Quantum dynamics and entanglement in open systems

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Part-A

Collective theory for Page Curve–like Dynamics of a Freely Expanding Fermionic Gas
Saha, **MK**, Dhar (PRL 2024)


Madhumita Saha (ICTS) Abhishek Dhar (ICTS)

Part-B

Page curve like dynamics in Interacting Quantum Systems, Ray, Dhar, **MK**
(arXiv:2504.14675)


Tamoghna Ray (ICTS) Abhishek Dhar (ICTS)

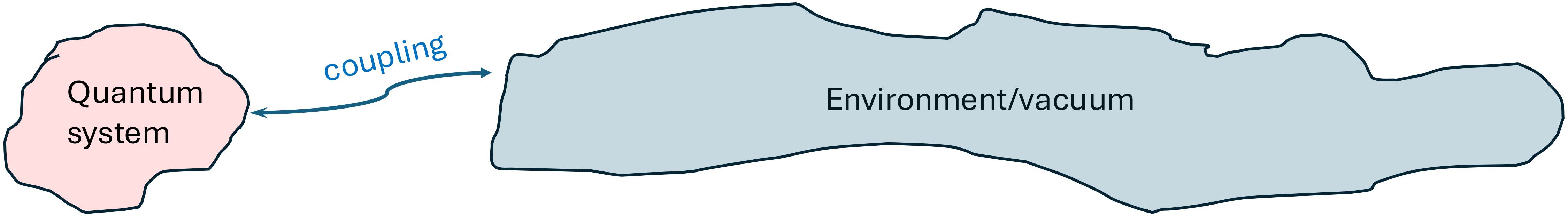
Part-C

Quantum injection of effectively or inherently interacting particles
Manuscript in preparation (2025)



Tamoghna Ray (ICTS) Katha Ganguly (IISER Pune) Bijay Agarwalla (IISER Pune)

Central Platform



- Number of particles that leave or enter the system as a function of time ?
- What about density profiles, currents, number fluctuations, higher cumulants ?
- How long does it take to empty or fill a system ?
- What about entanglement entropy between system and its complement (environment) ?
- Is there a field theory description / rate equation that captures essential features ?
- Is there some connection between entanglement entropy and Boltzmann entropy ?
- How imperfections (such as dephasing) or inherent interactions gets encoded in quantities mentioned above ?
- What about long-ranged models ?

Part A

Collective theory for Page Curve–like Dynamics of a Freely Expanding Fermionic Gas

Page (1993, 2013)

- Possible relevance to black holes

- Entanglement between black holes and the radiation starting from the unentangled initial state of just the back hole

- As the black hole radiates, the effective Hilbert space dimension of the radiation increases and there will be a corresponding increase in the entanglement entropy.

- However, this increase has to stop at some time when the black hole and radiation have the same Hilbert space dimensions. Beyond this time (referred to as the “Page time”), the entropy has to decrease.

Part A

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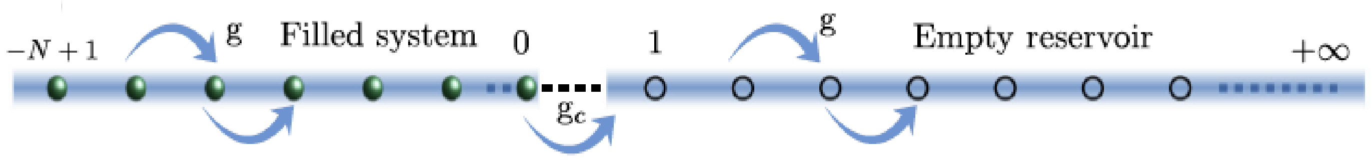

$$\hat{H} = \sum_{i,j=-N+1}^{\infty} h_{ij} \hat{c}_i^\dagger \hat{c}_j$$

$$h_{i,j} = -g(\delta_{i,j+1} + \delta_{i+1,j}) \ \forall\ i,\ j \neq 1,\ 0.$$

Calabrese, Cardy (2005)
Bertini, Fagotti, Piroli, Calabrese (2018)
Alba, Bertini, Fagotti, Piroli, Ruggiero (2021)
and many more..

Very recent: Kehrein (2024), Glatthard (2024,2025)

- The fact that g_c is not equal to g is why we call it "defect". We consider three types of defects: conformal, hopping, onsite.

Bilinear Hamiltonians

Length of reservoir

Total Hamiltonian $\hat{H} = \sum_{i,j=-N+1}^{N_b} h_{ij}\hat{c}_i^\dagger \hat{c}_j$

Correlation matrix

$C_{ij} = \langle c_i^\dagger c_j \rangle$

Dynamics

$C(t) = e^{iht} C(0) e^{-iht}$.

Quantities of interest that can be extracted from correlation matrix

Density $\rho(i) = \langle \hat{c}_i^\dagger \hat{c}_i \rangle$

Current $I = 2g_c \mathrm{Im}[\langle \hat{c}_0^\dagger \hat{c}_1 \rangle]$

For Gaussian initial states we can compute the following

Eigenvalues of part of correlation matrix

Von-Neumann Entanglement Entropy $S = \mathrm{tr_s}[\rho_s \log \rho_s] = -\sum_{l=1}^{N} [(1-m_l)\log[1-m_l] + m_l \log m_l]$

Particle number fluctuations in system $\kappa_2 = \langle \hat{\mathcal{N}}^2 \rangle - \langle \hat{\mathcal{N}} \rangle^2$ $\sum_{\ell=1}^{N} m_\ell (1-m_\ell)$

Generalized Hydrodynamic Description

Reviews: Doyon (2020)
Essler (2022)

- The evolution of integrable systems observed on large time and length scales is described by generalized hydrodynamics

- The idea is that the system is a gas of quasiparticles that carry fixed momentum labels k and has a phase-space density $n_t(x, k)$.

- These quasiparticles drift with velocities which is given by the derivative of dispersion relation.

Euler equation: $$(\partial_t + \sin[k]\partial_x)n_t(x, k) = 0.$$

$\sin[k]$ → in our case

$n_t(x,k)$ → Coarse grained wigner function

} Reminiscent of collisionless Boltzmann equation / kinetic theory

- This equation needs to be solved with appropriate boundary conditions

- From $n_t(x, k)$, quantities of interest can be extracted such as density, average current, “hydrodynamic” entropy

- We can essentially get analytical solutions

Solutions to Generalized Hydrodynamic Description

Recall: $(\partial_t + \sin[k]\partial_x)n_t(x, k) = 0.$

Solution on infinite line: $n_t(x, k) = n_0(x - t\sin(k), k)$ where $n_0(x, k) = \theta(-x) - \theta(-x - N)$
(boost the function)

Solutions to Generalized Hydrodynamic Description

Recall: $(\partial_t + \sin[k]\partial_x) n_t(x,k) = 0.$

Solution on infinite line: (boost the function) $n_t(x,k) = n_0(x - t\sin(k), k)$ where $n_0(x,k) = \theta(-x) - \theta(-x-N)$

For our case, it is crucial to consider boundary conditions which is going to be instrumental in capturing Page-Curve

Transmission Reflection

$$n_t(x>0,k) = n_t(x>0,k>0) = \sum_{s=0}^{\infty} T_k R_k^s \big(\theta(-x-2sN+t\sin[k]) - \theta(-x-2sN-2N+t\sin[k])\big).$$

$$n_t(x<0,k>0) = \sum_{s=0}^{\infty} R_k^s \big(\theta(-x-2sN+t\sin[k]) - \theta(-x-2sN-2N+t\sin[k])\big)$$

$$n_t(x<0,k<0) = \sum_{s=0}^{\infty} R_k^{s+1} \big(\theta(x-2sN-t\sin[k]) - \theta(x-2sN-N-t\sin[k])\big) + R_k^s \big(\theta(-x+2sN+t\sin[k]) - \theta(-x+2sN-N+t\sin[k])\big)$$

defect g_c is encoded in reflection and transmission coefficients

Quantities of interest

Density profile from hydrodynamics: $$\rho(x) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} n_t(x,k)$$

See also: Pandey, Bhat, Dhar, Goldstein, Huse, M. K, Kundu, Lebowitz (2023)

Hydrodynamic/Thermodynamic (Yang-Yang) entropy :
(not Von-Neumann entanglement entropy)

$$s_{\text{hydro}}(x) = -\int_{-\pi}^{\pi} \frac{dk}{2\pi} \left[n_t(x,k) \log(n_t(x,k)) + (1-n_t(x,k)) \log(1-n_t(x,k)) \right]$$

$$S_{\text{hydro}}^{(S)} = \int_{-N}^{0} dx\, s_{\text{hydro}}(x), \qquad S_{\text{hydro}}^{(R)} = \int_{0}^{\infty} dx\, s_{\text{hydro}}(x)$$

We will discuss more on this in next slide

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$$S^{(S)}_{\rm hydro} = \int_{-N}^{0} dx\, s_{\rm hydro}(x), \qquad S^{(R)}_{\rm hydro} = \int_{0}^{\infty} dx\, s_{\rm hydro}(x)$$

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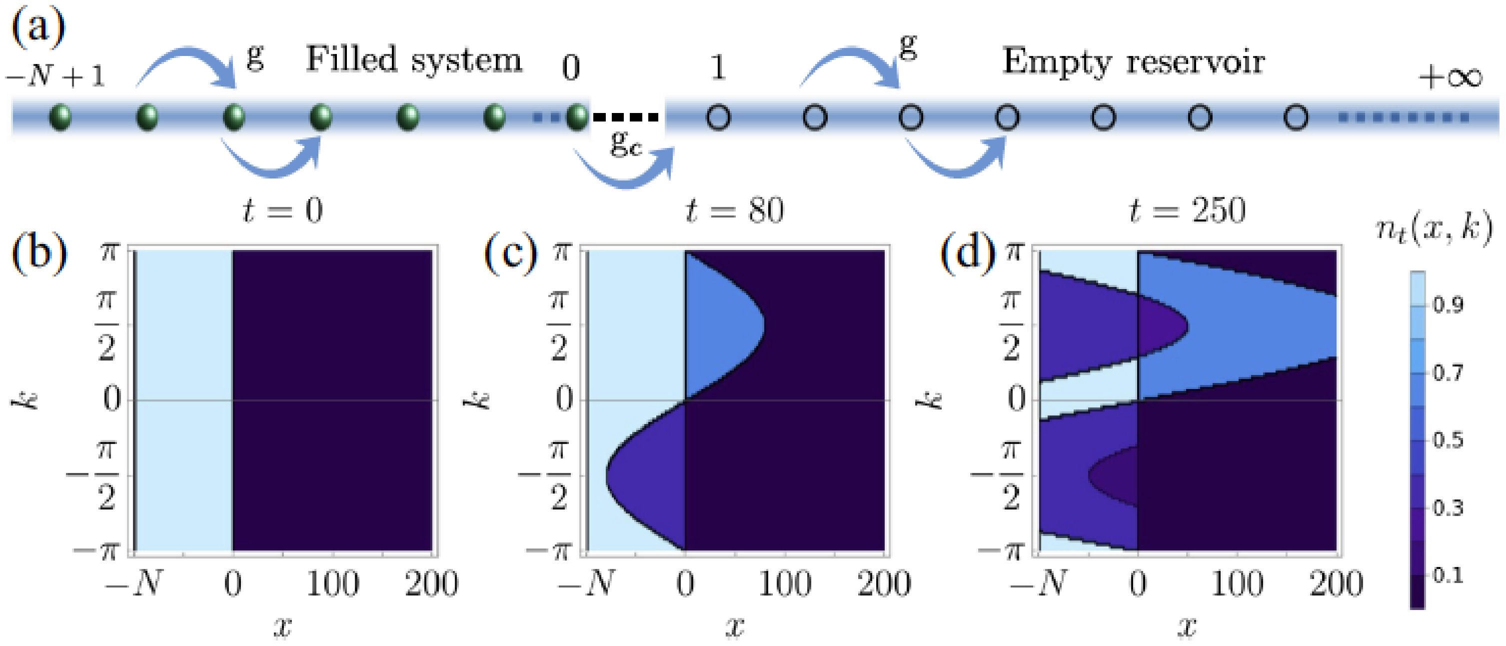

Phase-space dynamics

See also: **M.K.**, Mandal, Morita (2018)

Various Entropies

Recall

Eigenvalues of part of correlation matrix

Von-Neumann Entanglement Entropy $S = \mathrm{tr_s}[\rho_s \log \rho_s] = -\sum_{l=1}^{N} [(1-m_l)\log[1-m_l] + m_l \log m_l]$

Hydrodynamic/Thermodynamic (Yang-Yang) entropy :
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$$S^{(S)}_{\text{hydro}} = \int_{-N}^{0} dx\, s_{\text{hydro}}(x), \qquad S^{(R)}_{\text{hydro}} = \int_{0}^{\infty} dx\, s_{\text{hydro}}(x)$$

Yang-Yang entropy

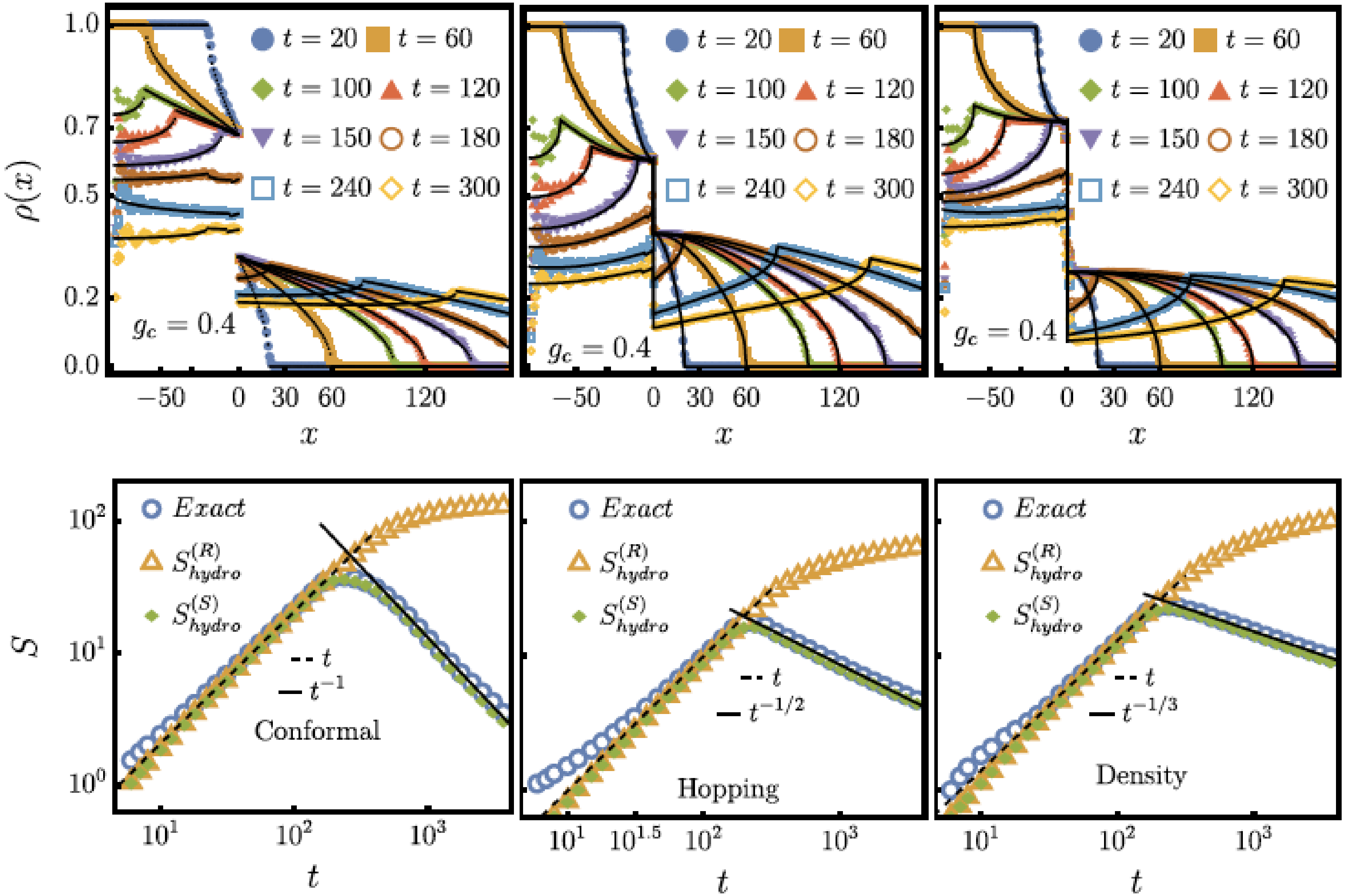
- The idea is to first look for all density matrices that satisfy a given constraint which in our case is $n(x,k,t) = Tr[\hat{n}(x,k)\rho_A] = Tr[\hat{n}(x,k)\tilde{\rho}]$
- In this space of density matrices find the one that maximizes $S = -Tr[\tilde{\rho}\ln\tilde{\rho}]$

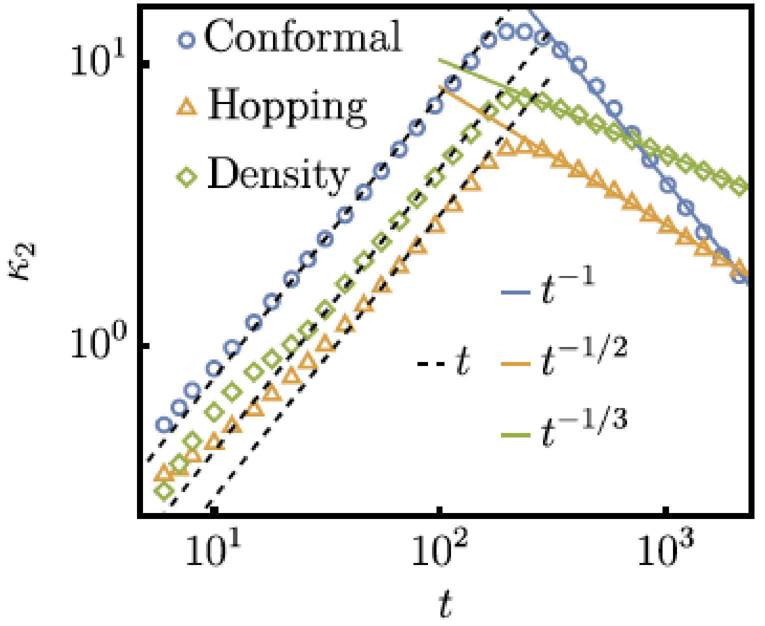
It can be shown that

$$S = -Tr[\tilde{\rho}_M \ln \tilde{\rho}_M] = -\int_{-N}^{0} dx \int \frac{dk}{2\pi} [n_t(x,k)\log(n_t(x,k)) + (1-n_t(x,k))\log(1-n_t(x,k))]$$

This is same as Yang-Yang entropy given in the box above

Density evolution and Page-Curve Entanglement





Recall:

$$S^{(S)}_{\text{hydro}} = \int_{-N}^{0} dx\, s_{\text{hydro}}(x), \qquad S^{(R)}_{\text{hydro}} = \int_{0}^{\infty} dx\, s_{\text{hydro}}(x)$$

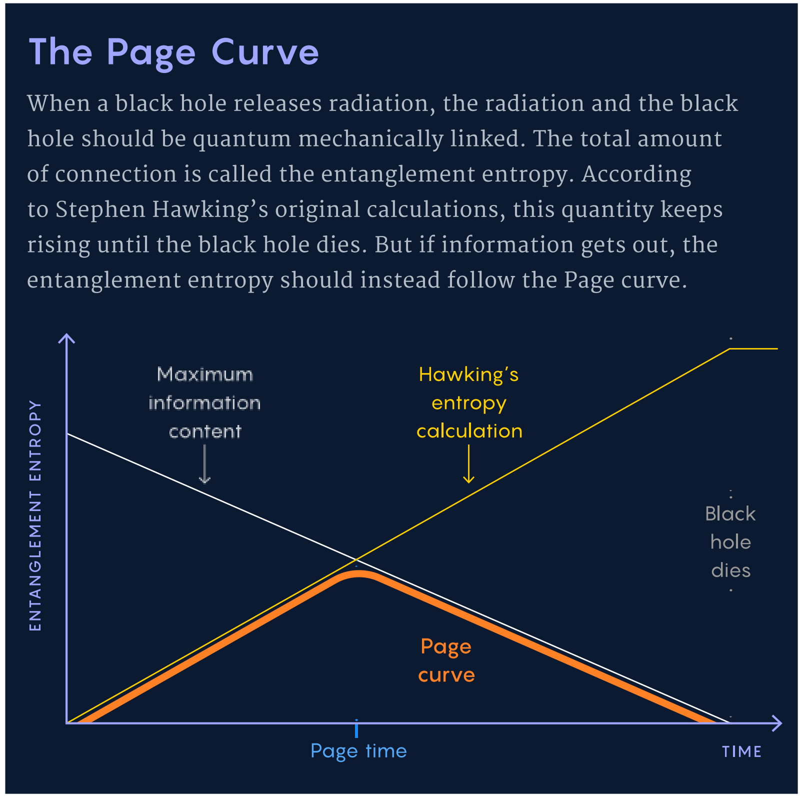
Exact: $S = -\sum_{l=1}^{N} [(1-m_l)\log[1-m_l] + m_l \log m_l]$

$$\kappa_2 = \langle \hat{\mathcal{N}}^2 \rangle - \langle \hat{\mathcal{N}} \rangle^2$$

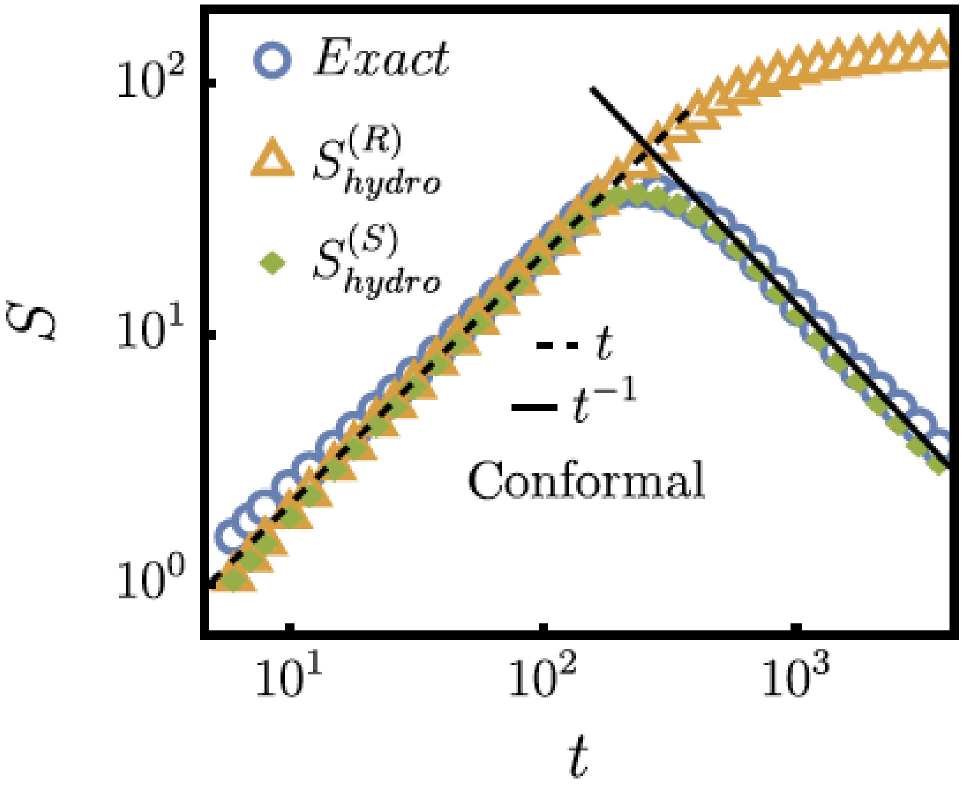
Can analytically extract early time (setting s=0) and also late time (using Poisson summations)

Conclusions to Part A

Saha, **MK**, Dhar (PRL 2024)



Samuel Velasco, Quanta Magazine



Review on entanglement in SYK and its generalizations: Zhang (2022)

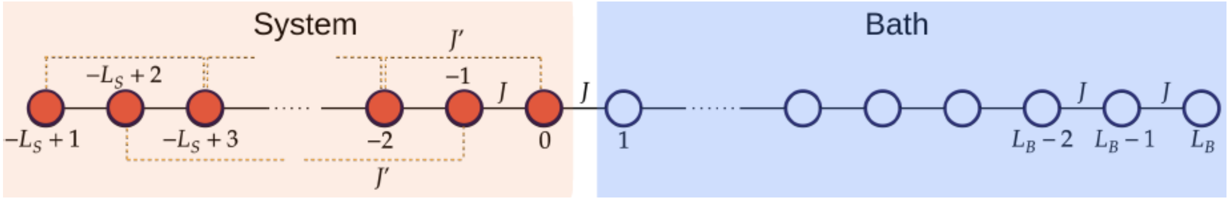
- Numerically and analytically amenable platform to capture essential features of a Page curve
- Semiclassics enables us to understand analytically the page curve, both early and late times
- Yang-Yang/thermodynamics entropy of the system remarkably agrees well with the von-Neumann entanglement entropy
- On the other hand, reservoir entropy only agrees at early times and keeps on increasing.
  (Black hole analogy: Radiation entropy computed from semi-classical theories keeps increasing ?)

PART B

Page curve like dynamics in Interacting Quantum Spin Chains

Ray, Dhar, **MK** (arXiv:2504.14675)

Large Bath



See also
Jha, Manmana, Kehrein, arXiv:2502.03563
Li, Kehrein, Gopalakrishnan, arXiv:2502.03524
Related: Fujimoto, Sasamoto (PRL, 2025)

$$H_{\rm sys} = J \sum_{i=-(L_S-1)}^{-1} \left[\frac{1}{2}\left(S_i^+ S_{i+1}^- + S_{i+1}^- S_i^+\right) + \Delta S_i^z S_{i+1}^z\right] + J' \sum_{i=-(L_S-1)}^{-2} S_i^z S_{i+2}^z$$

$$H_{\rm bath} = J \sum_{i=1}^{L_B-1} \left[\frac{1}{2}\left(S_i^+ S_{i+1}^- + S_{i+1}^- S_i^+\right) + \Delta S_i^z S_{i+1}^z\right]$$

in the talk, we will focus on interacting bath

System is either in -

- filled state - $|\uparrow \cdots \uparrow\rangle$
- Infinite temperature half-filled state

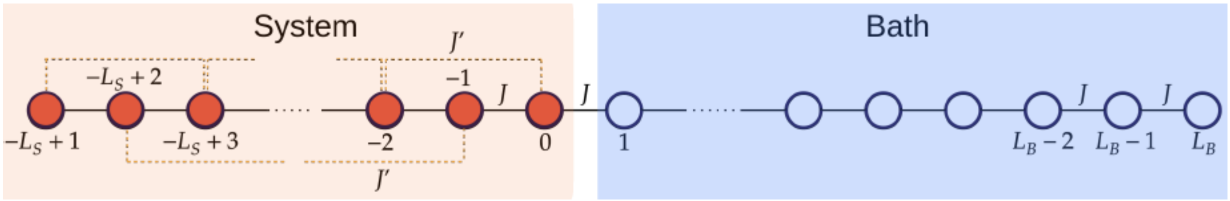
Bath is in the empty state - $|\downarrow \cdots \downarrow\rangle$

$$H_{\rm sys-bath} = J \left[\frac{1}{2}\left(S_0^+ S_1^- + S_1^- S_0^+\right) + \Delta S_0^z S_1^z\right]$$

(Equivalent to fermions in 1D with integrable and non-integrable interactions)

Basic Quantities of Interest

Large Bath



- Total Magnetization dynamics $S^z_{\mathrm{sys}} = \sum_{i=-(L_S-1)}^{0} S^z_i$ or equivalently total magnetization gained in bath
- $\mathrm{variance}(S^z_{\mathrm{sys}})$ or equivalently variance in bath
- von Neumann entanglement entropy $S_{vN} = -\mathrm{Tr}_A\,\rho_A \ln \rho_A = -\mathrm{Tr}_B\,\rho_B \ln \rho_B$ where $\rho_{A/B} = \mathrm{Tr}_{B/A}\,\rho$ → $\rho = |\psi\rangle\langle\psi|$ pure state
- Boltzman entropy

  (i) Define our coarse-grained description (macrostate) of the full setup.

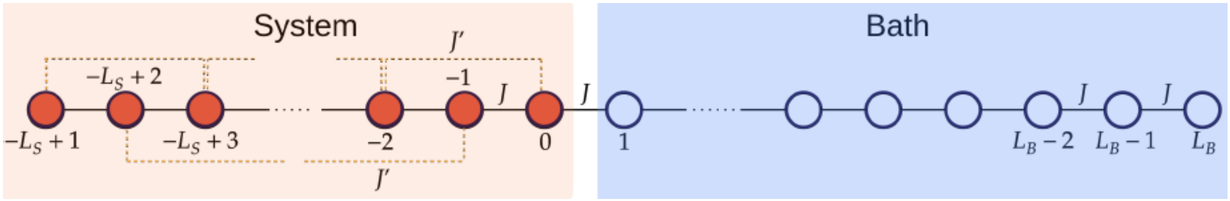
  (ii) The macrostate we consider is one where we divide the full setup into spatial cells of size L_S and specify the average number of particles (or magnetization) and the average energy in each cell.

  (ii) The Boltzmann entropy then essentially counts the number of microstates that correspond to this macrostate.

See also: Pandey, Bhat, Dhar, Goldstein, Huse, **MK**, Kundu, Lebowitz, J. Stat. Phys. (2023)

Procedure for computing the Boltzmann entropy

Large Bath



Boltzman entropy for the system

- Compute the state of the entire setup $|\psi(t)\rangle$ for all t (using TEBD)

- Using above compute average energy and average magnetization

$$E_{\text{sys}}(t) = \langle\psi(t)|H_{\text{sys}}|\psi(t)\rangle \qquad M_{\text{sys}}(t) = \langle\psi(t)|S^z_{\text{sys}}|\psi(t)\rangle$$

$$S^z_{\text{sys}} = \sum_{i=-(L_S-1)}^{0} S^z_i$$

- Find the the grand-canonical (GC) distribution

$$\rho^{\text{sys}}_{GC}(t) = \frac{1}{Z(t)} e^{-\beta(t)(H_{\text{sys}} - \mu(t) S^z_{\text{sys}})}$$

Yet to determine these two parameters for each time step

$$Z(t) = \text{Tr}\left[e^{-\beta(t)(H_{\text{sys}} - \mu(t) S^z_{\text{sys}})}\right]$$

- Solve for the two parameters numerically such that

$$\text{Tr}\left[H_{\text{sys}}\rho^{\text{sys}}_{GC}(t)\right] = E_{\text{sys}}(t),$$
$$\text{Tr}\left[S^z_{\text{sys}}\rho^{\text{sys}}_{GC}(t)\right] = M_{\text{sys}}(t).$$

Procedure for computing the Boltzmann entropy

Boltzman entropy for the system

- Using the numerical solutions for the two parameters compute $\rho_{GC}^{\rm sys}(t) = \frac{1}{Z(t)} e^{-\beta(t)(H_{\rm sys} - \mu(t) S_{\rm sys}^{z})}$

- Finally, compute the Boltzmann entropy of the system $S_B^{\rm sys} = -\,{\rm Tr}\, \rho_{GC}^{\rm sys} \ln \rho_{GC}^{\rm sys}$

Important: Unlike von-Neumann entropy for pure state, the Boltzman entropy for system and bath are different

Procedure for computing the Boltzmann entropy

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Important: Unlike von-Neumann entropy for pure state, the Boltzman entropy for system and bath are different

Boltzman entropy for the bath

- Divide the bath into spatial bins where each bin has an equal number of sites

- Compute total energy and total magnetization for each bin

$$E_{\text{bin}}(t) = \langle \psi(t) | H_{\text{bin}} | \psi(t) \rangle \qquad M_{\text{bin}}(t) = \langle \psi(t) | S_{\text{bin}}^z | \psi(t) \rangle$$

$$H_{\text{bin}} = J \sum_{i \in \text{bin}} \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z \right) \qquad S_{\text{bin}}^z = \sum_{i \in \text{bin}} S_i^z$$

- Follow similar procedure (as employed for the system) and then finally compute $S_B^{\text{bath}} = \sum_{\text{bins}} S_B^{\text{bin}}$

von-Neumann entanglement entropy at very early times

- **Recall:** We have two initial states for the system (i) Polarized state (ii) Infinite temperature state
- $|\psi(0)\rangle = |\psi_{\text{sys}}(0)\rangle \otimes |\psi_{\text{bath}}(0)\rangle$ Bath always $|\psi_{\text{bath}}(0)\rangle = |\downarrow, \ldots, \downarrow\rangle$
- Just a Taylor expansion $e^{-iHt}|\psi(0)\rangle = \left[\mathbb{I} - itH - \frac{t^2}{2}H^2 + O(t^3)\right]|\psi(0)\rangle$

Polarized state

$|\psi_{\text{sys}}(0)\rangle = |\uparrow, \ldots, \uparrow\rangle$

- Only sites that are affected by this dynamics are the 2 sites on either side of the system bath boundary.
- The rest of the system and the bath remain decoupled from these 4 sites, and dynamics is governed by the dynamics of these 4 sites.
- We get

$$S_{vN} = -\left(1 - \frac{t^2}{4}\right)\log\left(1 - \frac{t^2}{4}\right) - \frac{t^2}{4}\log\frac{t^2}{4}$$

$$\langle N_{\text{bath}}\rangle = \frac{t^2}{4}, \quad var(N_{\text{bath}}) = \frac{t^2}{4}$$

Magnetization and fluctuation at very early times

Infinite temperature state

- Even for very early times, computing von Neumann entropy for infinite temperature state is challenging
- But magnetization and variance is feasible
- Let us start with

$$|\phi(0)\rangle = \frac{1}{\sqrt{\mathbb{N}}} \sum_{k=1}^{\mathcal{N}} c_k \, |\chi_k\rangle \otimes |\downarrow\rangle^{\otimes L_B}$$

$\mathcal{N}$ → ${}^{L_S}C_{L_S/2}$

$\frac{1}{\sqrt{\mathbb{N}}}$ → Normalization

$|\chi_k\rangle$ → S^z product basis of the system in half-filled sector

c_k → complex numbers chosen from a distribution with mean zero and variance $\frac{1}{2}$.

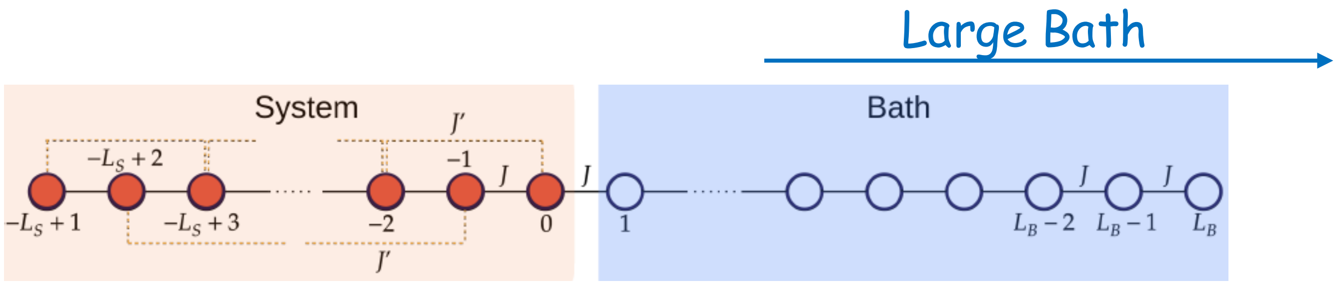
$$\langle N_{\text{bath}}\rangle = t^2 \langle \phi(0)|H N_{\text{bath}} H|\phi(0)\rangle = \frac{t^2}{2\mathbb{N}} \sum_{k,k'}{}' c_k c_{k'}^* \langle \chi_{k'}| \langle \downarrow|^{\otimes L_B} H \, |\tilde{\chi}_{k,\downarrow}\rangle \, |\uparrow\rangle \, |\downarrow\rangle^{\otimes L_B - 1} = \frac{t^2}{8}$$

$$var(N_{\text{bath}}) = \frac{t^2}{8}$$

We will now discuss numerical results that will capture, very early, intermediate and long times

Numerical Results

Recall



$$H_{\text{sys}} = J \sum_{i=-(L_S-1)}^{-1} \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z\right)$$

$$+ J' \sum_{i=-(L_S-1)}^{-2} S_i^z S_{i+2}^z,$$

integrability breaking in system

$$H_{\text{bath}} = J \sum_{i=1}^{L_B-1} \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z\right)$$

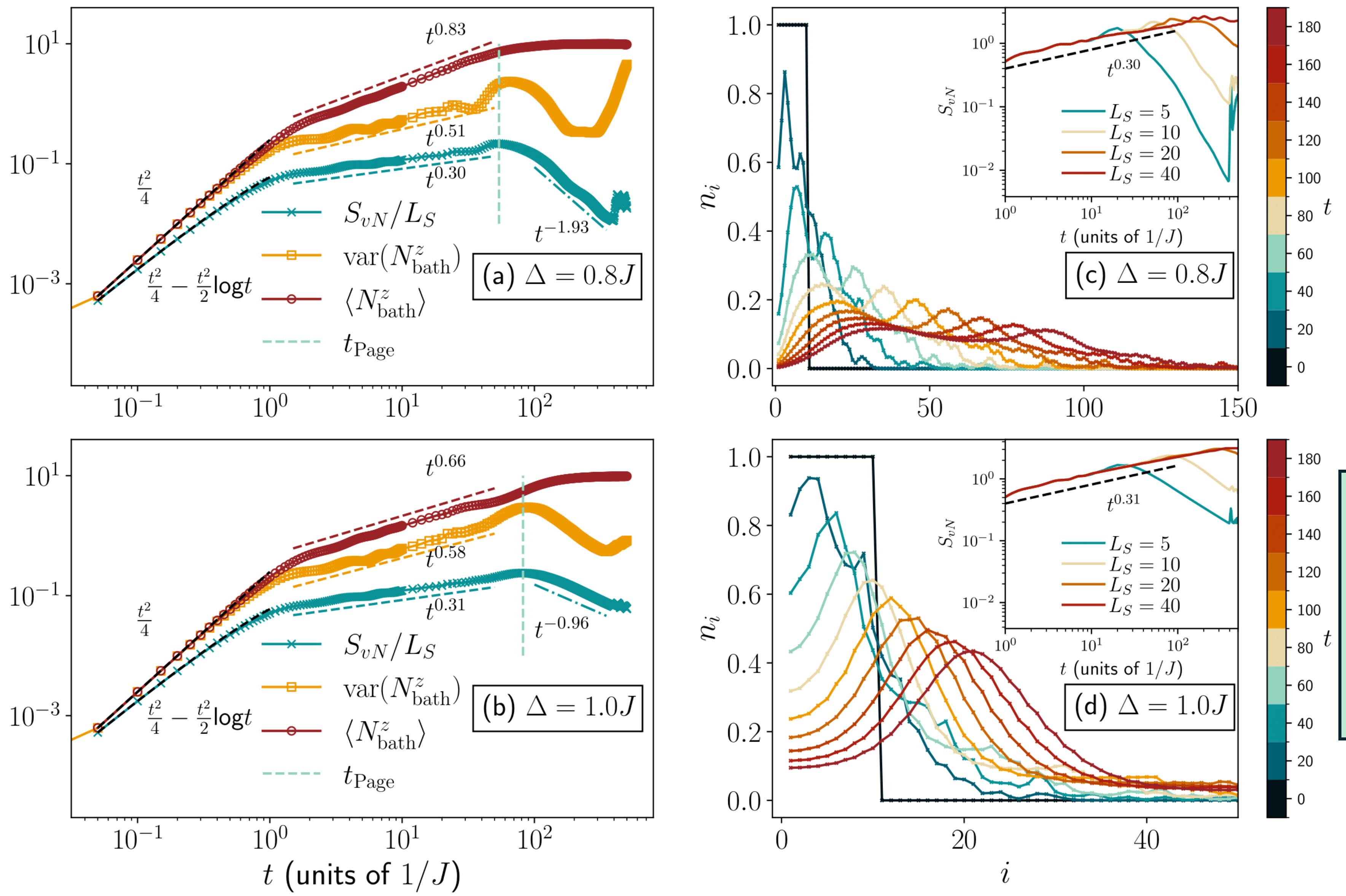
Bath is integrable in all cases

$$H_{\text{sys-bath}} = J \left(S_0^x S_1^x + S_0^y S_1^y + \Delta S_0^z S_1^z\right).$$

Method employed - Time evolution block decimation (TEBD) method, represent states as matrix product states (MPS), Bond dimension cutoff is 150.

We will present four cases in this talk (i) integrable polarized (ii) integrable infinite temperature (iii) non-integrable polarized (iv) non-integrable infinite temperature

Numerical Results for Integrable case – Domain Wall



$\Delta = 0.8J$

$S_{vN} \sim t^{0.30}$, $\langle N_{\mathrm{bath}} \rangle \sim t^{0.83}$, $var(N_{\mathrm{bath}}) \sim t^{0.51}$

$t_{\mathrm{Page}} \approx L_S^{1.7}$

Fraction escaped around page time - 0.74

$\Delta = 1.0J$

$S_{vN} \sim t^{0.31}$, $\langle N_{\mathrm{bath}} \rangle \sim t^{0.66}$, $var(N_{\mathrm{bath}}) \sim t^{0.58}$

$t_{\mathrm{Page}} \approx L_S^{2}$

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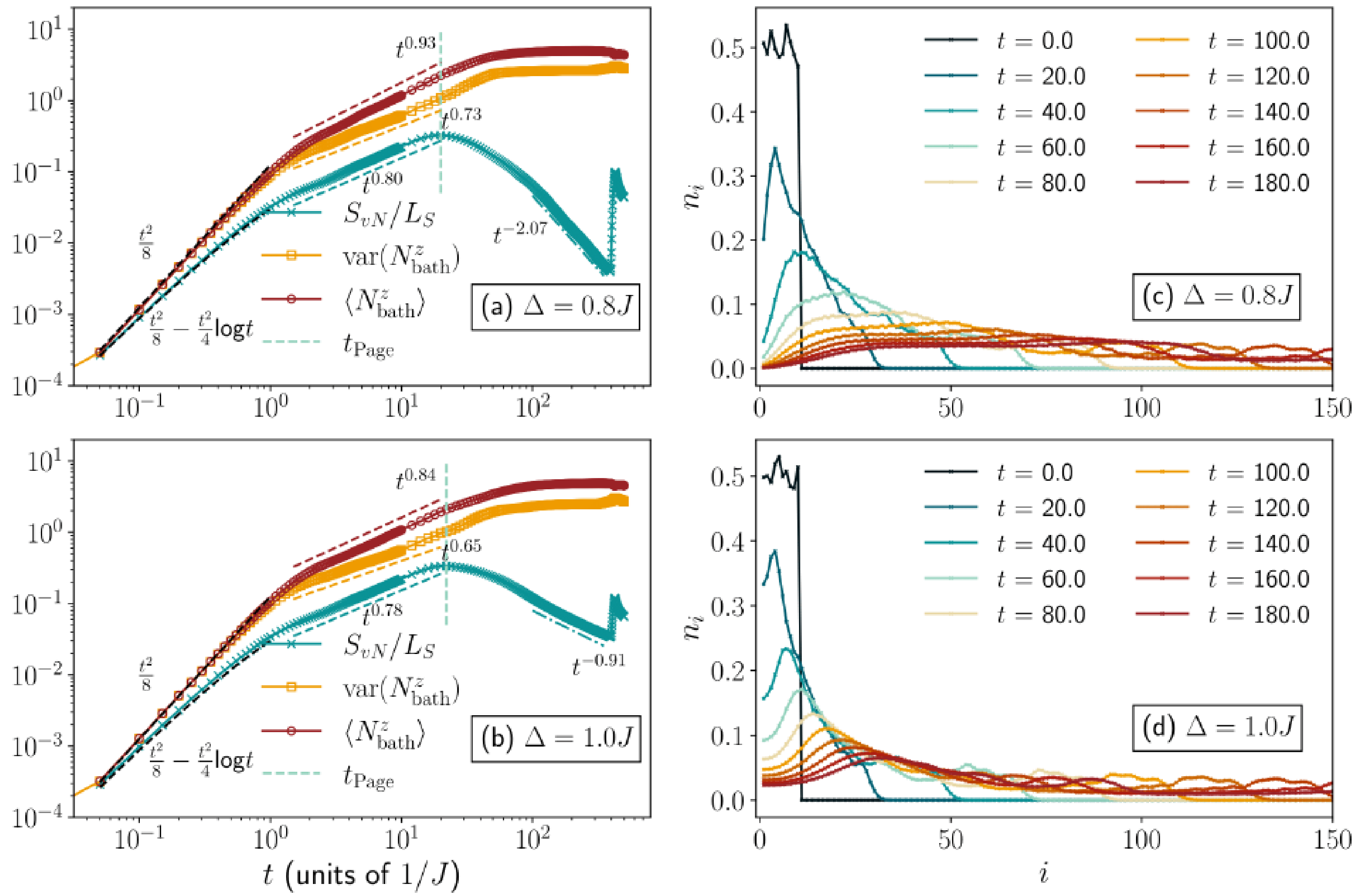
Numerical Results for Integrable case – Infinite temperature case



$\Delta = 0.8J$

$S_{vN} \sim t^{0.80}$, $\langle N_{\mathrm{bath}} \rangle \sim t^{0.93}$, $var(N_{\mathrm{bath}}) \sim t^{0.73}$

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$\Delta = 1.0J$

$S_{vN} \sim t^{0.78}$, $\langle N_{\mathrm{bath}} \rangle \sim t^{0.84}$, $var(N_{\mathrm{bath}}) \sim t^{0.65}$

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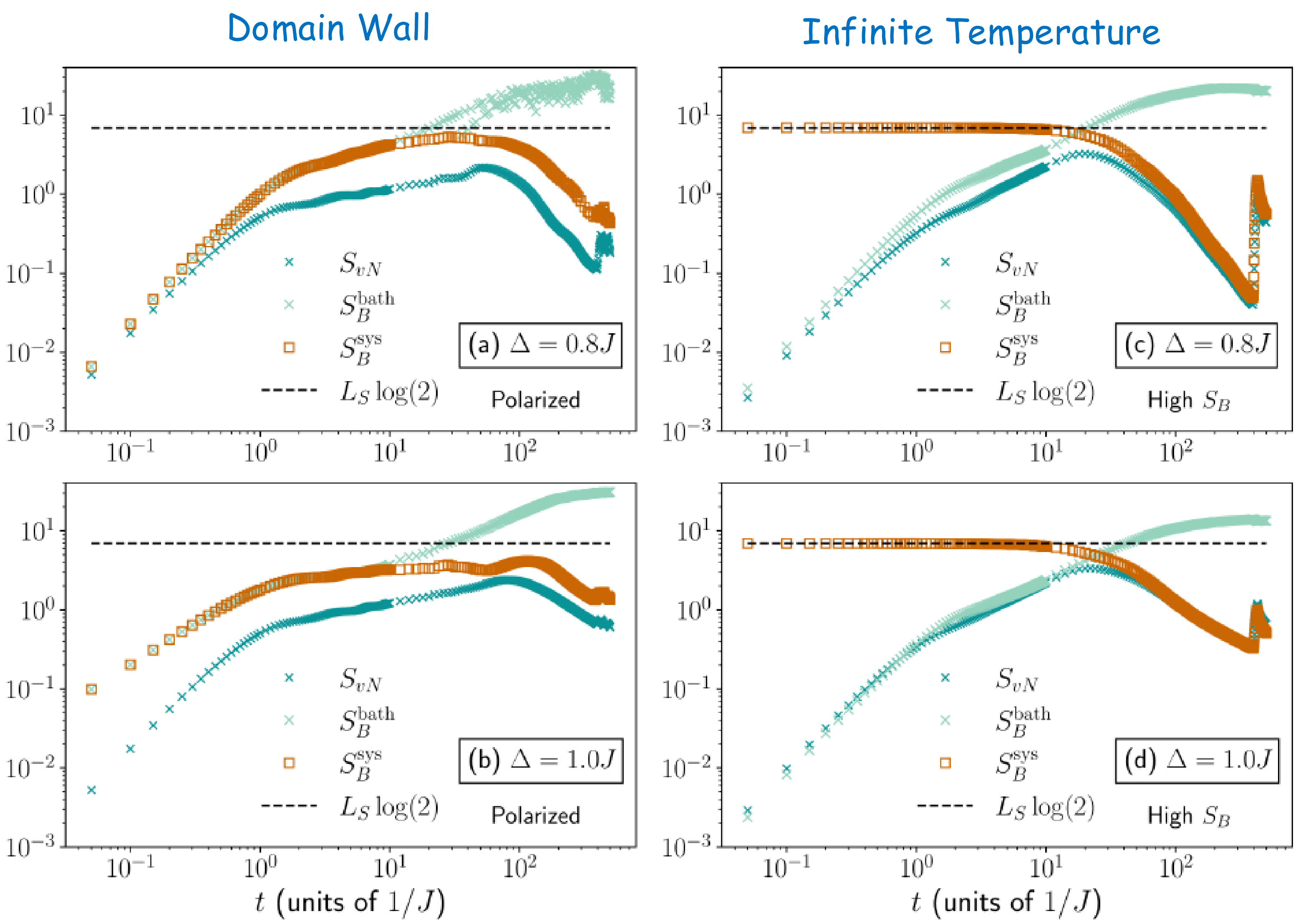
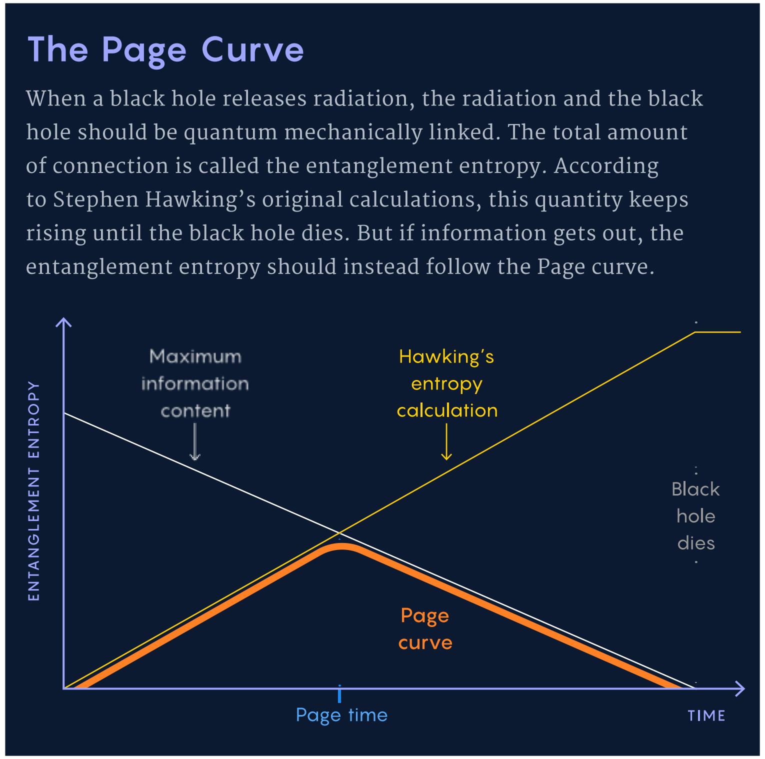
Numerical Results for Integrable case – Comparion with system and bath Boltzman entropy



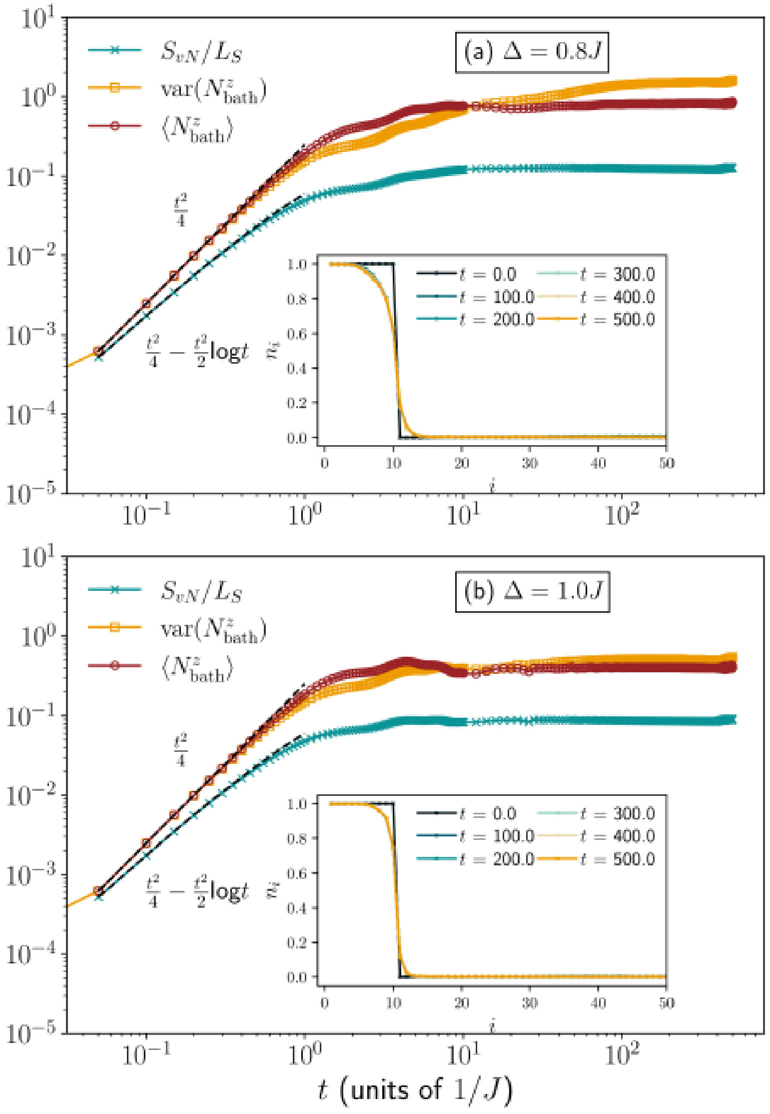
Reminiscent



Samuel Velasco, Quanta Magazine

Numerical Results for non-integrable case

Domain wall



Recall (non-integrable case)

$$H_{\mathrm{sys}} = J \sum_{i=-(L_S-1)}^{-1} \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z \right) + J' \sum_{i=-(L_S-1)}^{-2} S_i^z S_{i+2}^z,$$

(J') set to 1

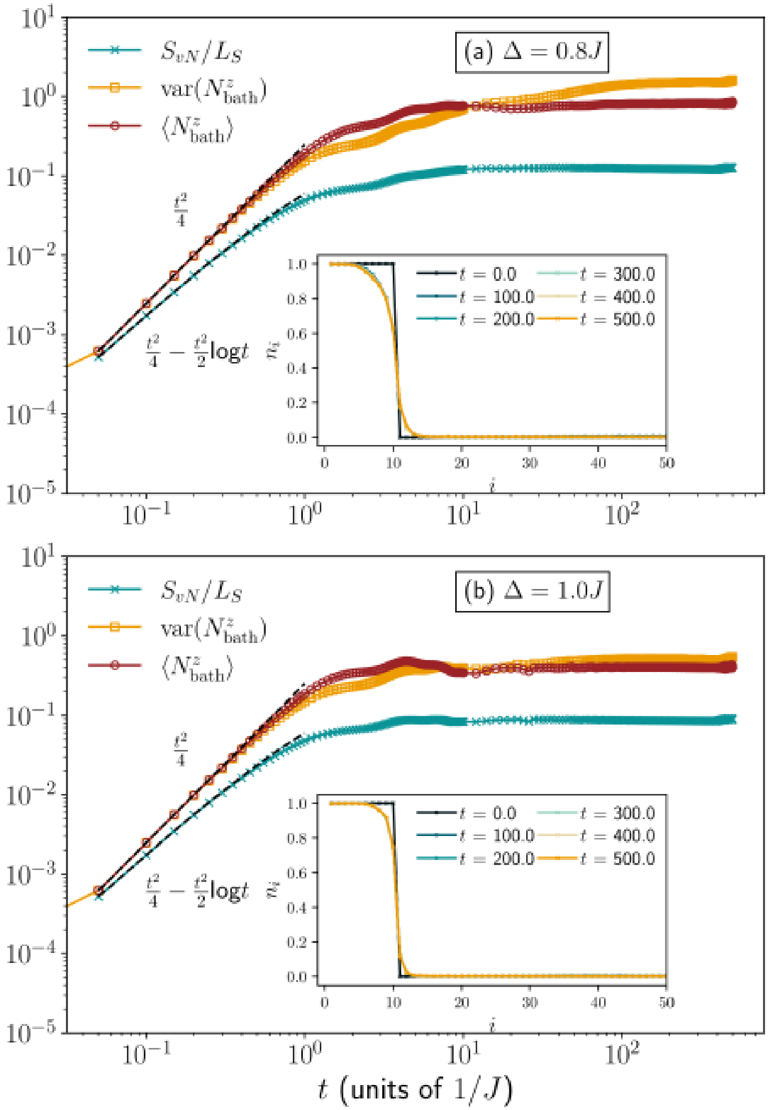
Numerical Results for non-integrable case

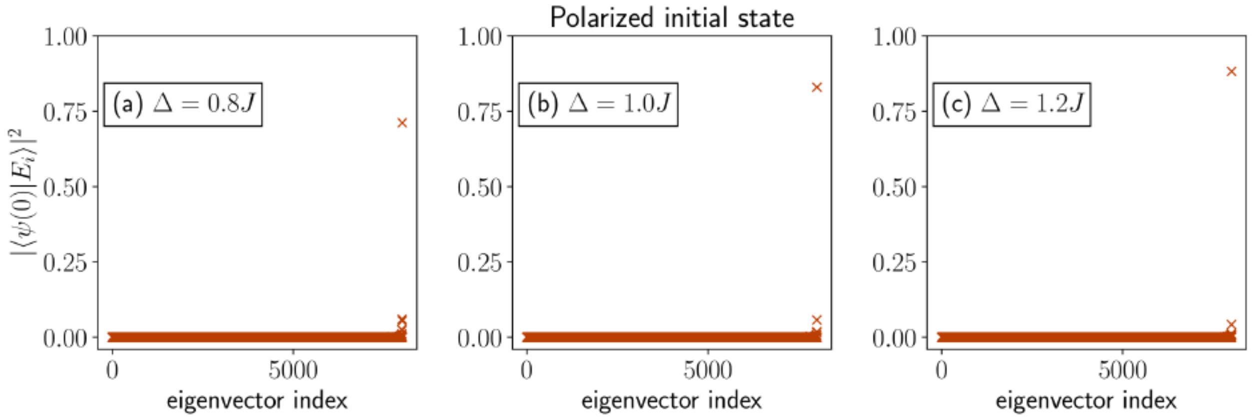
Recall (non-integrable case)

$$H_{\text{sys}} = J \sum_{i=-(L_S-1)}^{-1} \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z \right) + J' \sum_{i=-(L_S-1)}^{-2} S_i^z S_{i+2}^z,$$

J' → set to 1

Domain wall

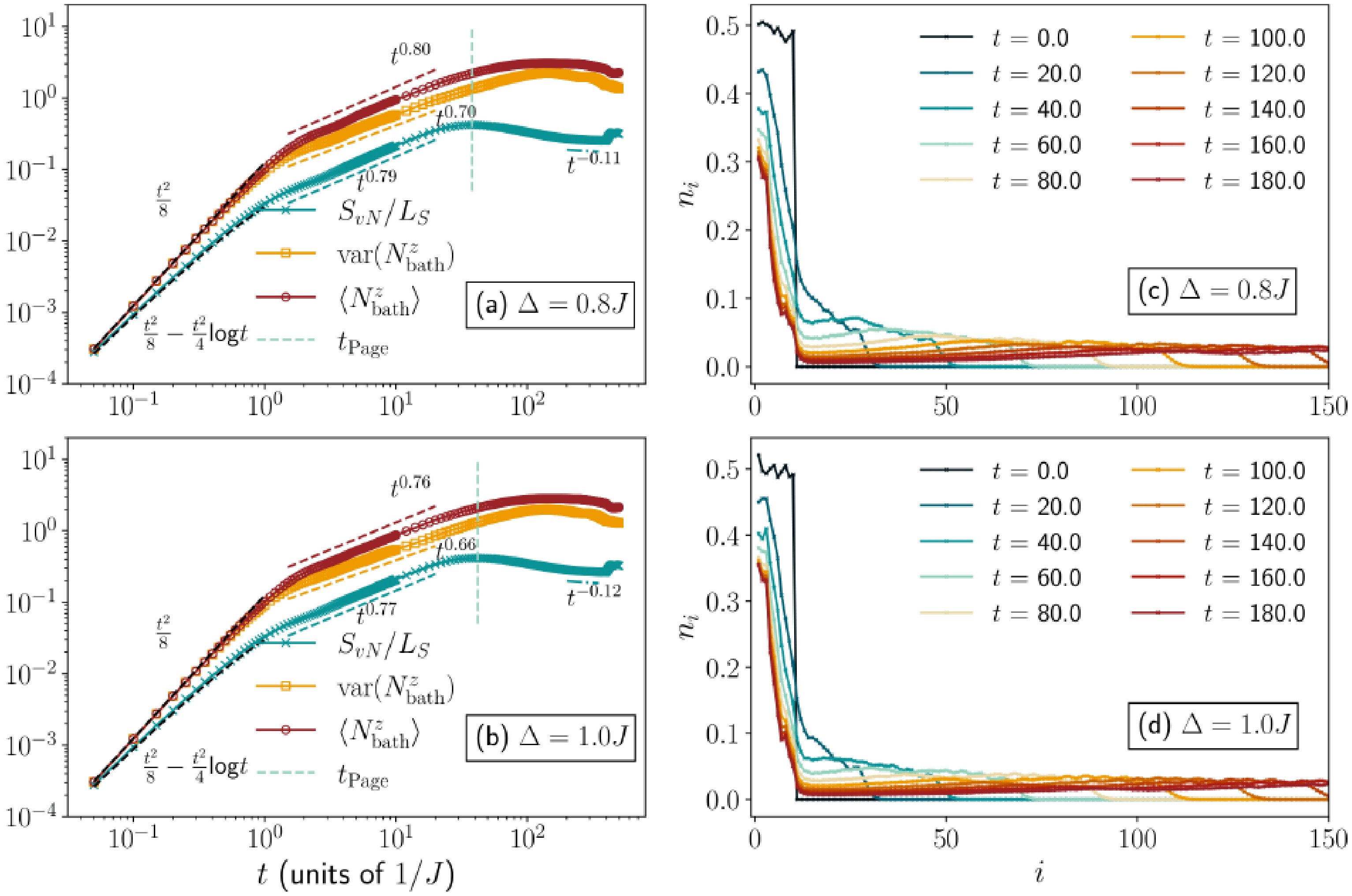




- Dynamics freezes (filled initial state has a high overlap with a single localized eigenstate of the entire setup (system and bath).
- After an initial growth, all quantities eventually seem to saturate

Numerical Results for non-integrable case

Infinite temperature



$\Delta = 0.8J$

$S_{vN} \sim t^{0.79}$, $\langle N_{\mathrm{bath}}\rangle \sim t^{0.80}$, $var(N_{\mathrm{bath}}) \sim t^{0.70}$

$\Delta = 1.0J$

$S_{vN} \sim t^{0.77}$, $\langle N_{\mathrm{bath}}\rangle \sim t^{0.76}$, $var(N_{\mathrm{bath}}) \sim t^{0.66}$

Ray, Dhar, **MK** (arXiv:2504.14675)

Conclusions to Part B

- Quantum dynamics of the von-Neumann entanglement entropy for an interacting system with and without integrability breaking interactions, connected to a bath
- Also, computed mean particle number and particle fluctuations as a function of time
- Two initial conditions, (i) domain wall and (ii) infinite temperature
- Thorough characterization of page curve and related exponents
- Coarse grained Boltzman entropy and its comparision with fine grained entanglement entropy
- It will be interesting to develop a thorough understanding from hydrodynamics / generalized hydrodynamics

Part C

Quantum Injection of particles

T. Ray, k. Ganguly, MK, Agarwalla
(manuscript in preparation)

Central question: What happens when we inject particles in a system that is (i) either itself subject to dephasing mechanism or (ii) is itself inherently interacting

Main quantities of interest in this part of the talk are spatial density profile and the total number of particles.

Part C

Quantum Injection of particles

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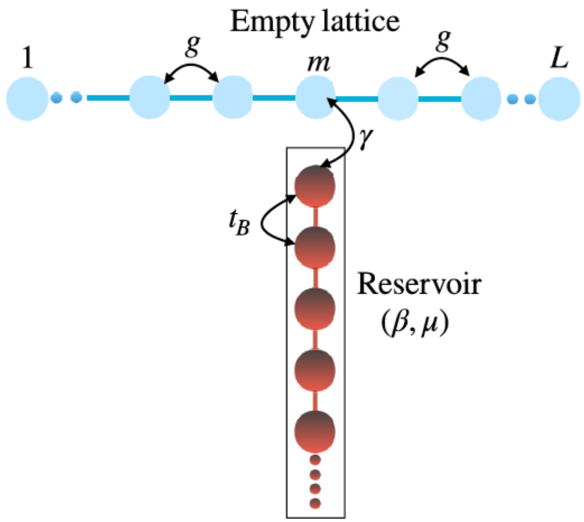
Central question: What happens when we inject particles in a system that is (i) either itself subject to dephasing mechanism or (ii) is itself inherently interacting

Main quantities of interest in this part of the talk are spatial density profile and the total number of particles.

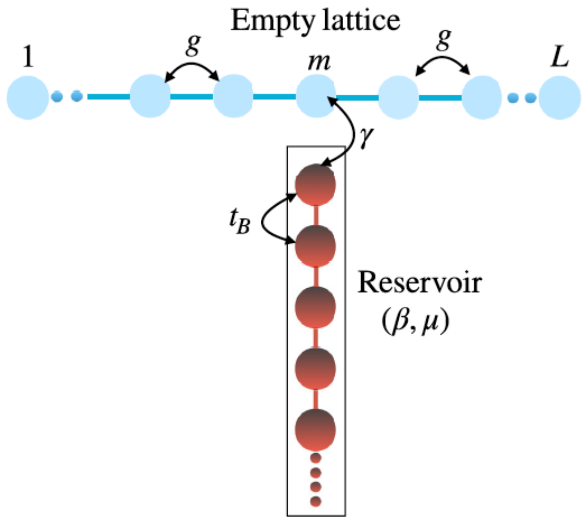
Before proceeding to the case (i) and (ii) mentioned above, we will discuss an earlier result, Trivedi, Gupta, Agarwalla, Dhar, Mk, Kundu, Sabhapandit (PRA 2023) on "Filling an empty lattice by local injection of quantum particles"



- Setup to study quantum dynamics of filling an empty lattice of size L by connecting it locally with an equilibrium bath that injects noninteracting bosons or fermions.
- We will mainly discuss the Lindblad approach

Krapivsky, Mallick, Sels (2019,2020)
Butz, Spohn (2010)

Many past and recent literature on "localized loss"



$$H_S = g \sum_{i=1}^{L-1} (a_i^\dagger a_{i+1} + a_{i+1}^\dagger a_i),$$

System

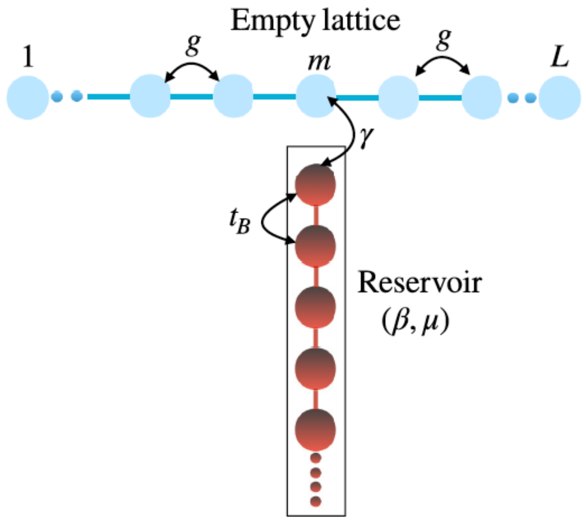
$$\dot{\rho}_{SS} = i[\rho_{SS}, H_S] + \Gamma_G[2a_m^\dagger \rho_{SS} a_m - \{a_m a_m^\dagger, \rho_{SS}\}] + \Gamma_L[2a_m \rho_{SS} a_m^\dagger - \{a_m^\dagger a_m, \rho_{SS}\}],$$

Gain of particles

Reduced density matrix of system

Loss of particles

m stands for middle injection point



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Gain of particles

Reduced density matrix of system

Loss of particles

m stands for middle injection point

We will write down equation for the correlation matrix $C_{i,j} = \langle a_i^\dagger a_j\rangle$

$$\frac{dC_{i,j}}{dt} = i\,g\,(C_{i-1,j} - C_{i,j+1} + C_{i+1,j} - C_{i,j-1}) - \Gamma'(\delta_{im} + \delta_{jm})\,C_{i,j} + 2\,\Gamma_G\delta_{mi}\delta_{mj}$$

$$\Gamma' = \Gamma_L \mp \Gamma_G$$

plus/minus stands for bosons/fermions respectively

Spatial density profile $n_i(t) = 2\,\Gamma_G \int_0^t d\tau\, |\tilde{S}_i(\tau)|^2$

$$\tilde{S}_i(\tau) = J_i(2\,g\,\tau) - \Gamma' \int_0^\tau d\bar{t}\, e^{-\Gamma'\bar{t}} \times \left(\frac{\tau - \bar{t}}{\tau + \bar{t}}\right)^{i/2} J_i[2\,g\sqrt{\tau^2 - \bar{t}^2}]$$

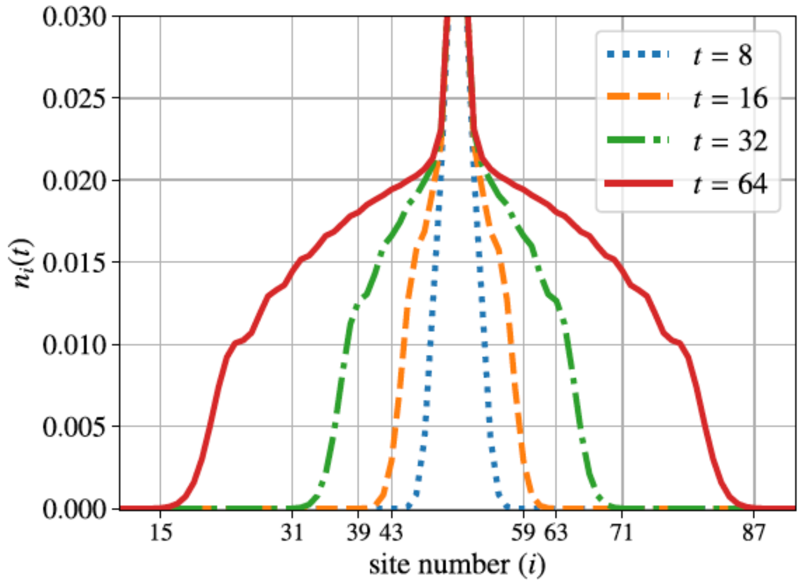
Interestingly, it turns out that $n_i(t)$ can admit an interesting scaling form. To do so, let us take the limits

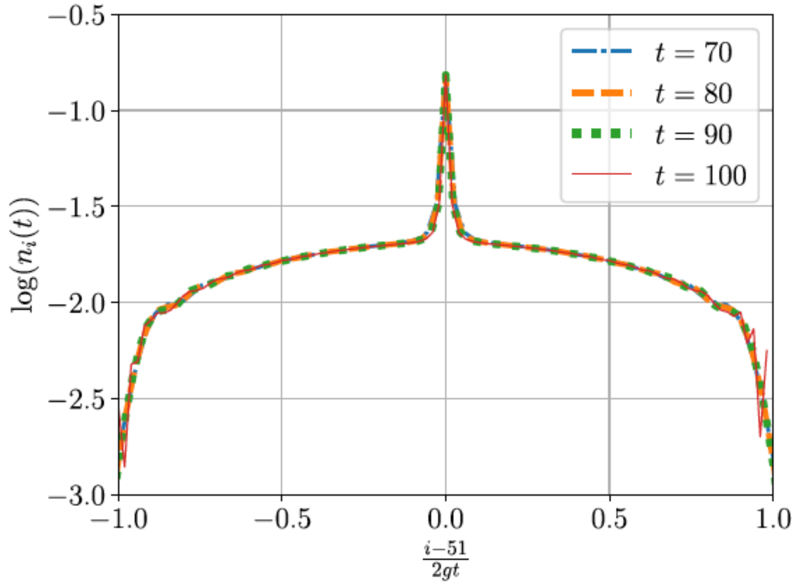
$$i \to \infty, \quad t \to \infty, \quad \nu = \frac{i}{2\,g\,t} \sim O(1)$$

$$n_i(t) = \Phi\left(\frac{i}{2gt}\right), \quad \text{where} \quad \Phi(\nu) = \frac{\tilde{g}\,(1+\nu\,\tilde{g})[\ln(1+\nu\,\tilde{g}) - \ln(\tilde{g}+\nu-\sqrt{(\tilde{g}^2-1)(1-\nu^2)})] - \sqrt{(\tilde{g}^2-1)(1-\nu^2)}}{(\tilde{g}^2-1)^{3/2}(\nu\,\tilde{g}+1)}, \quad 0<\nu<1$$

$$\tilde{g} = \frac{2g}{\Gamma'} = \frac{4g}{J(0)}$$

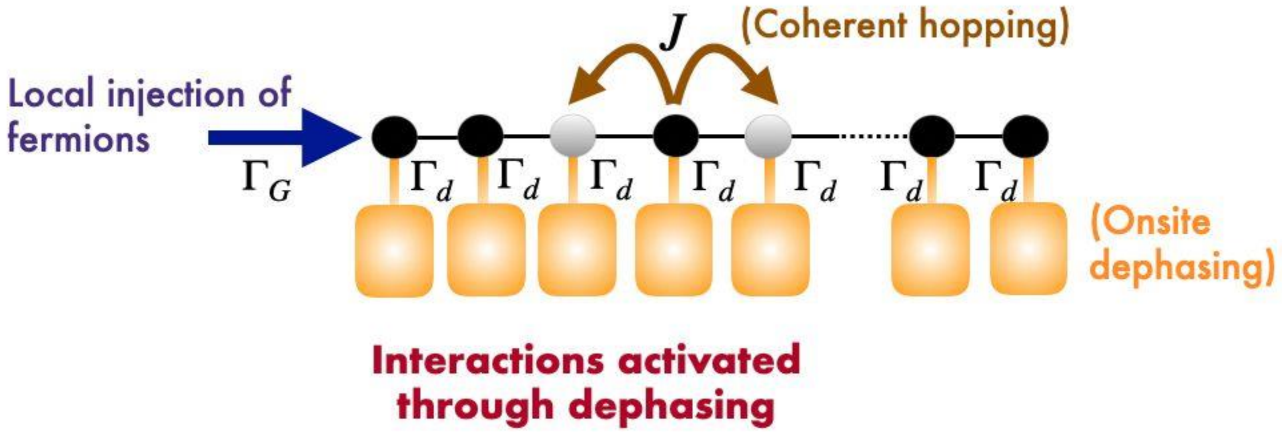
Total occupation $N(t) = -\dfrac{2\,\Gamma_G\,t}{\pi\,(1-\tilde{g}^2)}\left[2\tilde{g} - \pi\,(1-\tilde{g}^2) + 2(1-2\tilde{g}^2)\dfrac{\cos^{-1}(\tilde{g})}{\sqrt{1-\tilde{g}^2}}\right]$





What happens when we inject particles in a system that is (i) either itself subject to dephasing mechanism or (ii) is itself inherently interacting.

We will first discuss case (i)



$$H_S = -J\sum_{i=1}^{L} c_i^\dagger c_{i+1} + h.c.$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -i[H_S,\rho] + \Gamma_G\left[2c_1^\dagger \rho c_1 - \{c_1 c_1^\dagger, \rho\}\right] - \frac{\Gamma_d}{2}\sum_{i=1}^{L}[n_i,[n_i,\rho]]$$

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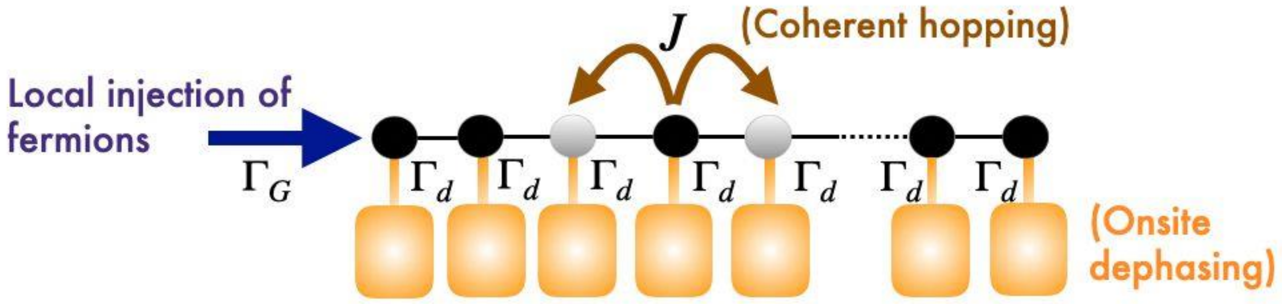


$$H_S = -J \sum_{i=1}^{L} c_i^\dagger c_{i+1} + h.c.$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -i[H_S, \rho] + \Gamma_G \left[2c_1^\dagger \rho c_1 - \{c_1 c_1^\dagger, \rho\} \right] - \frac{\Gamma_d}{2} \sum_{i=1}^{L} [n_i, [n_i, \rho]]$$

$$\frac{\mathrm{d}C_{m,n}(t)}{\mathrm{d}t} = -iJ\Big(C_{m-1,n}(t) + C_{m+1,n}(t) - C_{m,n-1}(t) - C_{m,n+1}(t)\Big) - \Gamma_G(\delta_{1m} + \delta_{1n})C_{m,n}(t) + \Gamma_d\,(\delta_{m,n} - 1)C_{m,n}(t) + 2\Gamma_G\,\delta_{1m}\delta_{1n}$$

where $C_{m,n}(t) = \langle c_m^\dagger(t) c_n(t) \rangle$

This equation is difficult to solve analytically.

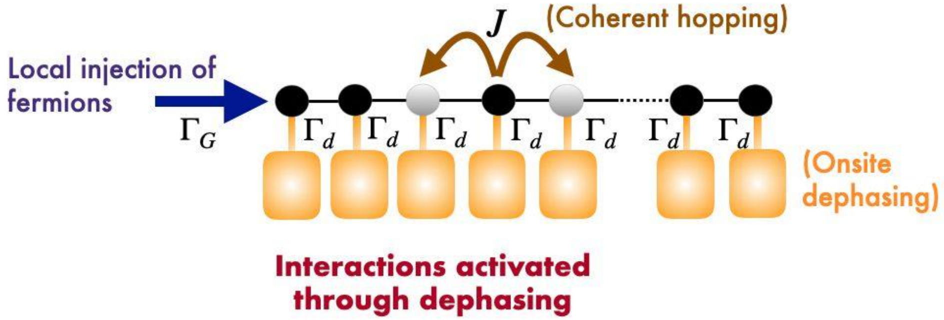
[without gain]
Ishiyama, Fujimoto, Sasamoto
J. Stat. Mech. (2025)

Recall

$$\frac{\mathrm{d}C_{m,n}(t)}{\mathrm{d}t} = -iJ\Big(C_{m-1,n}(t) + C_{m+1,n}(t) - C_{m,n-1}(t) - C_{m,n+1}(t)\Big) - \Gamma_G(\delta_{1m} + \delta_{1n})C_{m,n}(t) + \Gamma_d\,(\delta_{m,n} - 1)C_{m,n}(t) + 2\Gamma_G\,\delta_{1m}\delta_{1n}$$

where $C_{m,n}(t) = \langle c_m^\dagger(t) c_n(t)\rangle$

Before attempting an analytical solution, we will present below the numerical findings.



Recall

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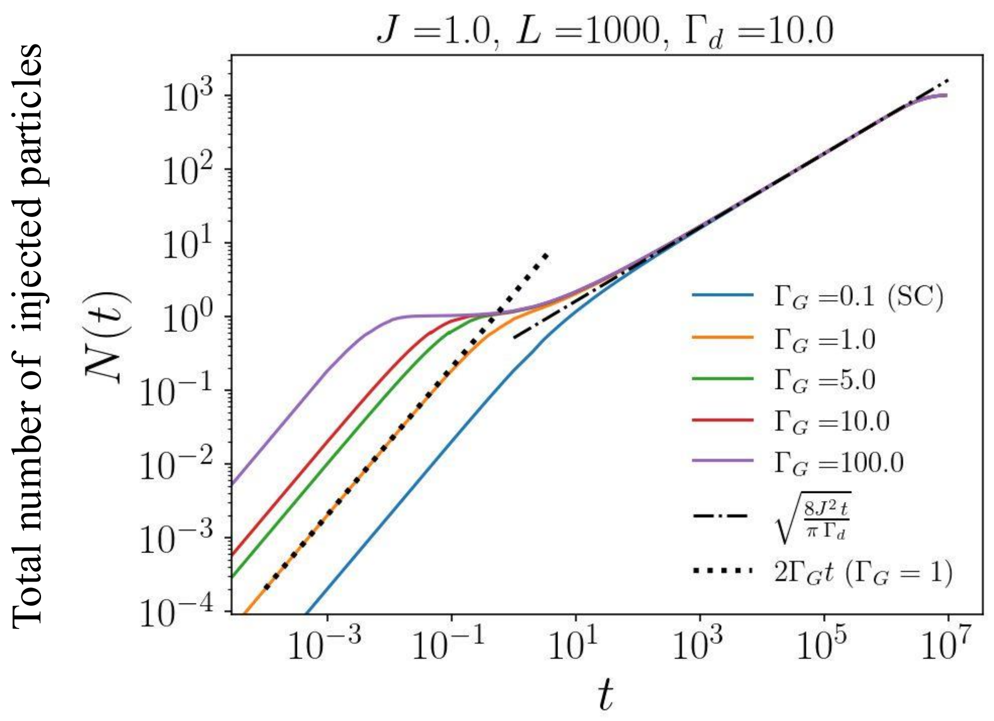
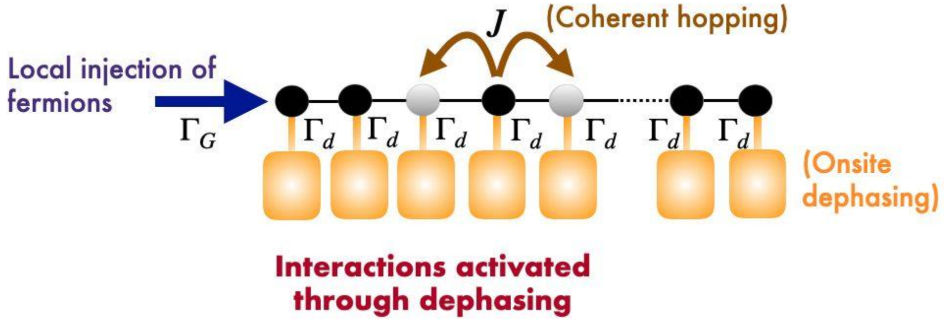


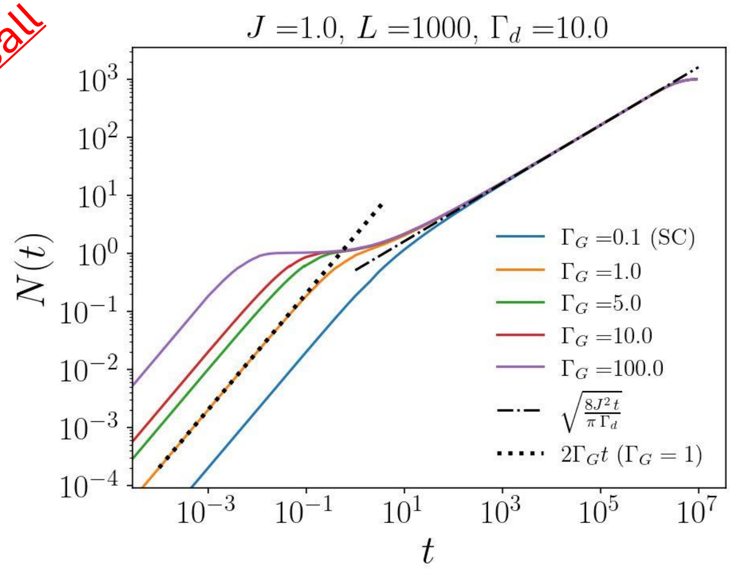


Key findings from numerics

- There is a linear early time behaviour that depends on injection rate.
- There is a square-root behaviour at later times with diffusion constant that depends on dephasing rate but is independent of injection rate
- Between these two time-scales the system goes trough a “congestion” which almost takes the shape of a plateau.

- We will now discuss some analytical results both at early and late times.
- For late-times, we will use the fact that the behaviour is independent of injection rate thereby enabling us to use a special value of injection rate that makes analytics more feasible.

Recall



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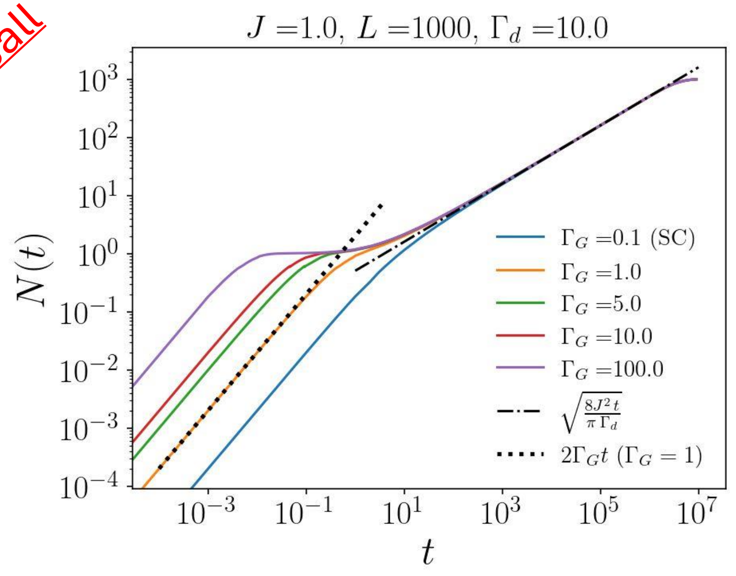


<u>Very early times:</u>

- This is a trivial regime where just at most one particle enters the system.
- During this time-scale even the hopping rate J does not play a role. Hence, coherences also donot develop
- Only the below equation for the first site matters

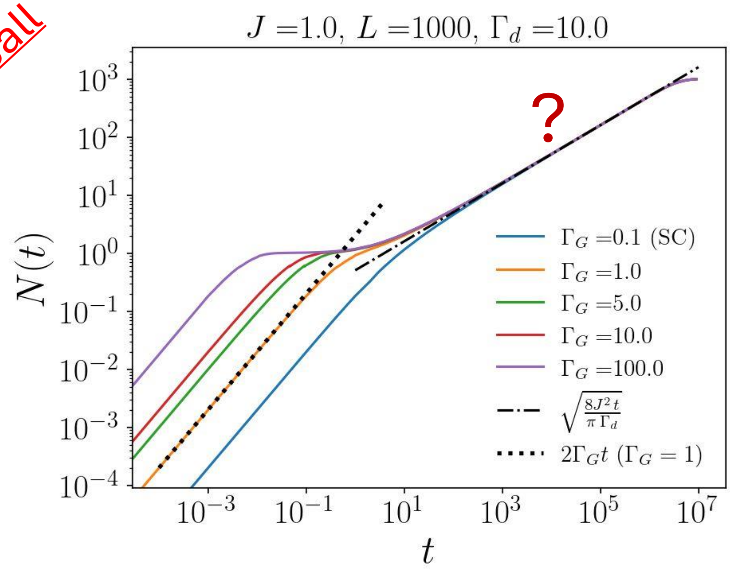
$$\frac{dC_{1,1}}{dt} = 2\Gamma_G(1 - C_{1,1}) \Longrightarrow C_{1,1}(t) = 1 - e^{-2\Gamma_G t} \Longrightarrow C_{1,1}(t) = 2\Gamma_G t$$ (short time expansion)

- This linear growth has been verified with exact numerics as seen in plot above

The late time square-root behavour is analytically more tricky which we will discuss next

Analytical understanding of the diffusive behaviour

Recall



We start with the equations for the correlation matrix

$$\frac{\mathrm{d}C_{m,n}(t)}{\mathrm{d}t} = -iJ\Big(C_{m-1,n}(t) + C_{m+1,n}(t) - C_{m,n-1}(t) - C_{m,n+1}(t)\Big) - \Gamma_G(\delta_{1m} + \delta_{1n})C_{m,n}(t) + \Gamma_d\,(\delta_{m,n} - 1)C_{m,n}(t) + 2\Gamma_G\,\delta_{1m}\delta_{1n}$$

We will do an "adiabatic approximation". This involves taking a large dephasing limit $\Gamma_d >> J$

The adiabatic approximation is about a separation of time-scales. The time scale at which the coherences relax is assumed to be much shorter than the times-scales of the population. This leads to

$$C_{m,n} = -\frac{\mathrm{i}J}{\Gamma_G(\delta_{1,m} + \delta_{1,n}) + \Gamma_d}\left(C_{m-1,n} + C_{m+1,n} - C_{m,n-1} - C_{m,n+1}\right) \qquad (m \neq n)$$

The EOM for the diagonal terms of the correlation matrix (densities) is given by

$$\dot{C}_{m,m} = -\mathrm{i}J\left(C_{m-1,m} + C_{m+1,m} - C_{m,m-1} - C_{m,m+1}\right) - 2\Gamma_G\delta_{1,m}C_{m,m} + 2\Gamma_G\delta_{1,m}$$

We will simplify this equation by using the equation above it and ignore second-neighbour correlations

We finally get

$$\dot{C}_{m,m} = \begin{cases} \frac{2J^2}{\Gamma_d}\left(C_{m-1} - 2C_m + C_{m+1}\right), & 3 \le m \le L-1 \\ \frac{2J^2}{\Gamma_G+\Gamma_d}\left(-C_m + C_{m+1}\right) - 2\Gamma_G(C_m - 1), & m = 1 \\ \frac{2J^2}{\Gamma_d}\left[\frac{1}{1+\Gamma_G/\Gamma_d}(C_{m-1} - C_m) + C_{m+1} - C_m\right], & m = 2 \\ \frac{2J^2}{\Gamma_d}\left(C_{m-1} - C_m\right), & m = L \end{cases}$$

Define

$C_m := C_{m,m}$

$$\frac{\mathrm{d}C_{\mathrm{diag}}}{\mathrm{d}t} = \mathbb{A}C_{\mathrm{diag}} + \mathbb{P}$$

where

$$\mathbb{A} = \begin{pmatrix} -\alpha_2 - 2\Gamma_G & \alpha_2 & 0 & 0 & \cdots & 0 & 0 \\ \alpha_2 & -(\alpha_1+\alpha_2) & \alpha_1 & 0 & \cdots & 0 & 0 \\ 0 & \alpha_1 & -2\alpha_1 & \alpha_1 & \cdots & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & & \cdots & & \alpha_1 & -2\alpha_1 & \alpha_1 \\ 0 & & \cdots & & & \alpha_1 & -\alpha_1 \end{pmatrix}$$

α_2: $2J^2/(\Gamma_G + \Gamma_d)$

α_1: $2J^2/\Gamma_d$

$$\mathbb{P} = [2\Gamma_G, 0, \cdots, 0],$$

which gives $C_{\mathrm{diag}}(t) = \left(e^{\mathbb{A}t} - \mathbb{I}\right)\mathbb{A}^{-1}\mathbb{P}$

The main task in now to analyse the matrix A

In the weak gain limit $\alpha_2 \approx \alpha_1$

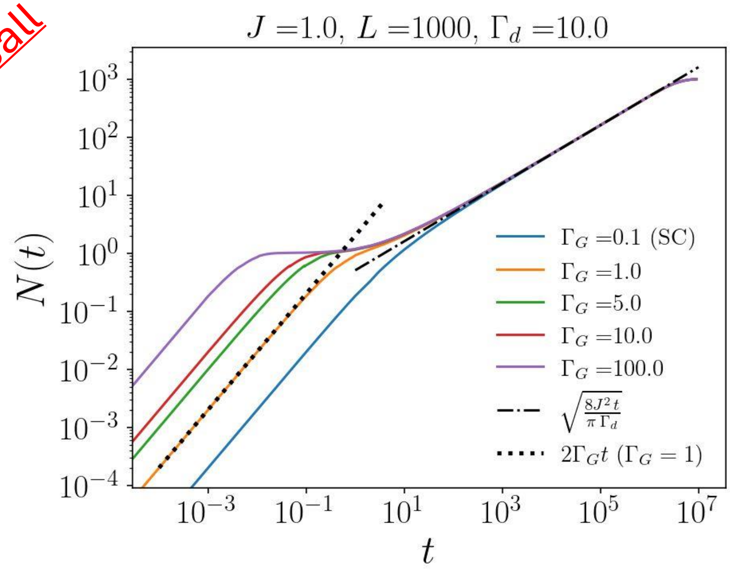
Even after this the matrix is not easy to analyse. So we go to a special case $\Gamma_G = \alpha_1/2$

Special case (SC) $\Gamma_G = \alpha_1/2$

$$\mathbb{A} = \alpha_1 \begin{pmatrix} -2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & & \cdots & & 1 & -2 & 1 \\ 0 & & \cdots & & & 1 & -1 \end{pmatrix}$$

$$C_{\text{diag}}(t) = \left(e^{\mathbb{A}t} - \mathbb{I}\right) \mathbb{A}^{-1} \mathbb{P}$$

Recall



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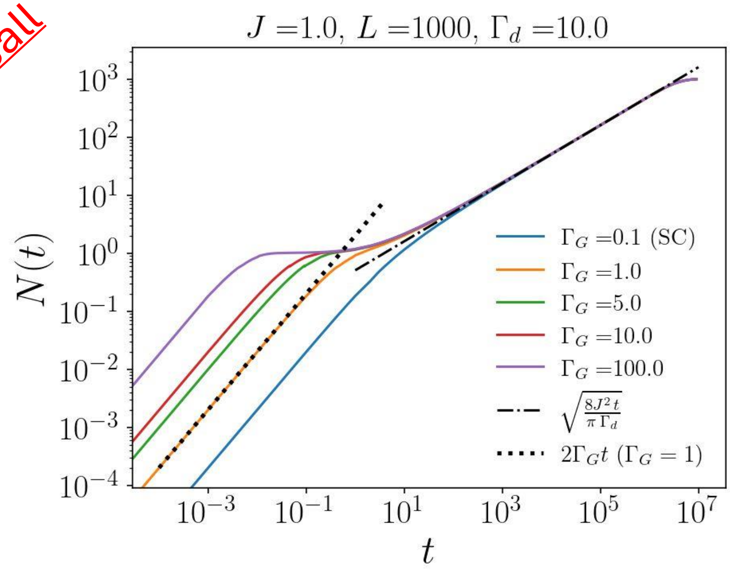
Recall



$$C_{\text{diag}}(t) = \left(e^{\mathbb{A}t} - \mathbb{I}\right) \mathbb{A}^{-1} \mathbb{P}$$

$$n_i(t) = -\frac{8\Gamma_G}{2L+1} \sum_{k=1}^{L} \frac{e^{-4\alpha_1 t \sin^2\left[\frac{(2k-1)\pi}{2(2L+1)}\right]} - 1}{4\alpha_1 \sin^2\left[\frac{(2k-1)\pi}{2(2L+1)}\right]} \sin\left[\frac{(2k-1)\pi}{2L+1}\right] \sin\left[\frac{(2k-1)i\pi}{2L+1}\right]$$

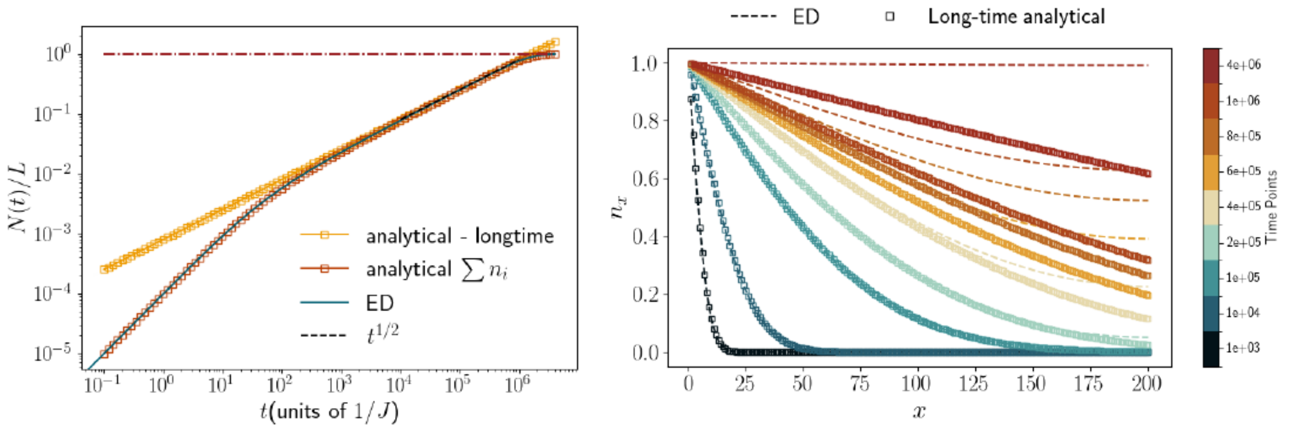
Converting summation to integration

$$n_x(t) = \frac{8\Gamma_G}{\pi} \int_0^{\pi/2} \mathrm{d}\tilde{k} \sin\left(2\tilde{k}\right) \sin\left(2\tilde{k}x\right) \frac{1 - e^{-4\alpha_1 t \sin^2 \tilde{k}}}{4\alpha_1 \sin^2 \tilde{k}}$$

In the limit $t \gg 1$ we can show spatial density profile and the total number of particles

$$n_x(t) = 1 - Erf\left(\frac{x}{2\sqrt{\alpha_1 t}}\right) \qquad N(t) = 2\sqrt{\frac{\alpha_1 t}{\pi}}$$

At special case



Direct numerics and analytics match

What happens when we inject into an interacting quantum system ?

We will study two types of interacting systems

Quasi-periodic XXZ chain

$$H_{\mathrm{QP-XXZ}} = J\sum_{i=1}^{L-1}\left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z\right) + W\sum_{i=1}^{L}\cos(2\pi b\, i + \phi) S_i^z$$

irrational number

$$\dot{\rho} = -i[H_{\mathrm{QP-XXZ}}, \rho] + \Gamma_G\left[S_1^+ \rho S_1^- - \frac{1}{2}\{S_1^- S_1^+, \rho\}\right]$$

Injection

$$|\rho(t)\rangle = e^{\mathscr{L}t}|\rho(0)\rangle$$

$$\mathscr{L} = -i\left(\mathbb{I}\otimes H_{\mathrm{QP-XXZ}} - H_{\mathrm{XXZ}}^T\otimes\mathbb{I}\right) + \Gamma_G\left[(S_1^-)^T\otimes S_1^+ - \frac{1}{2}\left[\mathbb{I}\otimes(S_1^-S_1^+) + (S_1^-S_1^+)^T\otimes\mathbb{I}\right]\right]$$

Next nearest neighbour XXZ spin chain

$$H_{\mathrm{NNN-XXZ}} = J\sum_{i=1}^{L-1}\left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z\right) + J'\sum_{i=1}^{L-2} S_i^z S_{i+2}^z$$

Next-nearest neighbor integrability breaking term

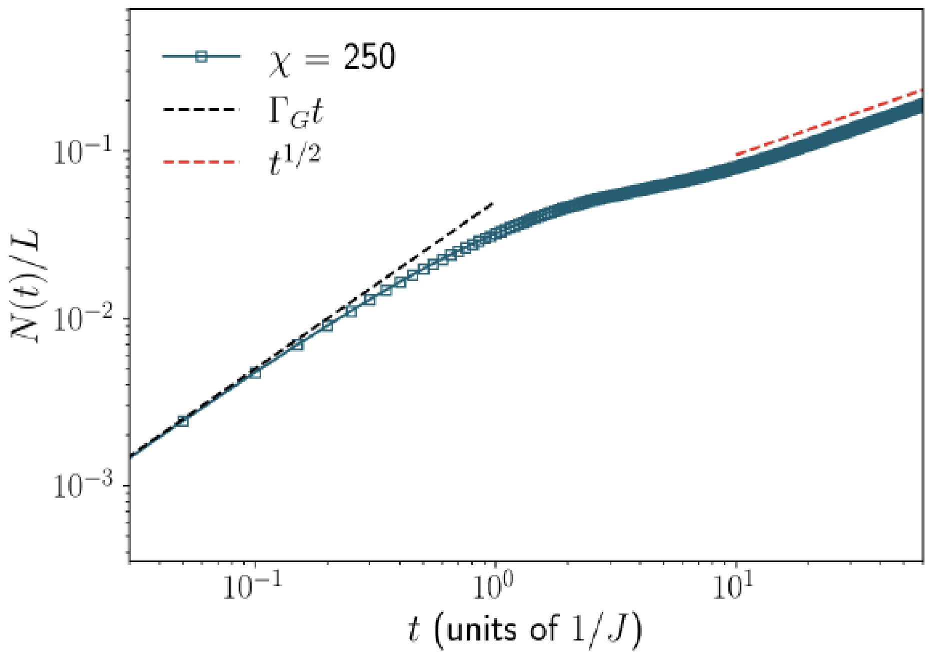
Since these are chaotic / non-integrable quantum systems, we would expect to see diffusive behaviour.
We now numerically look for evidence for it using TEBD algorithm.

What happens when we inject into an interacting quantum system ?

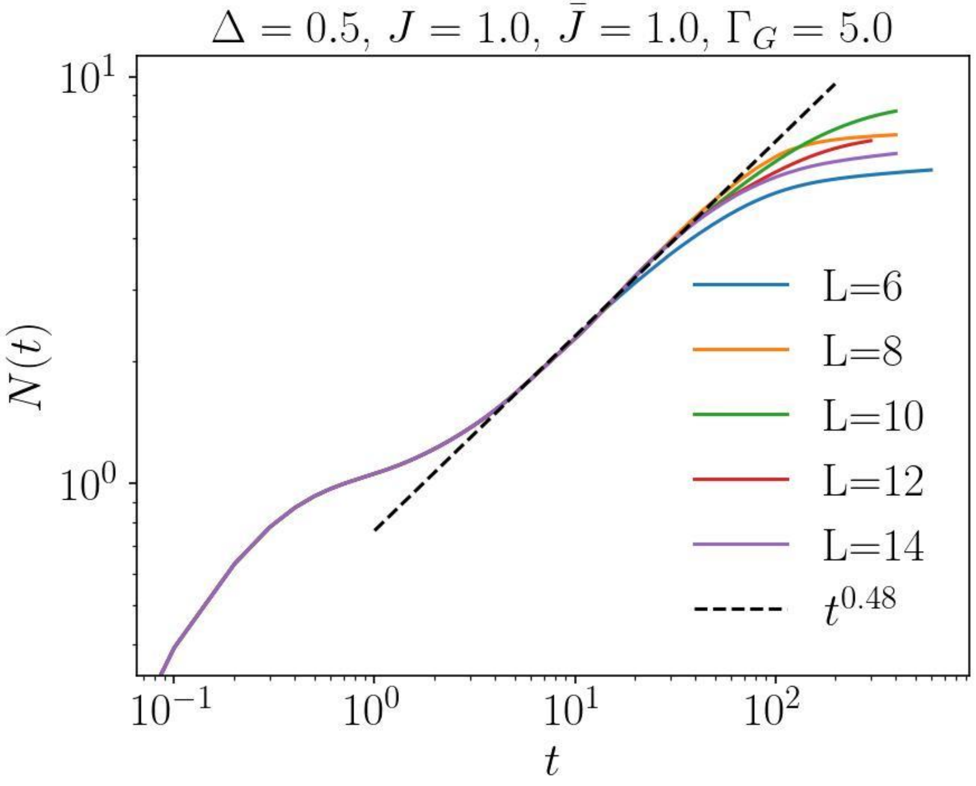
Quasi-periodic model

$$H_{\mathrm{QP-XXZ}} = J \sum_{i=1}^{L-1} \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z \right) + W \sum_{i=1}^{L} \cos(2\pi b i + \phi) S_i^z$$

$$H_{\mathrm{NNN-XXZ}} = J \sum_{i=1}^{L-1} \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z \right) + J' \sum_{i=1}^{L-2} S_i^z S_{i+2}^z$$



$L = 20, \Gamma_G = 1.0J, W = 1.0J$

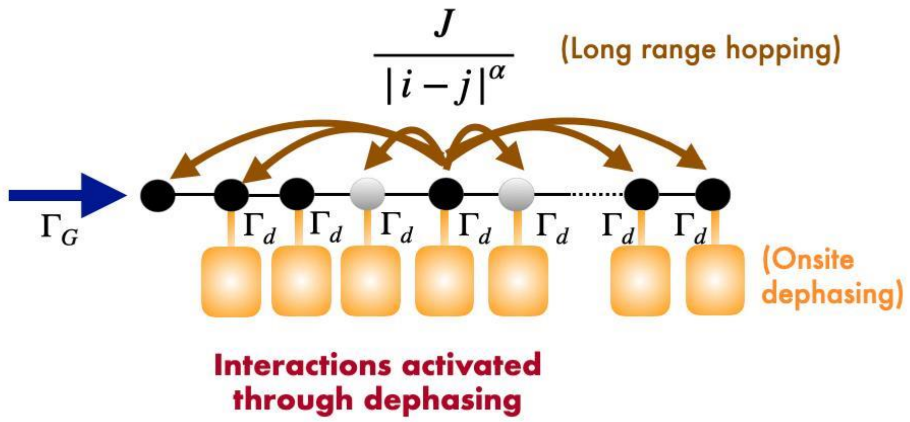


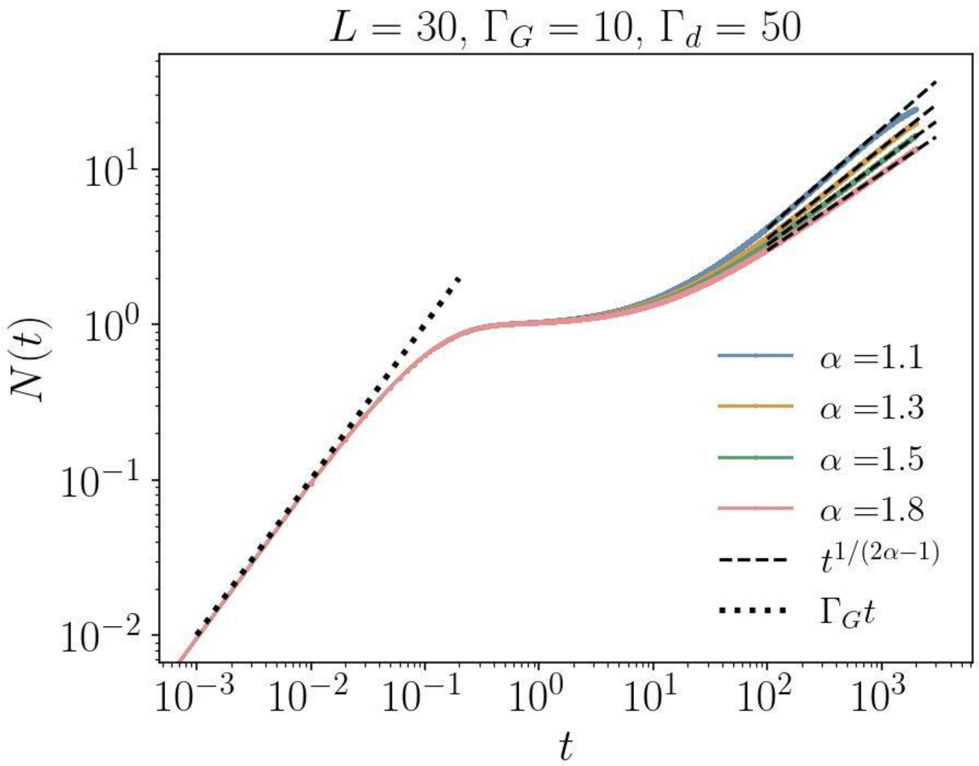
We see squareroot time behaviour but we are still awaiting better data

What happens when we inject long-ranged systems that are subject to dephasing ?

$$\hat{\mathcal{H}}_S = -\sum_{m=1}^{N} \frac{J}{m^\alpha} \left[\sum_{r=1}^{N-m} \hat{c}_r^\dagger \hat{c}_{r+m} + \hat{c}_{r+m}^\dagger \hat{c}_r \right]$$

$$\frac{d\rho}{dt} = -i[H_S, \rho] + \Gamma_G \left[2c_1^\dagger \rho c_1 - \{c_1 c_1^\dagger, \rho\} \right] - \frac{\Gamma_d}{2} \sum_{i=1}^{L} [n_i, [n_i, \rho]]$$





See,
Schuckert , Lovas, Knap (PRB 2020)
Dhawan, Ganguly, MK, Agarwalla (PRB 2024)
Nishikawa, Saito (2025)
Catalano et al, PRL (2025)

Key findings

$$N(t) \propto \begin{cases} t^{\frac{1}{2\alpha-1}} & \text{for} \quad \alpha \leq \frac{3}{2} \\ t^{\frac{1}{2}} & \text{for} \quad \alpha > \frac{3}{2} \end{cases}$$

These exponents of the injection problem seem to be the same as several long-ranged papers in slightly different contexts

Conclusions to Part C

- We studied injection of particles on lattices
- Two cases (i) either itself subject to dephasing mechanism or (ii) is itself inherently interacting.
- For the dephasing case, we provided numerical and analytical results.
- For the interacting case, we provided TEBD results showing square-root behaviour.
- We discussed long-ranged case subject to dephasing.

Thank you