Speed limits and optimal transport:
General bounds and maximal speed of particle transport

Tan Van Vu

YITP, Kyoto University

Vu and Saito, PRL **130**, 010402 (2023)
Vu, Kuwahara, and Saito, Quantum **8**, 1483 (2024)

Hydrodynamics of low-dimensional interacting systems
June 2-13 2025, YITP, Kyoto

Outline

- Speed of information propagation and particle transport
- Optimal transport theory & quantum speed limit
- General speed limit based on optimal transport theory
- Results on bosonic transport in closed quantum systems

Information propagation

- Upper limit on speed of information propagation

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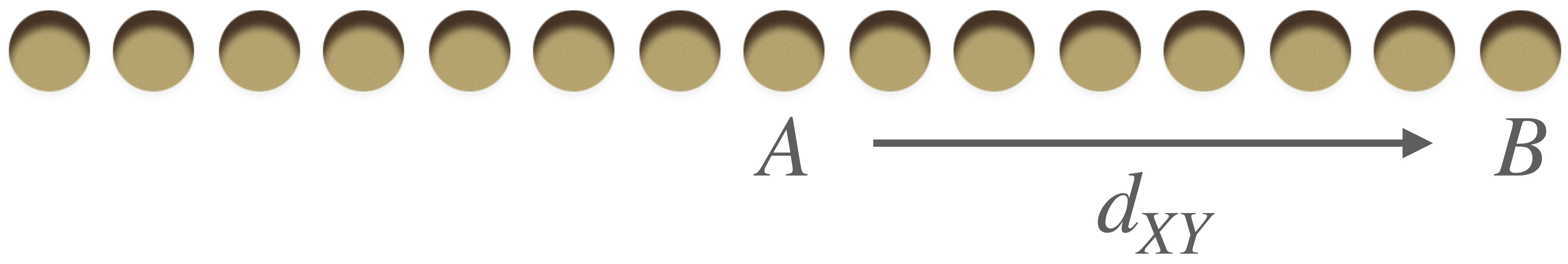


Information propagation

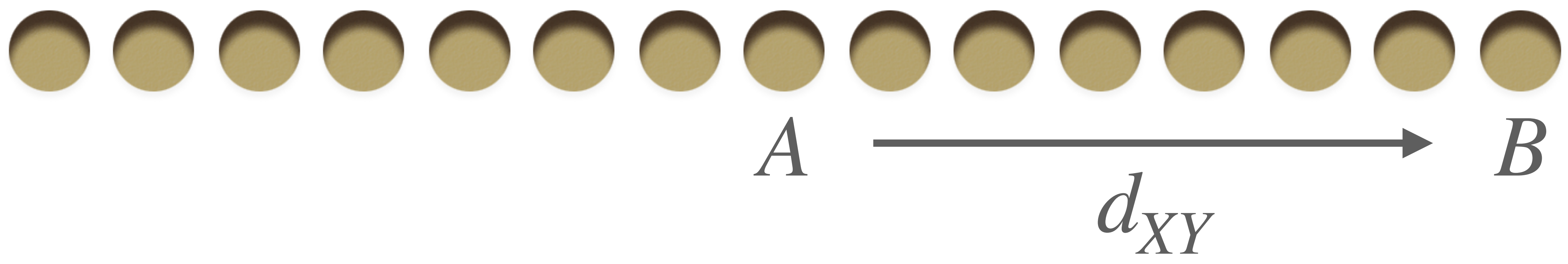
- Upper limit on speed of information propagation

$$\|[A(\tau), B]\| \leq c \exp[-a(d_{XY} - v\tau)]$$

Lieb and Robinson, Commun. Math. Phys. (1972)

A, B : local operators with supports X and Y

$A(t) := e^{iHt} A e^{-iHt}$: Heisenberg operator



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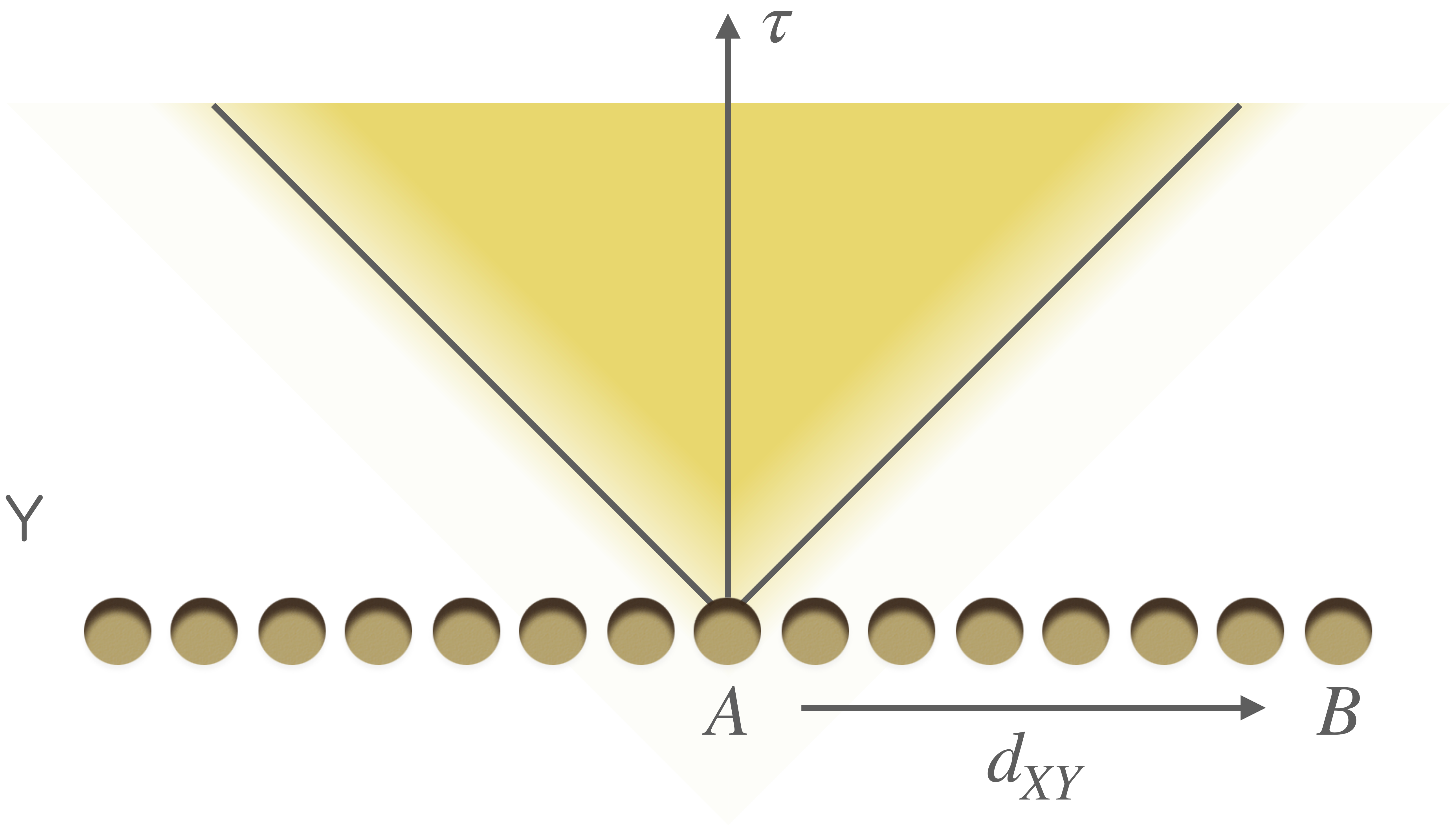

effective linear light cone $\tau \gtrsim d_{XY}$

Operator spreading & information propagation

- Quantum metrology perspective

Wysocki and Chwedenczuk, Phys. Rev. Lett. (2025)

Operator spreading & information propagation

- Quantum metrology perspective
  1. Encoding parameter $\varrho(\theta) = e^{-i\theta A_X} \varrho e^{i\theta A_X}$

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- Cramer-Rao inequality

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$$\mathrm{var}[\hat{\theta}] \geq \frac{1}{\mathcal{I}(\theta)}$$

$\mathcal{I}(\theta)$: quantum Fisher information

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$\mathcal{I}(\theta)$: quantum Fisher information

- Quantum Fisher information is upper bounded by operator spreading

$$\mathcal{I}(\theta) \leq \sum_{i=1}^{3} \| [O_{Y,i}(t), A_X] \|^2$$

Lieb-Robinson bound

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- Completely resolved for spin and fermion systems

$$\tau \gtrsim d_{XY}^{\min(1,\alpha-2D)}$$

Kuwahara and Saito, Phys. Rev. X (2020); Tran+, Phys. Rev. Lett. (2021)

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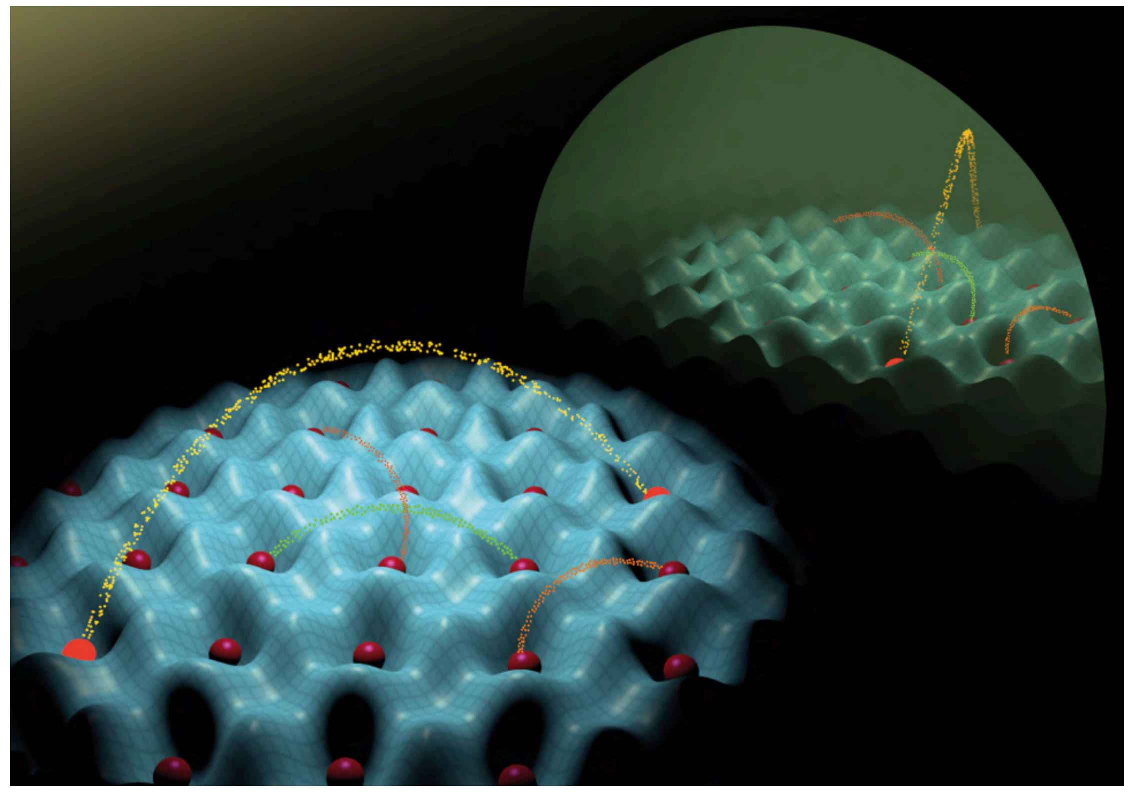

Defenu+, Rev. Mod. Phys. (2023)

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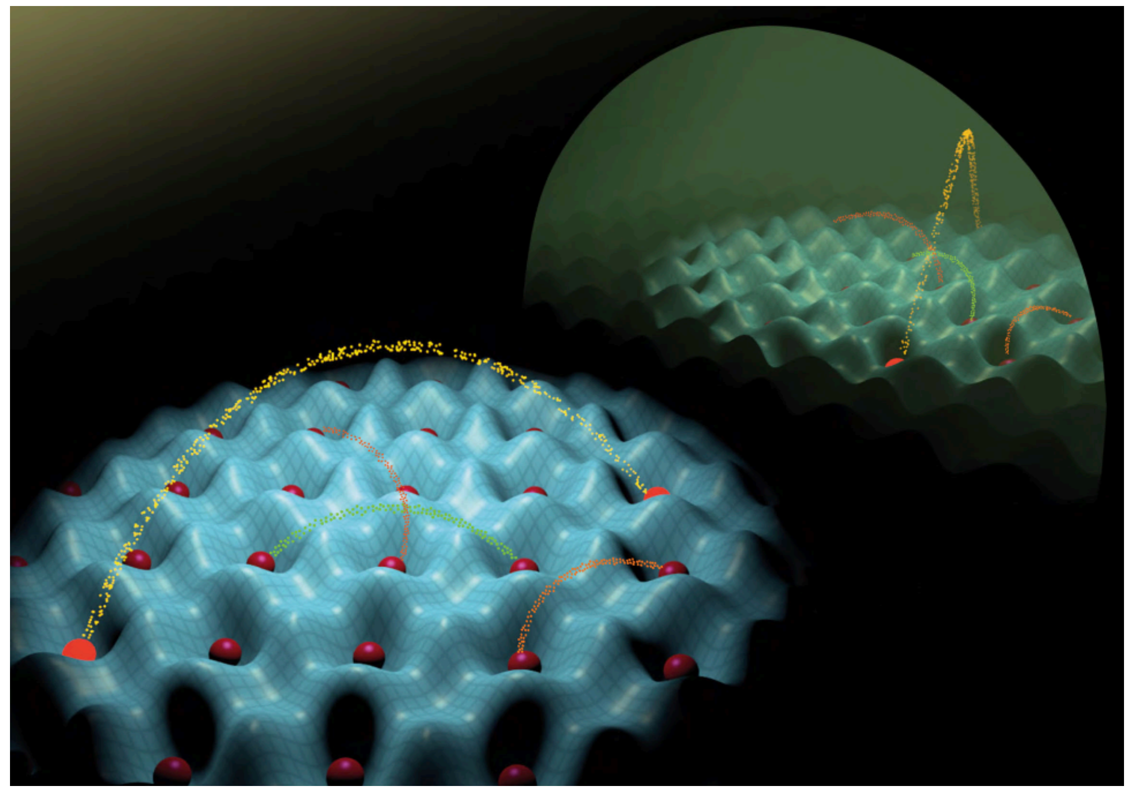

Defenu+, Rev. Mod. Phys. (2023)

speed of particle transport ➕ speed of information propagation

Macroscopic particle transport

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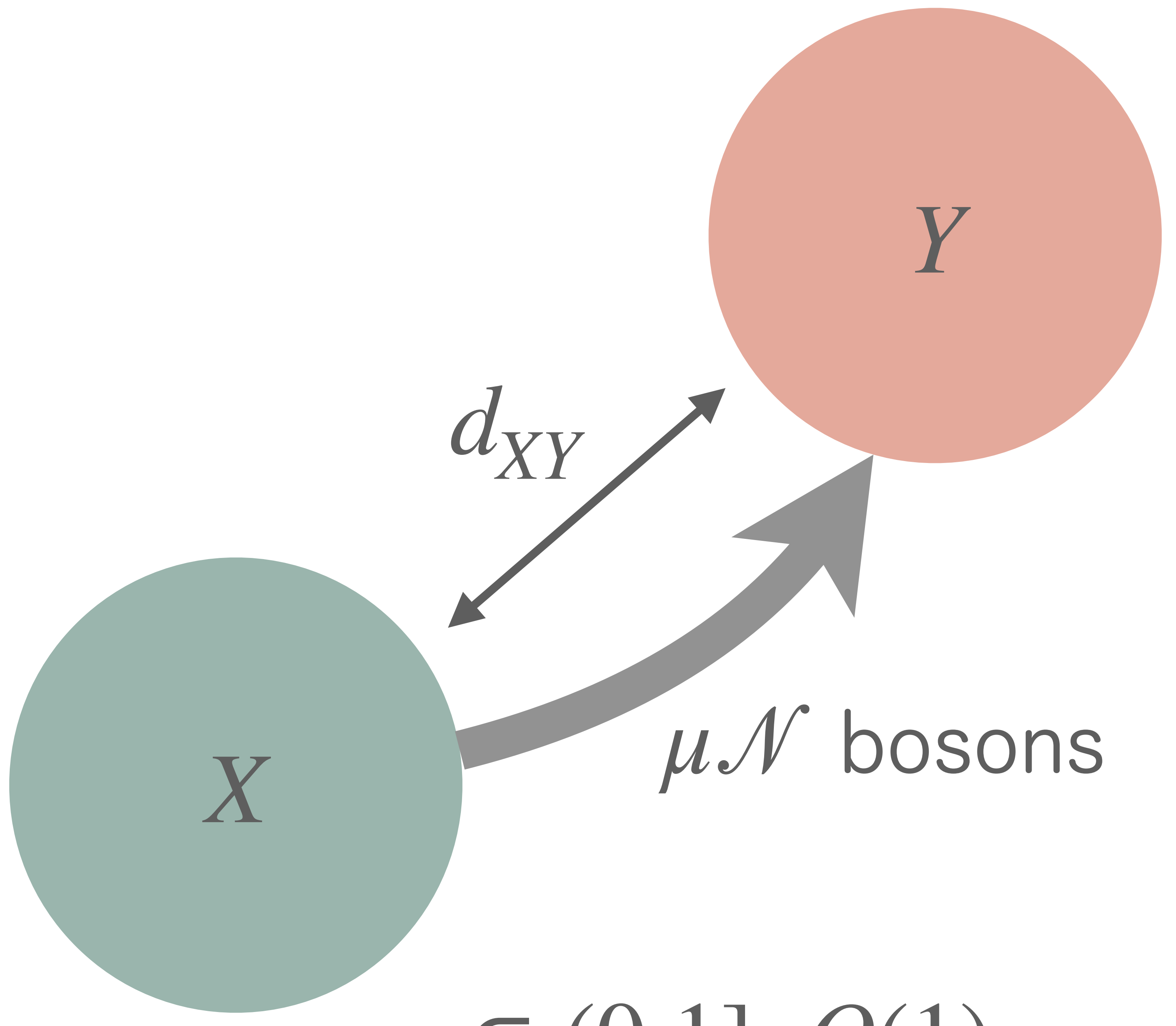

$\mu \in (0,1]$: $O(1)$ constant

$\mathcal{N}$: total number of bosons

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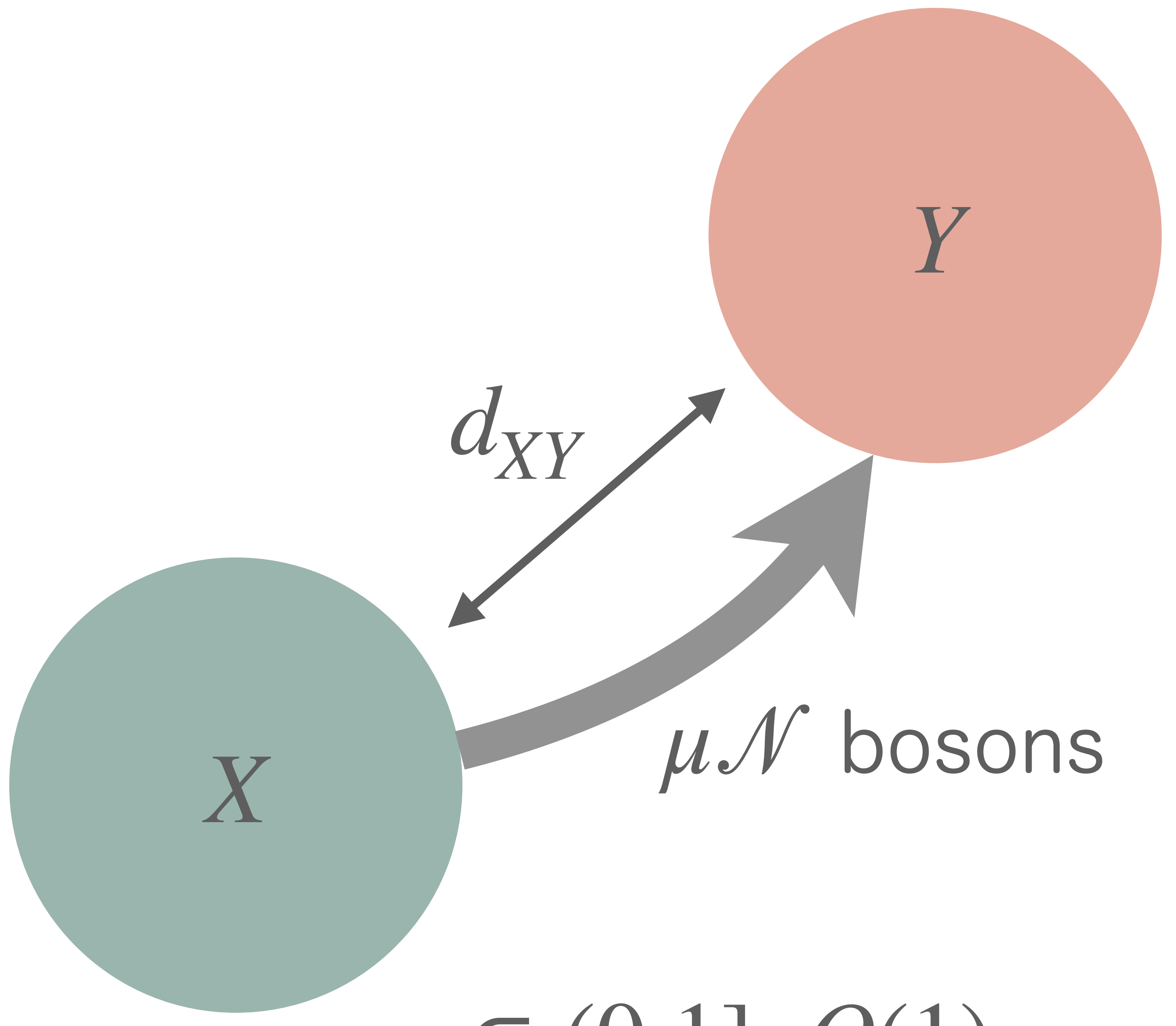
requirement : $n_Y(\tau) \geq n_{X^c}(0) + \mu\mathcal{N}$



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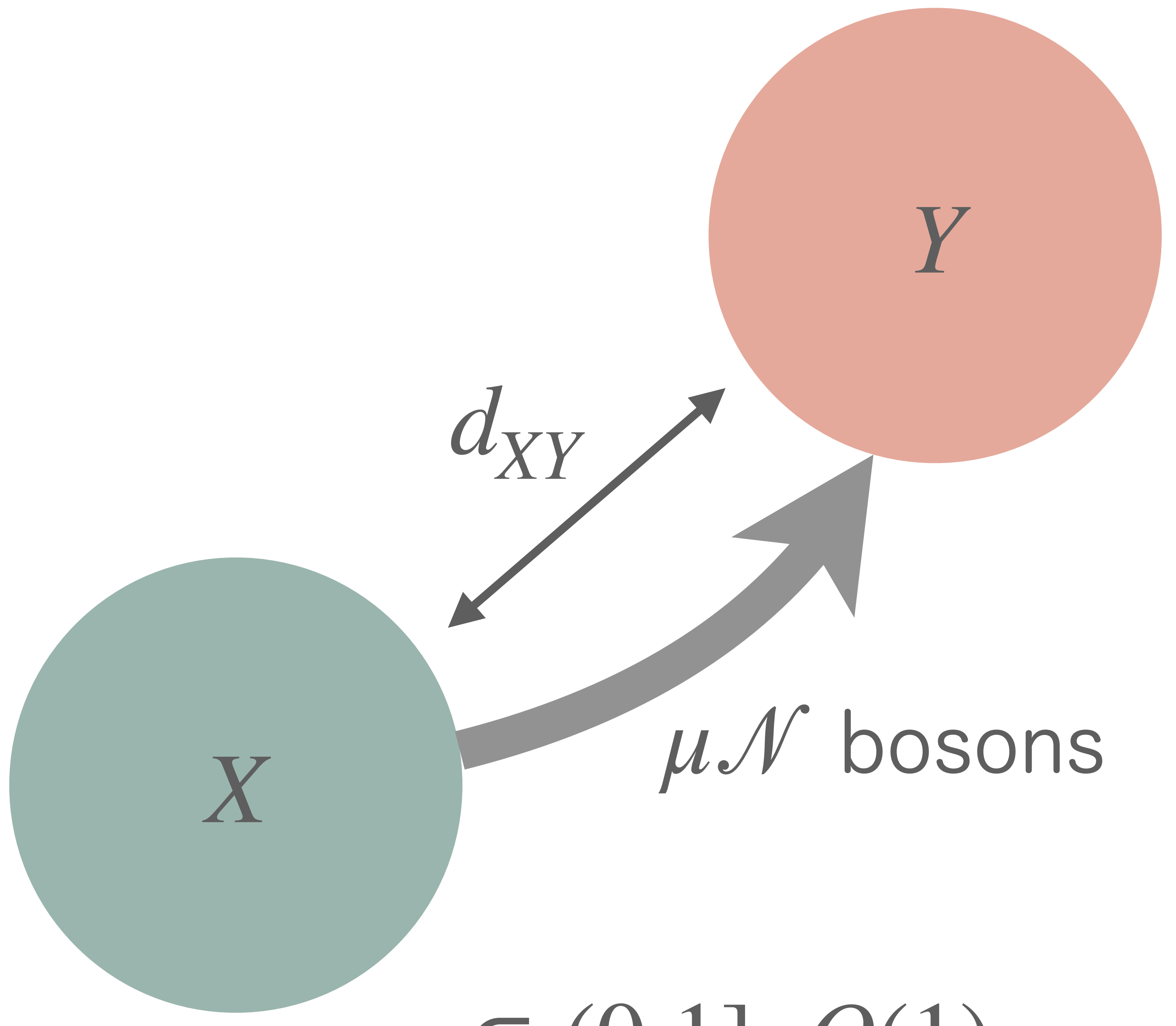

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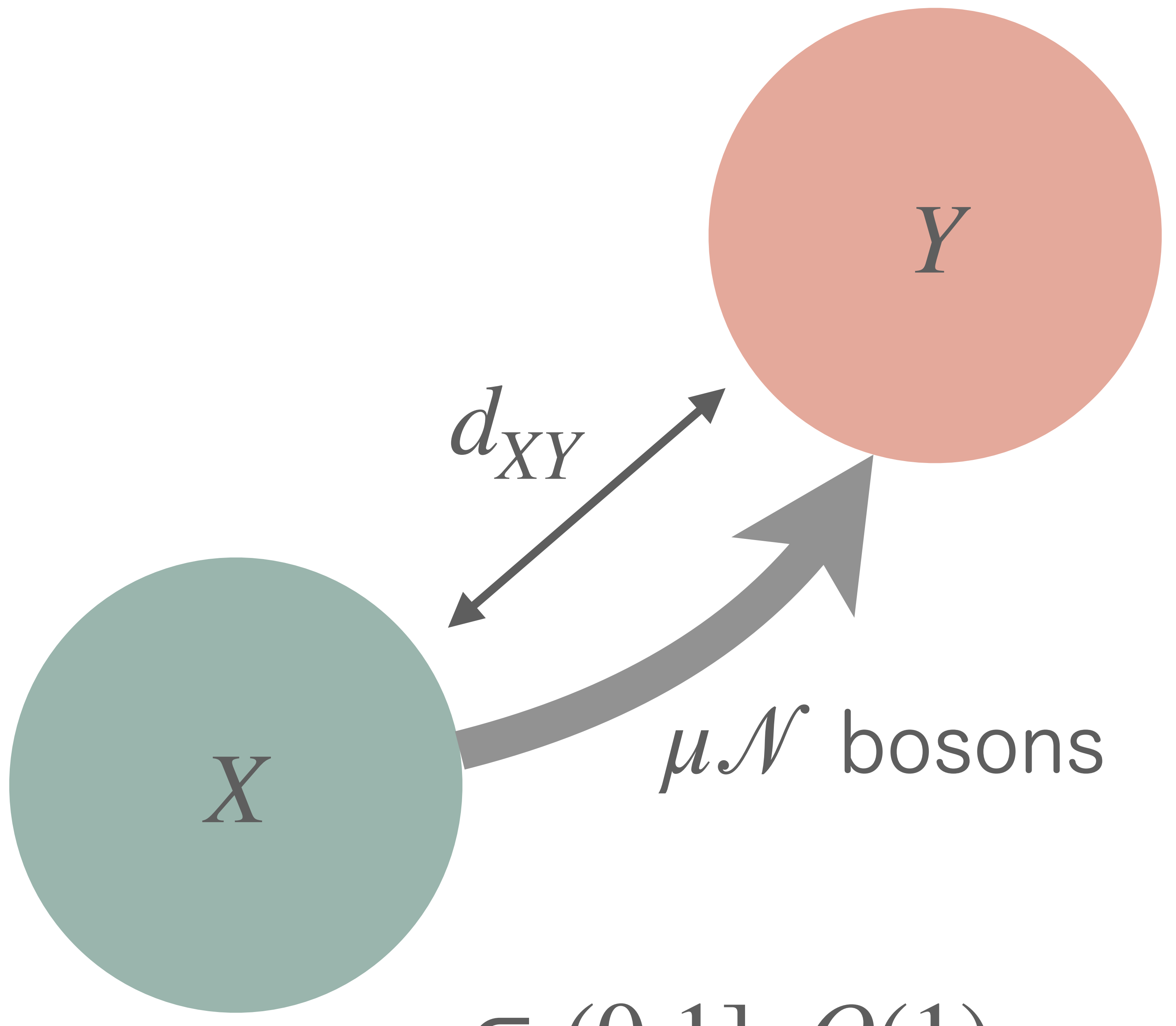

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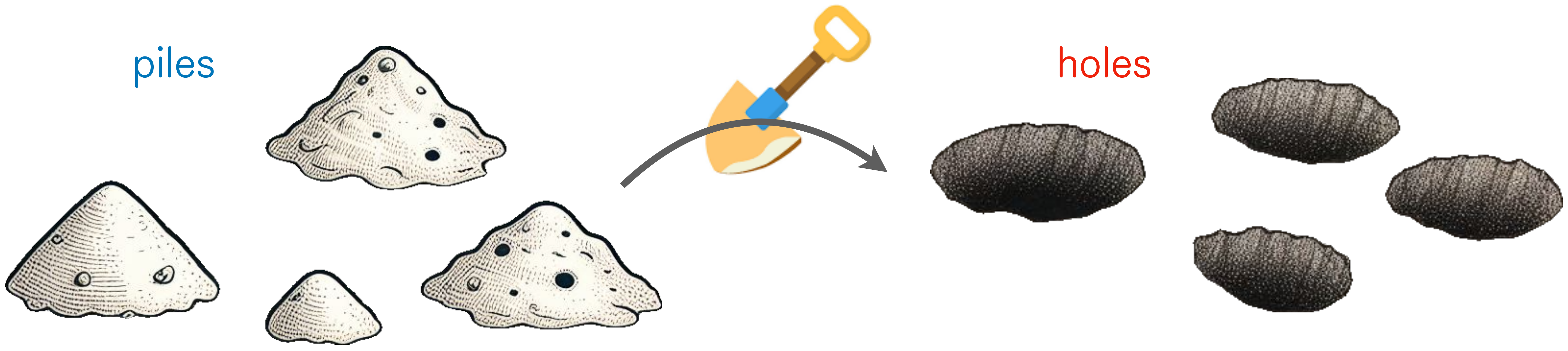
Our approach: Optimal transport theory & Quantum speed limit

Optimal transport

Optimal transport

- Problem of transporting a source distribution to a target distribution

Gaspard Monge (1781)



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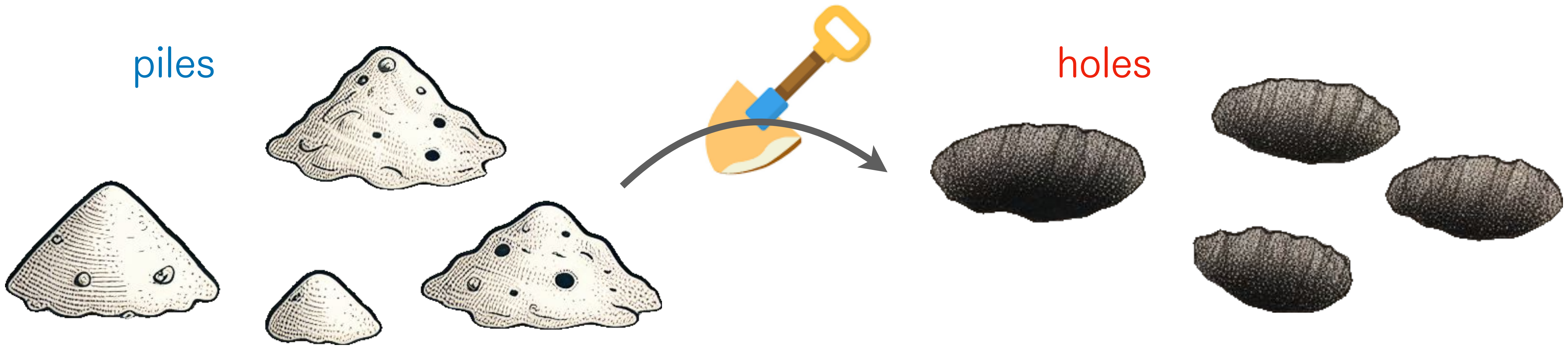

Optimal transport plan & optimal transport cost

Monge formulation

Optimal transport cost with respect to a cost function $c(x, y)$:

$$M(p^A, p^B) = \min_T \int dx\, c(x, T(x)) p^A(x)$$

T : one-to-one map that preserves the total probability

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- Non-existence of a valid transport map: T might not exist in discrete cases because no mass can be split

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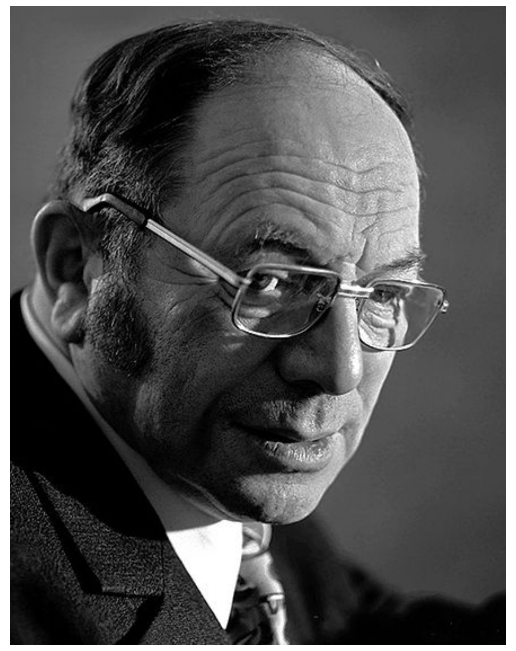
- Resolved by the relaxation of Kantorovich

Kantorovich formulation

Leonid Kantorovich
(1942)



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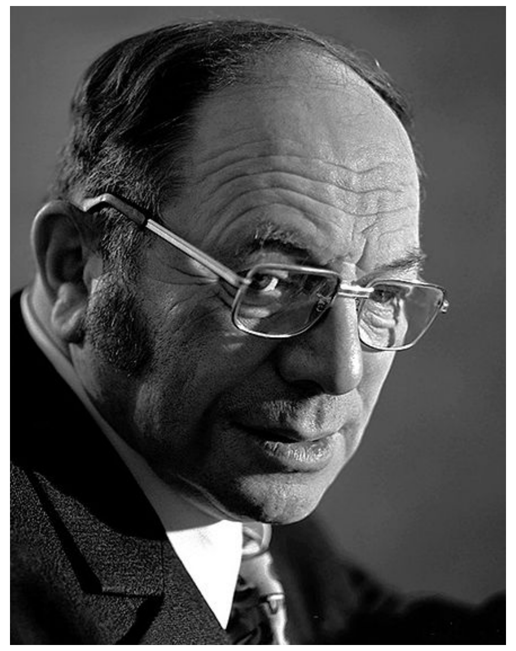

Optimal transport cost with respect to a cost function $c(x, y)$:

$$K(p^A, p^B) := \min_{\pi} \int dx\, dy\, c(x, y)\pi(x, y)$$

coupling π : joint probability distribution of p^A and p^B

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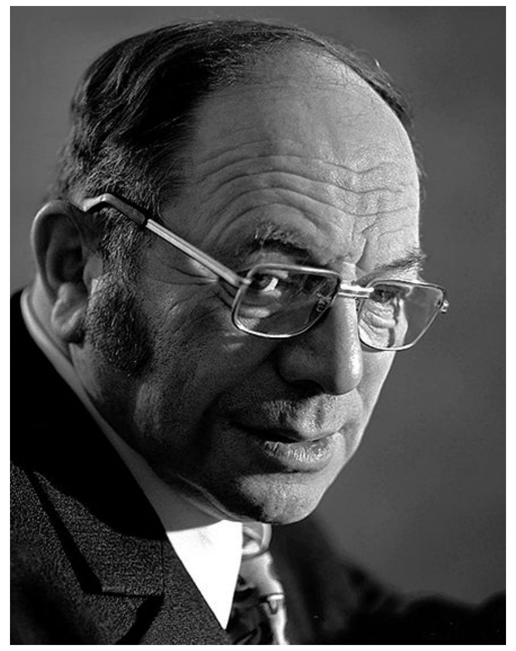

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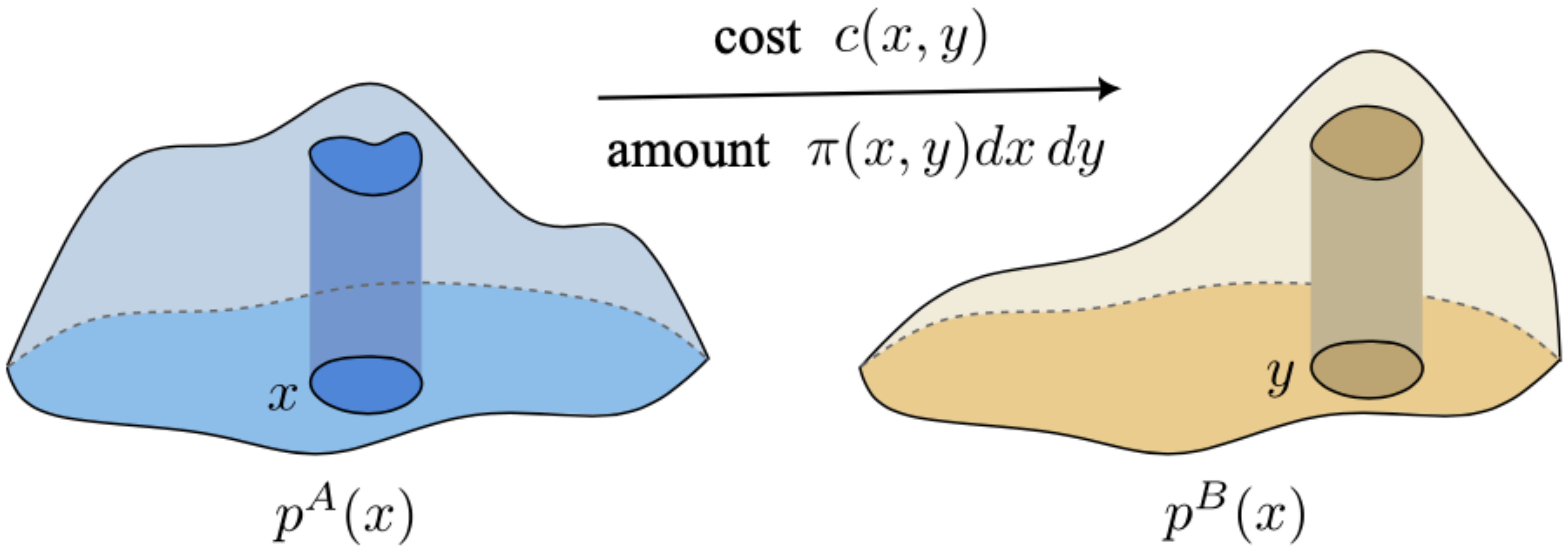

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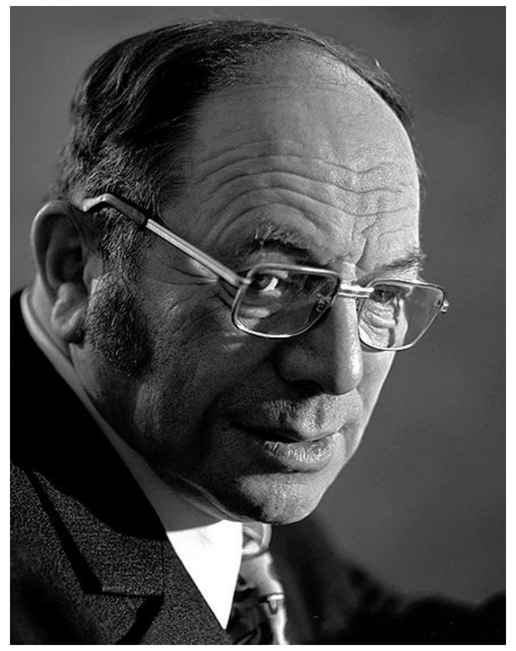

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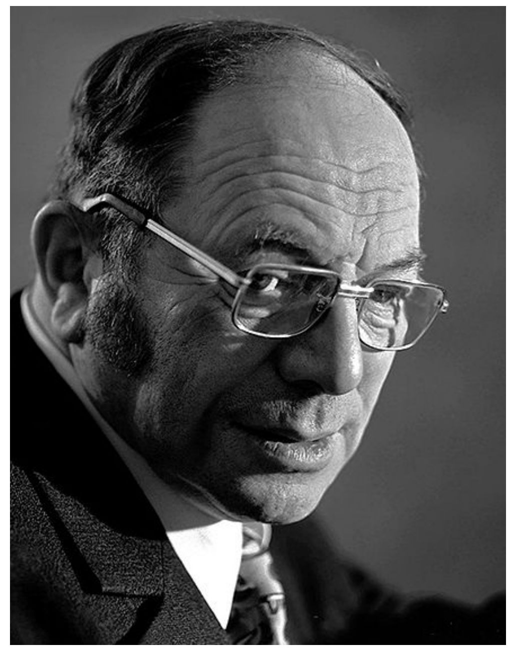

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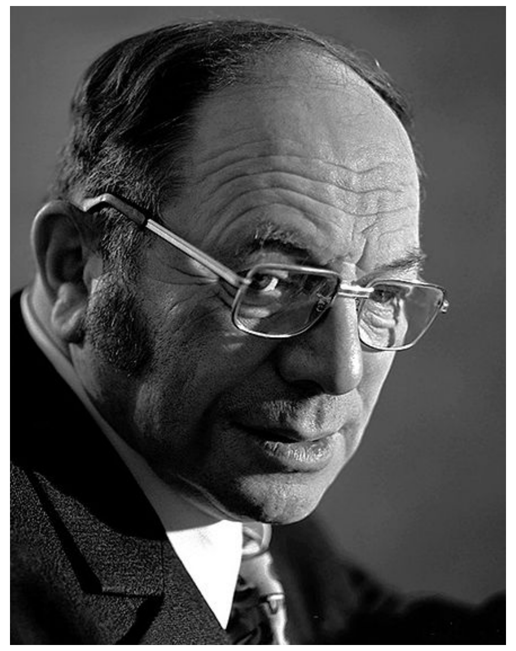

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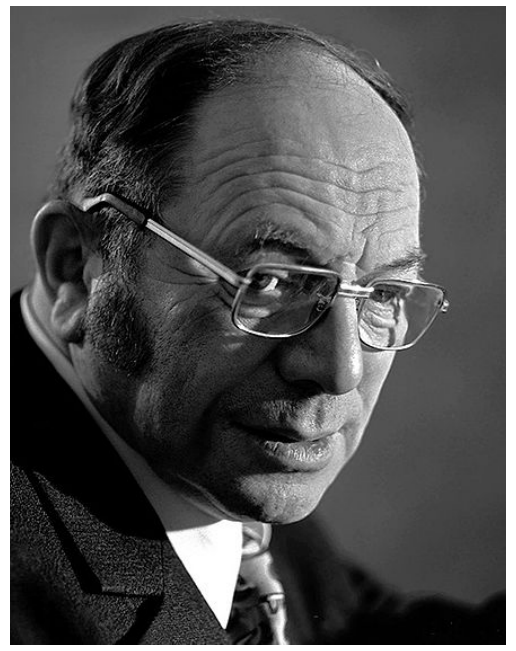

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- In general, $K(p^A, p^B) \leq M(p^A, p^B)$
- Optimal coupling of Kantorovich formulation always exists
- Two formulations are equivalent if p^A has no atom and c is continuous

Discrete optimal transport

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- Transport of goods

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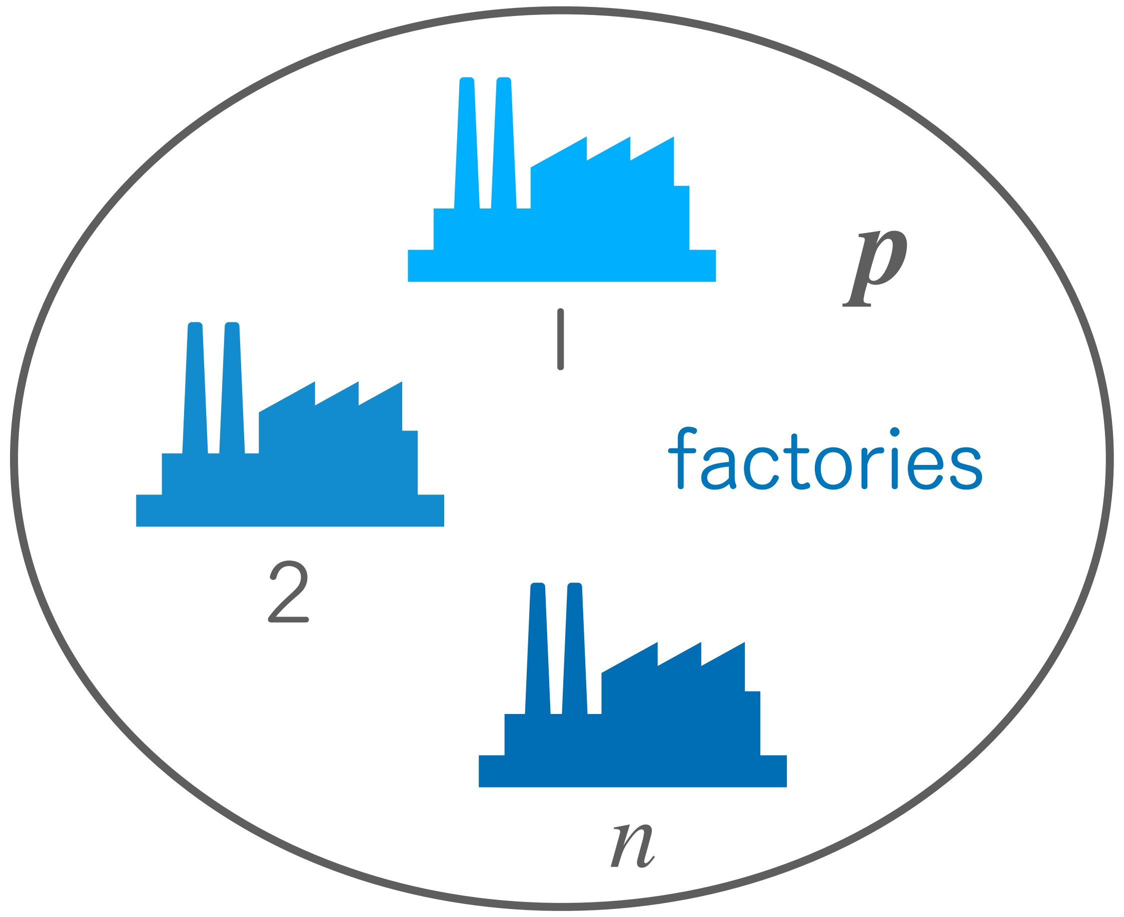

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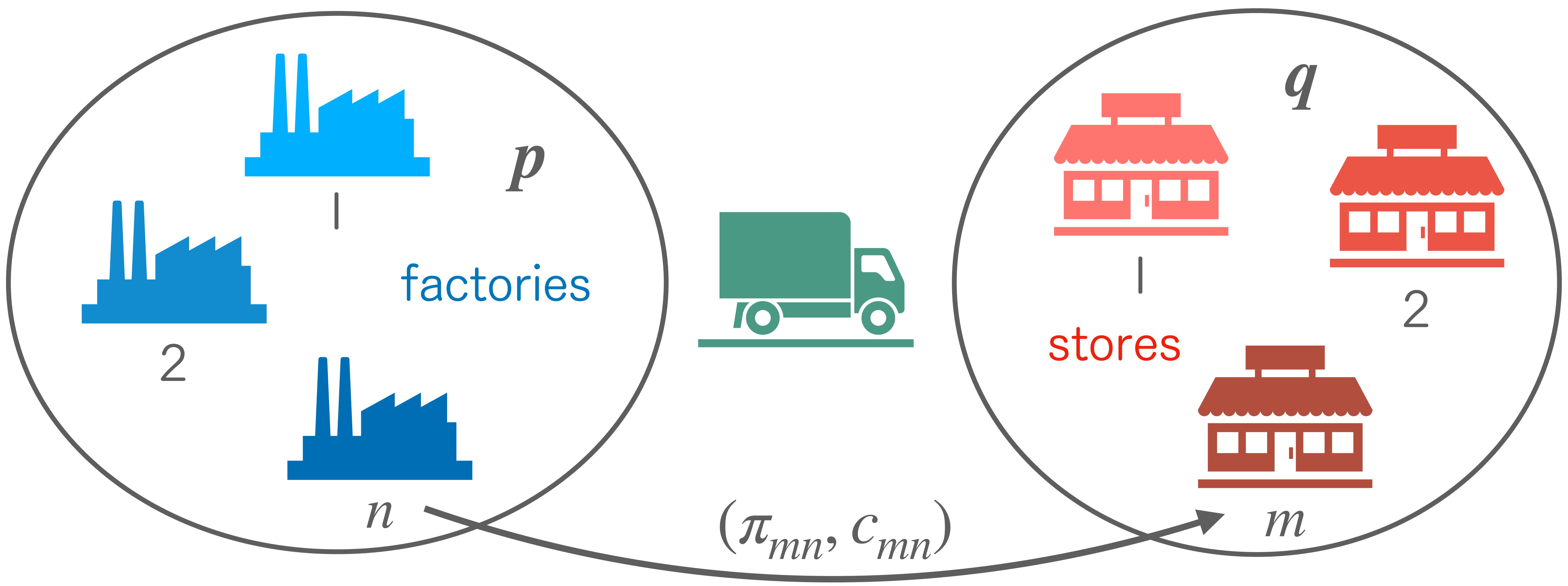


π_{mn} : fraction of goods from factory n to store m

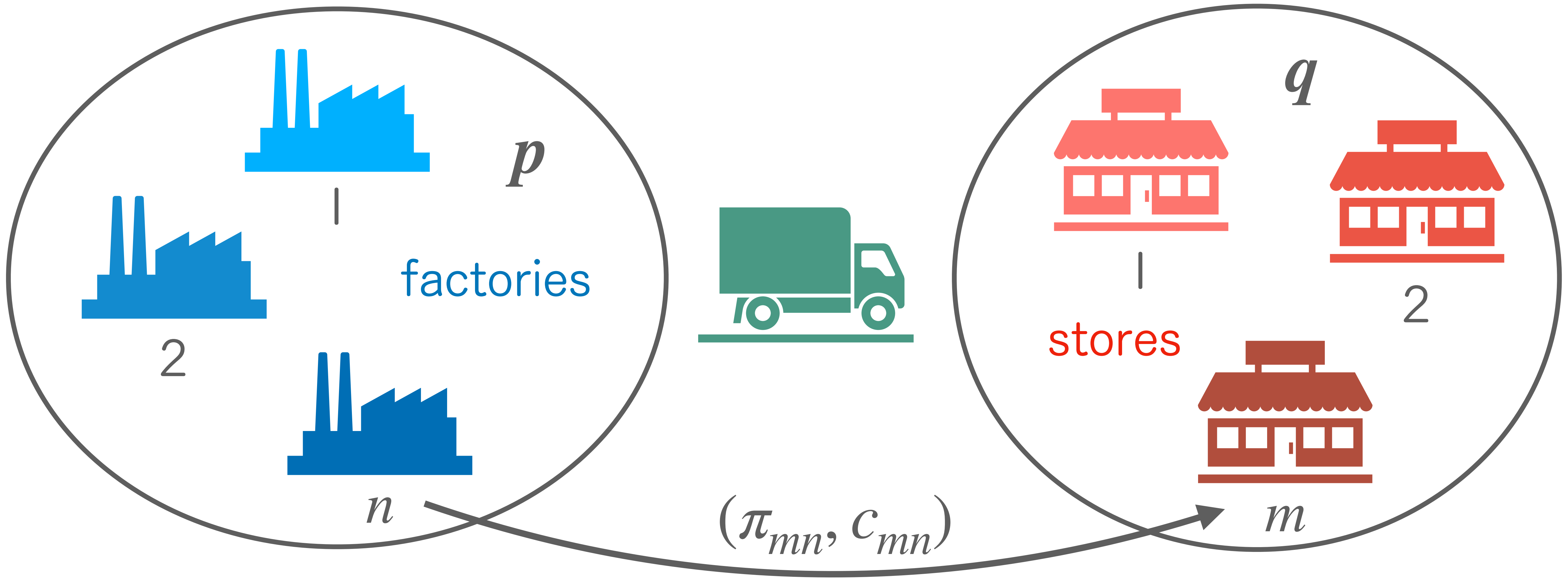
c_{mn} : cost per unit of goods from factory n to store m

Discrete optimal transport

- Transport of goods

Total transport cost

$$\sum_{m,n} \pi_{mn} c_{mn}$$



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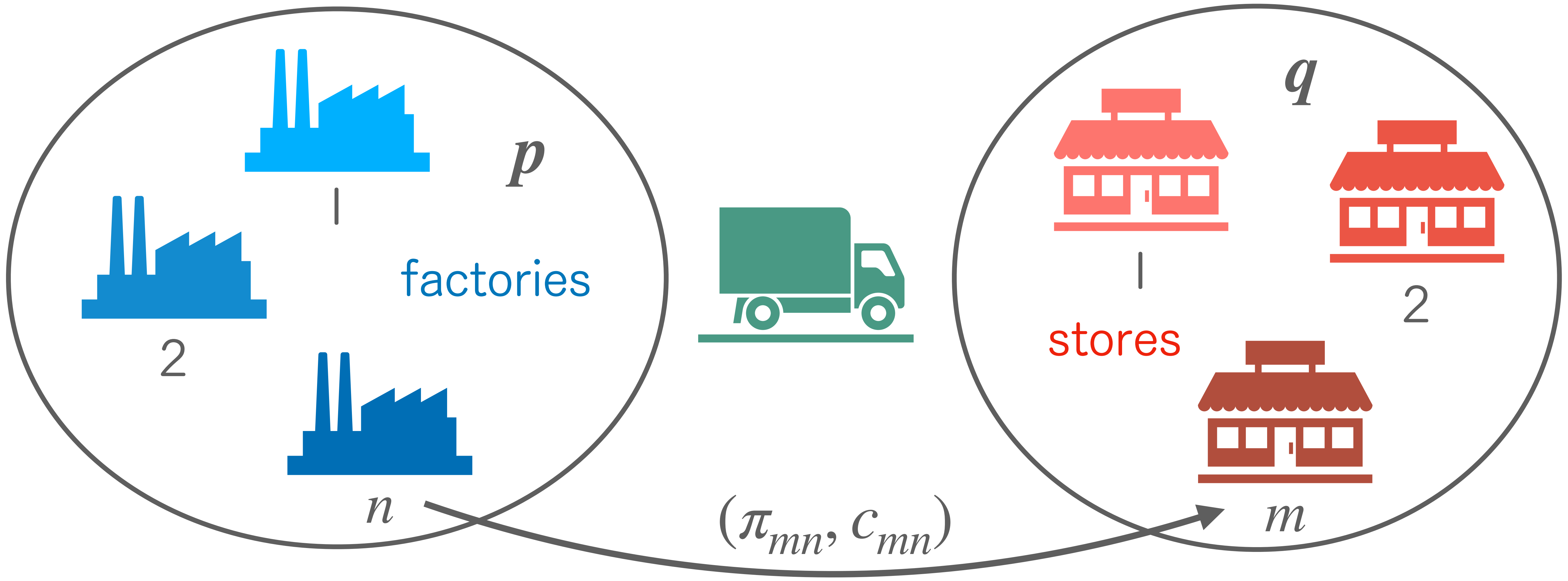

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- L^1-Wasserstein distance

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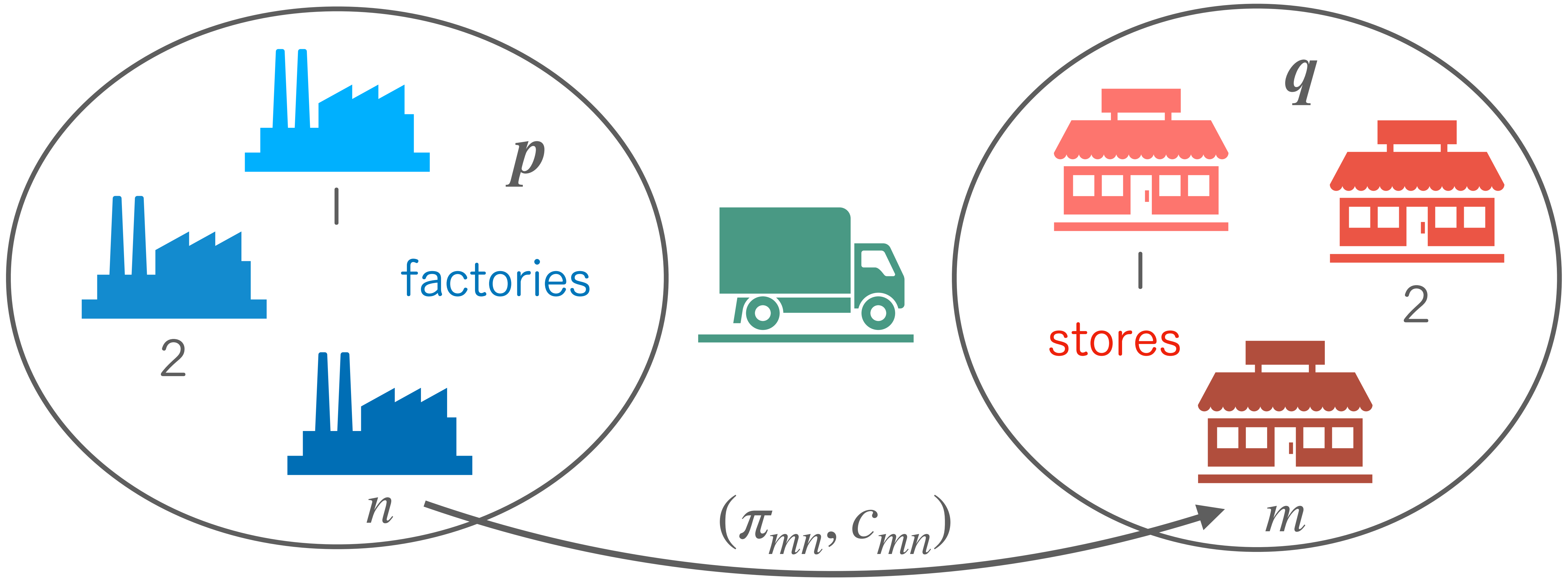

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transport plan π : $\sum_m \pi_{mn} = p_n, \quad \sum_n \pi_{mn} = q_m$

cost matrix $[c_{mn}]$: $c_{mn} + c_{nk} \geq c_{mk}$

Kantorovich-Rubinstein duality

Kantorovich-Rubinstein duality

- For symmetric cost matrix $c_{mn} = c_{nm}$

$$\mathscr{W}(\boldsymbol{p}, \boldsymbol{q}) = \max_{\boldsymbol{\phi}} \boldsymbol{\phi}^\top (\boldsymbol{p} - \boldsymbol{q})$$

maximum is over all vectors $\boldsymbol{\phi}$ satisfying $|\phi_m - \phi_n| \leq c_{mn}$

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Applications of optimal transport

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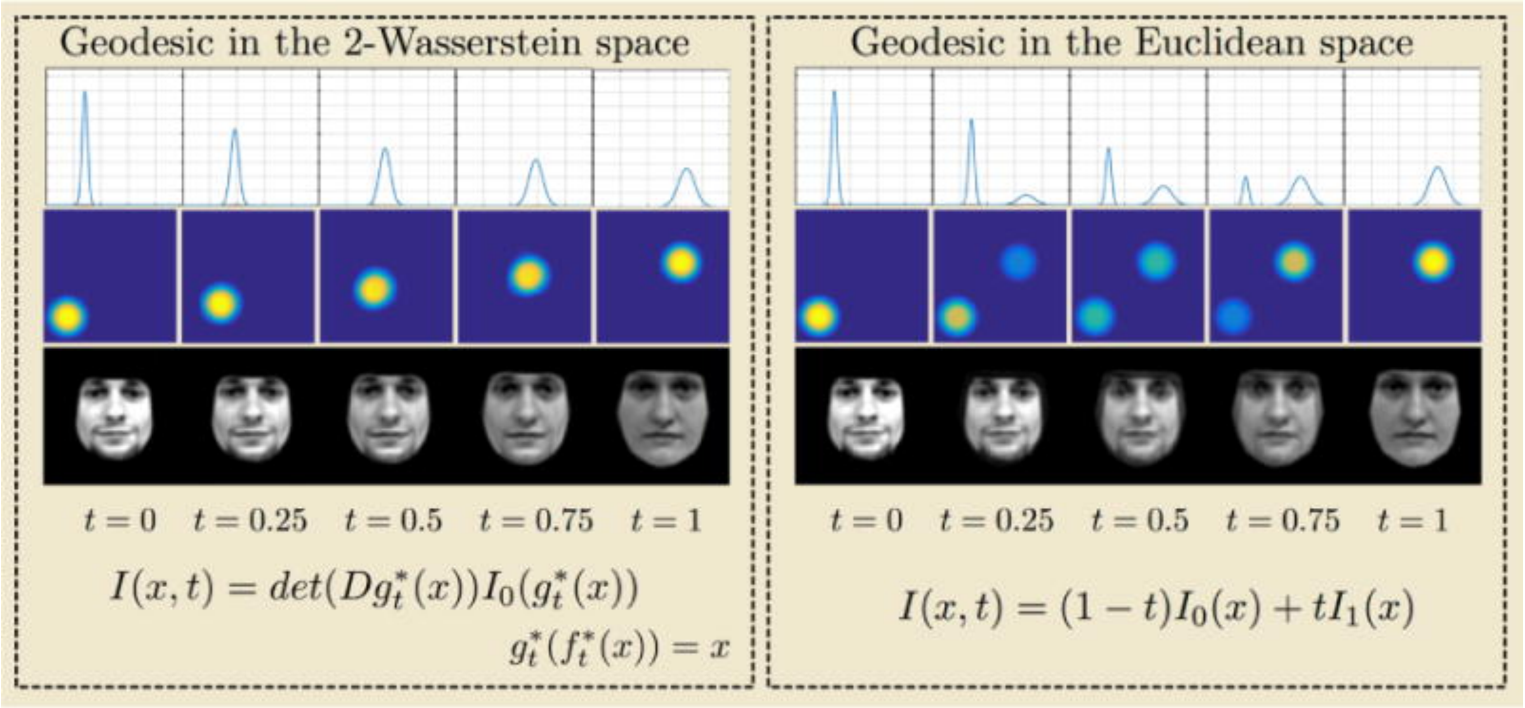
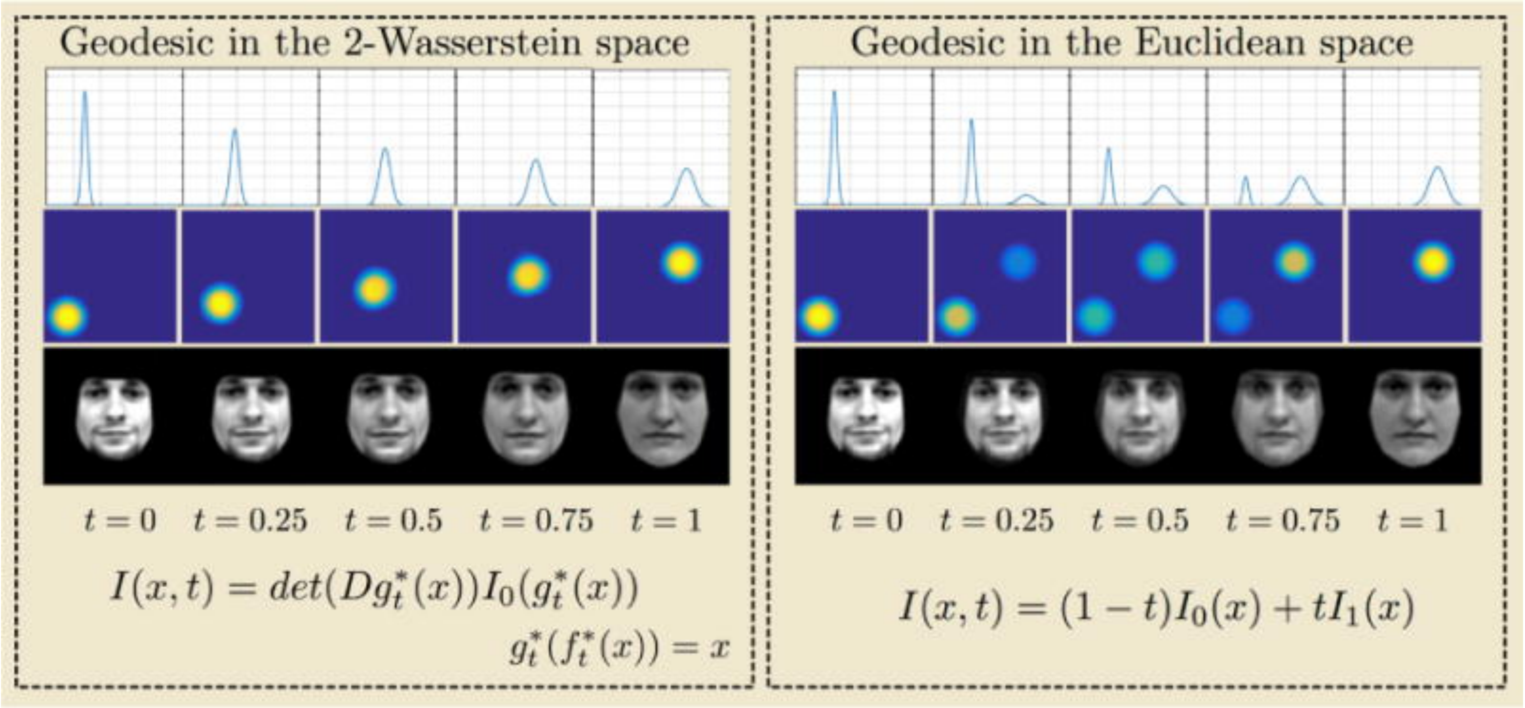


Kolouri+, IEEE Signal Process. Mag. (2017)

Applications of optimal transport



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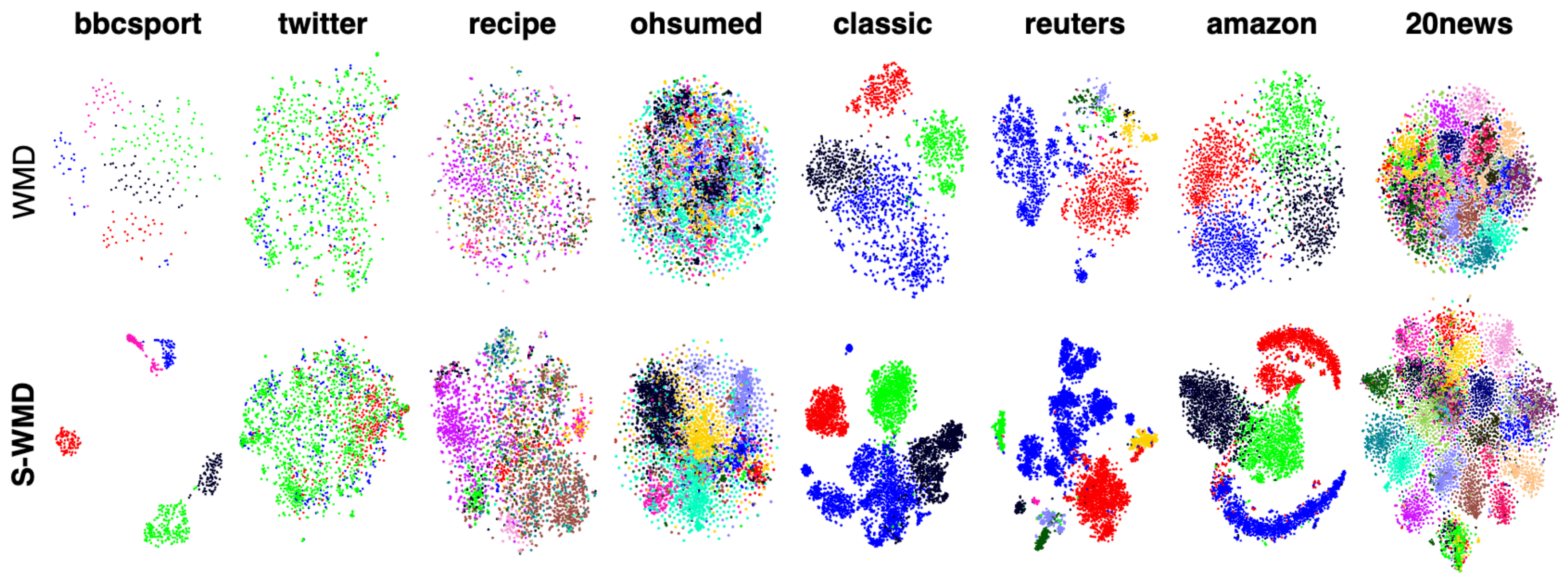

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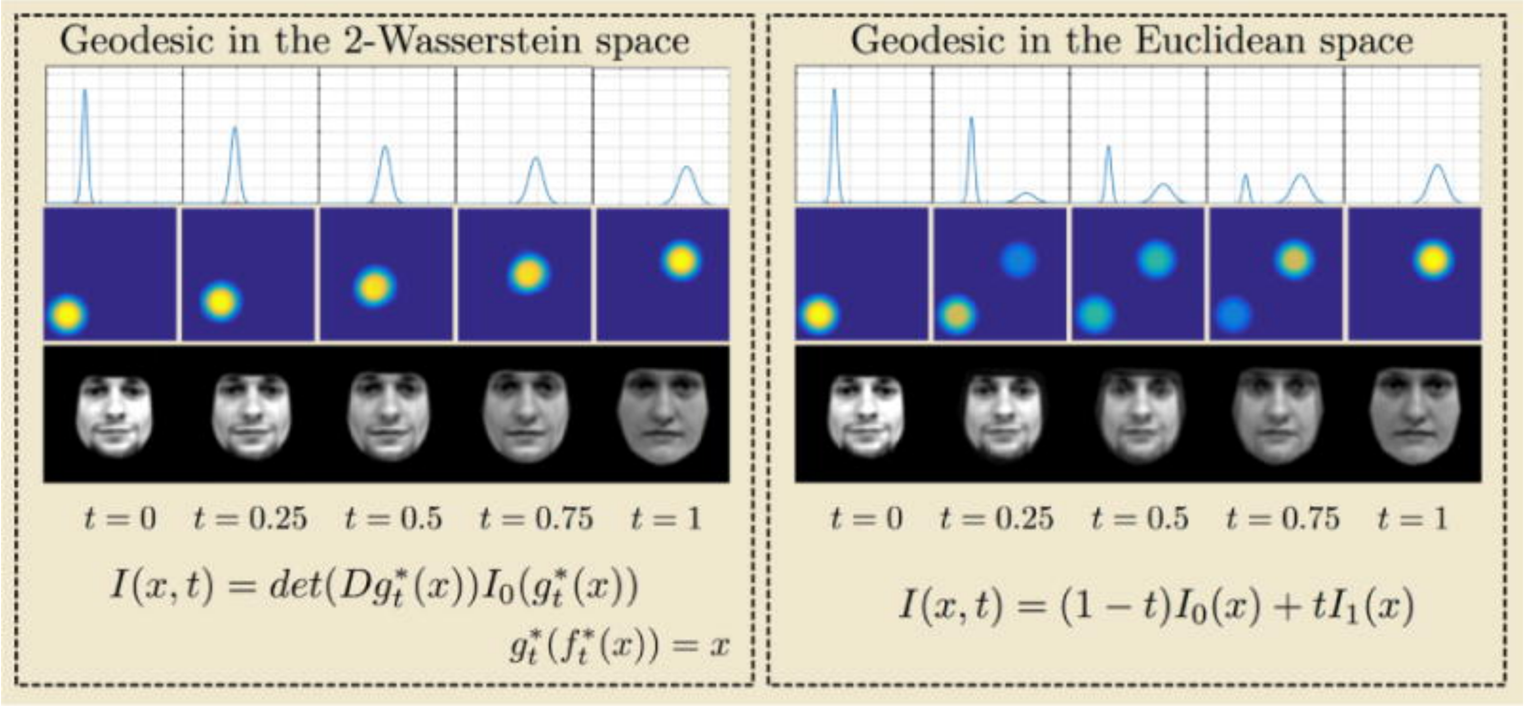

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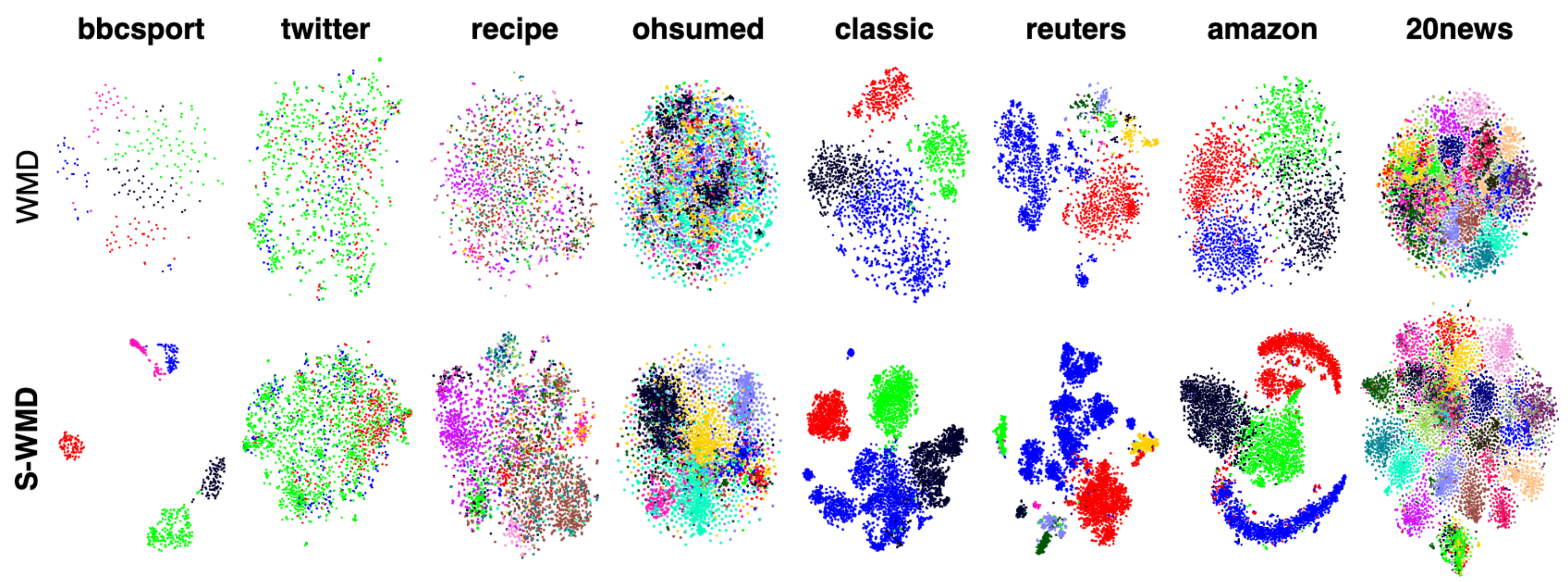

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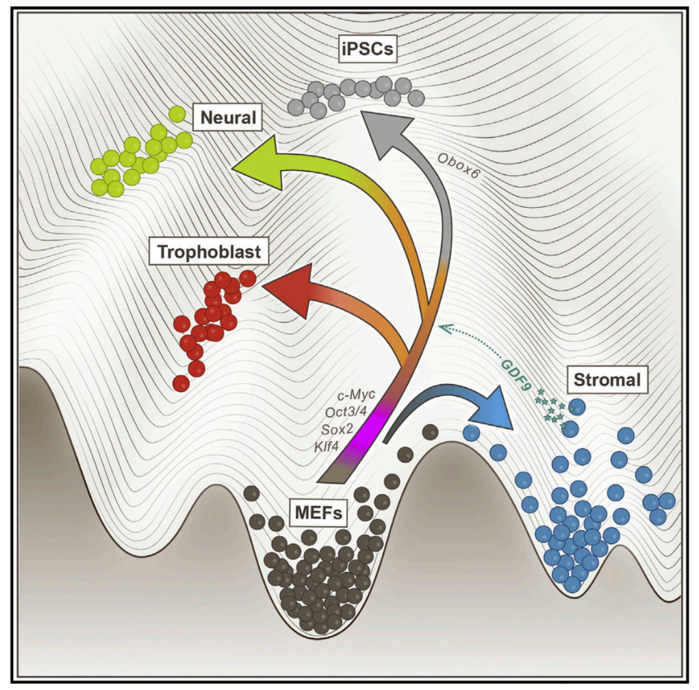

Schiebinger+, Cell (2019)

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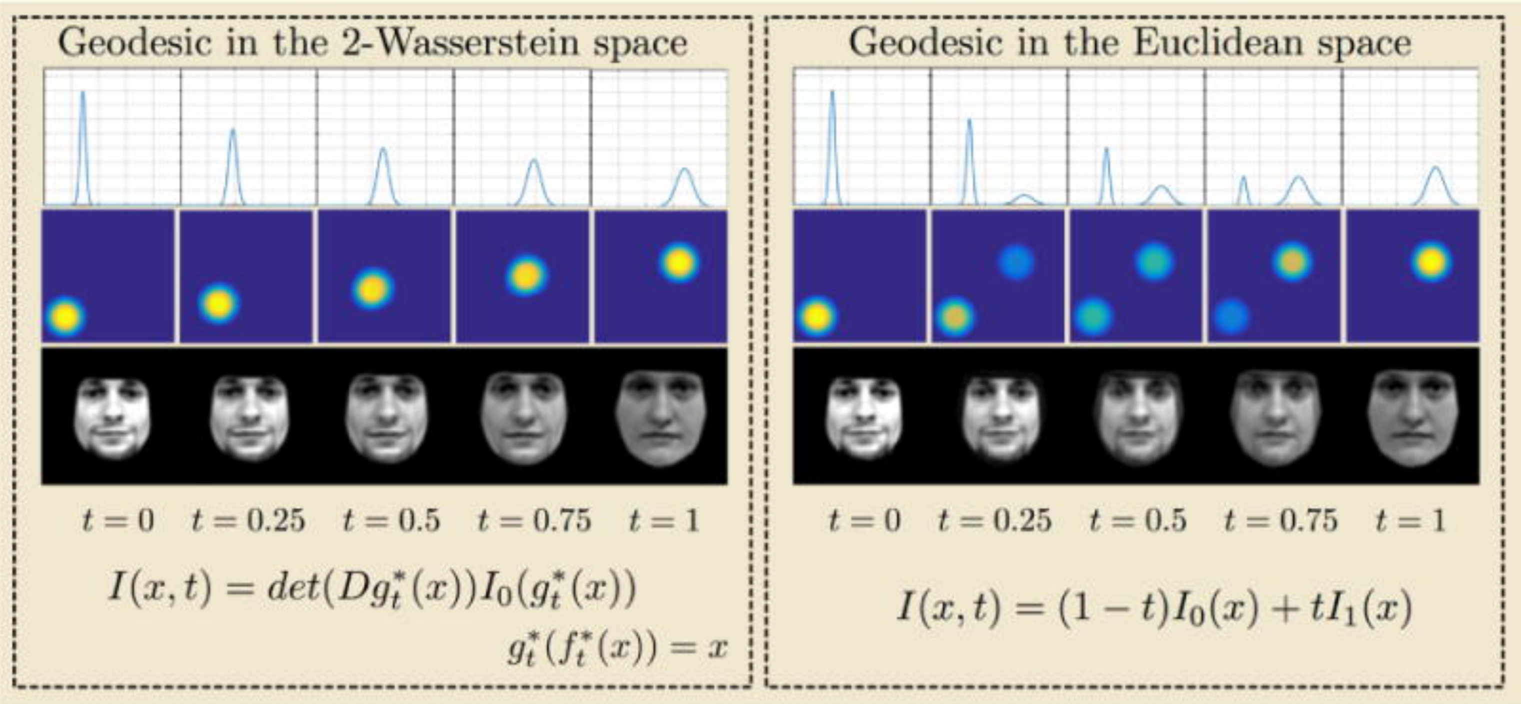

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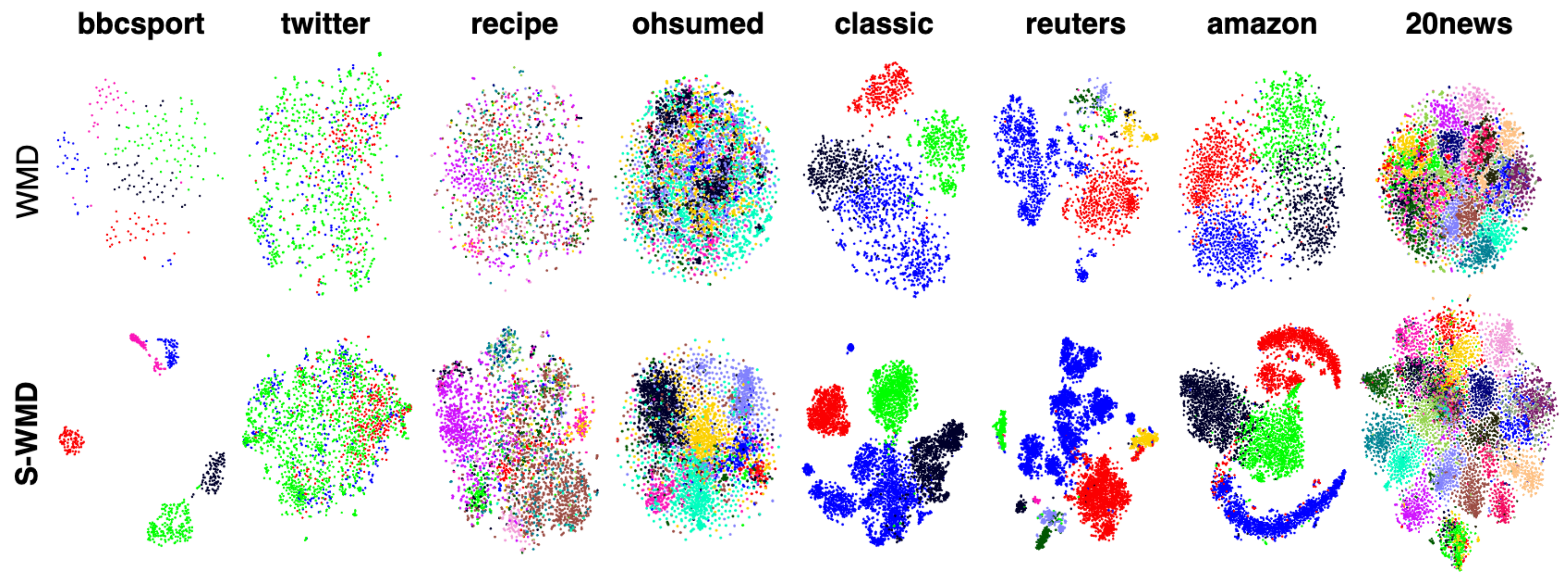

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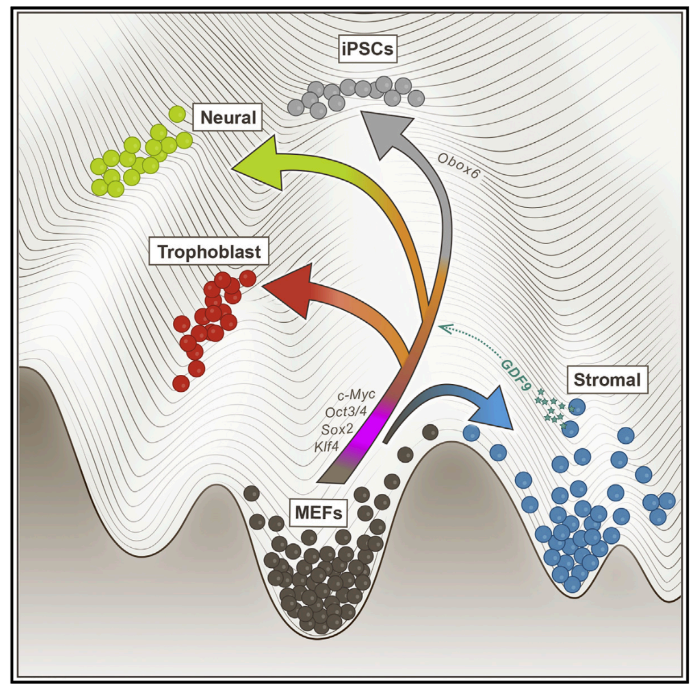

Schiebinger+, Cell (2019)

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Accurate Baryon Acoustic Oscillations Reconstruction via Semidiscrete Optimal Transport

Sebastian von Hausegger,[1,2,3,*] Bruno Lévy,[2] and Roya Mohayaee[1,3]
[1]*Rudolf Peierls Centre for Theoretical Physics, University of Oxford, Parks Road, Oxford OX1 3PU, United Kingdom*
[2]*Université de Lorraine, CNRS, Inria, LORIA, F-54000 Nancy, France*
[3]*Institut d'Astrophysique de Paris, CNRS, Sorbonne Université, 98bis Bld Arago, 75014 Paris, France*

Optimal transport and thermodynamics

PHYSICAL REVIEW X **13,** 011013 (2023)

Thermodynamic Unification of Optimal Transport: Thermodynamic Uncertainty Relation, Minimum Dissipation, and Thermodynamic Speed Limits

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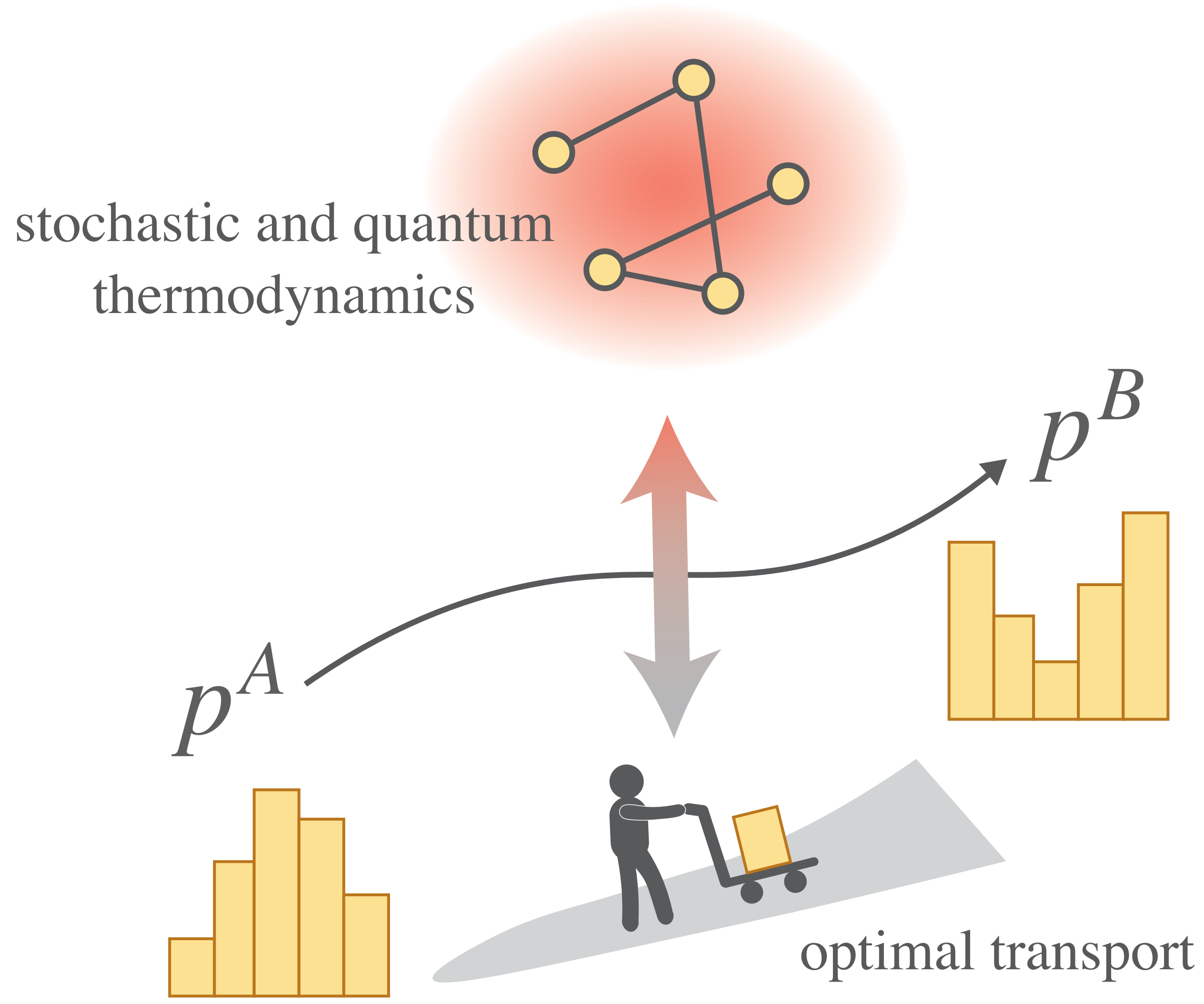
(Received 26 July 2022; revised 1 December 2022; accepted 13 December 2022; published 3 February 2023)

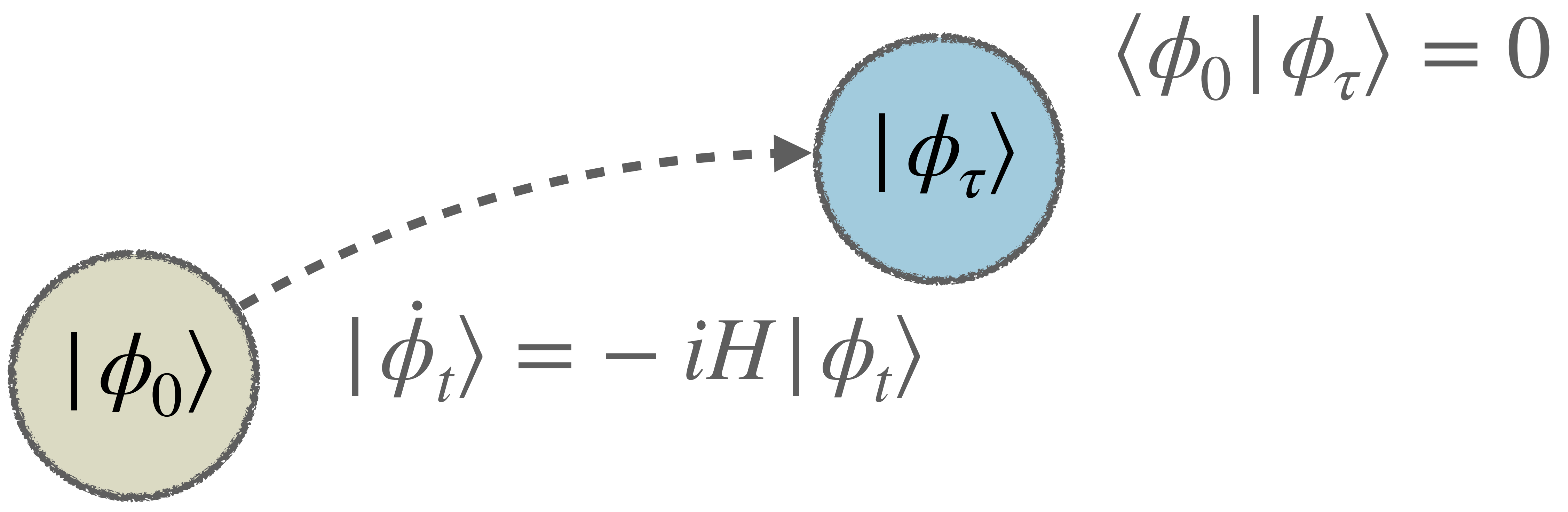
Discrete generalization of Benamou-Brenier formula

$$\mathscr{W}(\boldsymbol{p},\boldsymbol{q}) = \min\sqrt{\Sigma_\tau \mathscr{M}_\tau}$$

Σ_τ : entropy production
$\mathscr{M}_\tau$: state mobility



Quantum speed limit



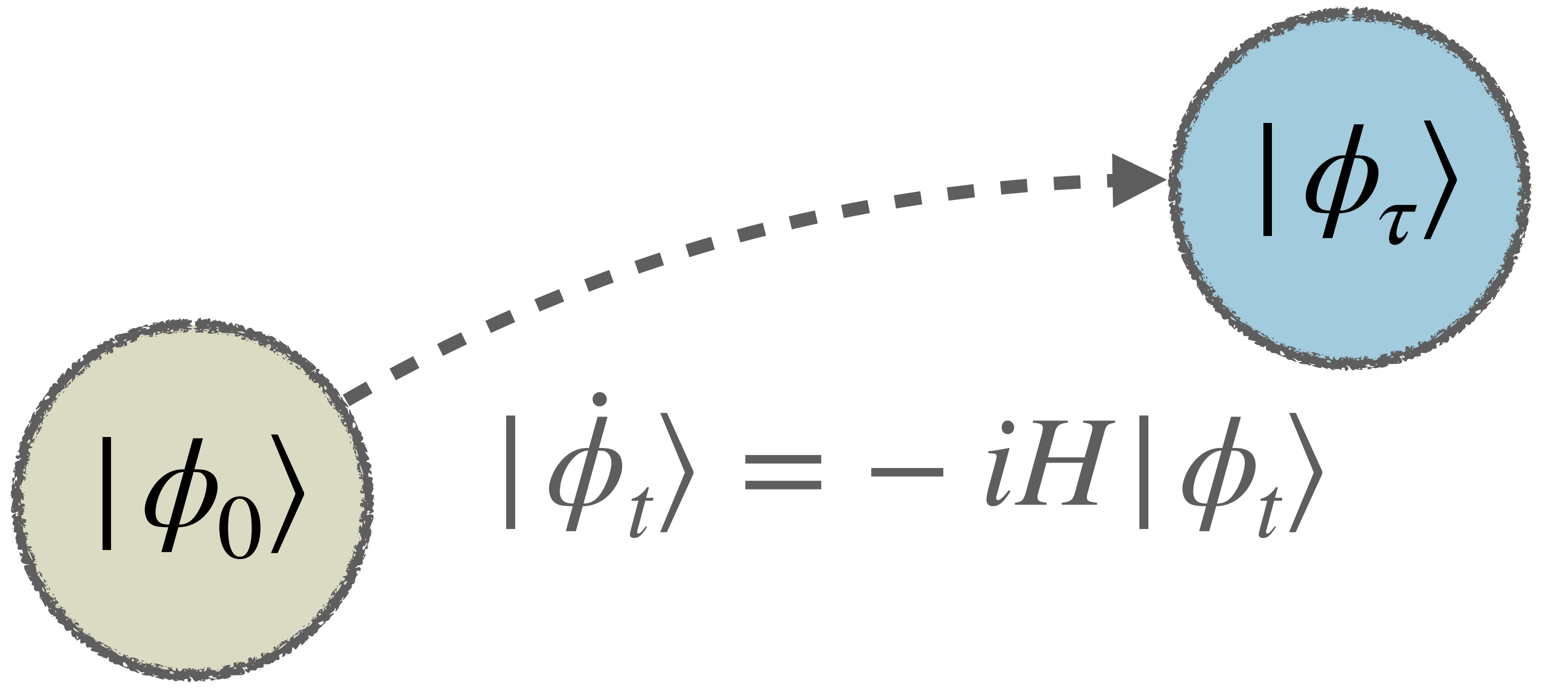
Quantum speed limit

- Minimum time for evolution between orthogonal states

$$\tau \geq \frac{\pi}{2} \max \left\{ \frac{1}{\Delta H}, \frac{1}{\langle H \rangle} \right\}$$

Mandelstam and Tamm, J. Phys. USSR (1945)
Margolus and Levitin, Physica (1998)

$$\langle \phi_0 | \phi_\tau \rangle = 0$$



$$|\dot{\phi}_t\rangle = -iH|\phi_t\rangle$$

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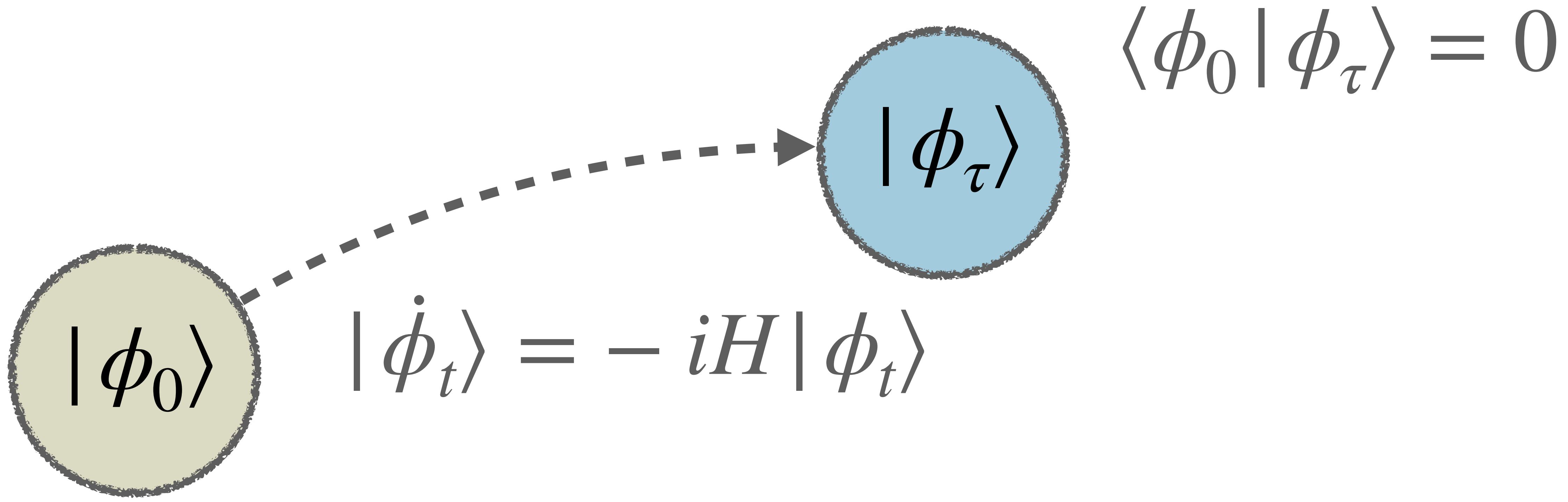

- Generalizations to arbitrary states and other dynamics

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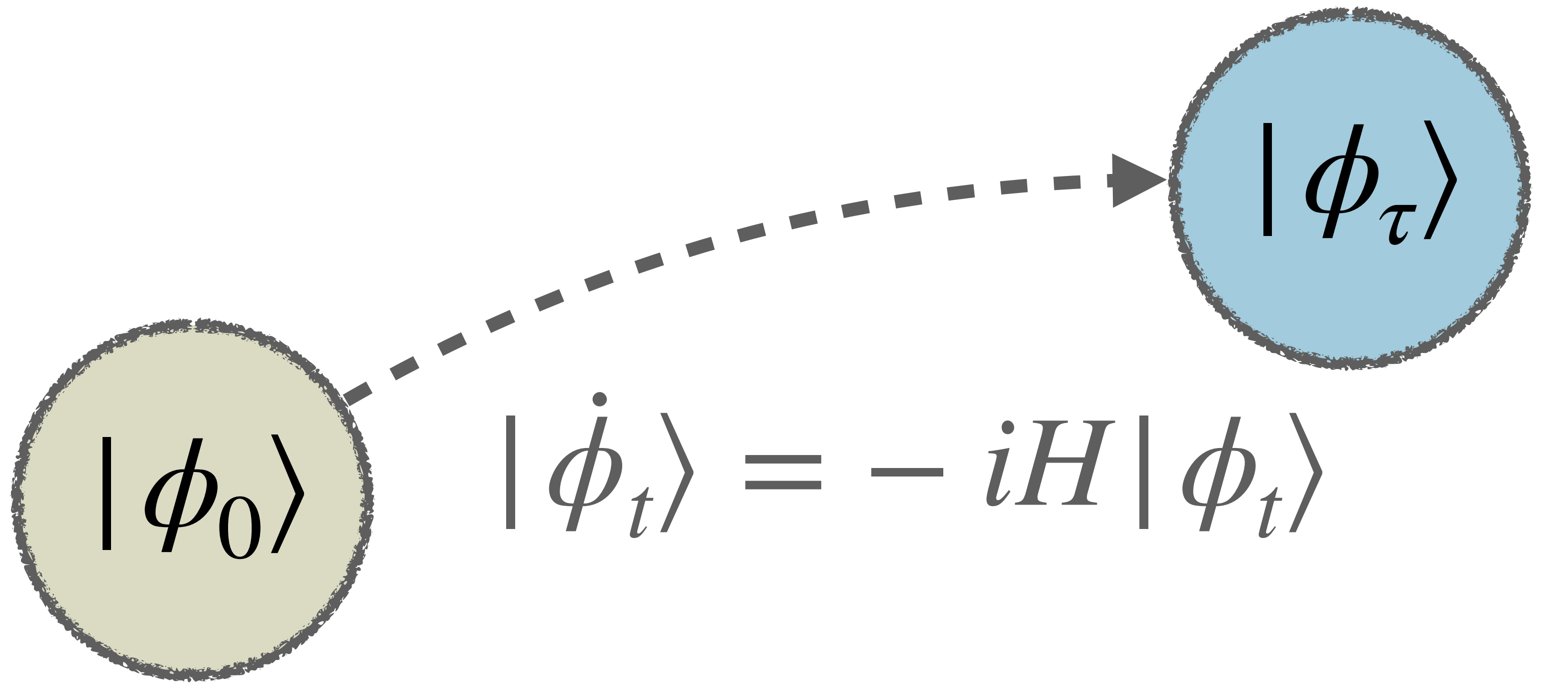

- Generalizations to arbitrary states and other dynamics

$$\tau \geq \frac{\mathscr{L}(\varrho_0, \varrho_\tau)}{\bar{v}}$$

$\mathscr{L}$: distance metric (Bures angle, Fisher information, etc.)

$\bar{v}$: maximal average velocity

Deffner and Campbell, J. Phys. A (2017) (review)

Strength and weakness

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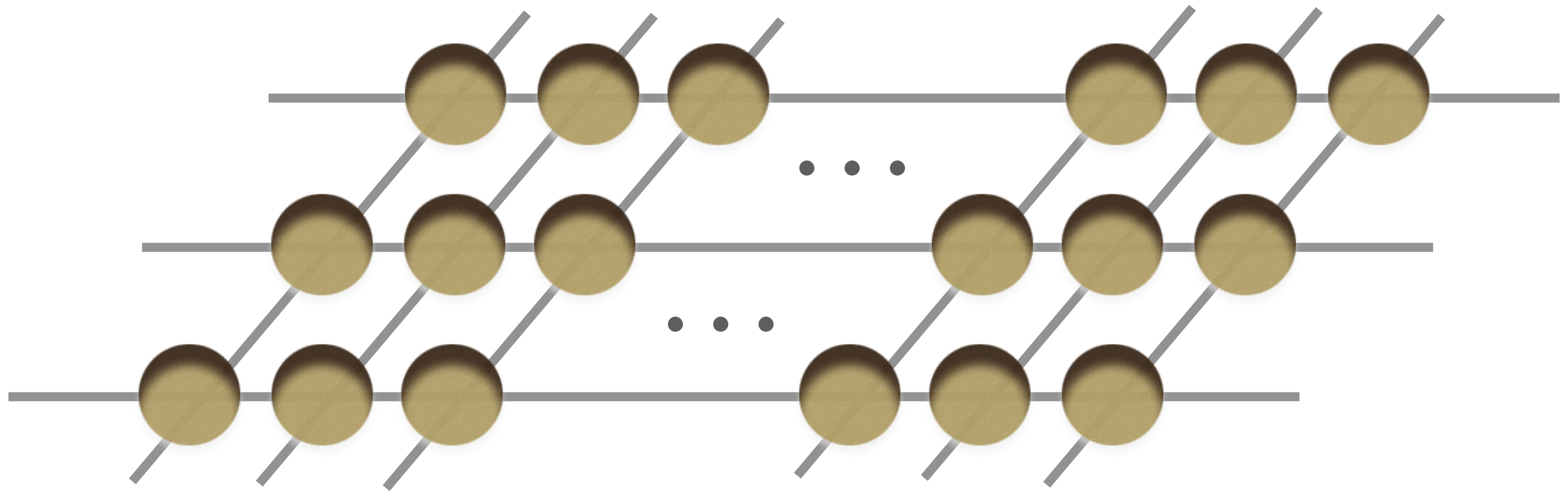
$$\mathscr{L}(\varrho_0, \varrho_\tau) = O(1)$$



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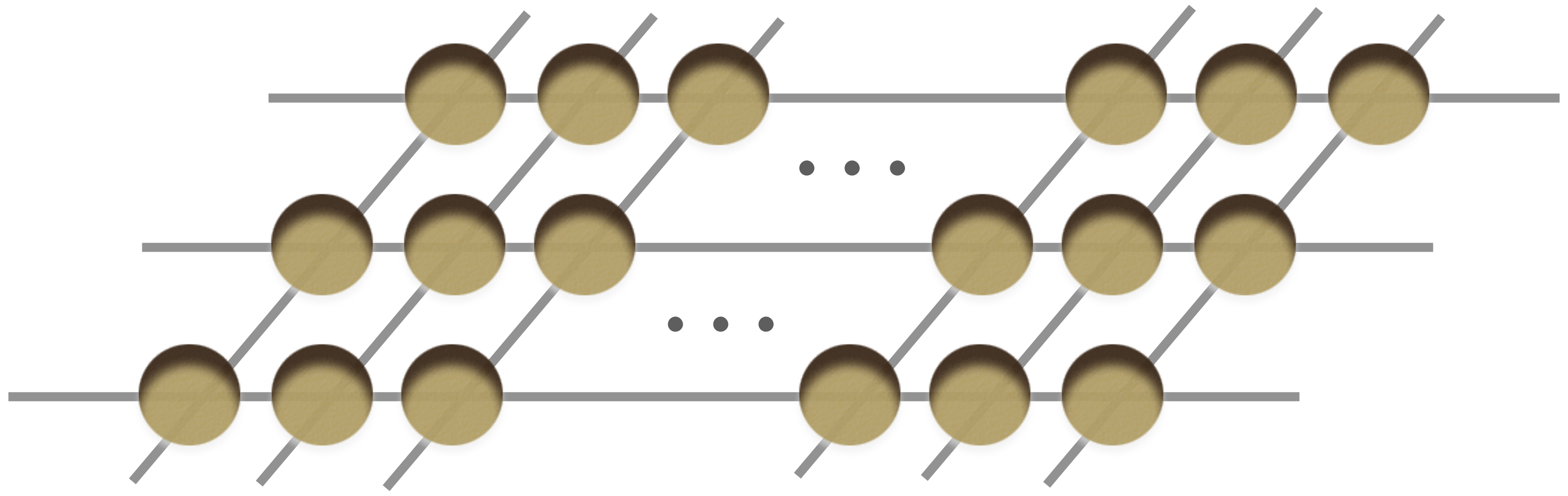


✓ speed limit between specific states

× not tight for many-body systems

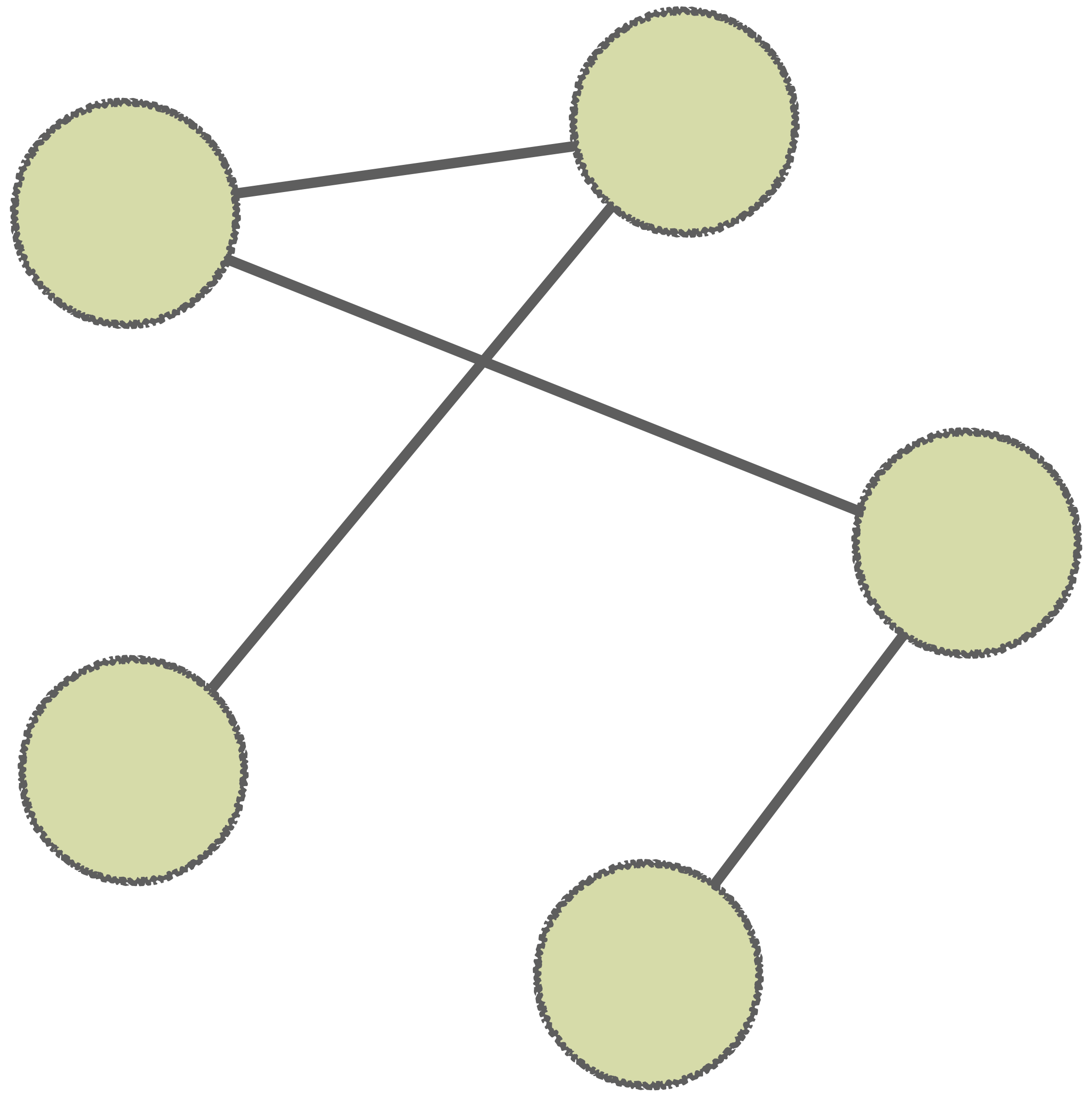
→ We need a speed limit that can incorporate the structure of many-body systems

General speed limit

- Undirected graph $G(V, E)$

$$V = \{1, \ldots, N\}$$

$$E = \{(i, j) \,|\, i \,\&\, j : \text{neighboring nodes}\}$$



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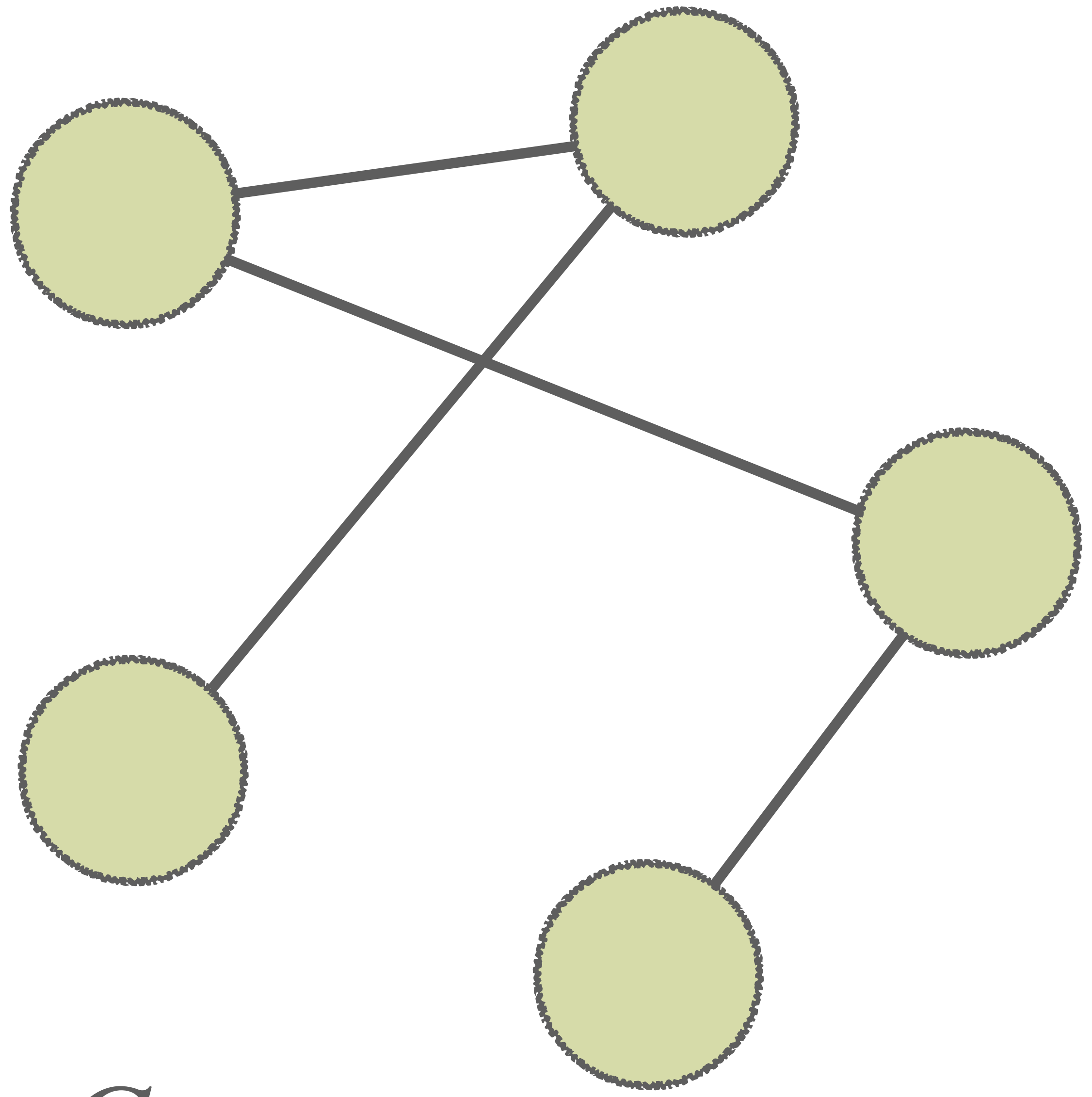

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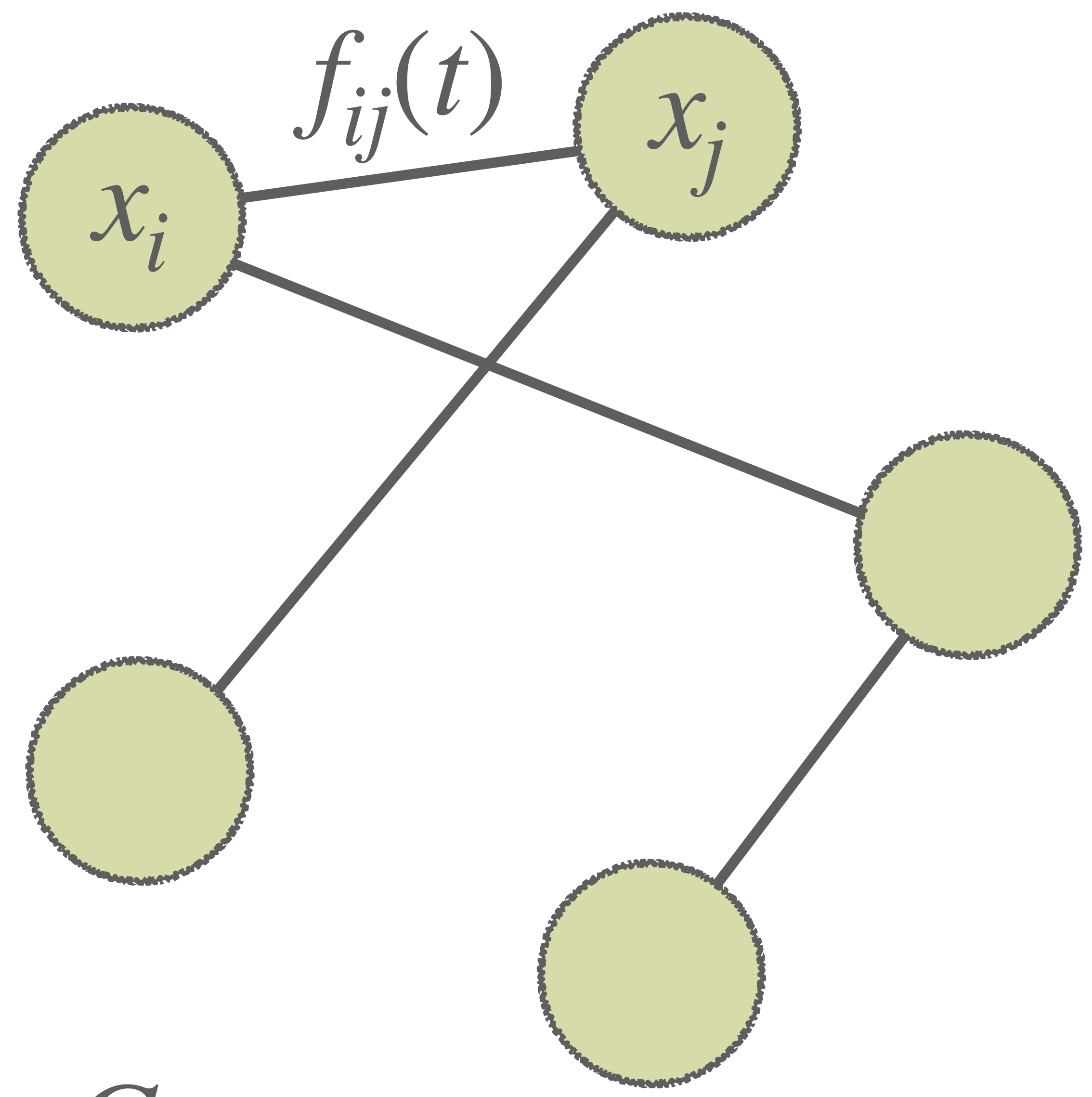

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$f_{ij}(t) = -f_{ji}(t)$: anti-symmetric flow

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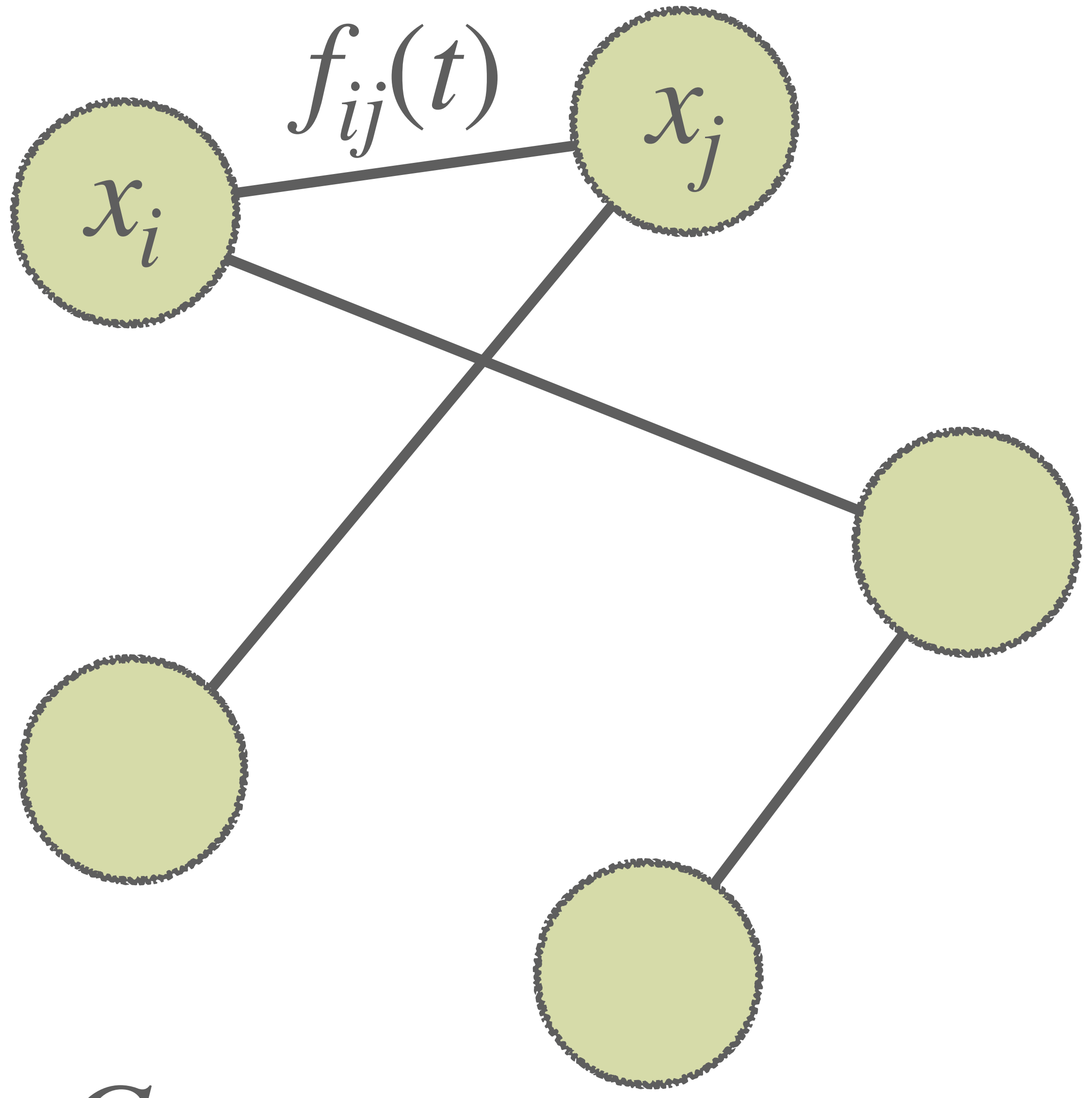

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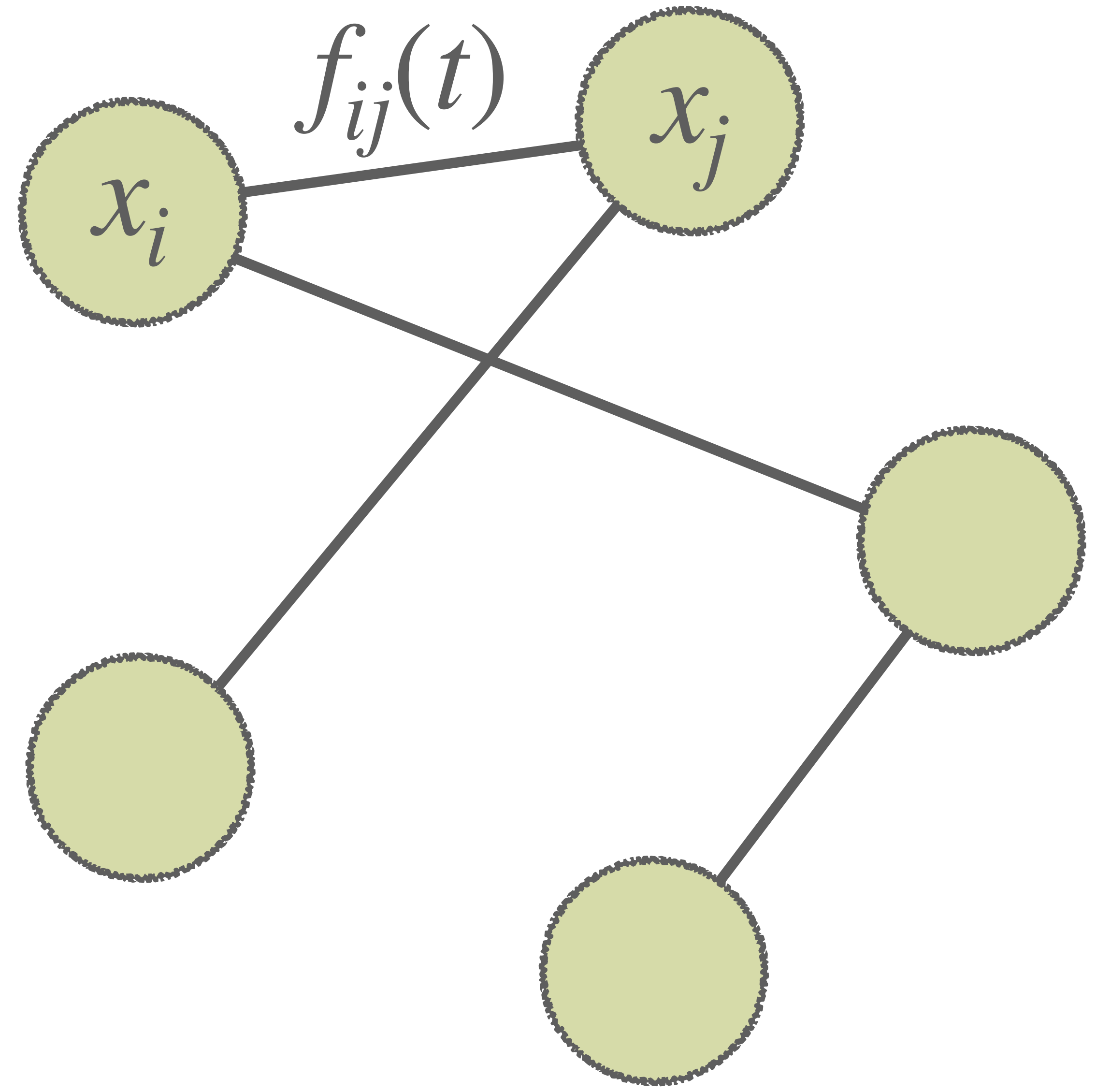

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General speed limit

- Minimum time required for changing states

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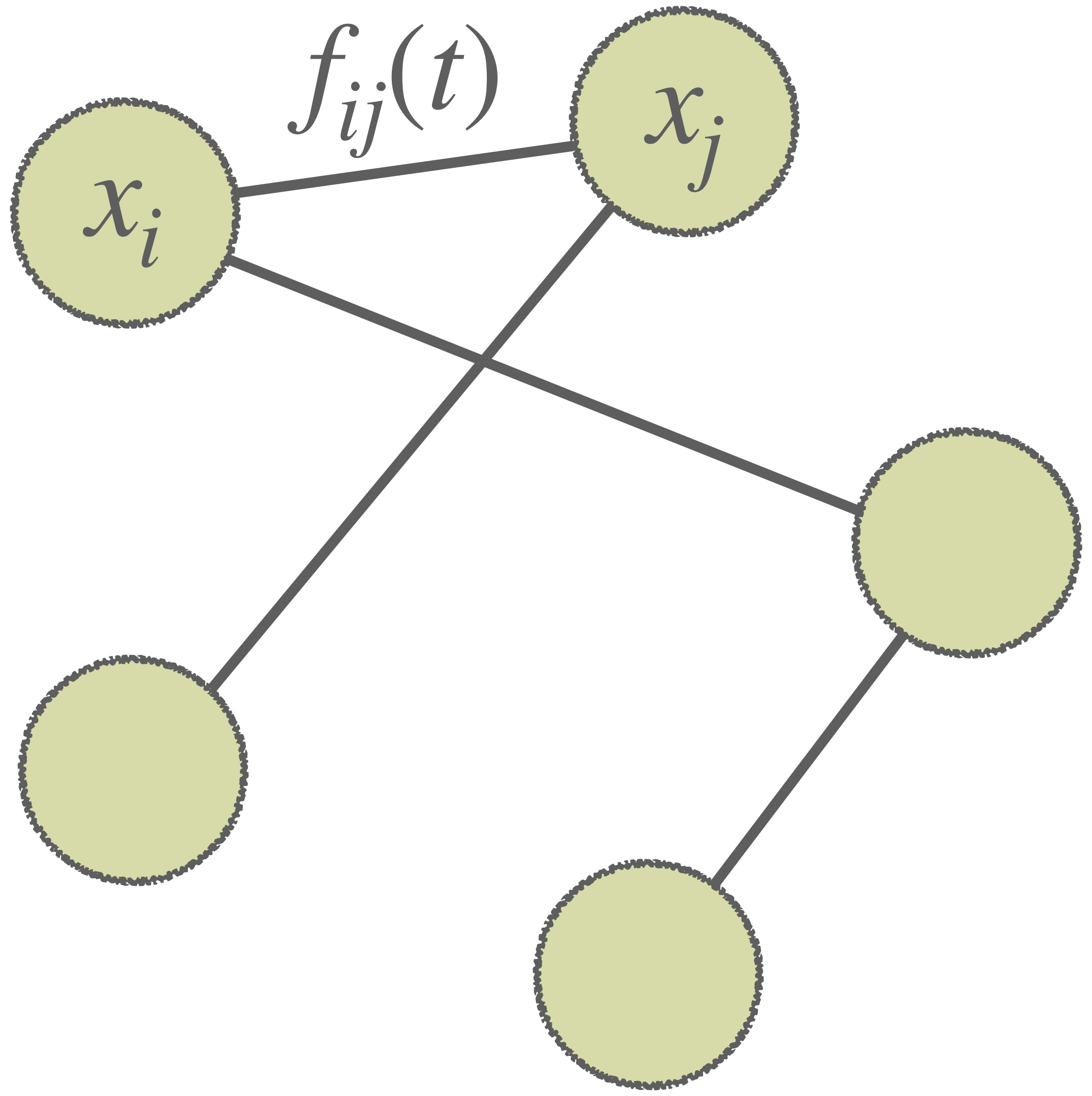
$$\overline{v} := \tau^{-1} \int_0^\tau dt \sum_{(i,j)\in E} c_{ij} |f_{ij}(t)|$$: average velocity



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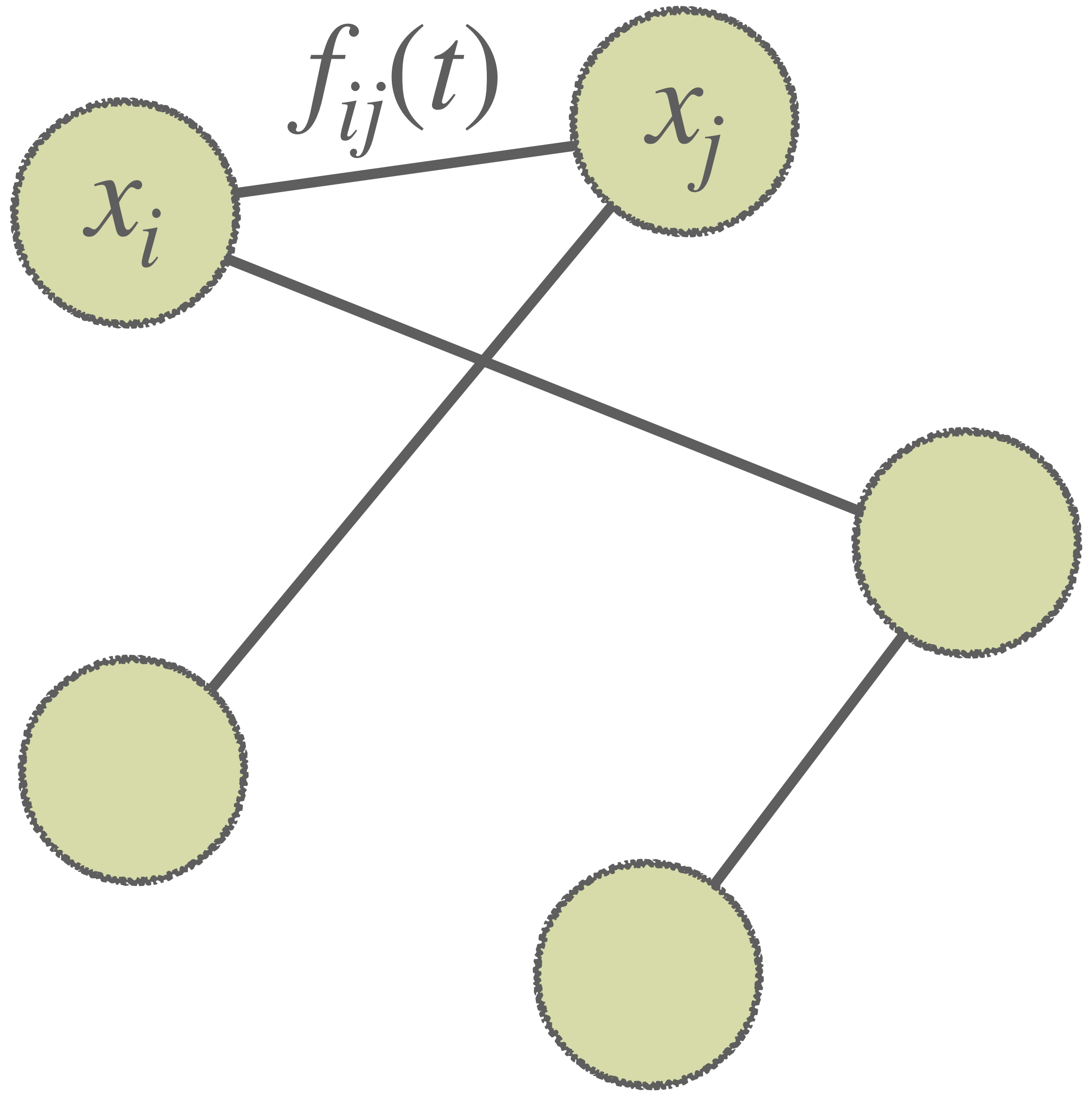

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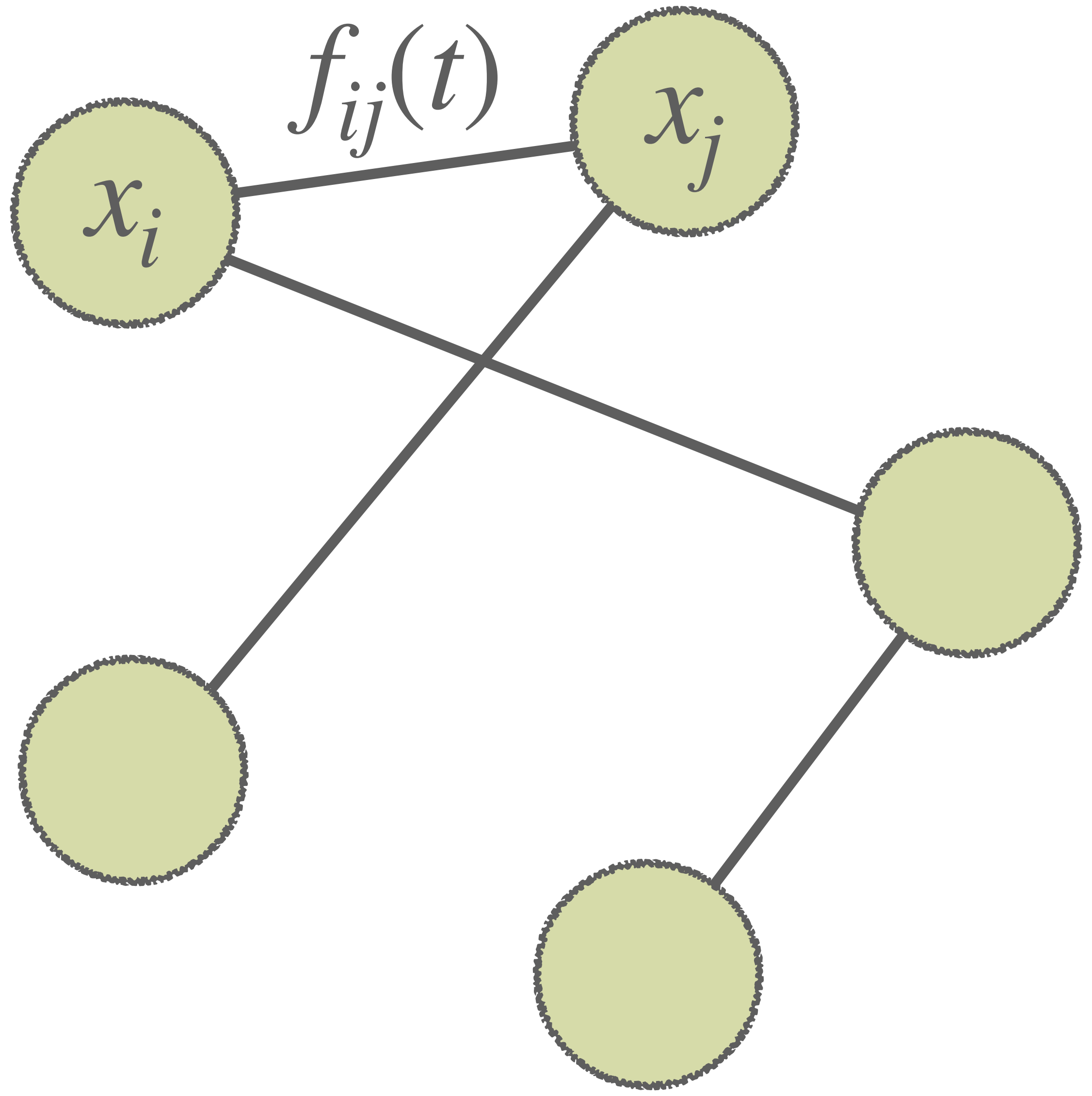

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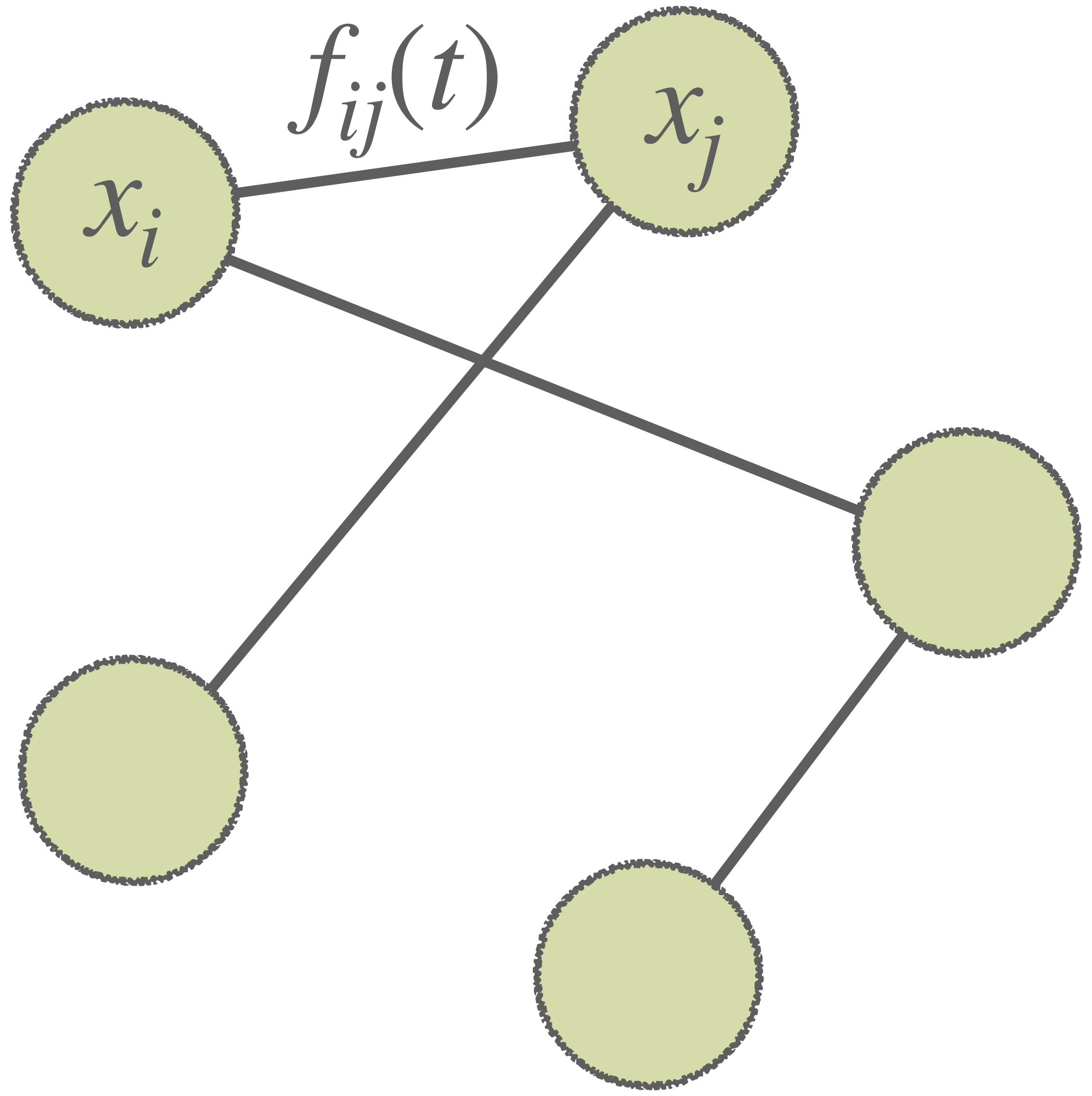

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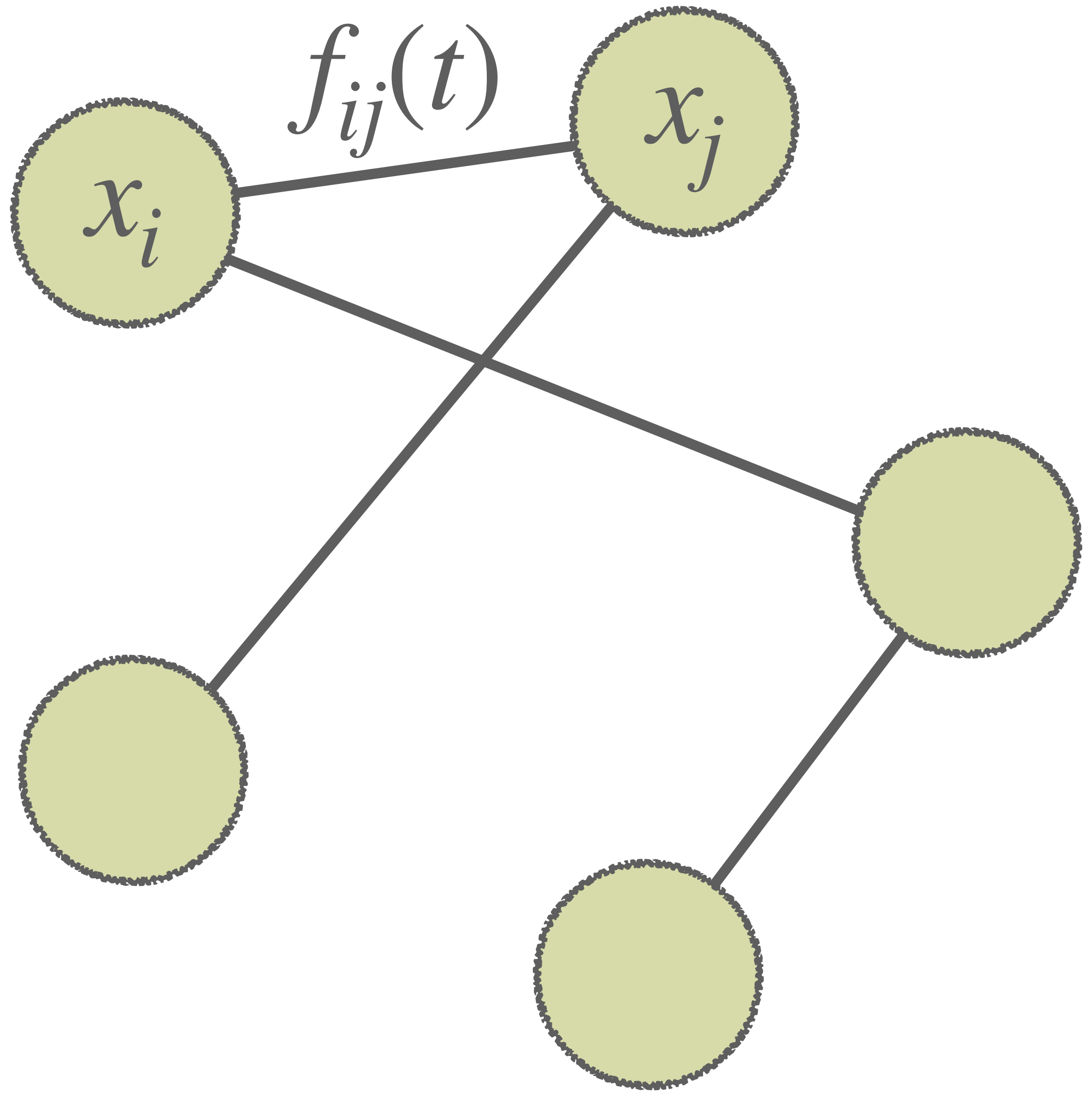

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✓ suitable choices of cost matrix lead to significant implications

Derivation of general speed limit

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- L^1-Wasserstein distance

$$\mathscr{W}(\boldsymbol{x}, \boldsymbol{y}) := \min_{\pi} \sum_{m,n} \pi_{mn} c_{mn}$$

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$$\mathscr{W}(\boldsymbol{x}, \boldsymbol{y}) = \max_{\boldsymbol{\phi}} \boldsymbol{\phi}^\top (\boldsymbol{x} - \boldsymbol{y})$$

$$\text{s.t. } |\phi_i - \phi_j| \le c_{ij} \ \forall i, j$$

↑

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$$\begin{aligned}
\mathscr{W}(\boldsymbol{x}_0,\boldsymbol{x}_\tau) &= \max_{\boldsymbol{\phi}} \boldsymbol{\phi}^\top(\boldsymbol{x}_\tau - \boldsymbol{x}_0)\\
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&= \max_{\boldsymbol{\phi}} \sum_{(i,j)\in E} (\phi_i - \phi_j) \int_0^\tau dt\, f_{ij}(t)\\
&\le \max_{\boldsymbol{\phi}} \sum_{(i,j)\in E} |\phi_i - \phi_j| \int_0^\tau dt\, |f_{ij}(t)|\\
&\le \int_0^\tau dt \sum_{(i,j)\in E} c_{ij} |f_{ij}(t)|
\end{aligned}$$

Consequences

General speed limit

Topological speed limit

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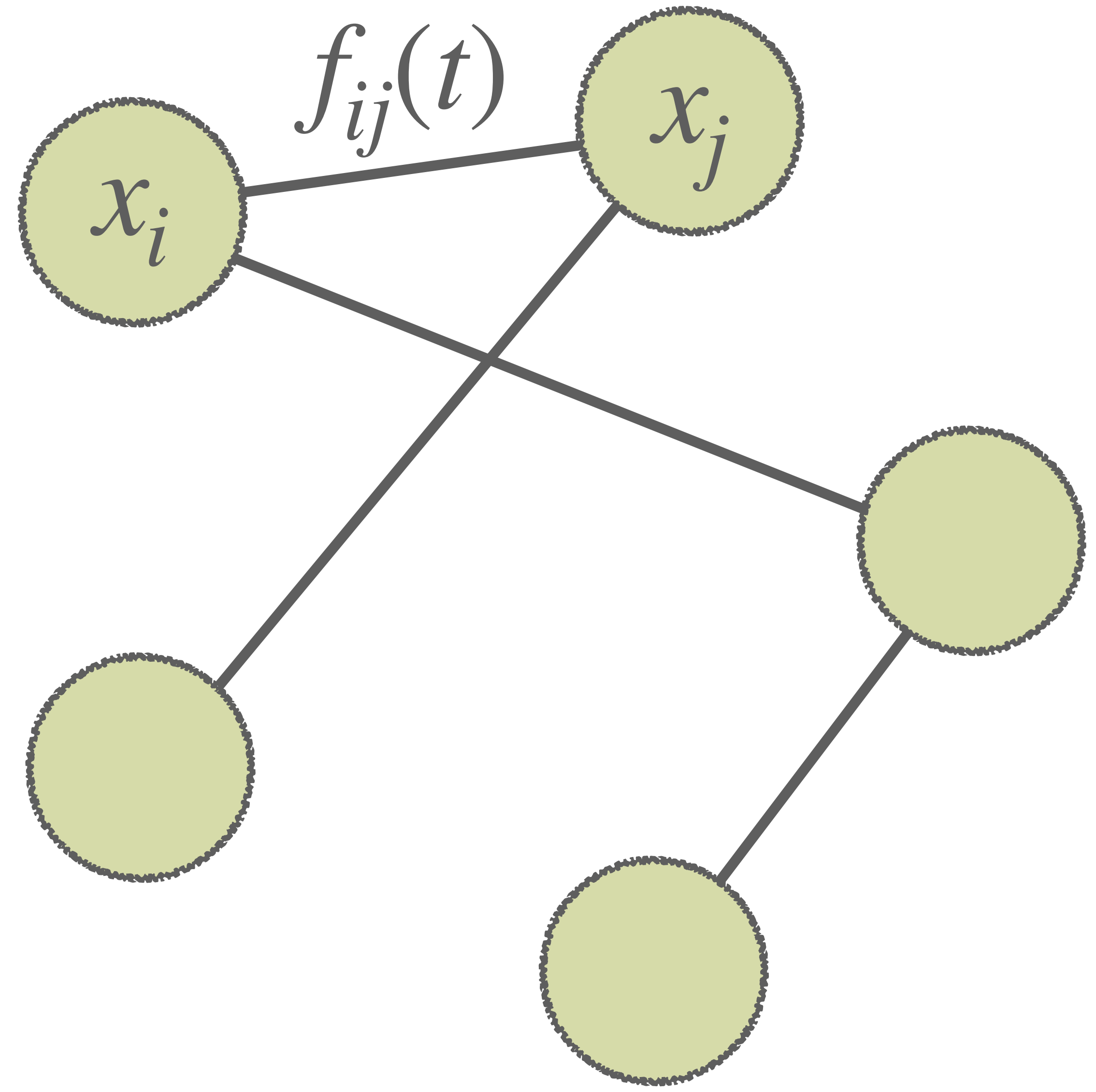
Topological speed limit



Topological speed limit

- Shortest-path distances $[c_{ij}]$

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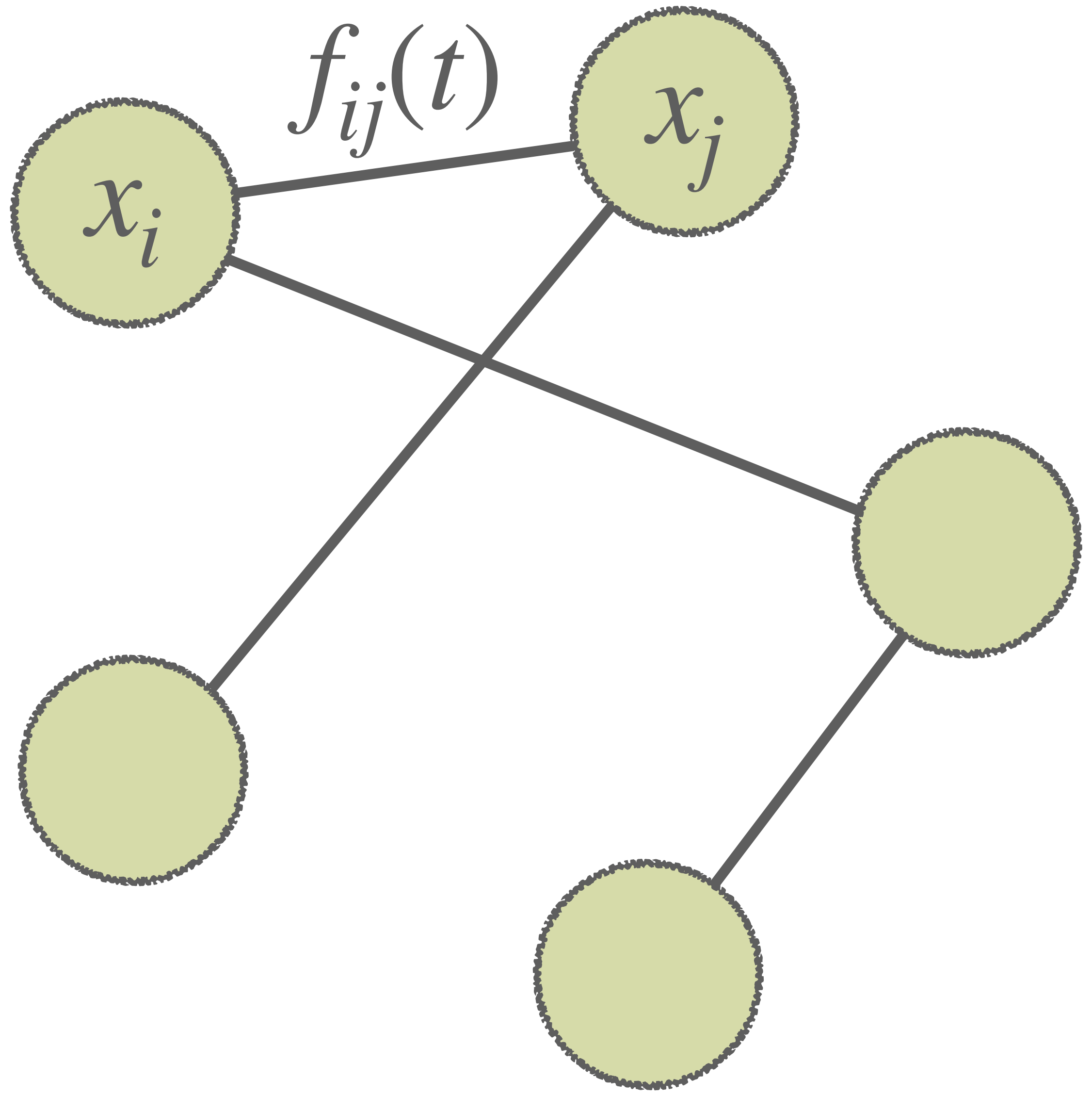

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Vu and Saito, Phys. Rev. Lett. (2023)

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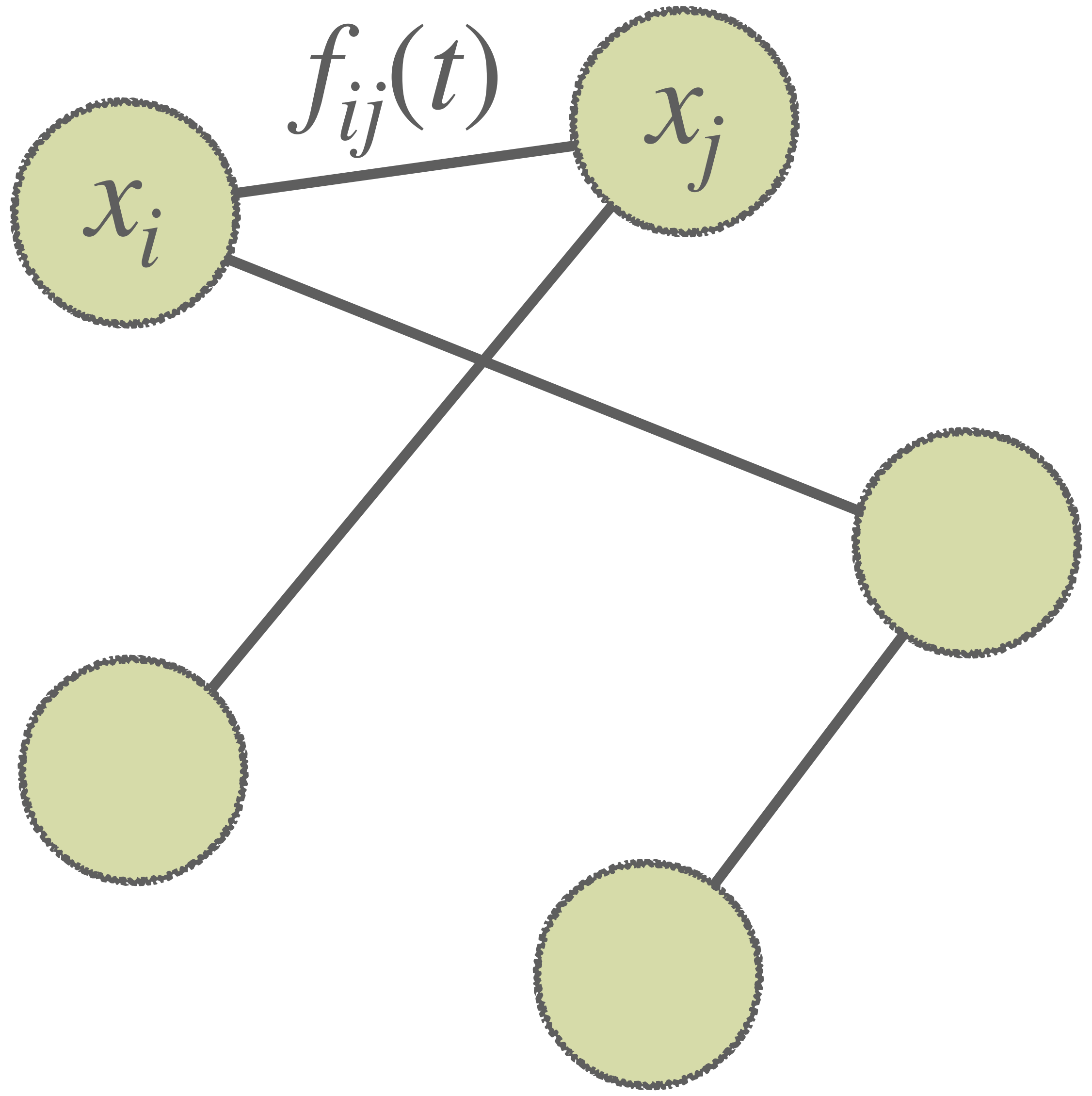
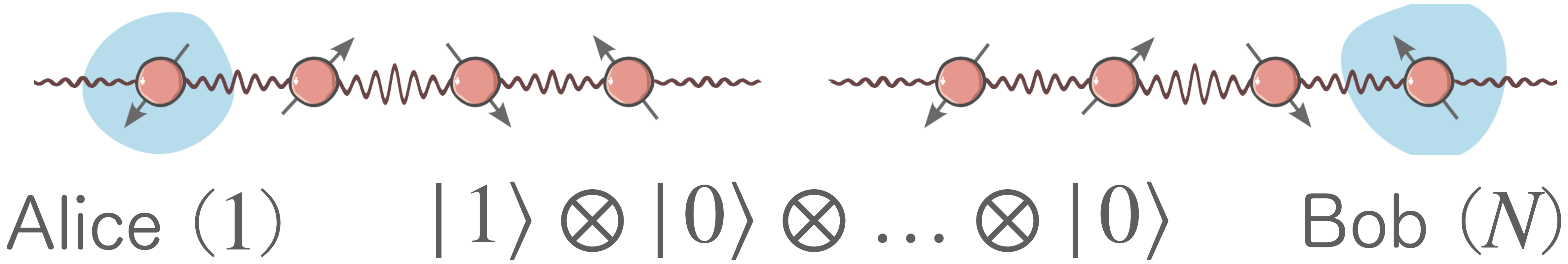


Applications

- Quantum communication through spin chains

Bose, Phys. Rev. Lett. (2003)

$$H_t = -\frac{\gamma}{2}\sum_{n=1}^{N-1}\vec{\sigma}_n\cdot\vec{\sigma}_{n+1} + \sum_{n=1}^{N} B_n(t)\sigma_n^z$$



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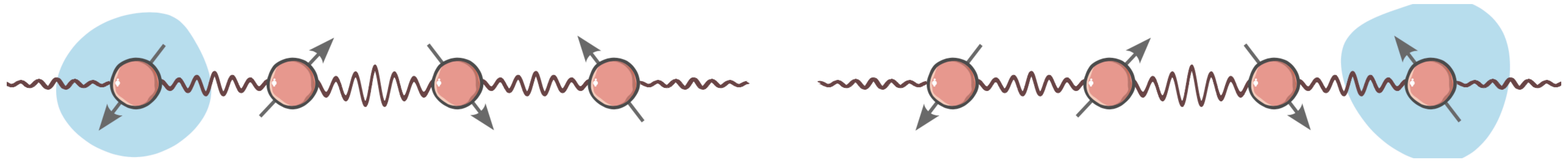

Alice (1) $|1\rangle \otimes |0\rangle \otimes \ldots \otimes |0\rangle$ Bob (N)

Time needed to transfer state from spin 1 to N

$$\tau \geq \frac{N-1}{2\gamma}$$

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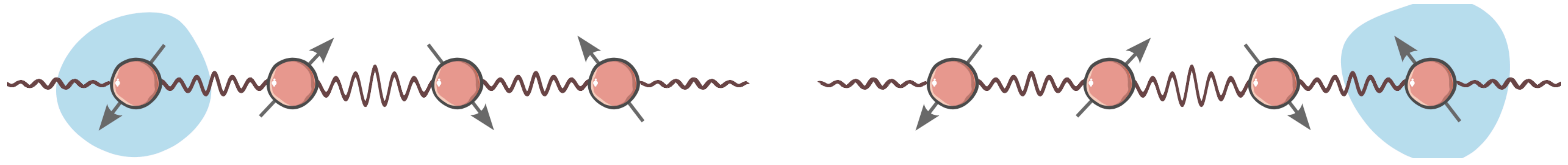

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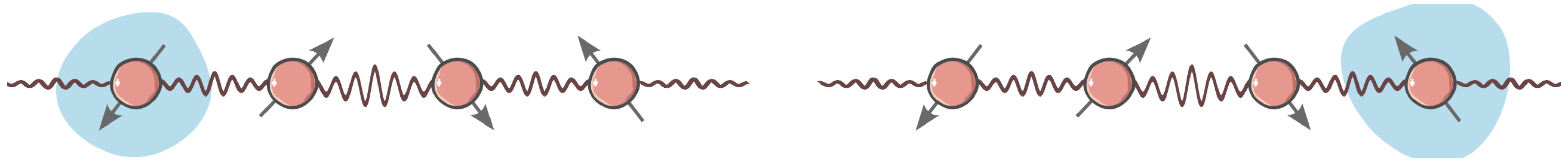

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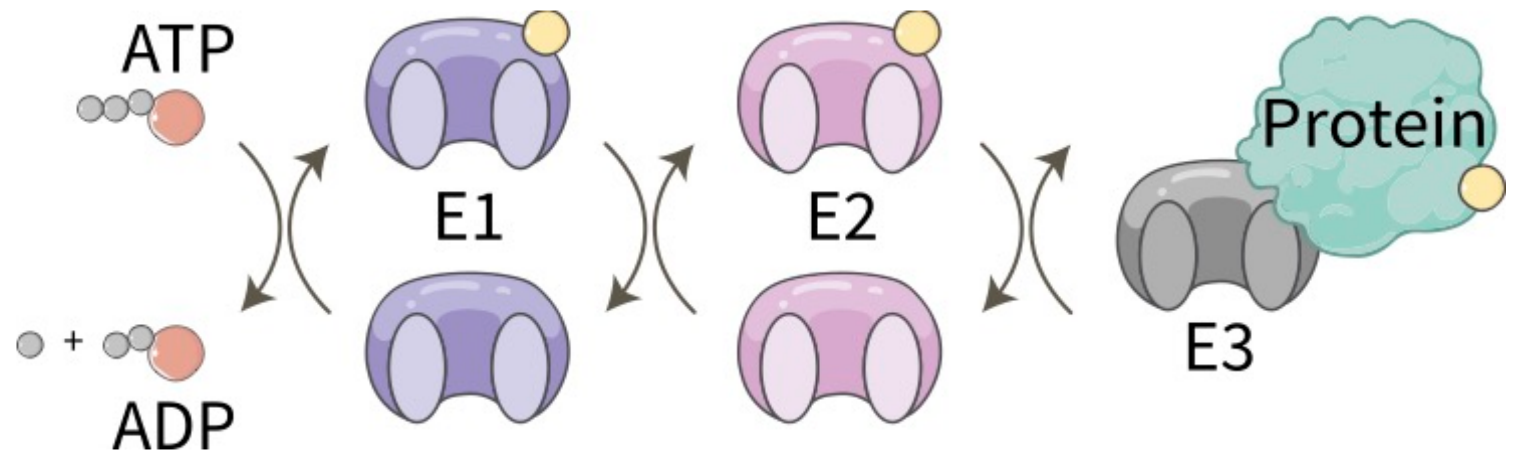

chemical reaction networks

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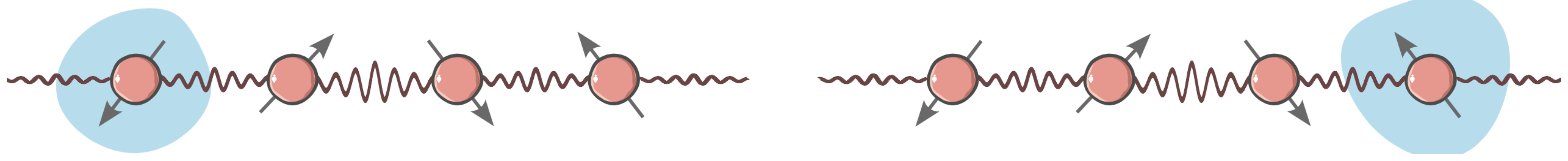

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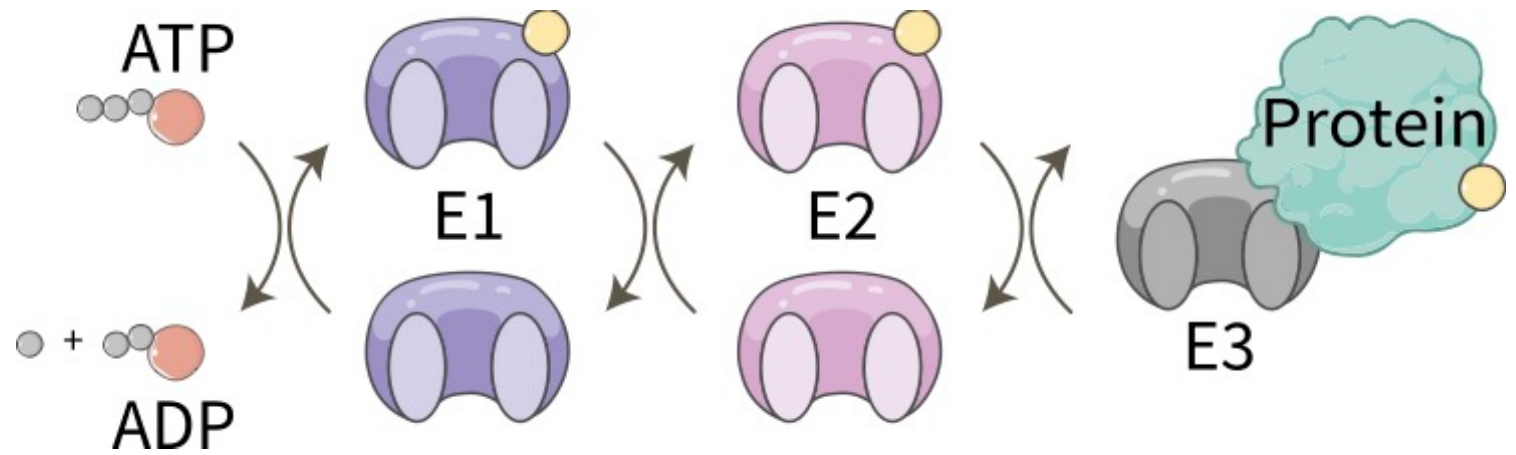

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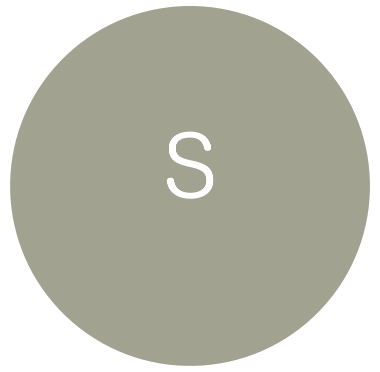

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isolated quantum systems

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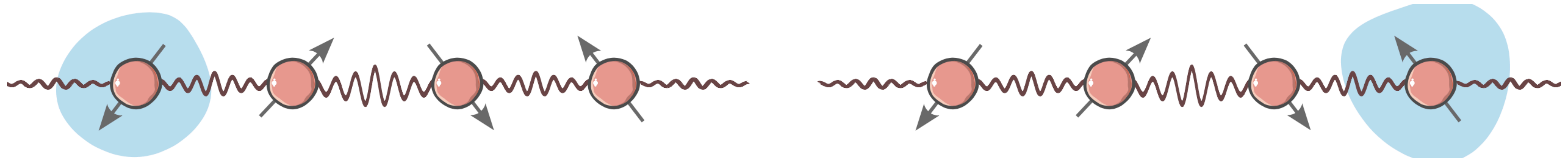

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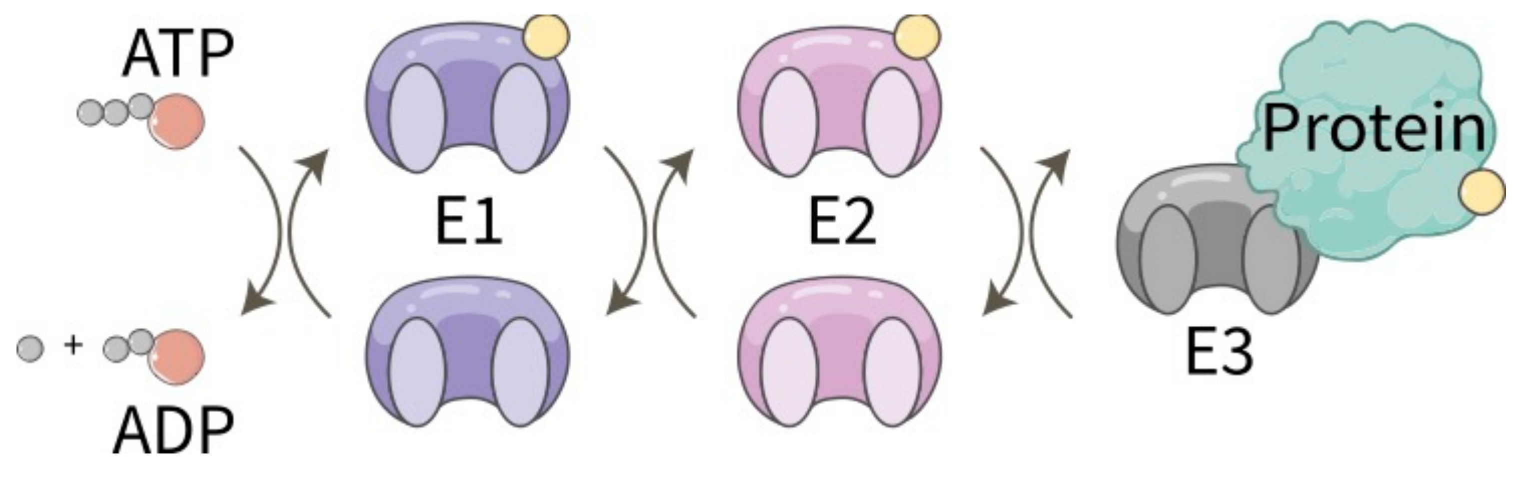

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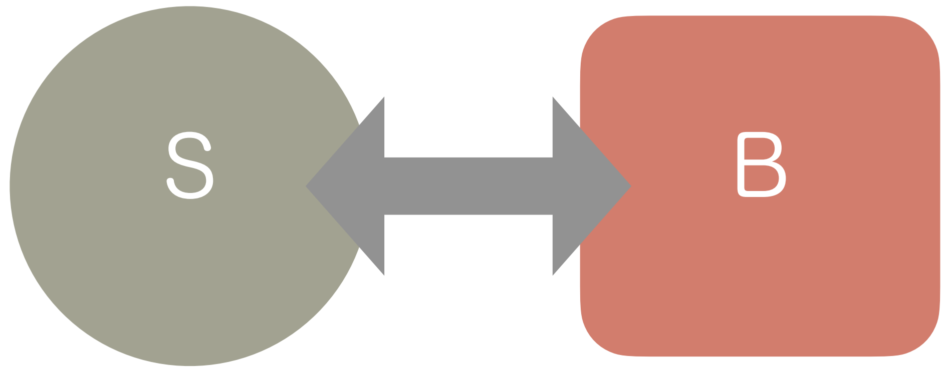

$$\dot{\varrho}_t = \mathscr{L}_t\varrho_t$$

open quantum systems

Consequences

General speed limit

Topological speed limit

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Particle transport in closed systems

- Boson system with long-range hopping and long-range interactions

$$H_t = \sum_{i \neq j} J_{ij}(t) \hat{b}_i \hat{b}_j^\dagger + \sum_{Z \subset \Lambda} h_Z(t)$$

$|J_{ij}(t)| \leq J/\|i-j\|^\alpha$: power-law decay $\alpha > D$ (spatial dimension)

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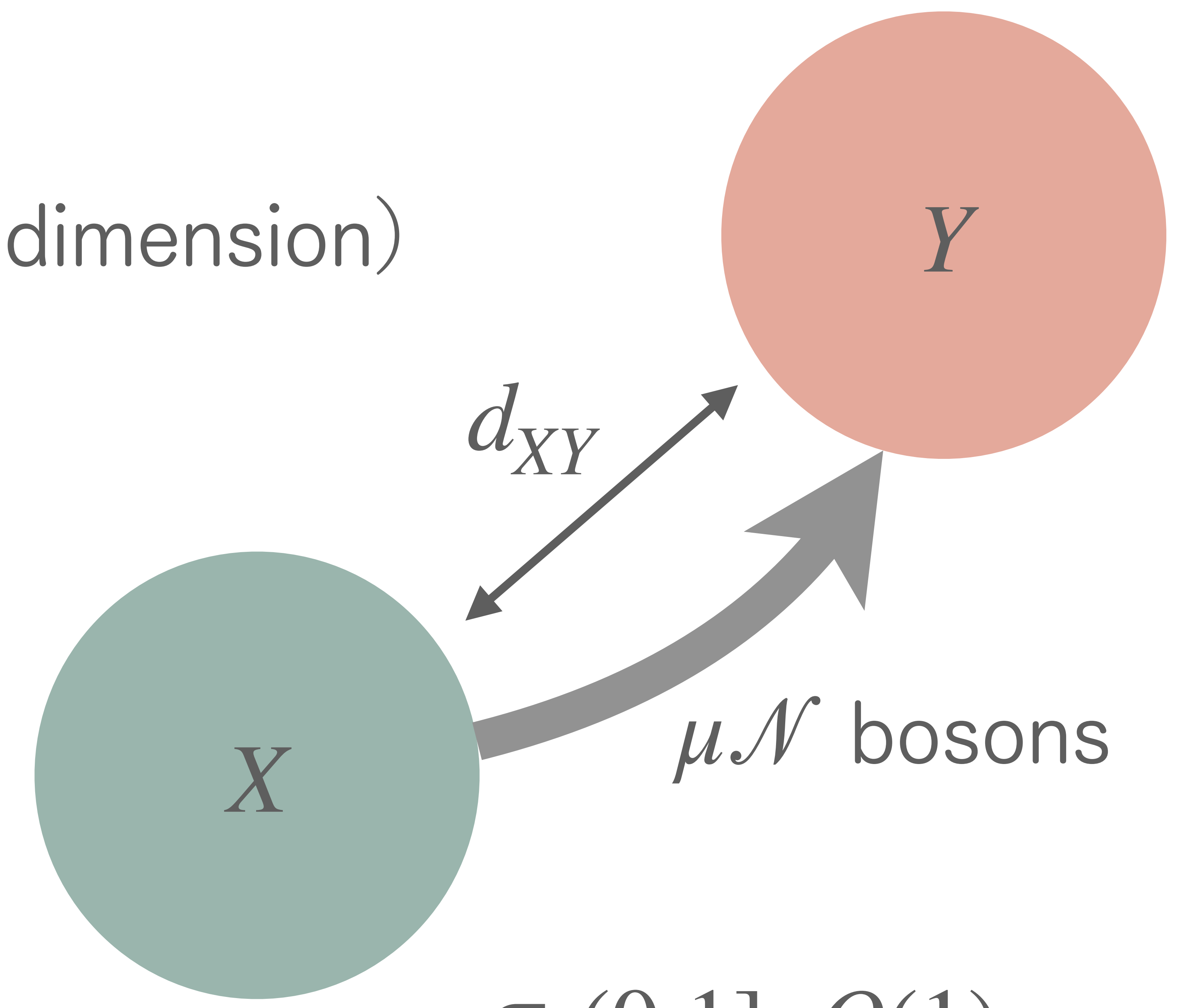

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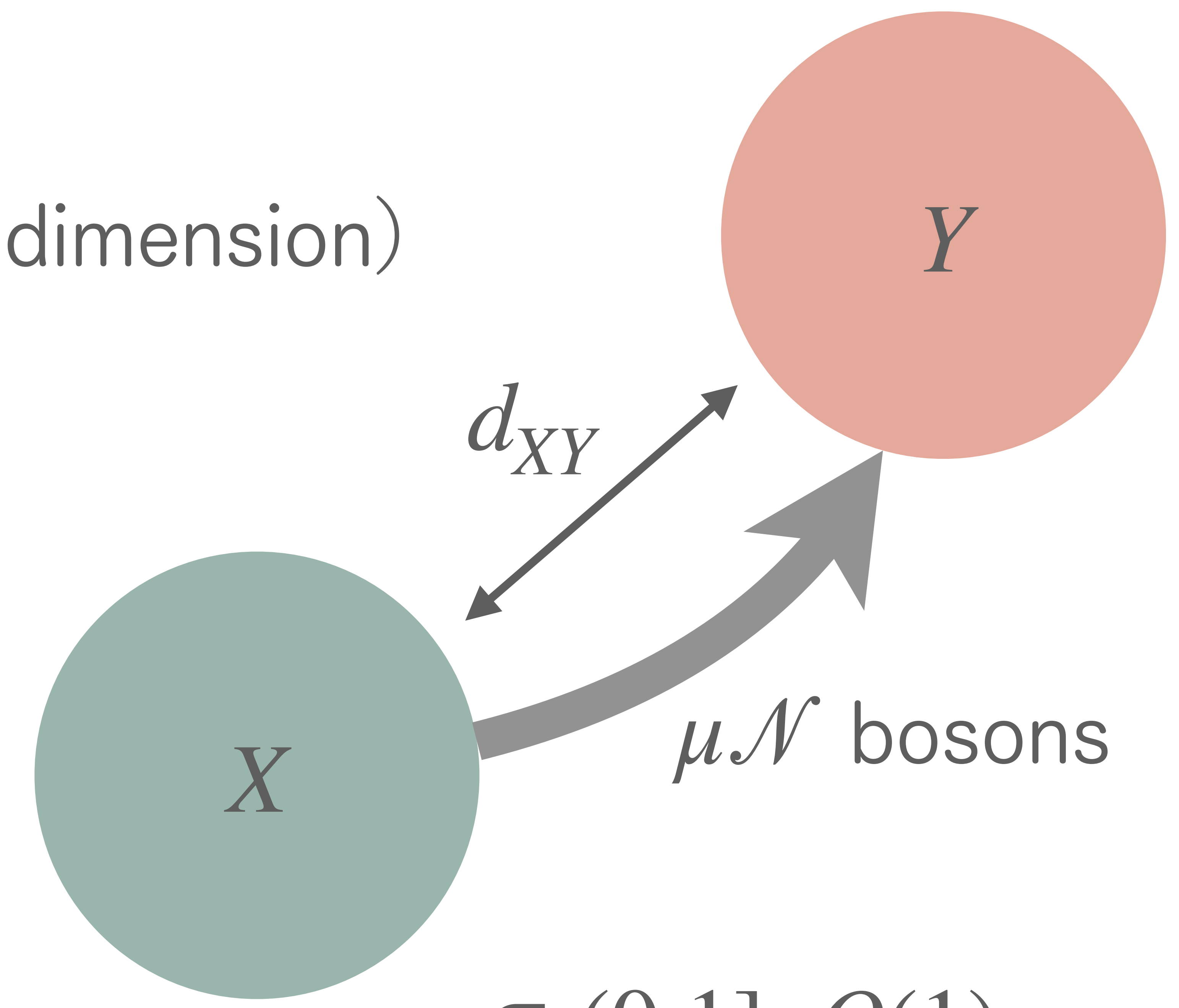
Faupin, Lemm, and Sigal,
Phys. Rev. Lett. (2022)



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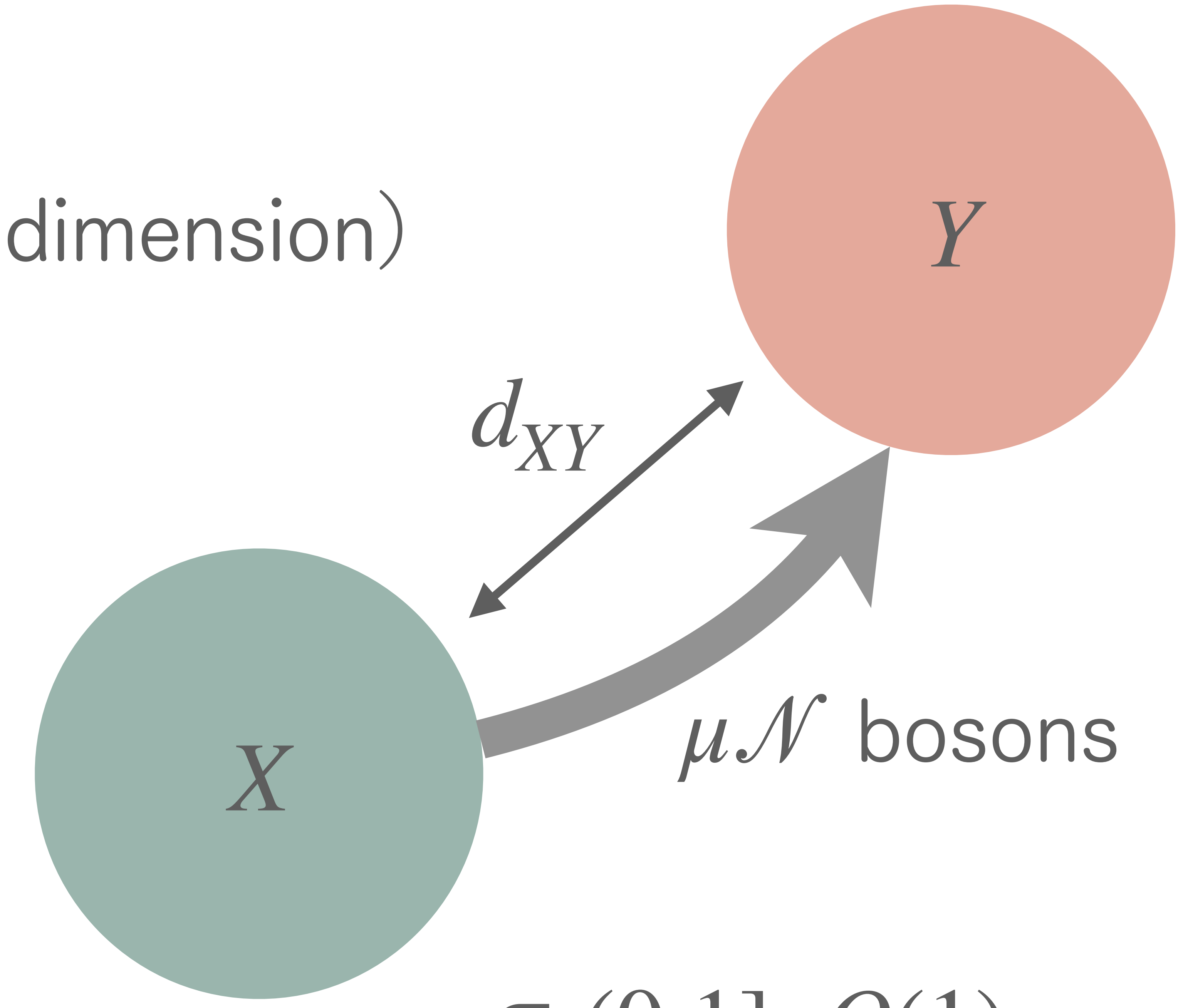
Faupin, Lemm, and Sigal, Phys. Rev. Lett. (2022)

× open problem for full range $\alpha > D$



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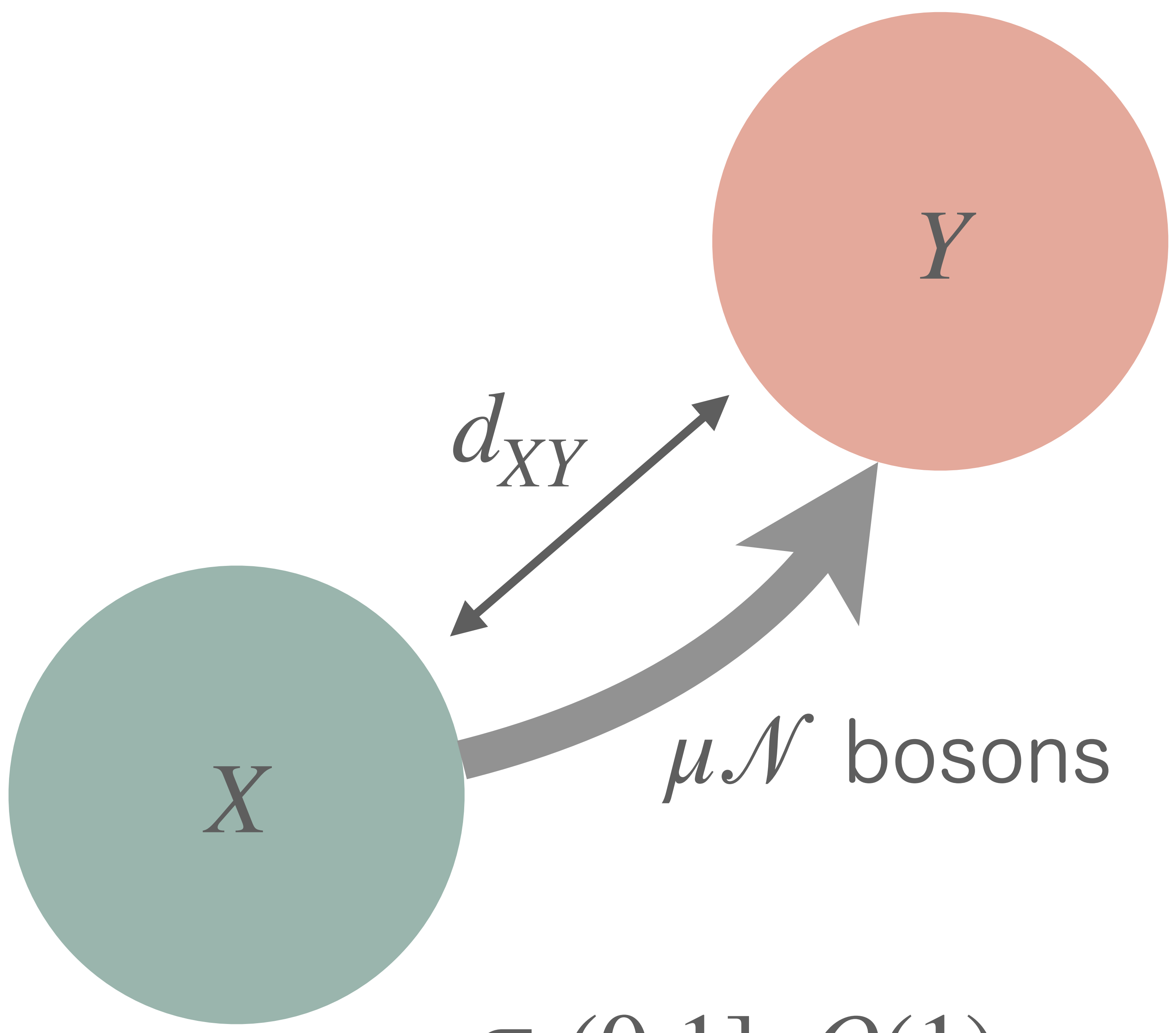

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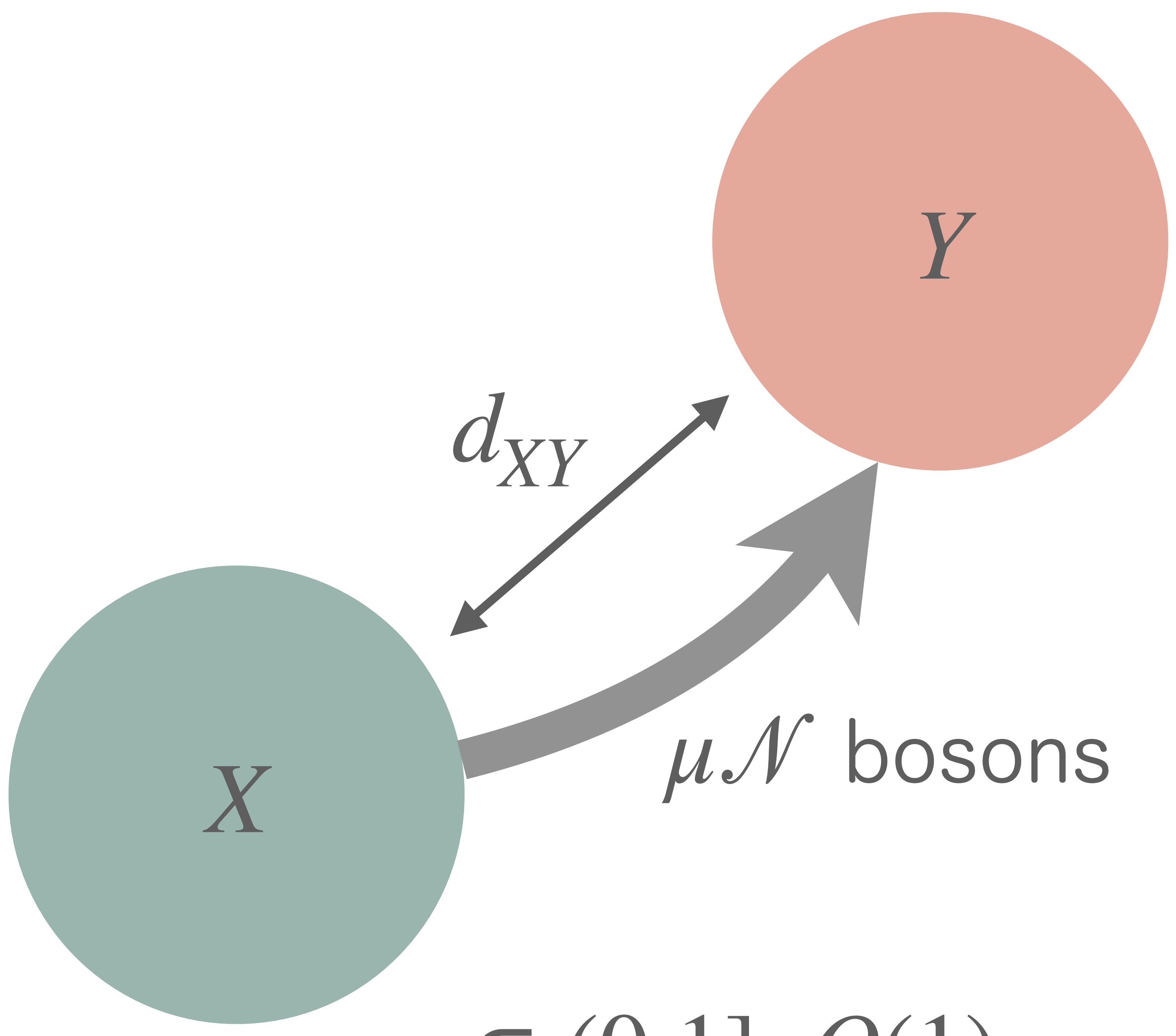
$$d_{XY} := \min_{i\in X,\, j\in Y} \|i - j\|$$

Resolution of particle transport

- Boson concentrations at each site

$$x_i(t) := \mathcal{N}^{-1}\mathrm{tr}\{\hat{n}_i \varrho_t\}$$



$\mu \in (0,1]$: $O(1)$ constant

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Resolution of particle transport

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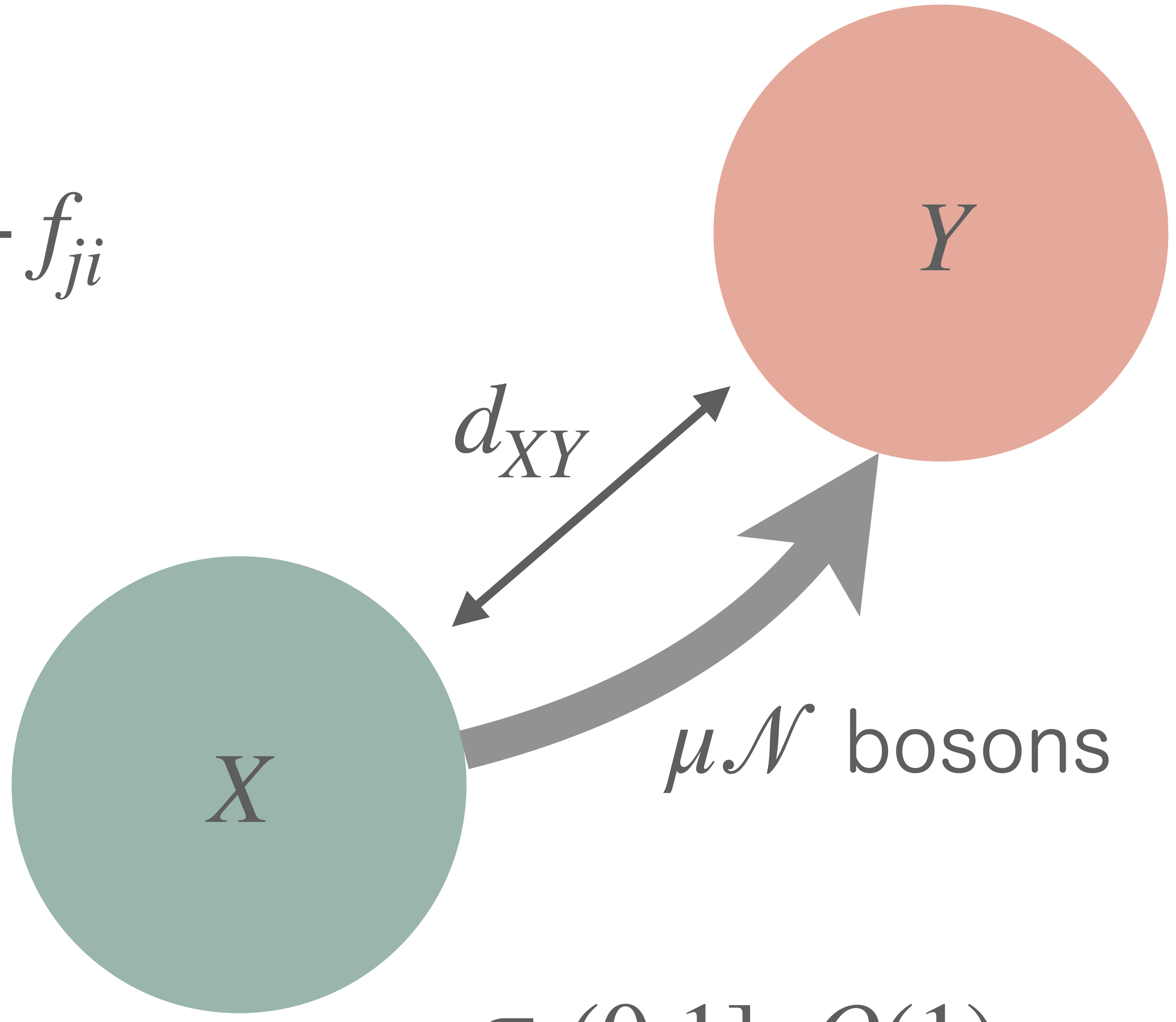

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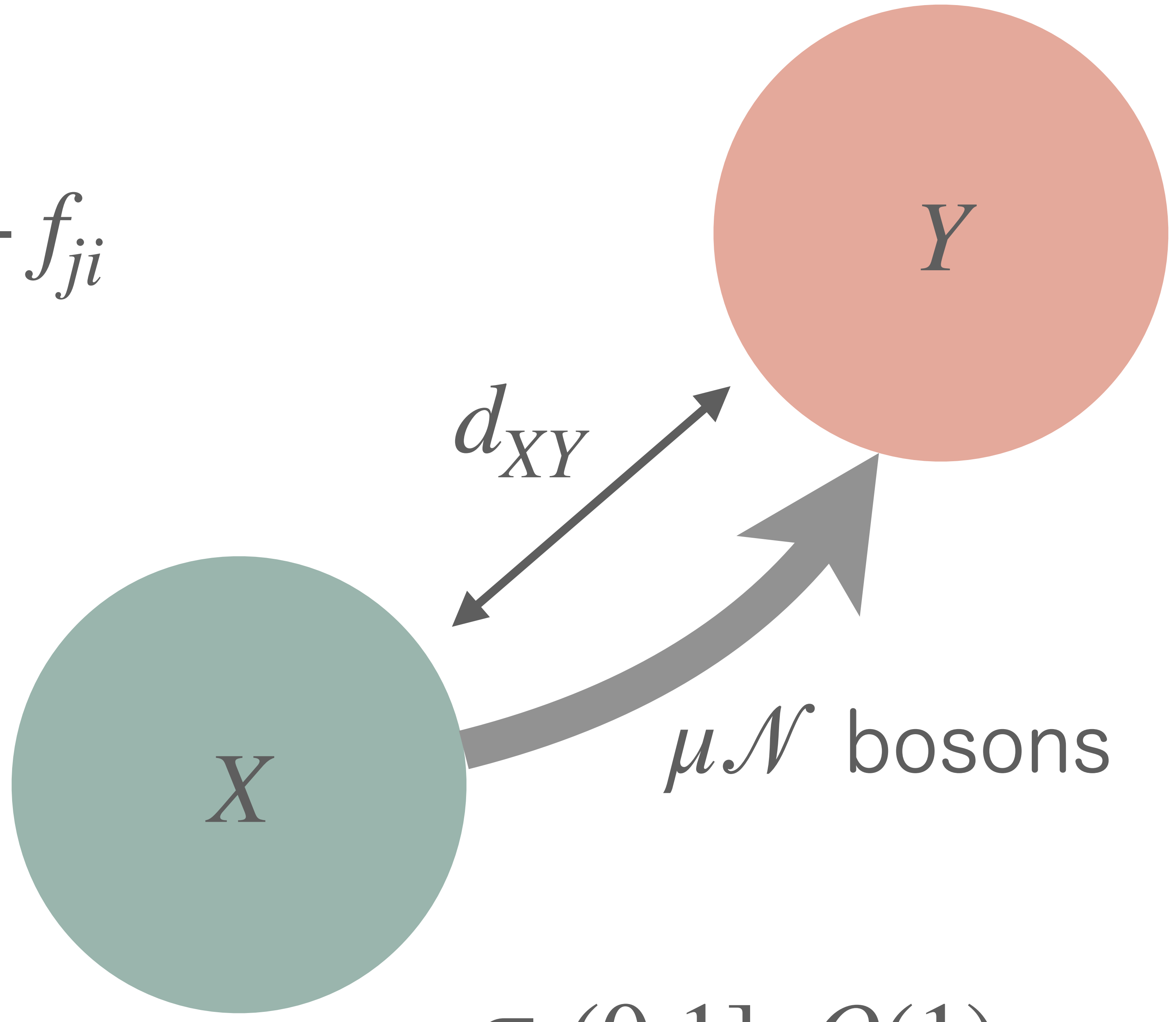

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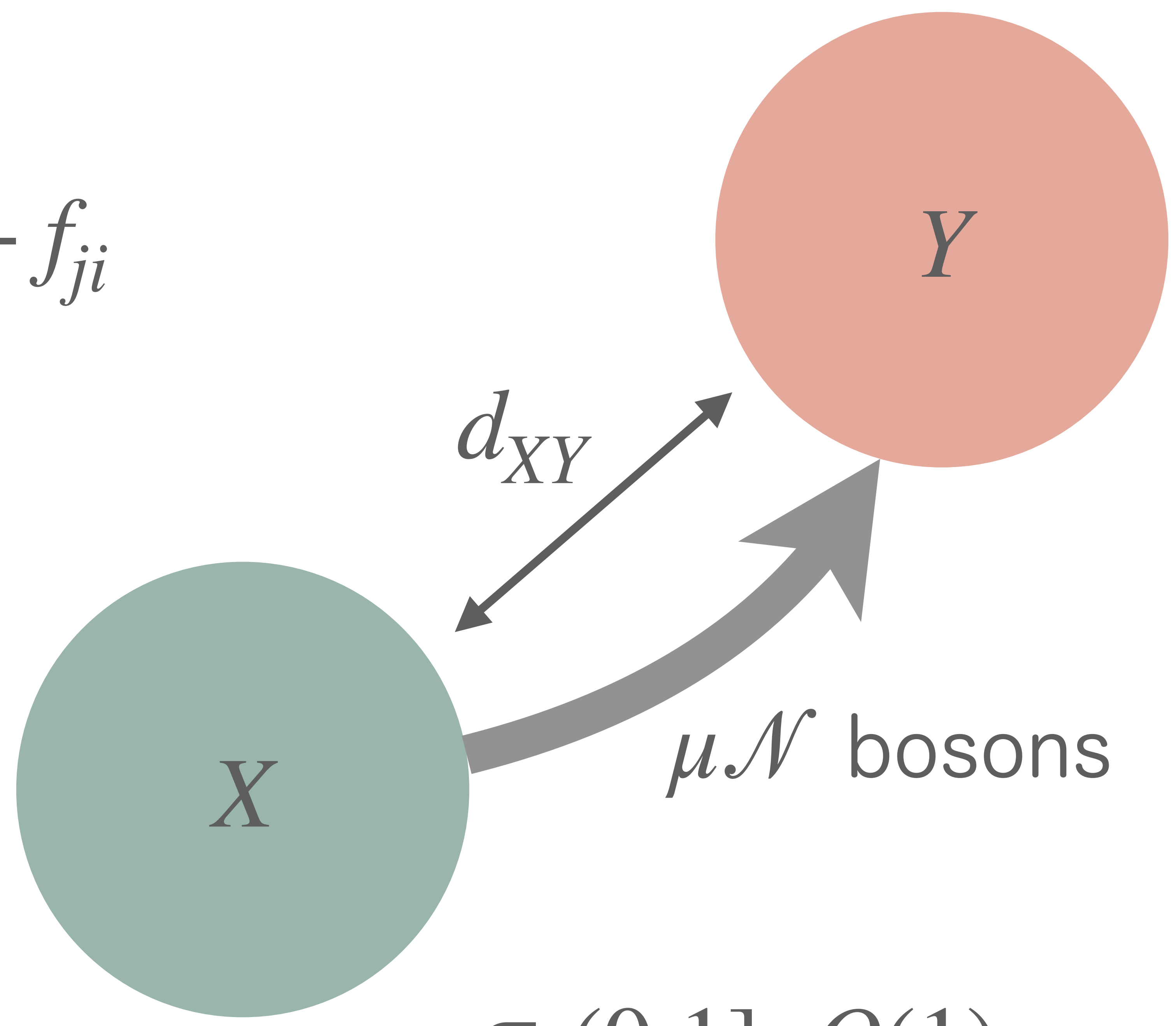
Vu, Kuwahara, and Saito, Quantum (2024)



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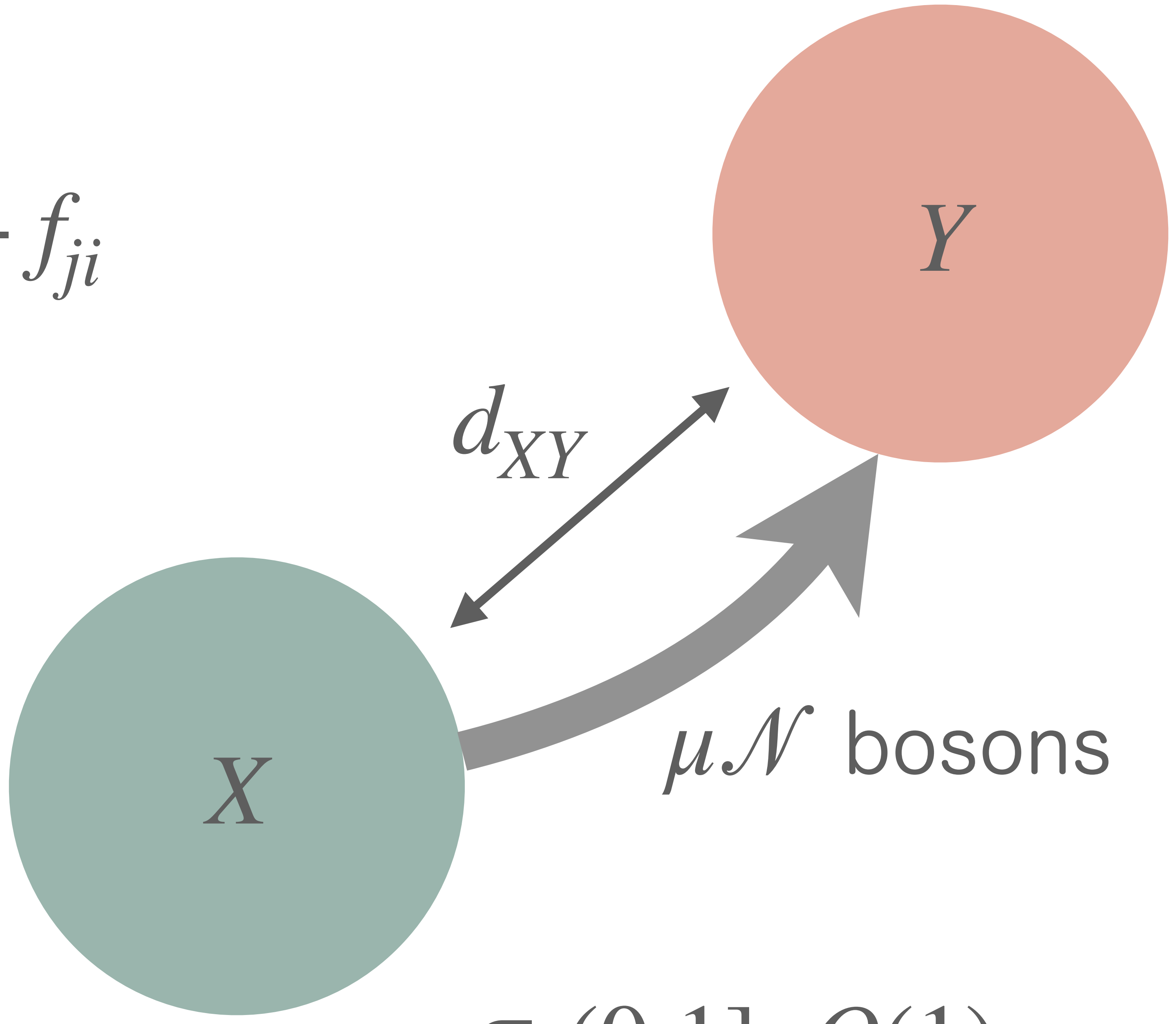

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Vu, Kuwahara, and Saito, Quantum (2024)

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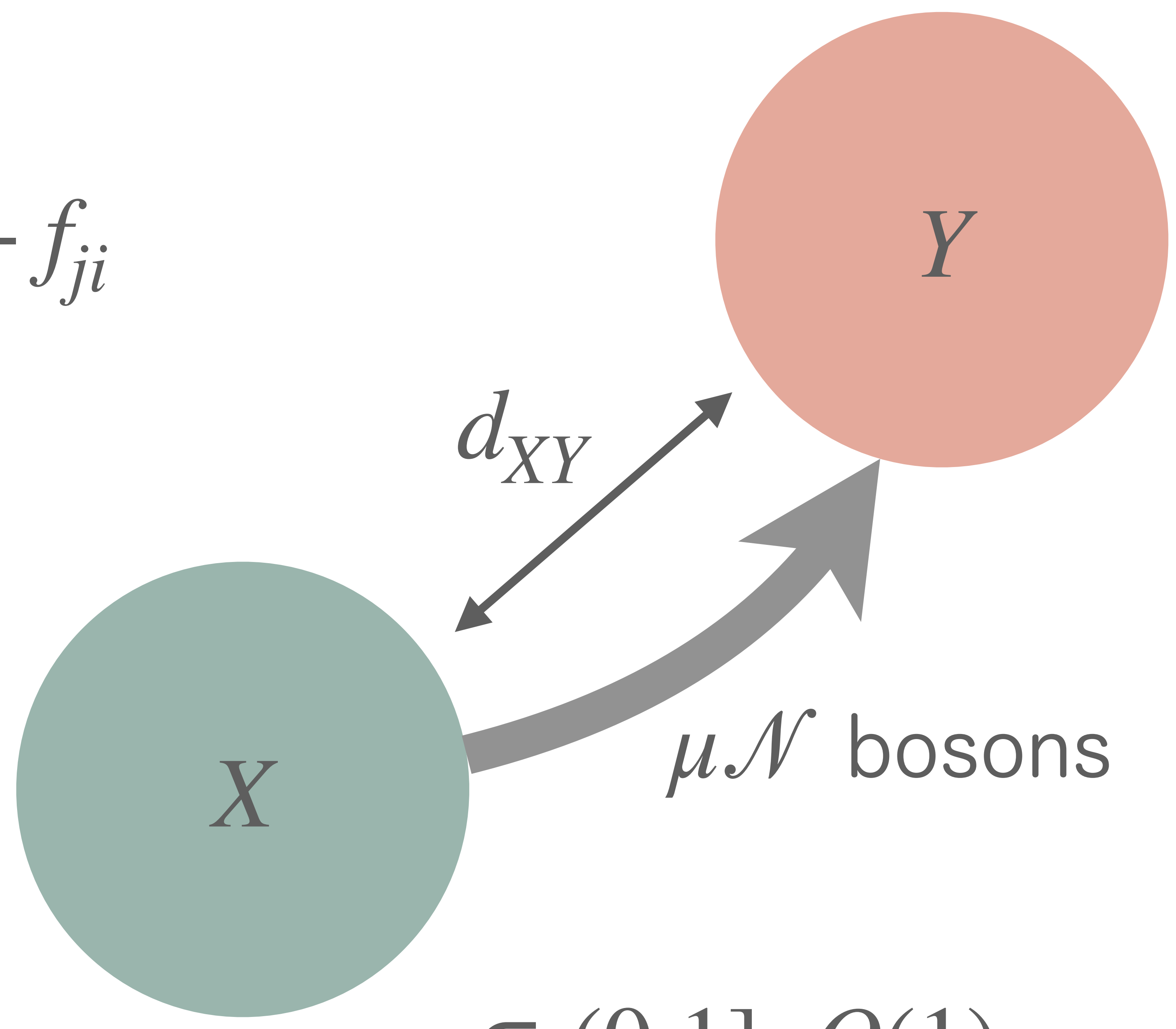

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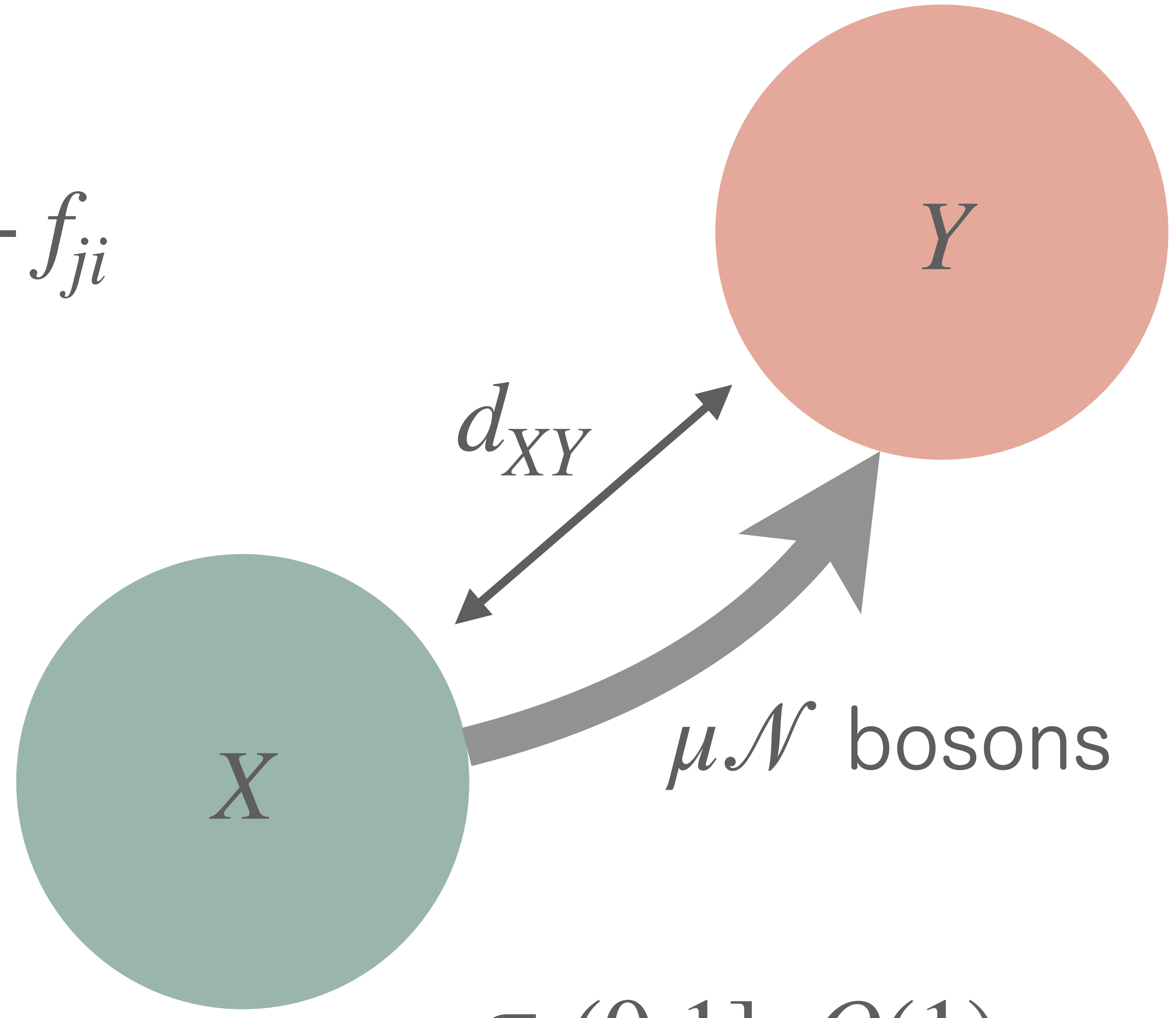
✓ bound is optimal



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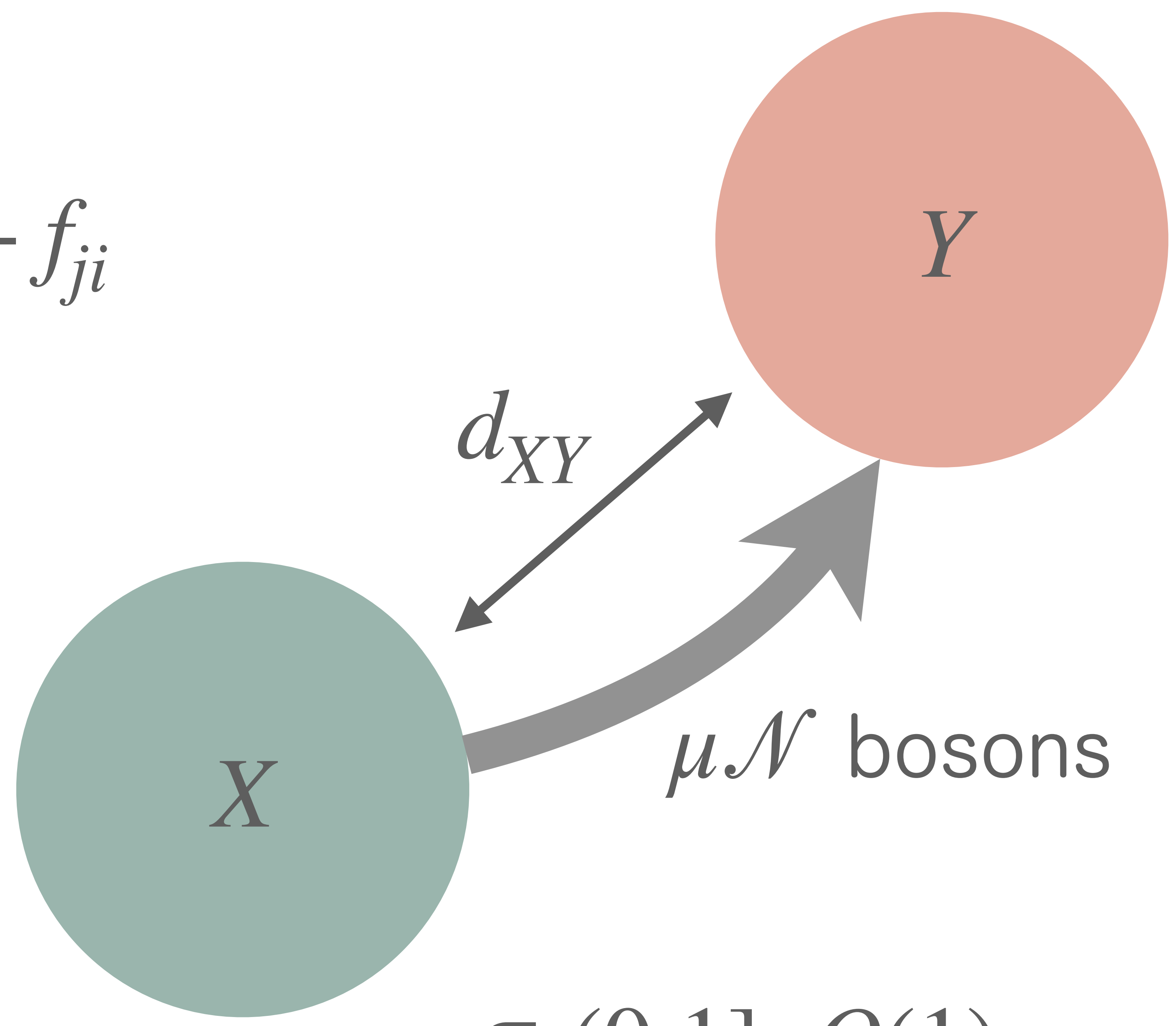
✓ bound is optimal

✓ resolve open problem of macroscopic particle transport



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Sketch proof

cost matrix $c_{ij} = ||i - j||^{\alpha_\epsilon}$

$\alpha_\epsilon := \min(1, \alpha - D - \epsilon)$

$$\tau \geq \frac{\mathscr{W}(\boldsymbol{x}_0, \boldsymbol{x}_\tau)}{\overline{v}} \quad \Rightarrow \quad \tau \gtrsim d_{XY}^{\min(1, \alpha - D)}$$

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- Observation #1: Wasserstein distance is bounded by transport distance

$$n_Y(\tau) \geq n_{X^c}(0) + \mu\mathscr{N} \Rightarrow \mathscr{W}(\boldsymbol{x}_0, \boldsymbol{x}_\tau) \geq \mu d_{XY}^{\alpha_\epsilon}$$

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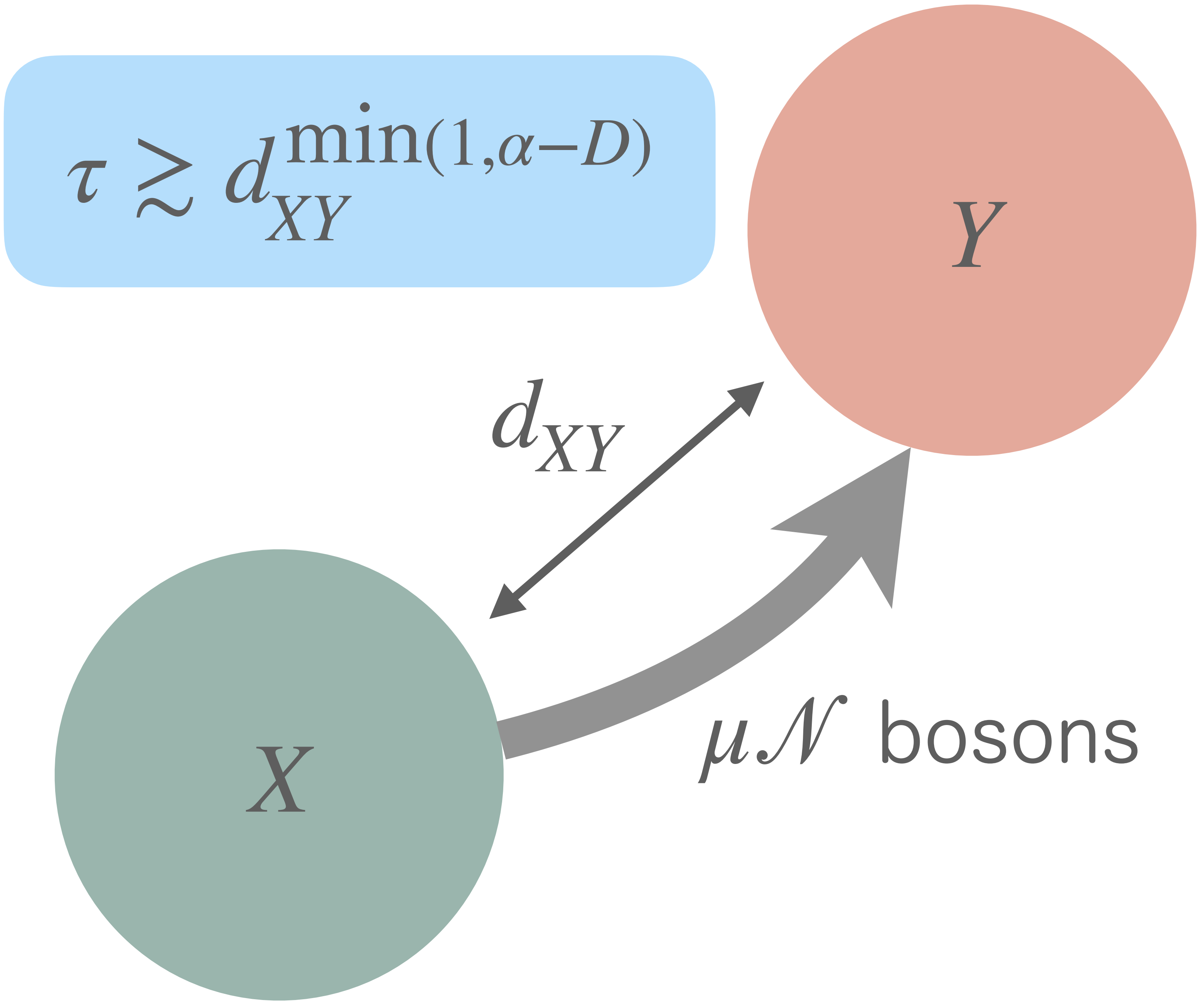
- Observation #2: velocity $\bar{v}$ is finite

$$\bar{v} \leq J\gamma\zeta(\alpha - \alpha_\epsilon - D + 1)$$

ζ : zeta function

Beyond Bose-Hubbard-type models



$$H_t = \sum_{i\neq j} J_{ij}(t)\hat{b}_i\hat{b}_j^\dagger + \sum_Z h_Z(t)$$

Beyond Bose-Hubbard-type models

- Interaction-induced tunneling terms

Sowinski+, Phys. Rev. Lett. (2012)

$$H_t = \sum_{|i-j|=1} J_{ij}(t)\hat{b}_i^\dagger \hat{b}_j + \sum_{|i-j|=1} T_{ij}(t)\hat{b}_i^\dagger(\hat{n}_i + \hat{n}_j)\hat{b}_j + \frac{1}{2}\sum_i U_i(t)\hat{n}_i(\hat{n}_i - 1) + \sum_i \mu_i(t)\hat{n}_i$$

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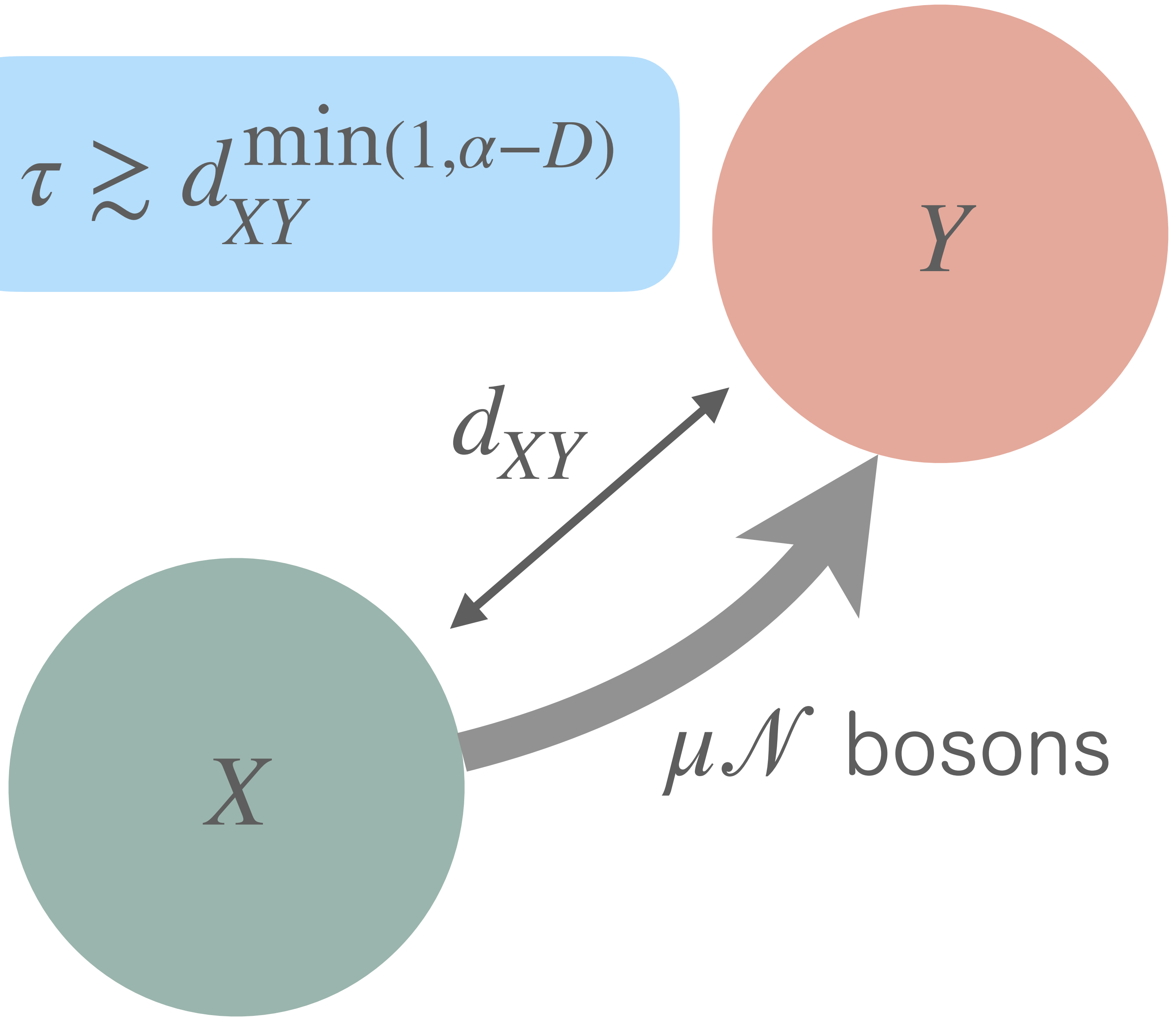

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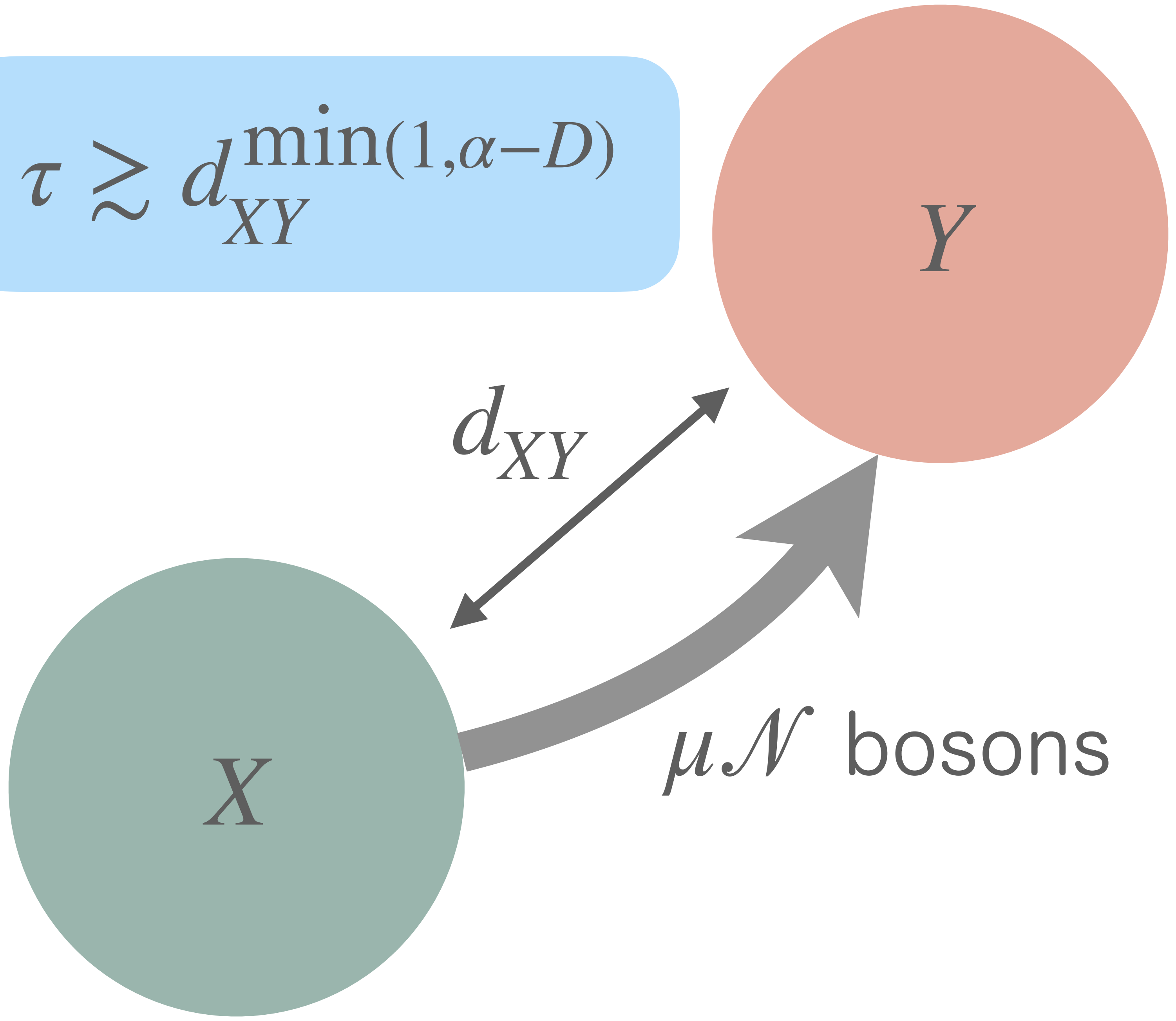

- Speed of macroscopic particle transport?

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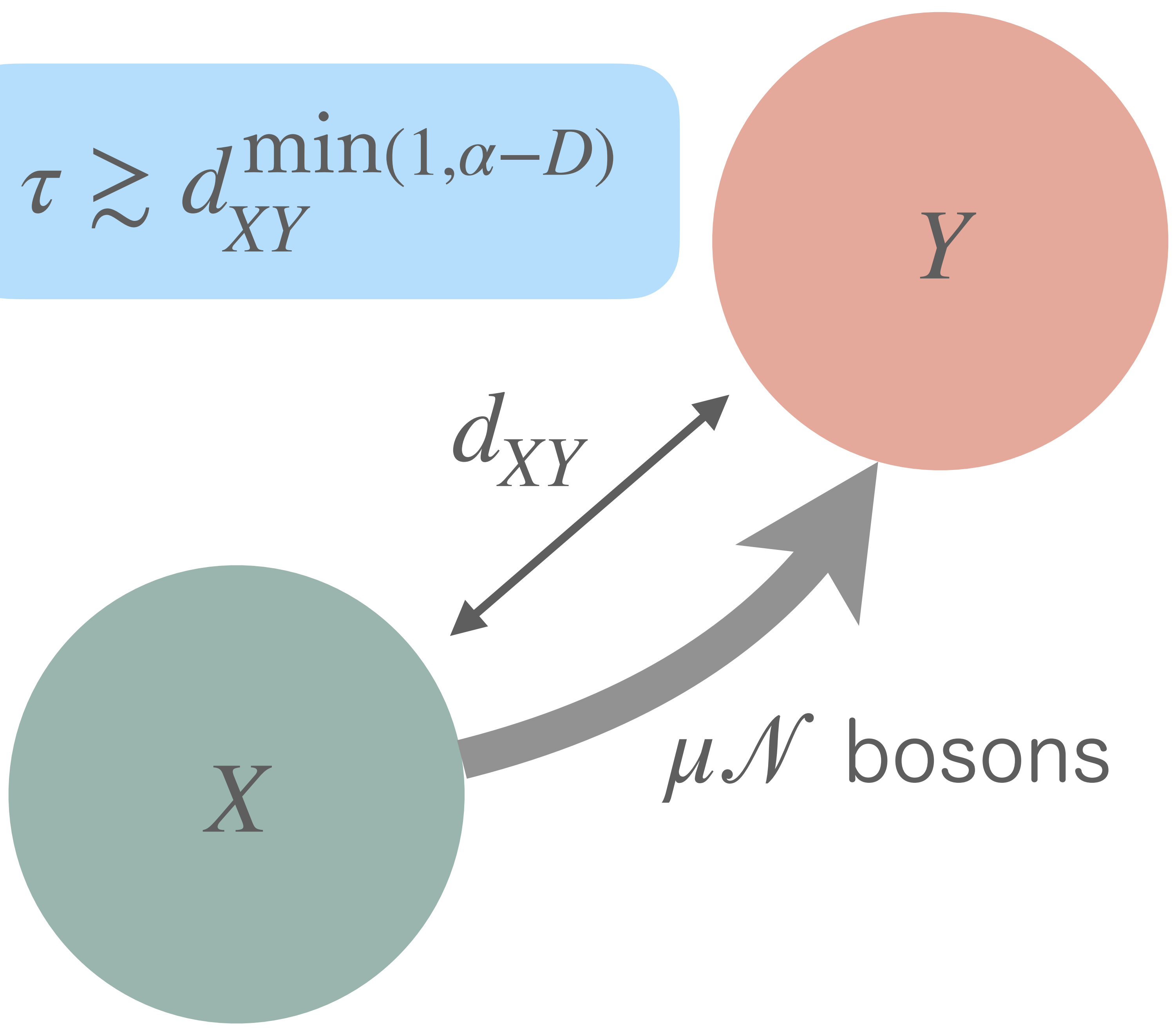

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  ✓ speed can be infinite

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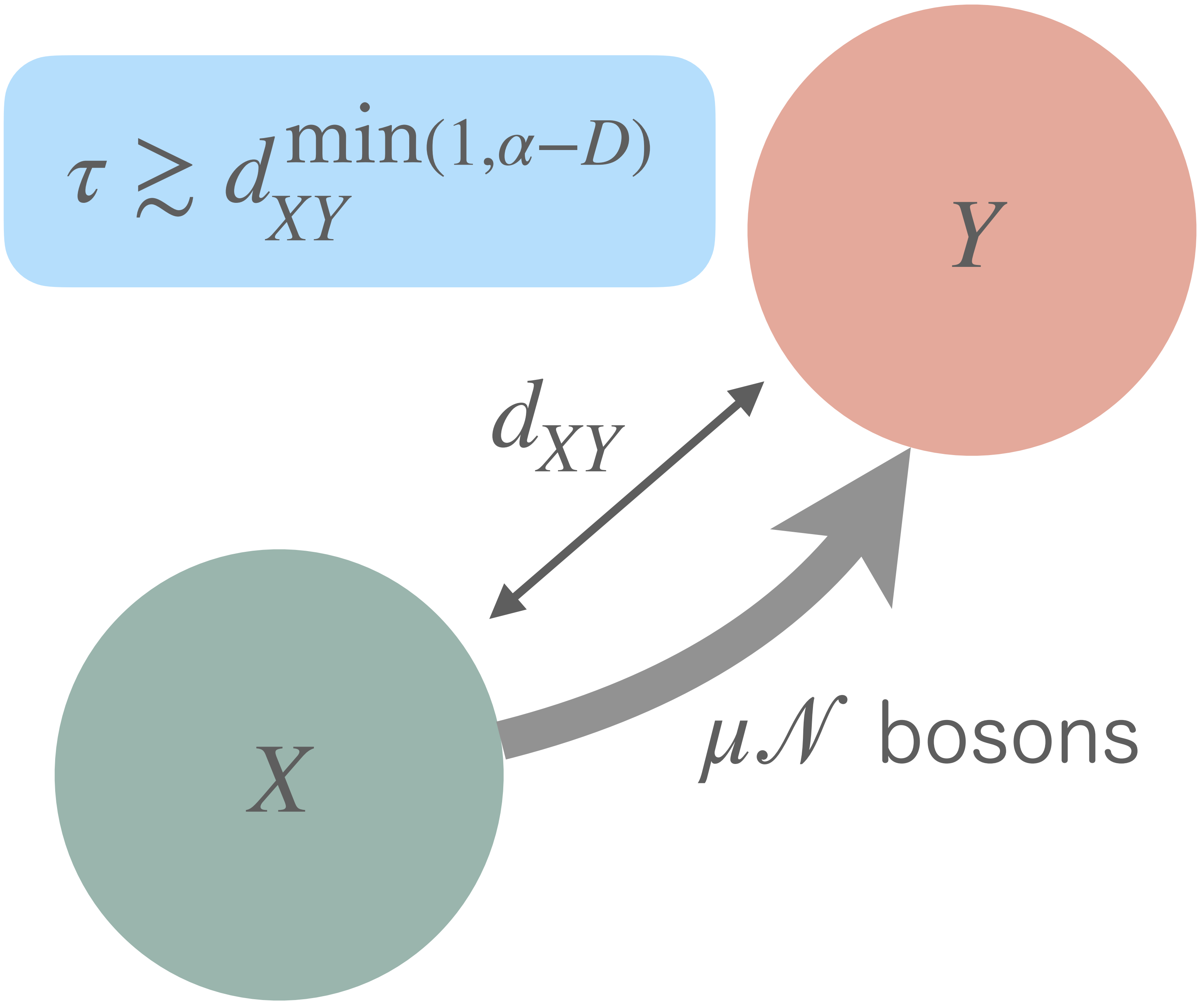

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$$H_t = \sum_{i\neq j} J_{ij}(t)\hat{b}_i\hat{b}_j^\dagger + \sum_Z h_Z(t)$$

✓ speed can be infinite

➡ all bosons can be transported within a constant of time

Beyond Bose-Hubbard-type models

- Initial Mott state

$$|L\rangle \otimes |0\rangle \otimes \ldots \otimes |0\rangle \to |0\rangle \otimes |0\rangle \otimes \ldots \otimes |L\rangle$$

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- Protocol for transporting bosons

step $k = 1,\ldots,L-1 : |L\rangle_k \otimes |0\rangle_{k+1} \to |0\rangle_k \otimes |L\rangle_{k+1}$

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3 stages

$$|L\rangle_k \otimes |0\rangle_{k+1} \to |L-1\rangle_k \otimes |1\rangle_{k+1} \to |1\rangle_k \otimes |L-1\rangle_{k+1} \to |0\rangle_k \otimes |L\rangle_{k+1}$$

Beyond Bose-Hubbard-type models

- Protocol for each stage

Beyond Bose-Hubbard-type models

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$|L-1\rangle_1 \otimes |1\rangle_2 \rightarrow |1\rangle_1 \otimes |L-1\rangle_2$ within time $t = \pi/(2JL)$

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Beyond Bose-Hubbard-type models

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$|L\rangle \otimes |0\rangle \otimes \ldots \otimes |0\rangle \to |0\rangle \otimes |0\rangle \otimes \ldots \otimes |L\rangle$ ➡ transport to distant place within a constant of time

Summary

- Reveal maximal speed of macroscopic particle transport using optimal transport theory for both closed systems

General speed limit $\tau \geq \dfrac{\mathscr{W}(\boldsymbol{x}_0, \boldsymbol{x}_\tau)}{\overline{v}}$ → **Resolution of bosonic transport**

$$\tau \gtrsim d_{XY}^{\lambda}$$

Summary

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- Future works: exploit full quantum states to uncover new constraints

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Hongchao's talk on generalization to open systems (next Monday)

Li, Shang, Kuwahara, and Vu, arXiv:2503.13731

Thank you for your attention!