
    KPZ?

# Real Surprise:

[Prosen group: Ljubotina et al., PRL 122, 210602 (2019)]

Equilibrium spin 2pt function $C_2(j,t) \equiv \langle S_0^z(0) S_j^z(t) \rangle$

agrees with stationary KPZ 2pt func $\langle \frac{\partial h_0}{\partial x}(0,0) \frac{\partial h_0}{\partial x}(x,t) \rangle$




relation b/w $S_i^z(t)$ & $h(x,t)$

spin = interface slope

$S_j^z(t) = -\frac{\partial h_0}{\partial x}(x,t)$

height = integrated spin current

$$h(x,t) = \int_0^t J^z(x,\tau)\, d\tau$$

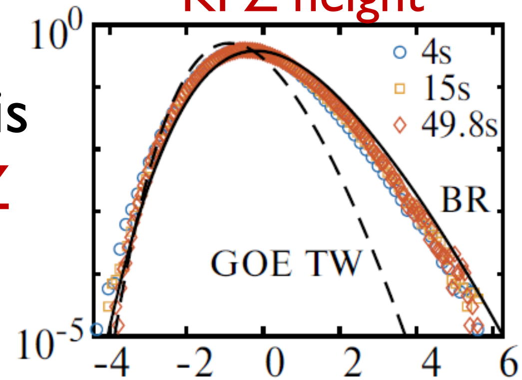
- Valid for isotropic integrable spin chains (both quantum/classic)

[e.g., Ye et al. PRL 129, 230602 (2022); Das et al. PRE 100, 042116 (2019)]

# A Pain in the Neck

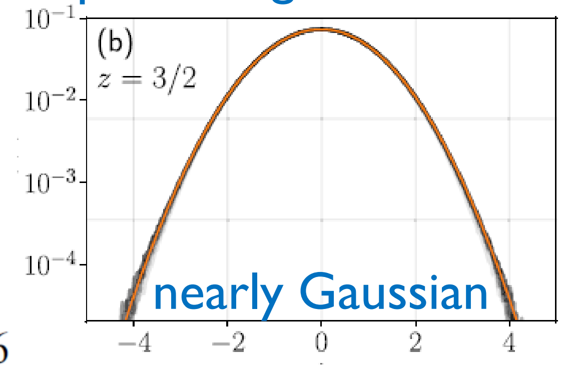
- Height distribution is asymmetric for KPZ symmetric for spins

KPZ height




[Iwatsuka et al., PRL 2020]

spins' integrated current




[Krajnik et al., PRL 2022]

- Fluctuating hydrodynamics proposed by De Nardis et al., (PRL 2023)

$m$: magnetization, $\phi$: spin fluid velocity

$$\boxed{\begin{aligned}&\partial_t m + \partial_x\left(m\phi - D_m\partial_x m - \sqrt{2D_m\chi}\xi_m\right) = 0,\\ &\partial_t\phi + \partial_x\left(\lambda_m\frac{m^2}{2} + \lambda_\phi\frac{\phi^2}{2} - D_\phi\partial_x\phi - \sqrt{2D_\phi\chi}\xi_\phi\right) = 0.\end{aligned}}$$

$t \to \infty$

Decoupled to 2 independent Burgers eqs for $u_\pm \equiv m \pm \phi$ (each $u_\pm$ behaves like $\mp\partial_x h_{\mathrm{KPZ}}$)

$$\boxed{\text{then } m = \tfrac{1}{2}(u_+ + u_-)}$$

- Kurtosis

| KPZ stationary | hydro 2023 | spins |
|---|---|---|
| $\mathrm{Ku[BR]} = \mathbf{0.289}$ | $\frac{1}{2}\mathrm{Ku[BR]} = \mathbf{0.145}$ | $\approx \mathbf{0.02?}\ \ (\mathbf{0?})$ |

Hydro 2023 doesn't work, at least as is...

[Krajnik et al PRL 2024; Rosenberg et al. Science 2024]

# Is it really KPZ?

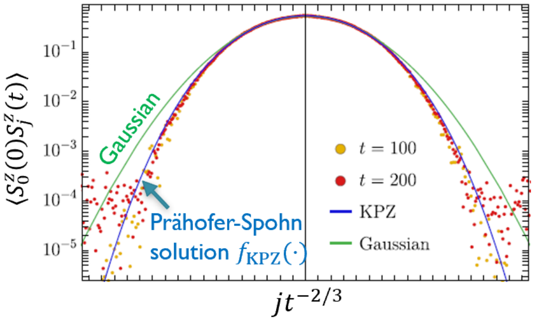
- Kurtosis

| KPZ stationary | hydro 2023 | spins |
|---|---|---|
| $\mathrm{Ku[BR]} = \mathbf{0.289}$ | $\frac{1}{2}\mathrm{Ku[BR]} = \mathbf{0.145}$ | $\approx \mathbf{0.02}$ |

[Krajnik et al PRL 2024;
Rosenberg et al. Science 2024]

- Agreement so far: ***only*** exponents & Prähofer-Spohn (PS) 2pt func.

- Agreement on PS 2pt function was claimed with arbitrarily tuned prefactors




Evidence for KPZ has been weak.
A new universality class was called for.

Let's determine its fate
by using a full body of knowledge from KPZ exact solutions

[K. A. Takeuchi *et al.*, Phys. Rev. Lett. **134**, 097104 (2025)]

# Model

※ simplest lattice Landau-Lifshitz is not integrable
$\dot{\vec{S}}_j = \vec{S}_j \times (\vec{S}_{j+1} + \vec{S}_{j-1})$

Quantum: isotropic Heisenberg chain (XXX)

Classic: Ishimori chain [Ishimori, JPSJ 1982]

(integrable variant of isotropic lattice Landau-Lifshitz model)

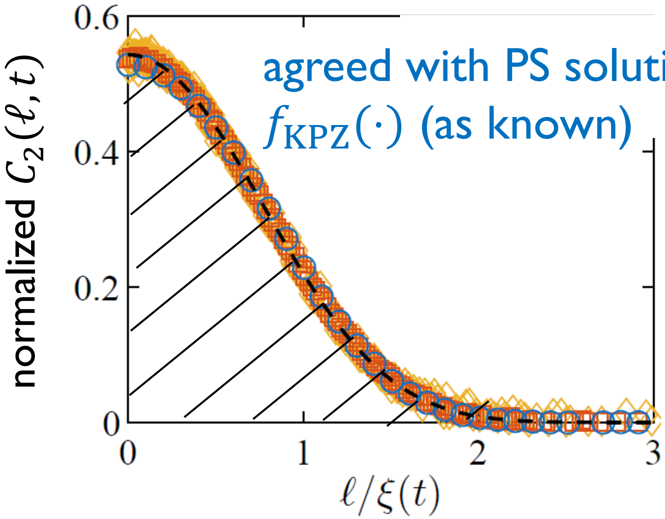
$$\boxed{\frac{\partial \vec{S}_j}{\partial t} = \frac{\vec{S}_j \times \vec{S}_{j+1}}{1 + \vec{S}_j \cdot \vec{S}_{j+1}} + \frac{\vec{S}_j \times \vec{S}_{j-1}}{1 + \vec{S}_j \cdot \vec{S}_{j-1}}}$$

Discretized in a specific manner that maintains the integrability
[Krajnik et al., SciPost Phys 11, 051 (2021)]

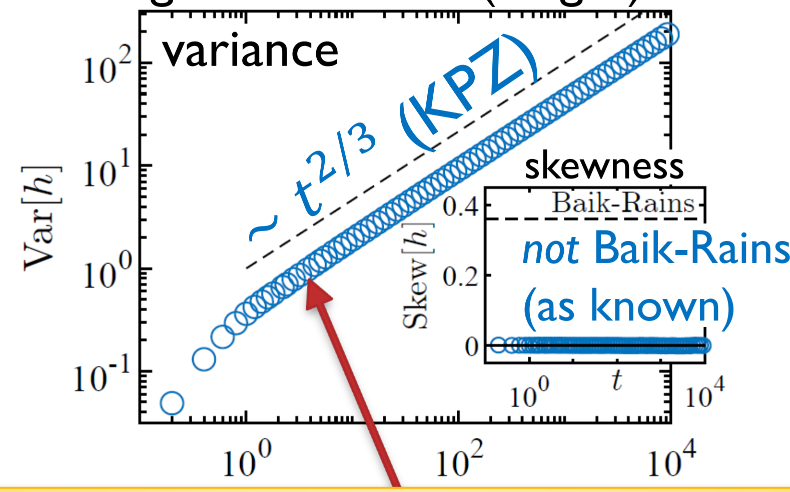
- Classic spins → large-scale simulations. We'll mainly use it. (quantum case is also checked)
- Initial state: infinite-temperature equilibrium state (each $\vec{S}_j$ drawn from uniform distribution on a unit sphere)
- Parameters: 40000 sites, time step 0.1, 10000 realizations

# 2pt Function & Integrated Current (Height)

2pt function $C_2(\ell,t) \equiv \langle S_j^z(0) S_{j+\ell}^z(t) \rangle$




cumulants of
integrated current (height) $h$




Baik-Rains controls the coeff of $t^{2/3}$ law!
(even though BR dist doesn't appear)

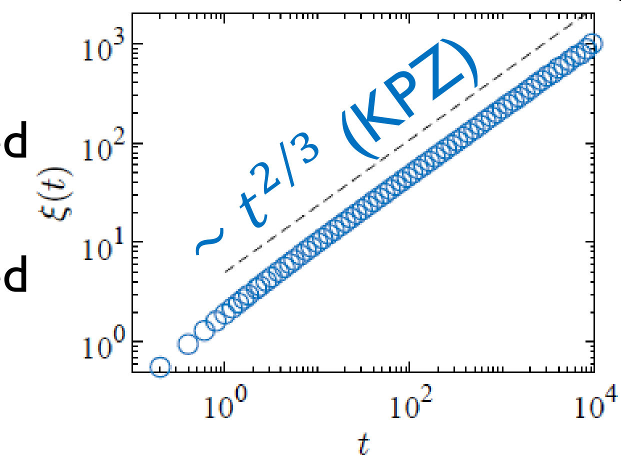
## Conservations

$\boxed{\sum_j S_j^z(t) \text{ cons'd} \Rightarrow \int C_2(\ell,t)d\ell \text{ cons'd}}$ ⇨ correlation length $\xi(t)$ determined

## KPZ scaling laws

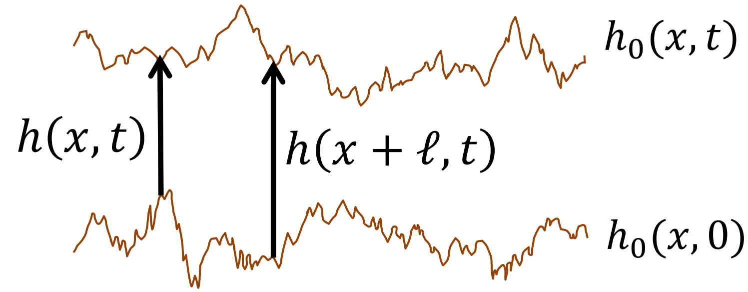
- $\boxed{C_2(x,t) \simeq \frac{2At^{2/3}}{\xi(t)^2} f_{\mathrm{KPZ}}\left(\frac{x}{\xi(t)}\right)}$ ⇨ $A$ determined $\equiv A_1$
- $\boxed{\mathrm{Var}[h] \simeq At^{2/3}\,\mathrm{Var}[\mathrm{BR}]}$ ⇨ $A$ determined $\equiv A_2$

Baik-Rains variance ≈ 1.15

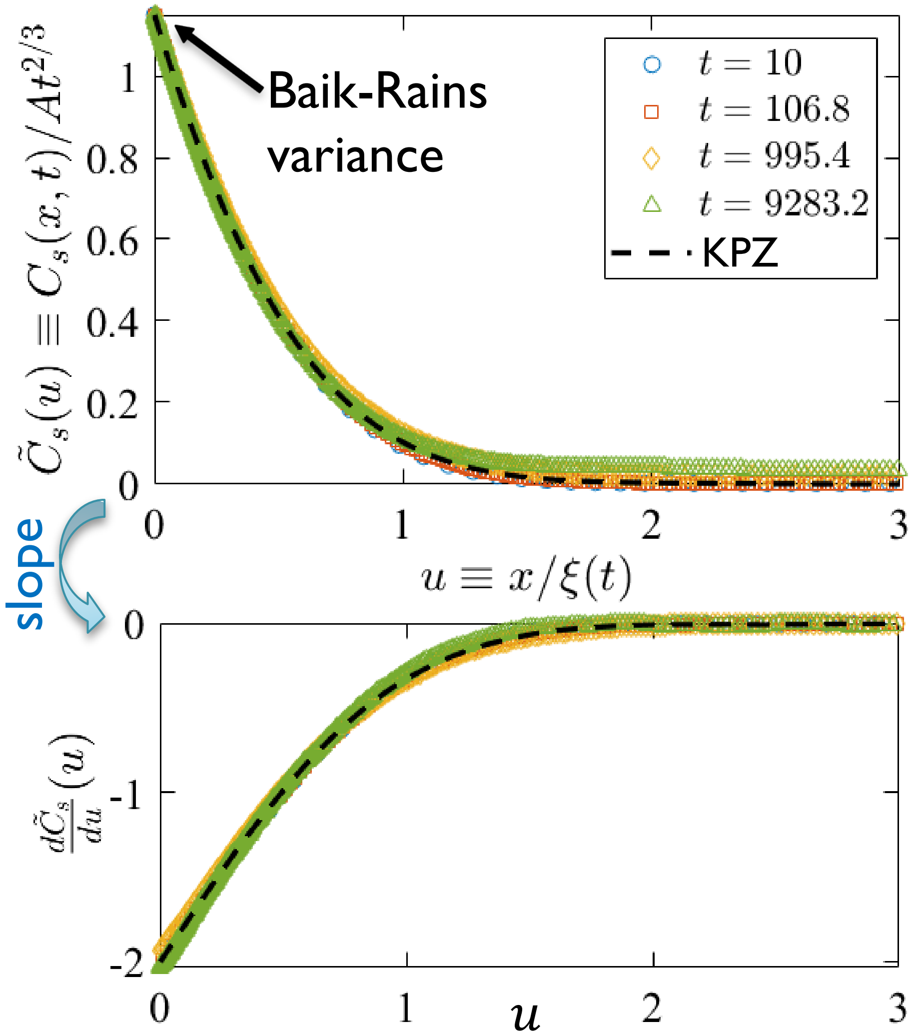


# Spatial Correlation of Integrated Current

$$\boxed{C_s(\ell,t) \equiv \langle \delta h(x+\ell,t)\,\delta h(x,t)\rangle}$$




$$\delta h(x,t) \equiv h(x,t) - \langle h(x,t)\rangle$$




- Precise agreement (w/o fit) between spins & KPZ!

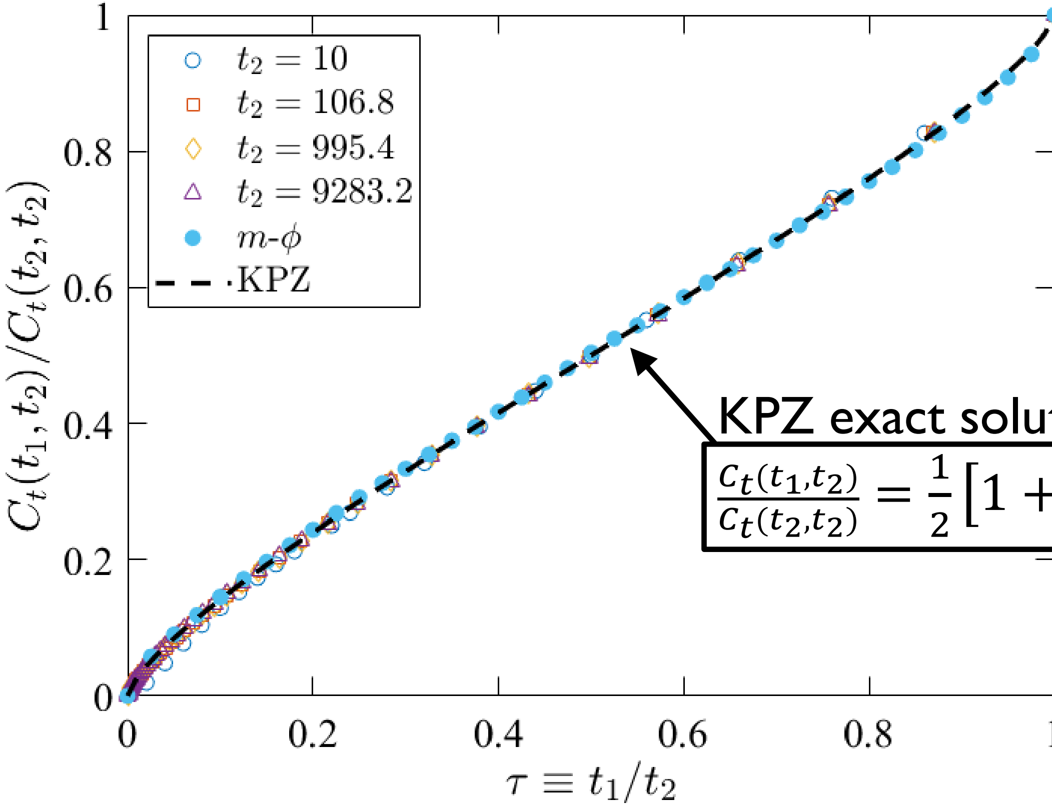
※ No study on this quantity in KPZ, so we formulated: "Airy$_0$ process".

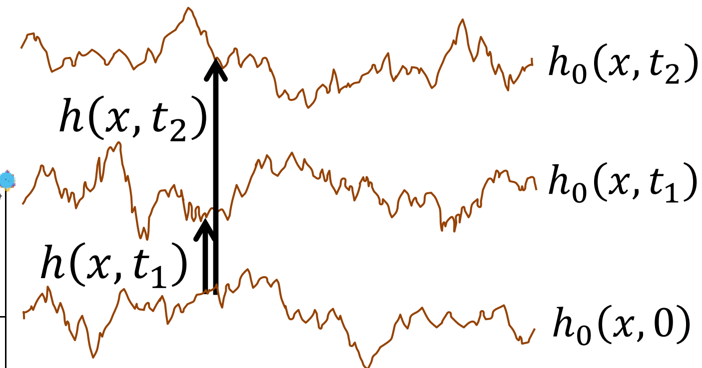
- $\frac{d\tilde{C}_s}{du}(0) = -2$ proven mathematically (Busani & Ferrari)

※ In KPZ language, this implies loss of correlation of two stationary interfaces $h_0(x,0), h_0(y,t)$ for $y \to x$

# Time Correlation of Integrated Current

$$C_t(t_1, t_2) \equiv \langle \delta h(x, t_1) \delta h(x, t_2) \rangle$$




KPZ exact solution (Ferrari & Spohn, SIGMA 2016)

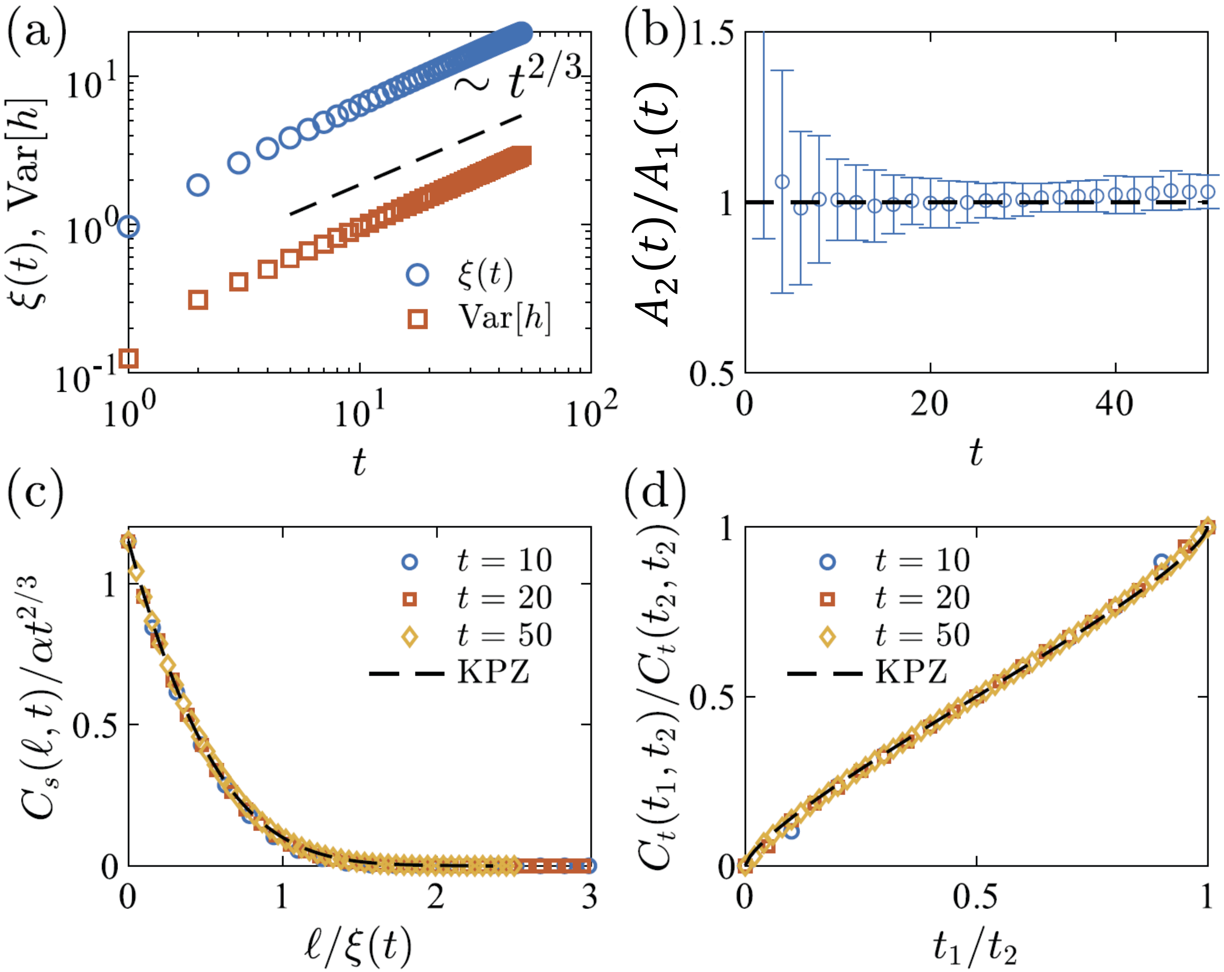
$$\frac{C_t(t_1,t_2)}{C_t(t_2,t_2)} = \frac{1}{2}\left[1 + \tau^{2/3} - (1-\tau)^{2/3}\right]$$

KPZ scaling holds! (again, w/o fitting parameter)

# Quantum Case

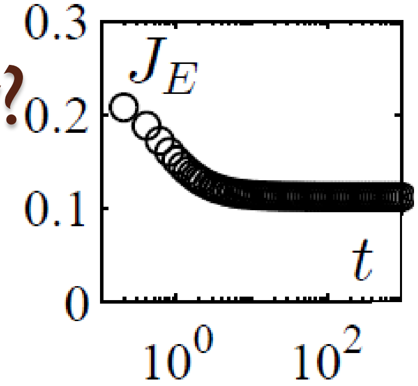
Kazuaki Takasan

Quantum Heisenberg model, numerically solved by TEBD




Same conclusion for the quantum case too

# What if there's no Left/Right Symmetry?

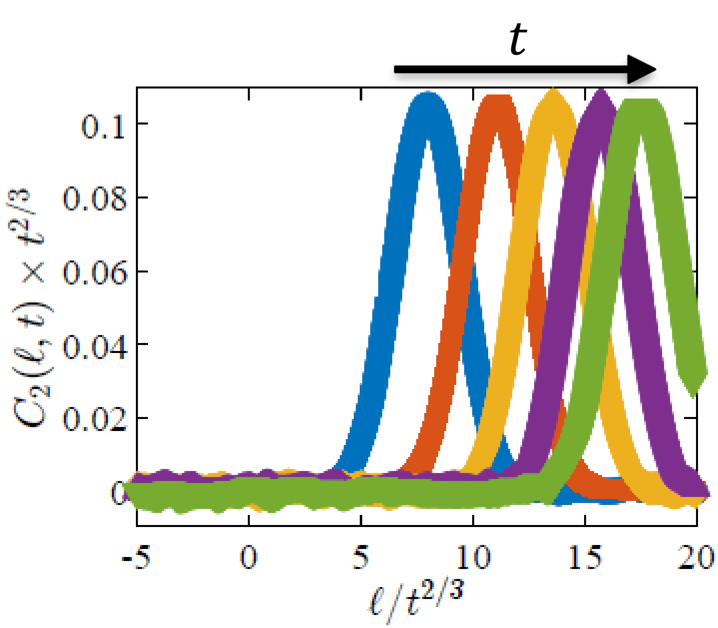



Let's study the case with finite energy current

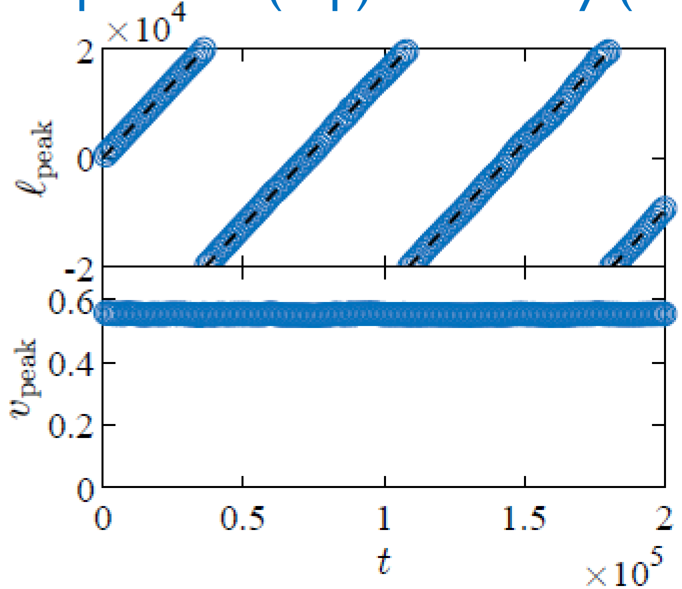
- Initial condition generated by weight $e^{J_E}$

  with total energy current $J_E \equiv -\sum_j \vec{S}_j \cdot (\vec{S}_{j+1} \times \vec{S}_{j+2})$

→ 2pt function $C_2(\ell, t) \equiv \langle S_j^z(0) S_{j+\ell}^z(t) \rangle$ now propagates!




peak position (top) & velocity (bottom)

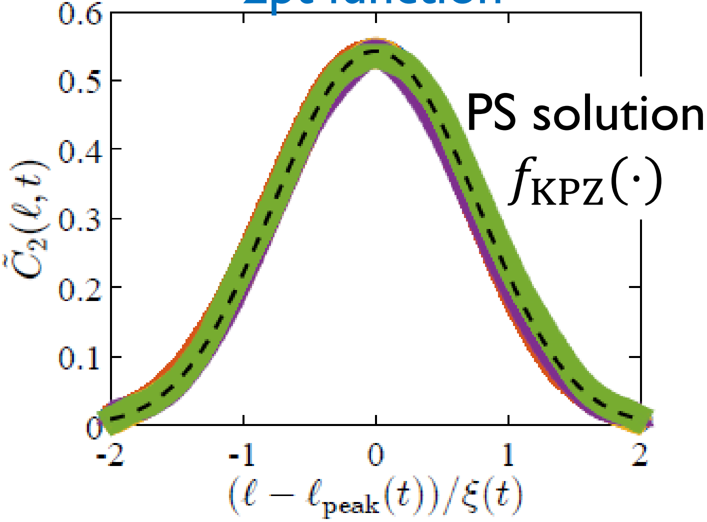



→ Let's consider a frame comoving at velocity $v_{\mathrm{peak}}$

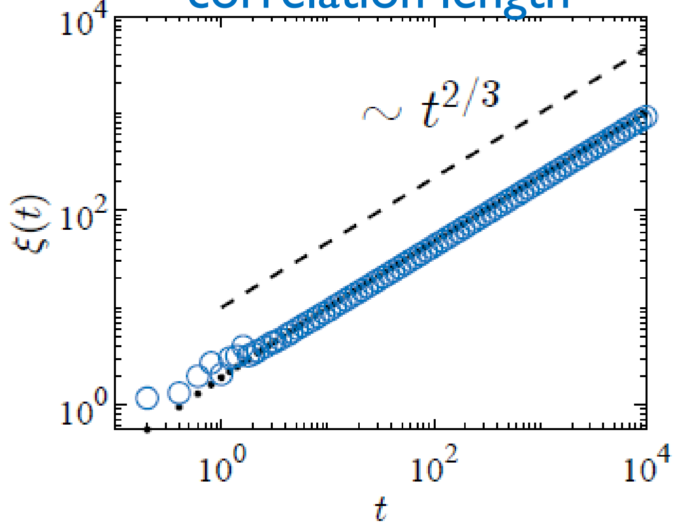
# KPZ is Intact in Comoving Frame!

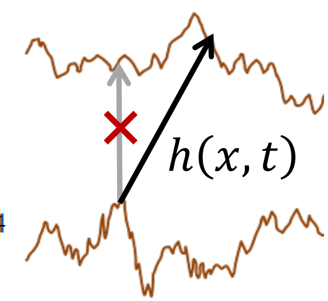
(only about 2pt quantities, just like the case w/o energy current)

2pt function

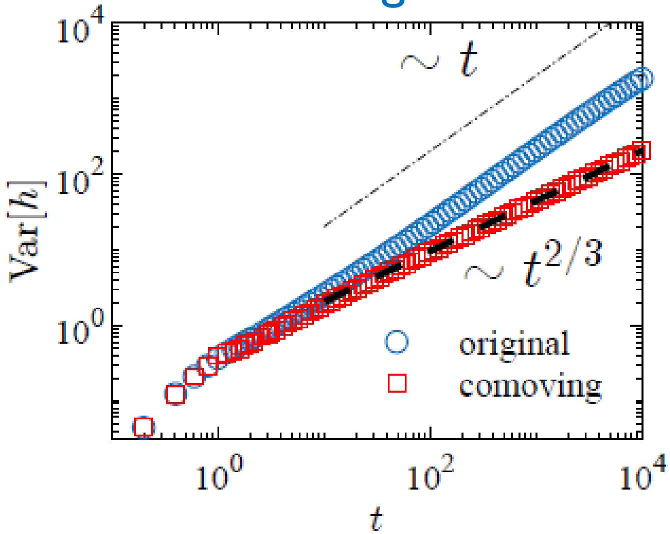


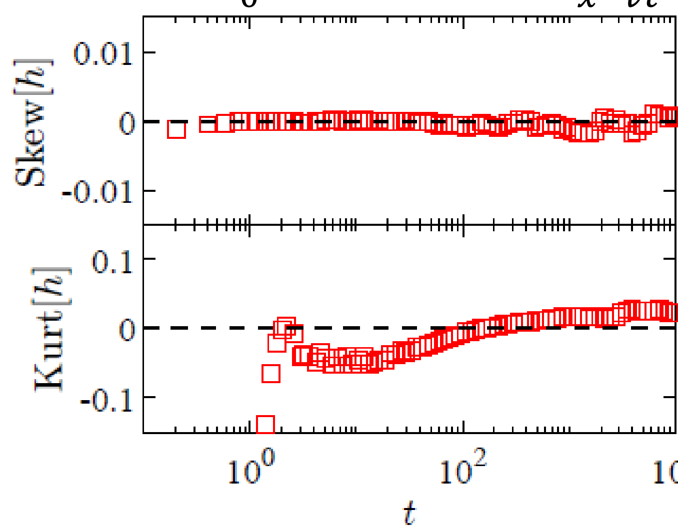

correlation length




cumulants of integrated current $h(x,t) \equiv \int_0^t J^z(x,t')dt' - \int_{x-vt}^{x} S^z(x',0)dx'$



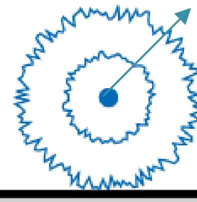
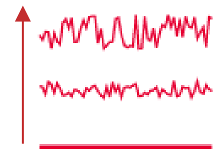
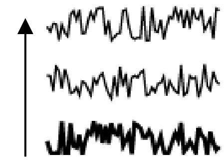




$\approx 0$ ($\neq$ Baik-Rains skewness 0.359)

$\approx 0.02$ ($\neq$ BR kurtosis 0.289, but consistent w/ case w/o current)

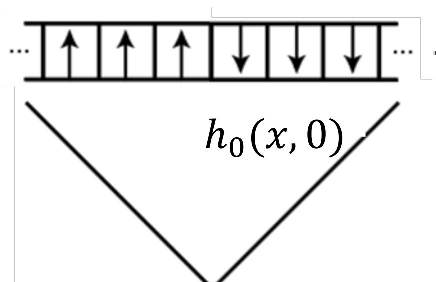
# Universality Subclasses?

[review: Takeuchi, Physica A 504, 77 (2018)]

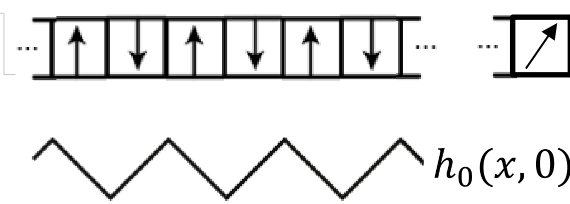
| KPZ interfaces | Circular | Flat | Stationary |
|---|---|---|---|
| exponents | height fluctuation $\delta h \sim t^{1/3}$, correlation length $\xi(t) \sim t^{2/3}$ | | |
| height distribution | GUE Tracy-Widom | GOE Tracy-Widom | Baik-Rains |
| Difference is set by initial condition |  |  |  |

**for spins** possibly related initial condition (recall: $S_j^z = -\frac{\partial h_0}{\partial x}$)

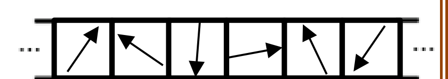
domain wall



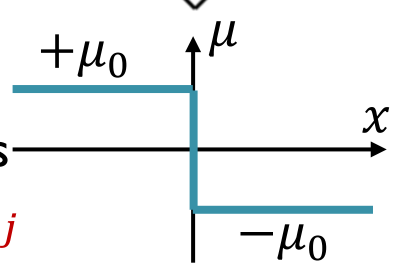

Néel




**equilibrium**



Randomness required for deterministic models → chem. potential $\mu_j$ favoring $S_j^z = +1$




Initial spins generated by prob. density $\rho\left(S_j\right) \propto e^{\mu_j S_j^z}$

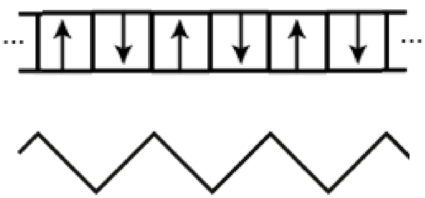
KPZ destroyed by broken isotropy (non-Abelian sym.) [Krajnik et al., PRL 2022]

at time $\sim \mathcal{O}(\mu_0^{-3})$ [Gopalakrishnan et al., PNAS 2019]

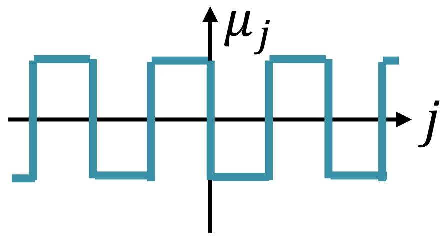
# Flat Case? What Initial Condition to Use?

chemical potential for initial states

Néel



alternating




Ornstein-Uhlenbeck-like

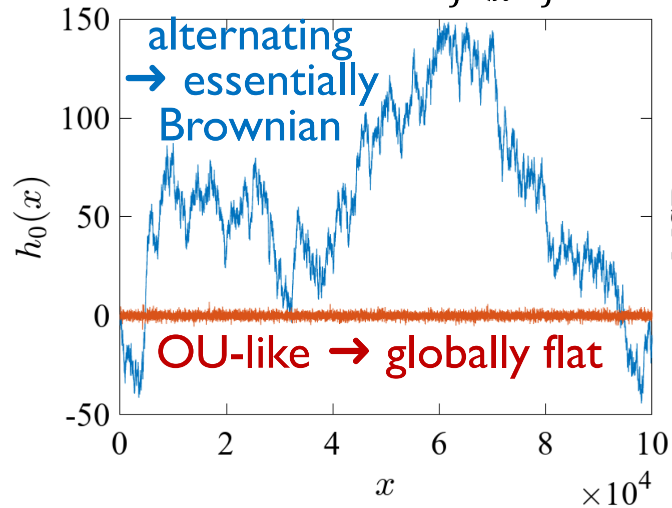
$\vec{\mu}_j$ favors $\vec{S}_j \parallel \vec{\mu}_j$ ($\rho \propto e^{\vec{S}_j \cdot \vec{\mu}_j}$)

$\vec{\mu}_0 = 0$ ($\vec{S}_0$ random)

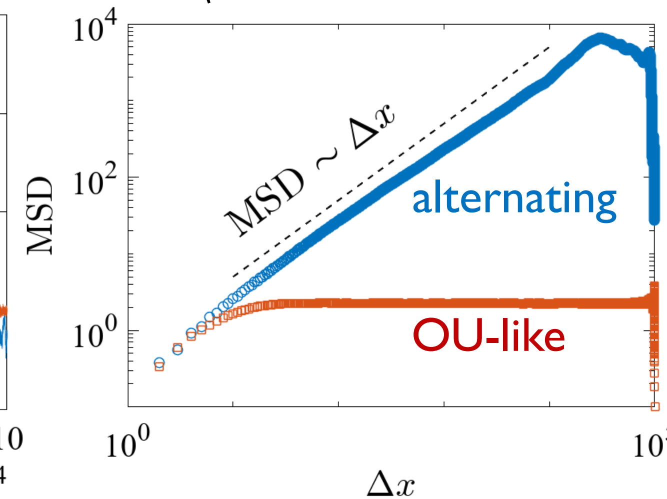
$\vec{\mu}_j$ s.t. $|\vec{\mu}_j| = \mu$ & $\vec{\mu}_j \parallel -\sum_{i<j} \vec{S}_i$

initial interface profile

$h_0(x) = -\sum_{j<x} S_j^z$




MSD, $\langle [h_0(x+\Delta x) - h_0(x)]^2 \rangle$




Let's use this Ornstein-Uhlenbeck-like initial condition!

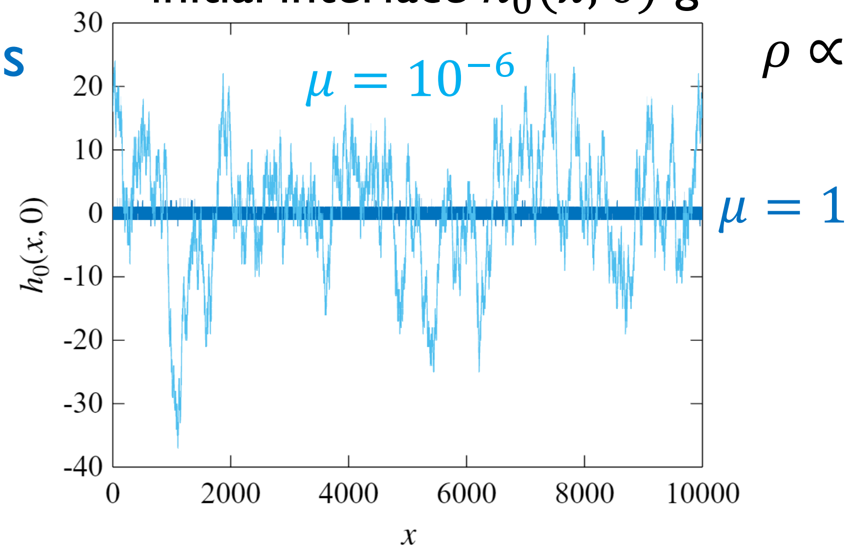
(note: isotropy is broken only locally, kept globally)

# Ornstein-Uhlenbeck Initial Condition

KPZ interfaces (TASEP)

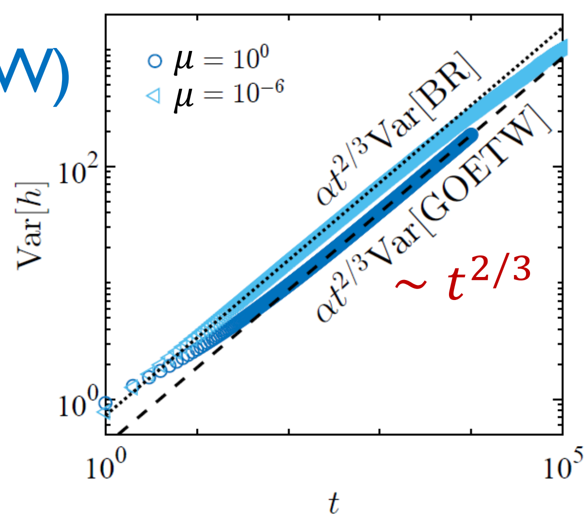
initial interface $h_0(x,0)$ generated by

$\rho \propto \exp\left(-\mu h_0(x,0)^2\right)$

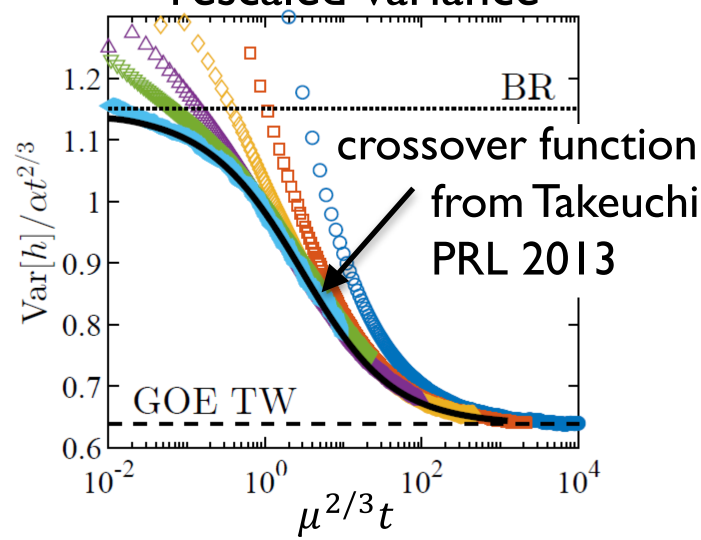



stationary (BR)
→ flat (GOE TW)
crossover

variance vs time




rescaled variance



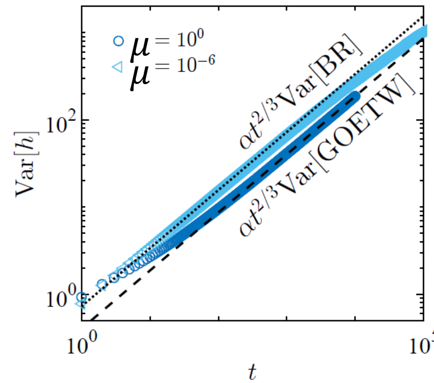
# Ornstein-Uhlenbeck Initial Condition

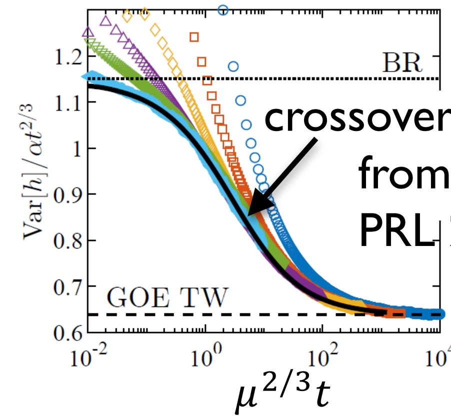
KPZ interfaces

stationary (BR)

→ flat (GOE TW)

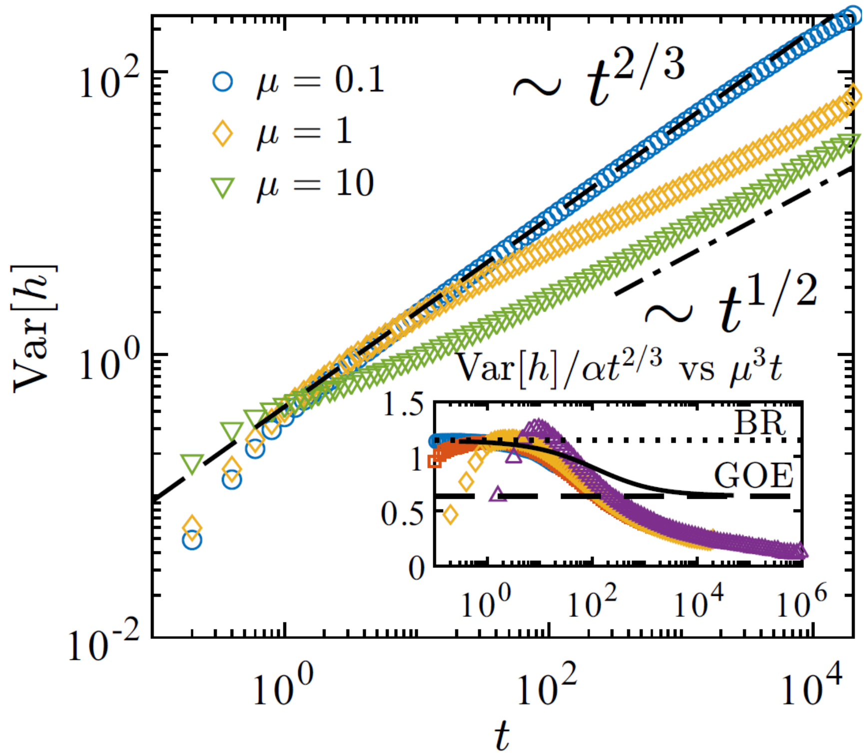
crossover




crossover function from Takeuchi PRL 2013

Spins




KPZ is destroyed even though isotropy was only locally broken

KPZ flat subclass is still elusive in spin chains.

K. A. Takeuchi *et al.*, Phys. Rev. Lett. **134**, 097104 (2025)

# Summary

## Partial yet definite emergence of KPZ in isotropic integrable spin chains!

- All 2pt quantities agreed with KPZ, including the prefactors, w/o fitting!
  (spin-spin corr., integrated current variance and its space & time 2pt corr.)
- Nevertheless non-2pt quantities are totally different
  (in particular, 1pt distribution, i.e., mean, skewness, kurtosis, ...)
- How to understand? (RG fixed point? Is there a quantity that fully captures KPZ?)
- How robust? Finite energy current → KPZ intact. Other cases?
- Non-equilibrium case? (in particular, circular & flat subclasses?)

| | Isotropic spin chains | KPZ class |
|---|---|---|
| Exponents | $\xi(t) \sim t^{2/3}, \mathrm{Var}[\delta h] \sim t^{2/3}$ | |
| Spin-spin corr. $C_2(x,t)$ | Prähofer-Spohn solution ※prefactors agreement found | |
| Space corr. of int'd current | $\mathrm{Airy}_0$ correlation function | |
| Time corr. of int'd current | Ferrari-Spohn solution | |
| 1pt dist. of int'd current | nearly Gaussian (symmetric) | Baik-Rains (asymmetric) |

new results