Non-equilibrium fluctuations and correlations in hard rod gas

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Hydrodynamics of low-dimensional interacting systems: Advances, challenges, and future directions

YITP, Kyoto
June 2-13, 2025

ICTS International Centre for Theoretical Sciences
Tata Institute of Fundamental Research

DAE

SERB India

Thermalization and hydrodynamics in an interacting integrable system: the case of hard rods
Singh, Dhar, Spohn, Kundu. Jstat. Phys. 191 (6), 66, (2024)

Conserved densities of hard rods: microscopic to hydrodynamic solutions
MJ Powdel, A Kundu, Jstat. Mech. 2024 (12), 123205

Dynamics of tagged quasiparticles in hard rod gas,
S. Chahal, I. Mukherjee, A. Dhar, H. Spohn, A. Kundu, Arxiv: 2506.xxxx

Macroscopic fluctuation theory of correlations in hard rod gas
A kundu, Arxiv: 2504.09201

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Abhishek Dhar (ICTS, Bangalore)
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Seema


Indranil

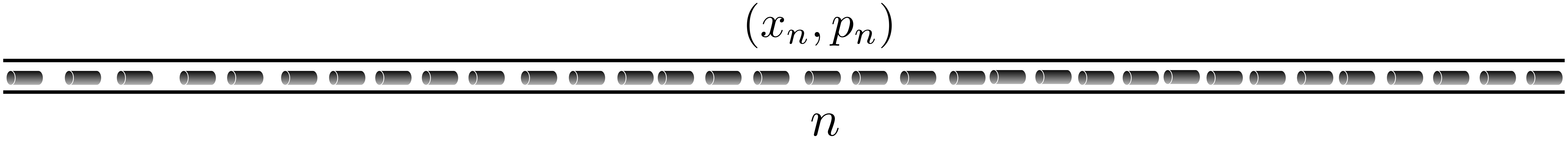

Sahil


Mrinal


Abhishek

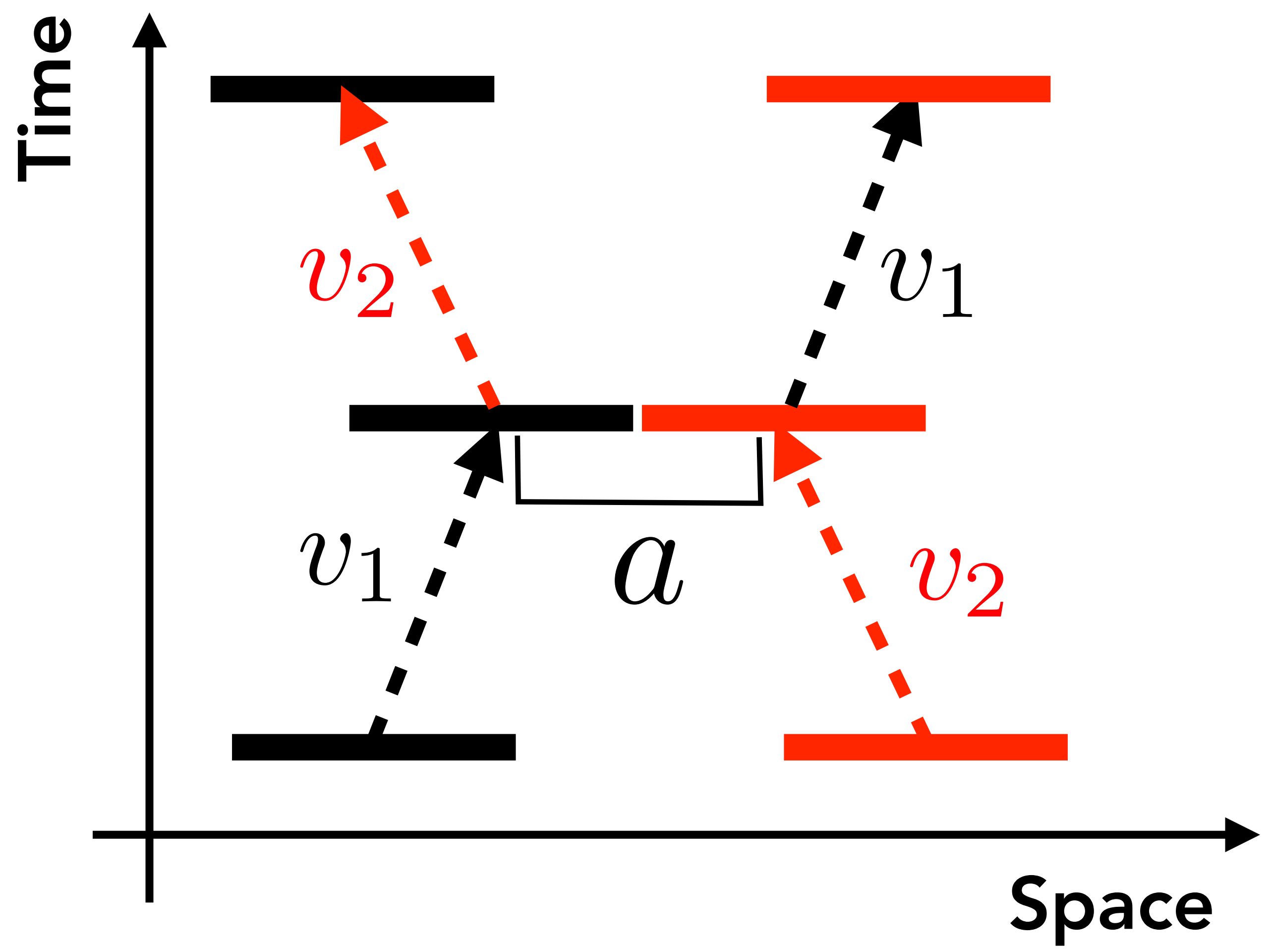

Herbert

A gas of hard rods in one dimension



$$H = \sum_{n=1}^{N} \frac{p_n^2}{2} + \sum_{n=1}^{N} \begin{cases} \infty & \text{if } |x_{n+1} - x_n| < a \\ 0 & \text{if } |x_{n+1} - x_n| \geq a \end{cases}$$

$$\begin{aligned} \text{Mass} &= 1 \\ \text{Length} &= a \end{aligned}$$



Two hard-rods exchange velocities at collision

Thermal equilibrium

L. Tonks (1936)

NOVEMBER 15, 1936 PHYSICAL REVIEW VOLUME 50

The Complete Equation of State of One, Two and Three-Dimensional Gases of Hard Elastic Spheres

LEWI TONKS, *Research Laboratory, General Electric Company, Schenectady, N. Y.*
(Received August 3, 1936)

Lord Rayleigh (1891)

ON THE VIRIAL OF A SYSTEM OF HARD COLLIDING BODIES.

Hard rod gas is a solvable model of one-dimensional fluid in equilibrium

[1] RAYLEIGH, LORD, 1891, *Nature, Lond.*, **45**, 80.
[2] TONKS, L., 1936, *Phys. Rev.*, **50**, 955.
[3] SALSBURG, Z. W., KIRKWOOD, J. G., and ZWANZIG, R. W., 1953, *J. chem. Phys.*, **21**, 1098.
[4] MÜNSTER, A., 1969, *Statistical Thermodynamics* (Springer), §§4.9–4.11, 6.11, 6.12.
[5] PERCUS, J. K., 1976, *J. statist. Phys.*, **15**, 505.
[6] ROBLEDO, A., 1980, *J. chem. Phys.*, **72**, 1701. ROBLEDO, A., and VAREA, C., 1981, *J. statist. Phys.*, **26**, 513.
[7] PERCUS, J. K., 1982, *The Liquid State of Matter*, edited by E. W. Montroll and J. L. Lebowitz (North-Holland), p. 31.
[8] LEFF, H. A., and COOPERSMITH, M. J., 1967, *J. math. Phys.*, **8**, 306.
[9] FLICKER, M., 1968, *J. math. Phys.*, **9**, 171.

Thermal equilibrium

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Lord Rayleigh (1891)

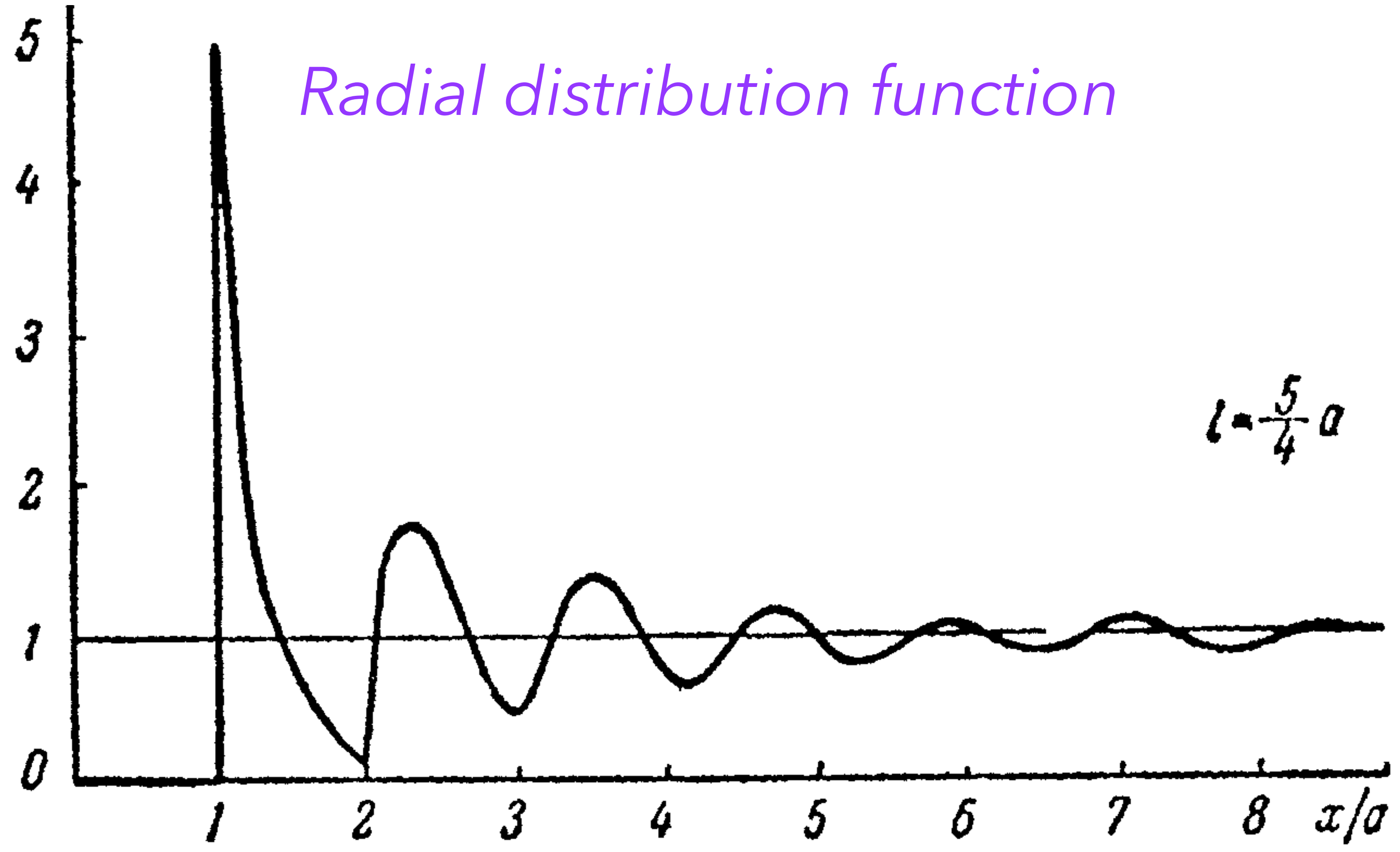
ON THE VIRIAL OF A SYSTEM OF HARD COLLIDING BODIES.

- *Analytical derivation of equation of state of interacting gas*
- *Radial distribution function*

- *Classical fluid in external potential*

J. K. Percus, JStat. Phys., Vol. 15, No. 6, 1976

J. K. Percus, JStat. Phys.,Vol. 28, No. 1, 1982



Fisher I.Z. - Statistical Theory of Liquids

Chicago Univ (1964)

Non-equilibrium properties : Kinetic equations

J. L. Lebowitz and J. K. Percus (1967)

PHYSICAL REVIEW VOLUME 155, NUMBER 1 5 MARCH 1967

Kinetic Equations and Density Expansions: Exactly Solvable One-Dimensional System*

J. L. Lebowitz, J Percus and J. Sykes (1969)

PHYSICAL REVIEW VOLUME 188, NUMBER 1 5 DECEMBER 1969

Kinetic-Equation Approach to Time-Dependent Correlation Functions

J. K. Percus (1969)

THE PHYSICS OF FLUIDS VOLUME 12, NUMBER 8 AUGUST 1969

Exact Solution of Kinetics of a Model Classical Fluid

J. K. Percus

- Boltzmann equation for hard rod fluid
- Time dependent distribution functions
- Distribution function of labelled particle
- Velocity auto-correlation functions
- Self diffusion

Non-equilibrium properties : Hydrodynamic equations & transport coefficients

Boldrighini C, Dobrushin R and Sukhov Y M 1983 J. Stat. Phys. 31 577
Boldrighini C and Suhov Y M 1997 Commun. Math. Phys. 189 577

- *Rigorous derivation of hydrodynamic Equations, both at ballistic and diffusive scales*

Spohn H 1982 Ann. Phys., NY 141 353
Spohn H 1991 Large Scale Dynamics of Interacting Particles (New York: Springer)
Doyon B and Spohn H 2017 J. Stat. Mech. 073210

- *Equilibrium current-current correlation*

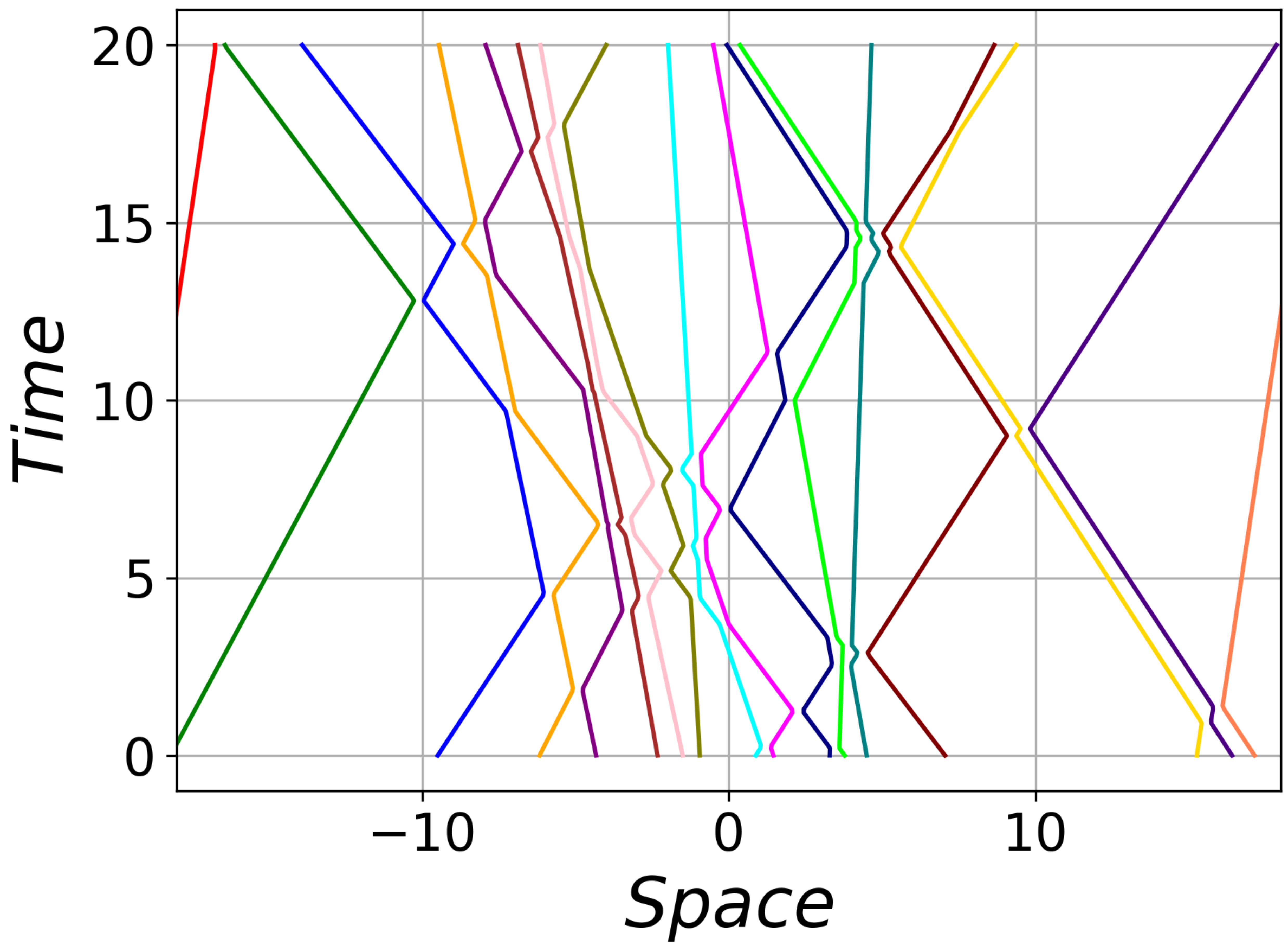
Vir B Bulchandani J. Stat. Mech. (2024) 043205

- *Revised Enskog equation for hard rods*

F. Hübner, L. Biagetti, J. De Nardis & B. Doyon, arXiv: 2503:07794

- *Diffusive hydrodynamics of hard rods from microscopics*

Hard rod systems, being one of the simplest integrable systems in one dimension, has recently drawn a lot of interest. It is due to the development of generalised hydrodynamics (GHD) that describes large scale motion in integrable systems.

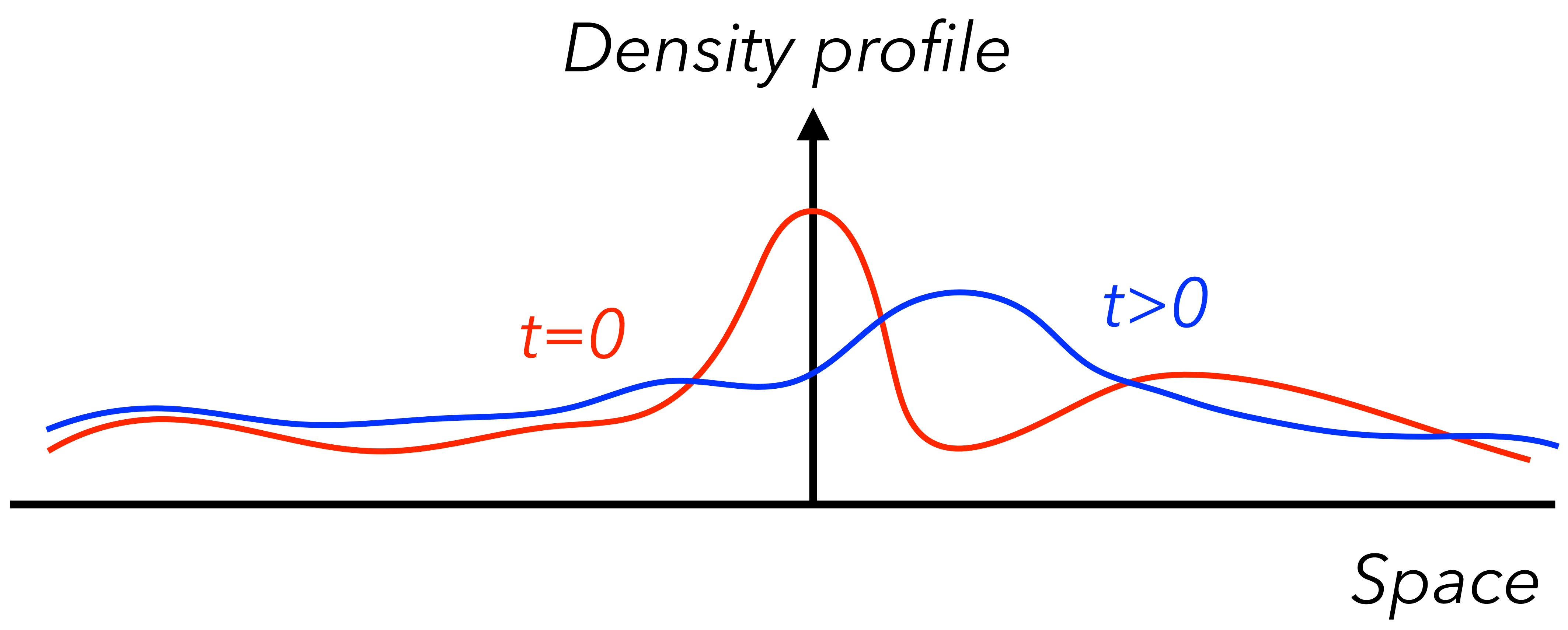


Broad goal: Large scale motion, fluctuation and correlation in hard rod gas

Main topics

- Generalised hydrodynamics of hard rods

  - Application to large scale dynamics



- Microscopic characteristics: Stochastic motion of quasiparticles

- Fluctuations in hydrodynamics

- Long-range correlation

Integrable systems

- ✤ Integrable systems have extensive number of local conservation laws (integrals of motion) and they fail to thermalize to Gibbs state.

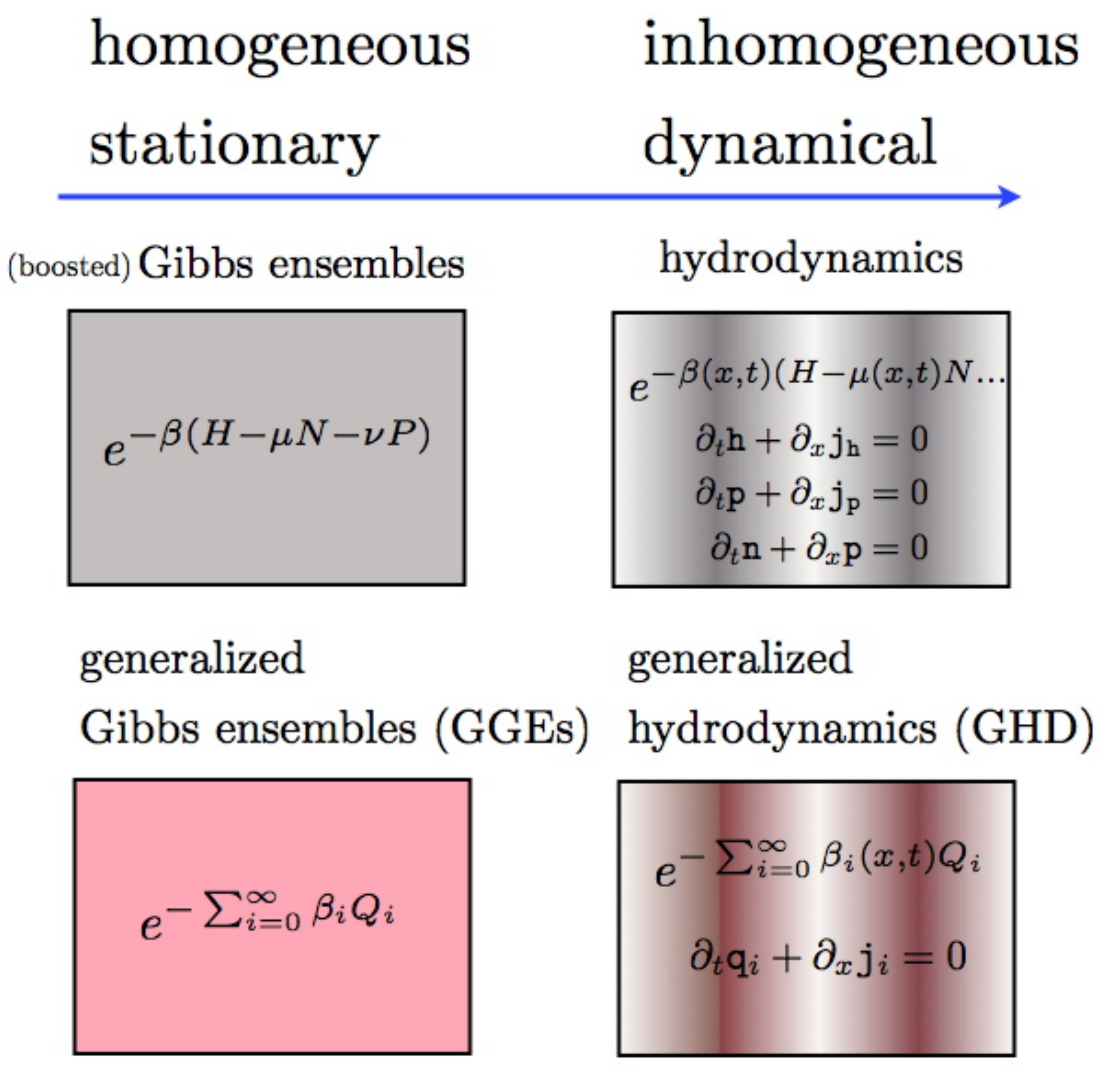
- ✤ It is believed that such systems with local conservation laws thermalize to generalized Gibbs state (GGE) upon coarse graining.

Generalised Gibbs ensemble (GGE):

Non-integrable systems	Integrable systems
$H(\{x_i, p_i\})$: Conserved quantities Number : N Energy : E Momentum : P $\mathbb{P}_{\text{GE}} = \frac{1}{Z} \exp\left[-\beta(E - \mu N - \nu P)\right]$	$H(\{x_i, p_i\})$: Conserved quantities $Q_\alpha(\{x_i, p_i\}),\ \alpha = 1, 2, \ldots$ $\mathbb{P}_{\text{GGE}}(\{x_i, p_i\}) = \frac{1}{Z} \exp\left(-\sum_\alpha \mu_\alpha Q_\alpha\right)$

Herbert' talk, Benjamin's talk

These systems have hydrodynamics different from non-integrable systems



Non-integrable

finite number of conservation laws

e.g.- Number N, Momentum P, Energy H

"generalization"

Integrable

infinite number of conservation laws

$H(\{x_i, p_i\})$

Conserved quantities $Q_\alpha(\{x_i, p_i\}),\ \alpha = 1, 2, \ldots$

Image taken from: Doyon, Lecture notes (2019)

Generalised hydrodynamics

$$\partial_t \mathfrak{q}_\alpha + \partial_x \mathfrak{j}_\alpha(\bar{\mathfrak{q}}(x,t)) = 0, \quad \alpha = 1, 2, ...$$

✤The hydrodynamics can be written concisely in terms of quasiparticle densities

$$\partial_t f(x,v,t) + \partial_x \left(v_{\text{eff}}(x,v,t)\ f(x,v,t)\right) = 0$$

$f(x,v,t) =$ Phase space density of quasi-particles

B. Doyon, Lecture notes (2019)

Castro-Alvaredo O A, Doyon B and Yoshimura T , PRX, (2016)

Bertini B, Collura M, De Nardis J and Fagotti M , PRL, (2016)

Conserved quantities:

$$Q_\alpha = \sum_{i=1}^{\infty} h_\alpha(v_i), \ \alpha = 1, 2, 3, ...$$

For hard rods $h_\alpha(v) = v^\alpha$

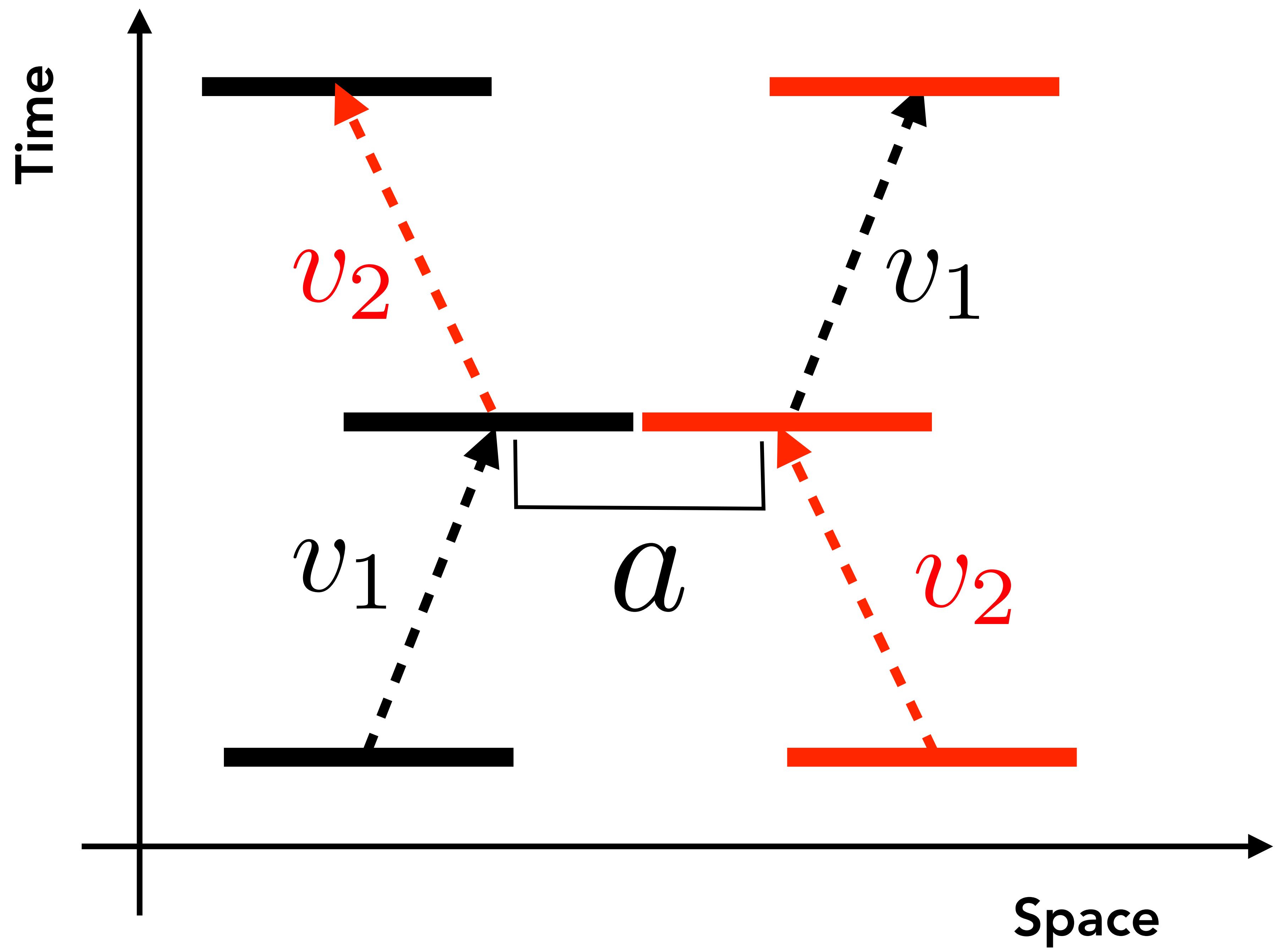
$$\langle Q_\alpha \rangle = \left\langle \sum_i h_\alpha(v_i) \right\rangle = \int dx \int dv\ h_\alpha(v)\ \left\langle \sum_i \delta(x - x_i(t))\delta(v - v_i(t)) \right\rangle$$

$$= \int dx \int dv\ h_\alpha(v)\ f(x,v,t) = \int dx\ \mathfrak{q}_\alpha(x,t)$$

$$\mathfrak{j}_\alpha(x,t) = \int dv\ h_\alpha(v) v_{\text{eff}}(x,v,t) f(x,v,t)$$

A collection of hard-rods moving in one dimension is an integrable system.

Recall: At collision two hard-rods exchange velocities, do not mix the velocities

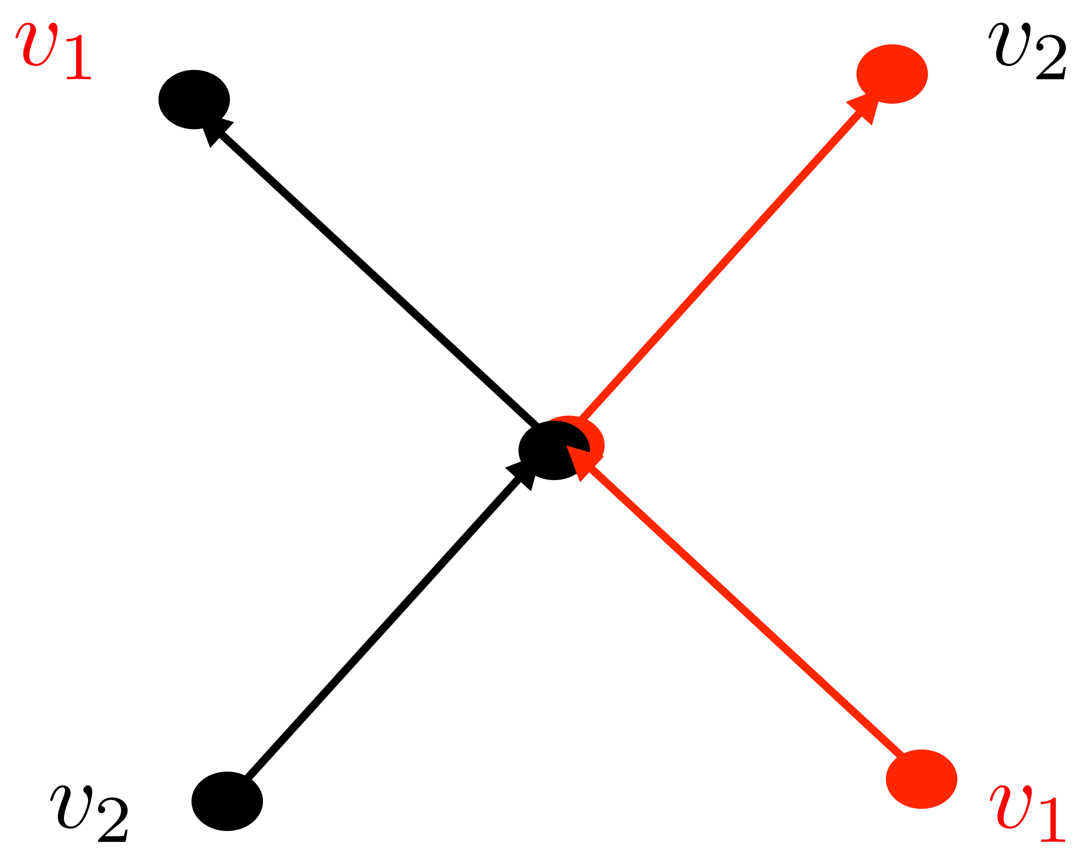


Conserved quantities:

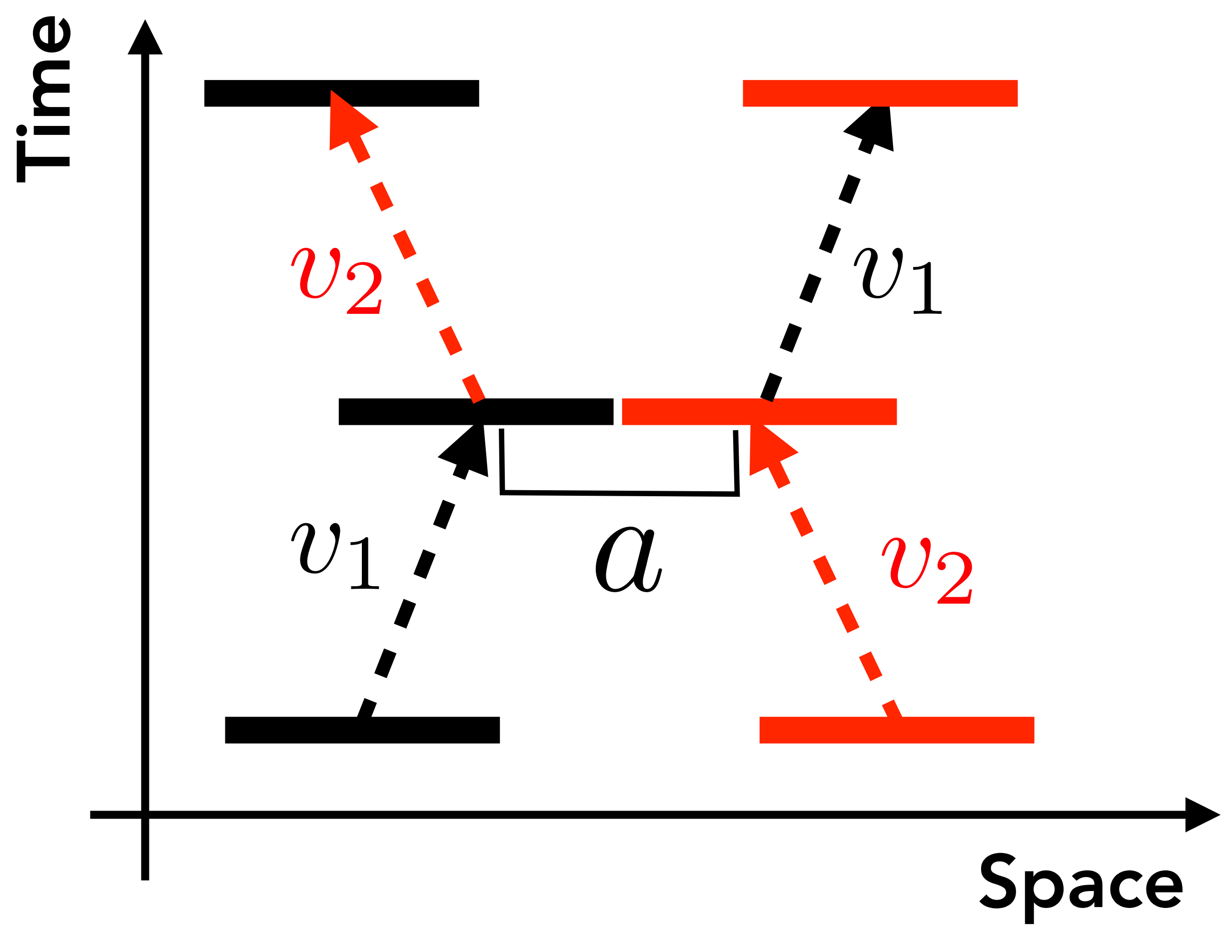
$$Q_\alpha = \sum_{i=1}^{\infty} v_i^\alpha, \;\; \alpha = 1, 2, 3, \ldots$$

Dynamics is similar to hard point gas

Hard point gas ($a = 0$)

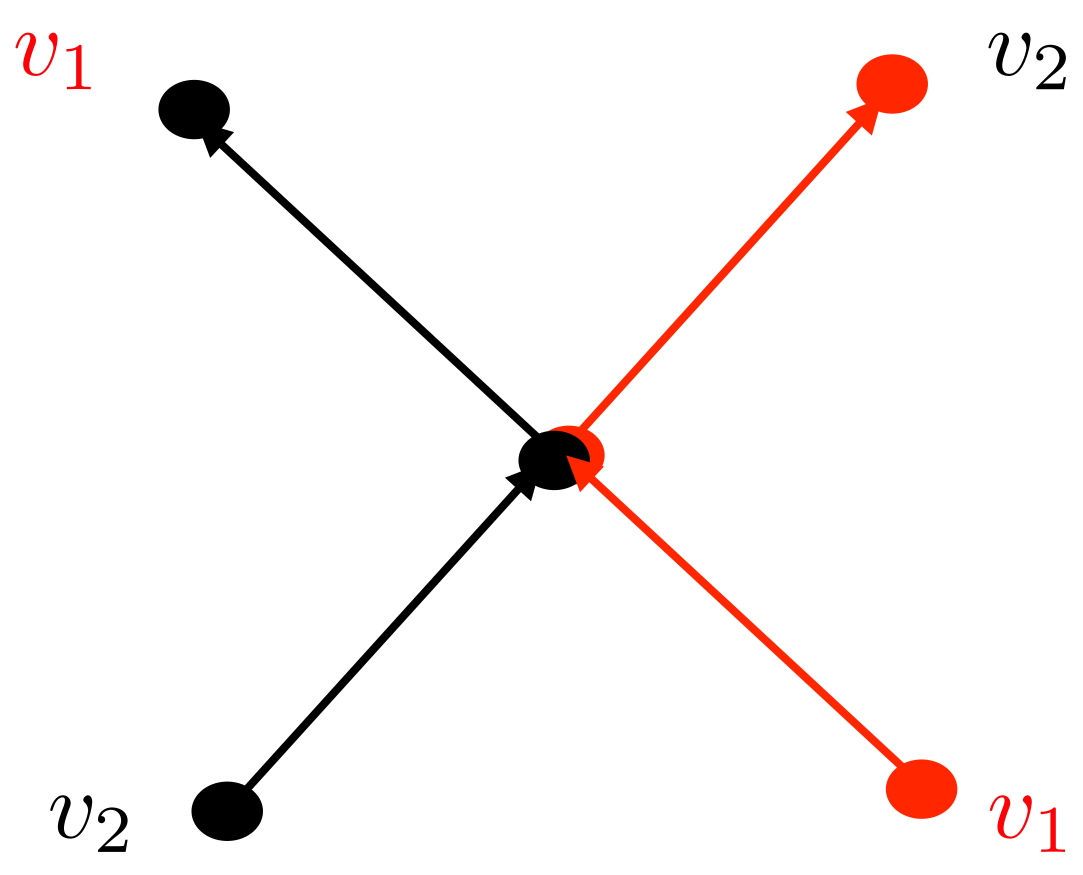


Exchange velocities at collision

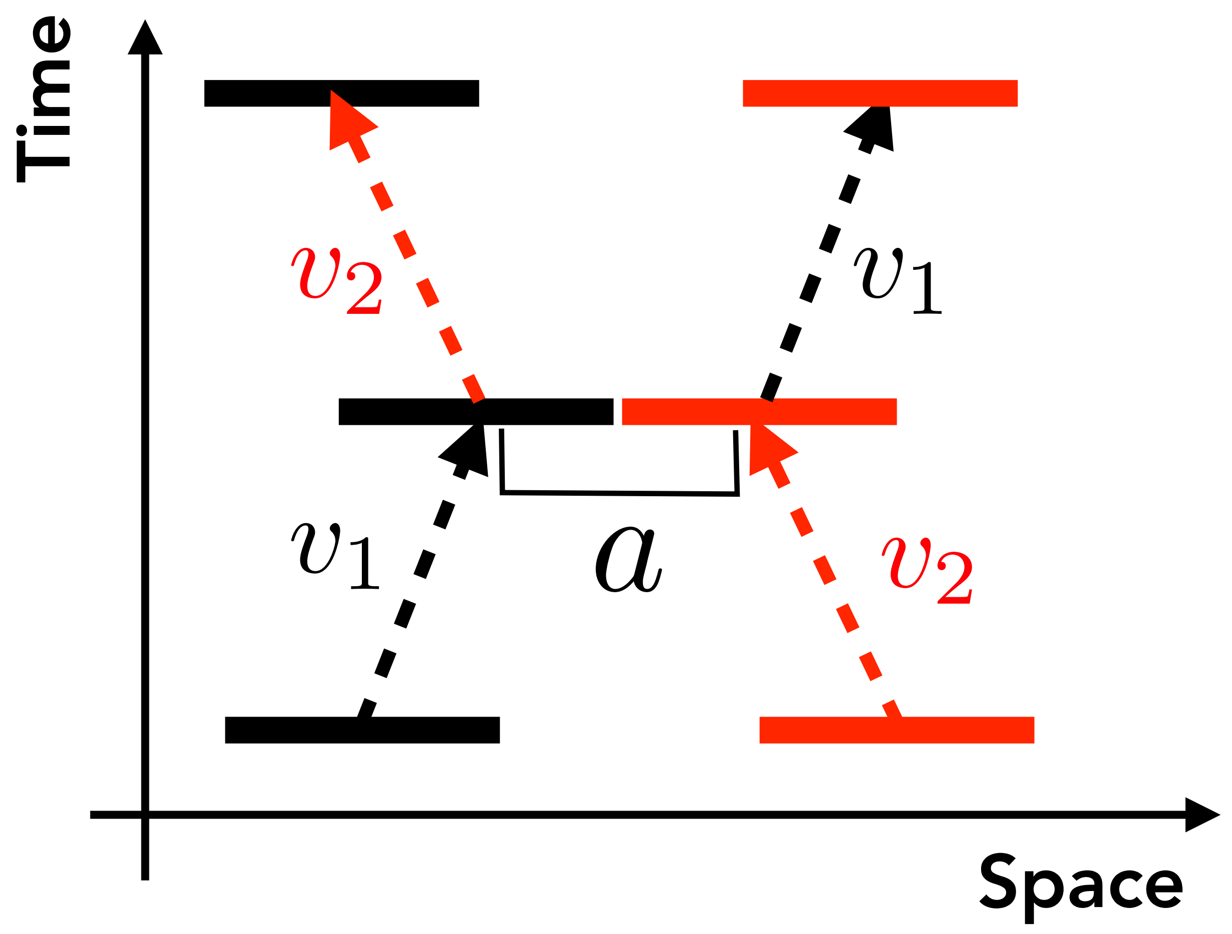


Dynamics is similar to hard point gas

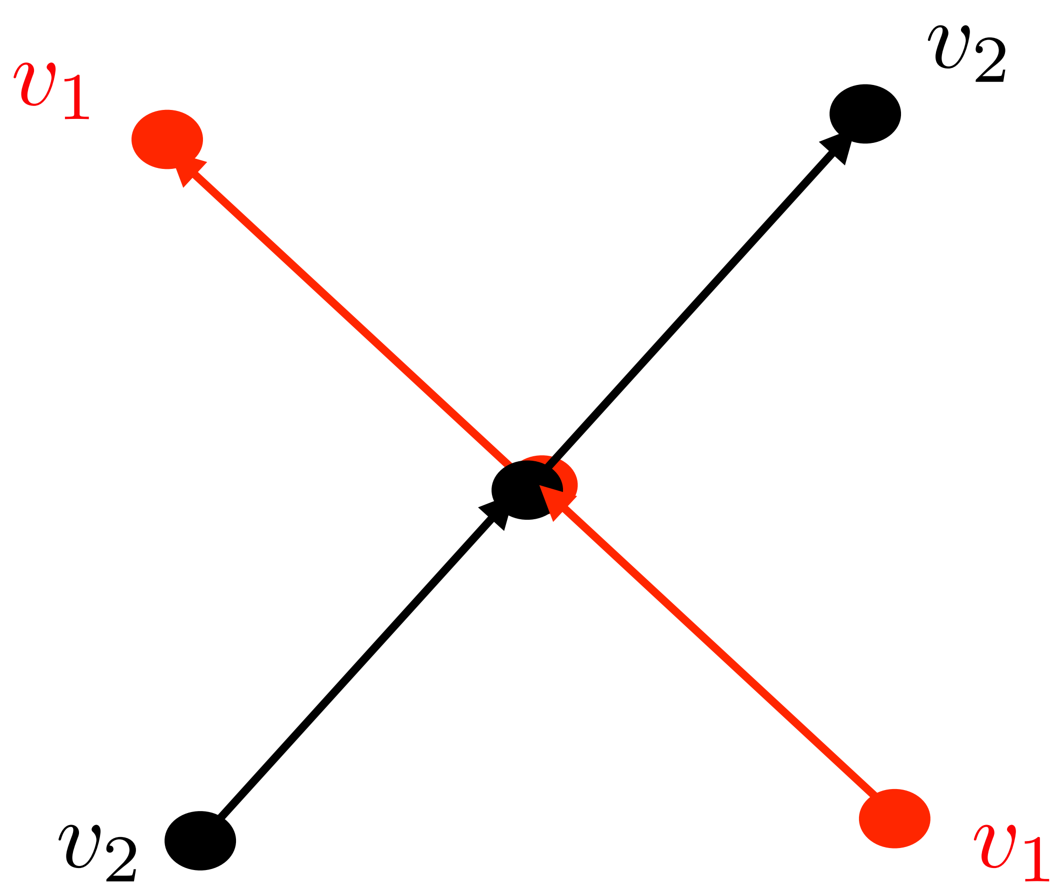
Hard point gas ($a = 0$)



Exchange velocities at collision



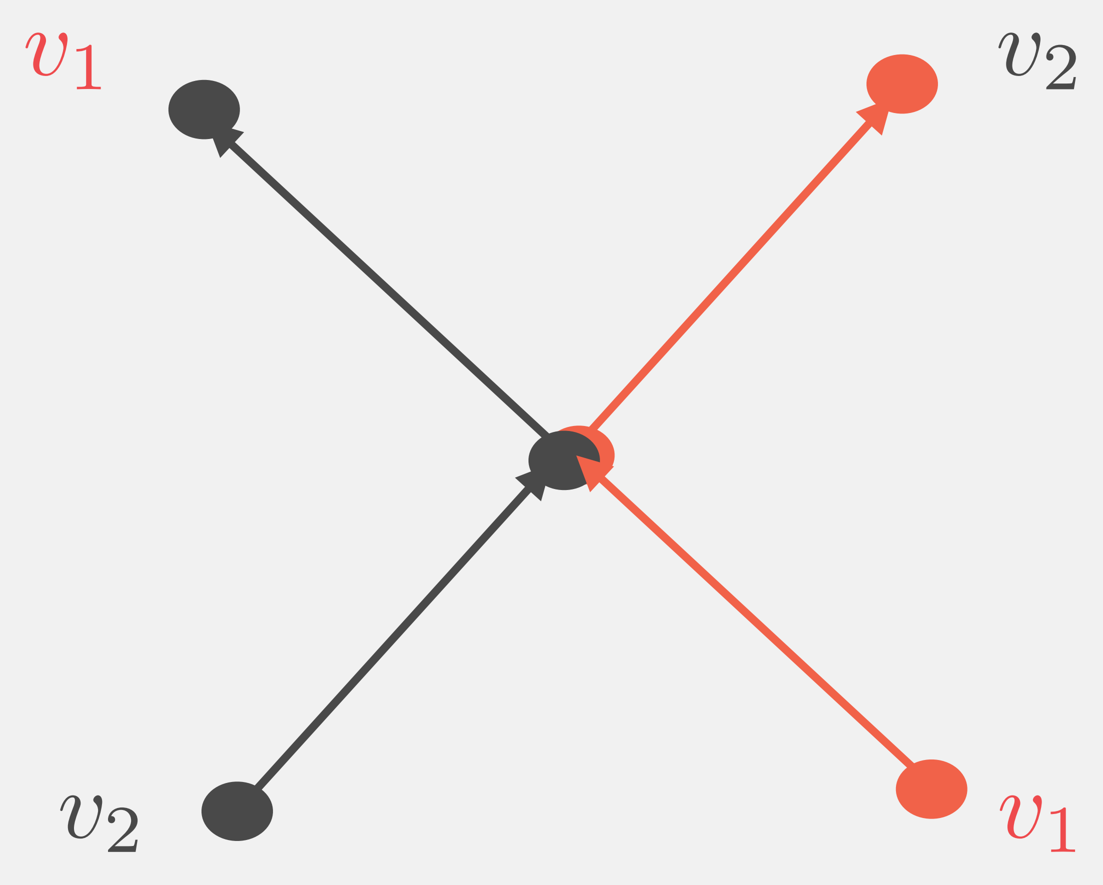
Non-interacting gas



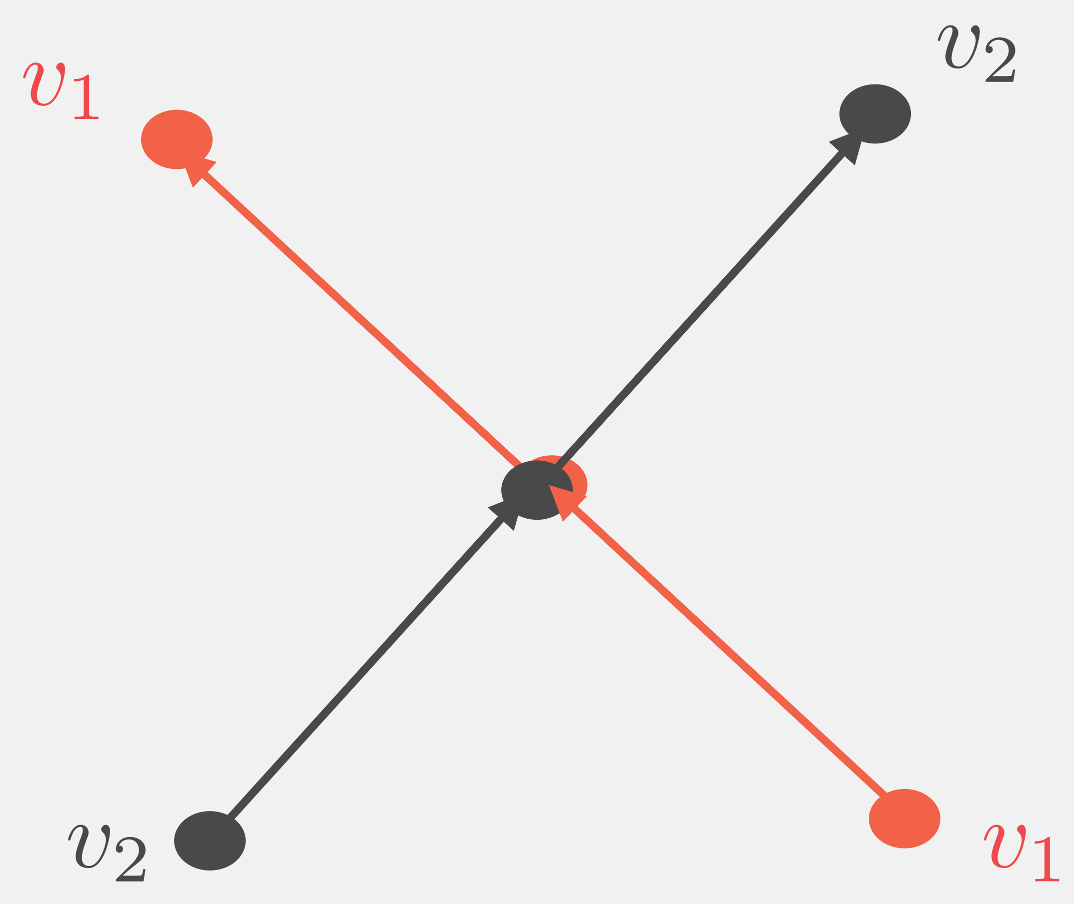
Exchange the labels instead of velocities

Dynamics is similar to hard point gas

Hard point gas ($a = 0$)

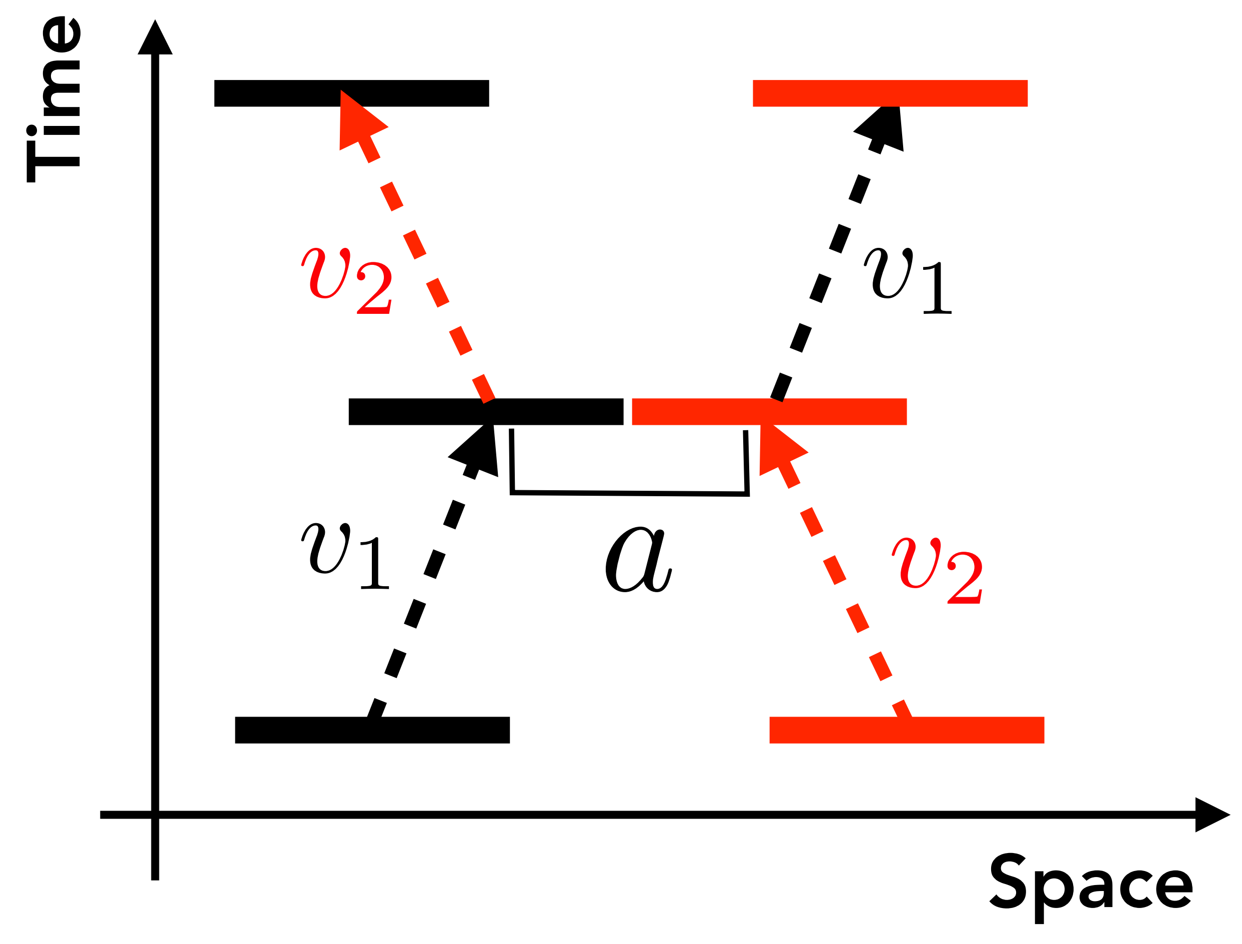


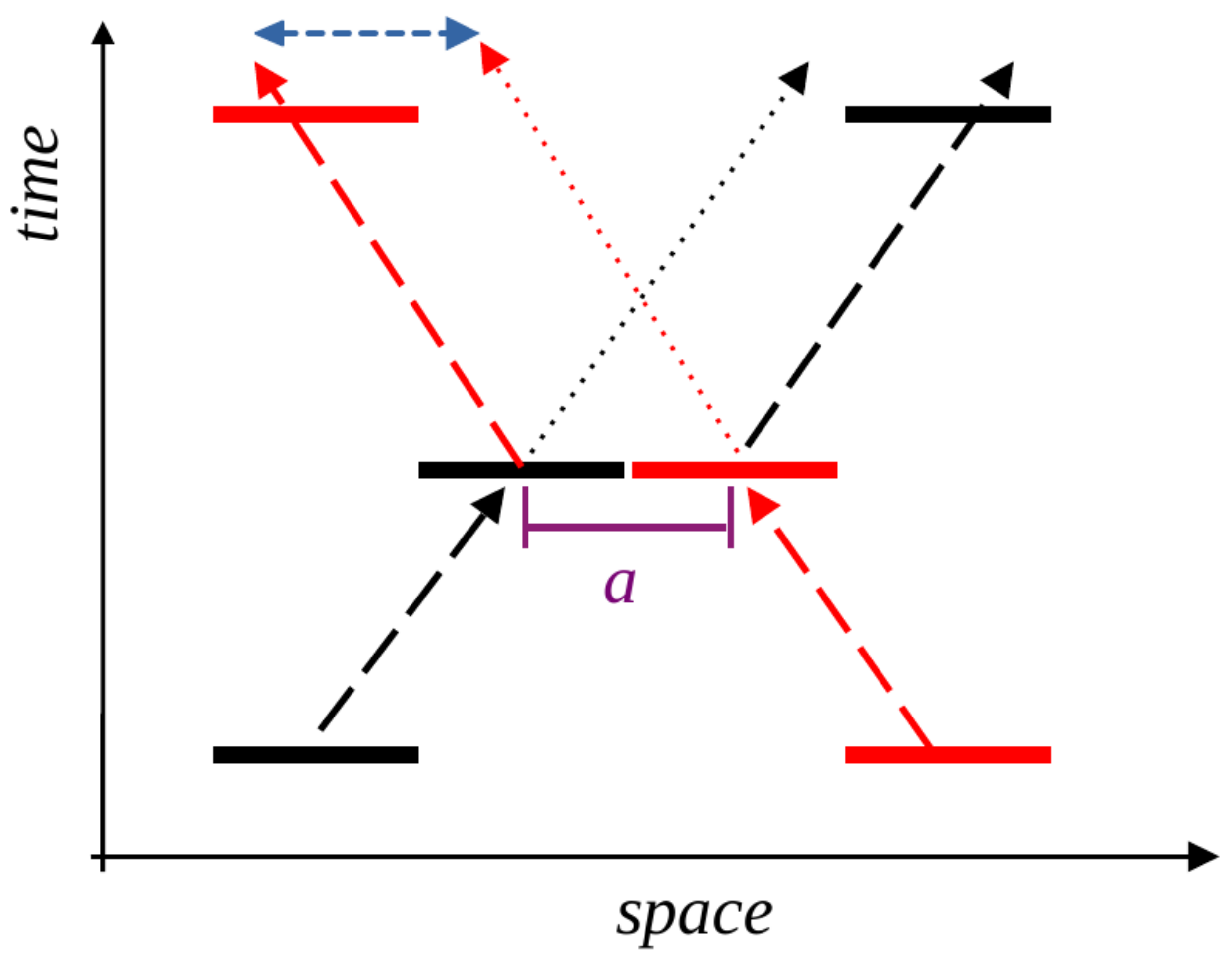
Non-interacting gas



Exchange the labels instead of velocities

Exchange velocities at collision

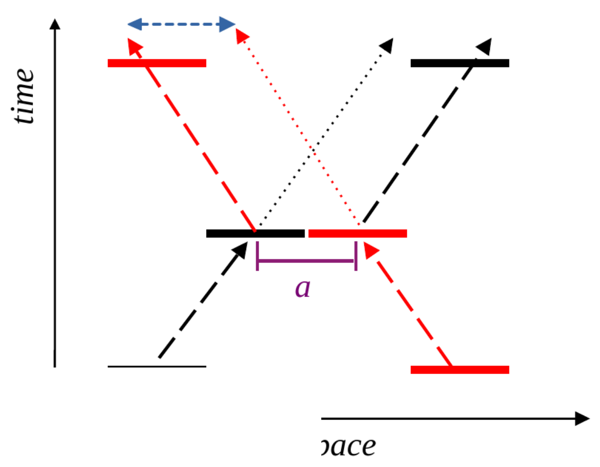




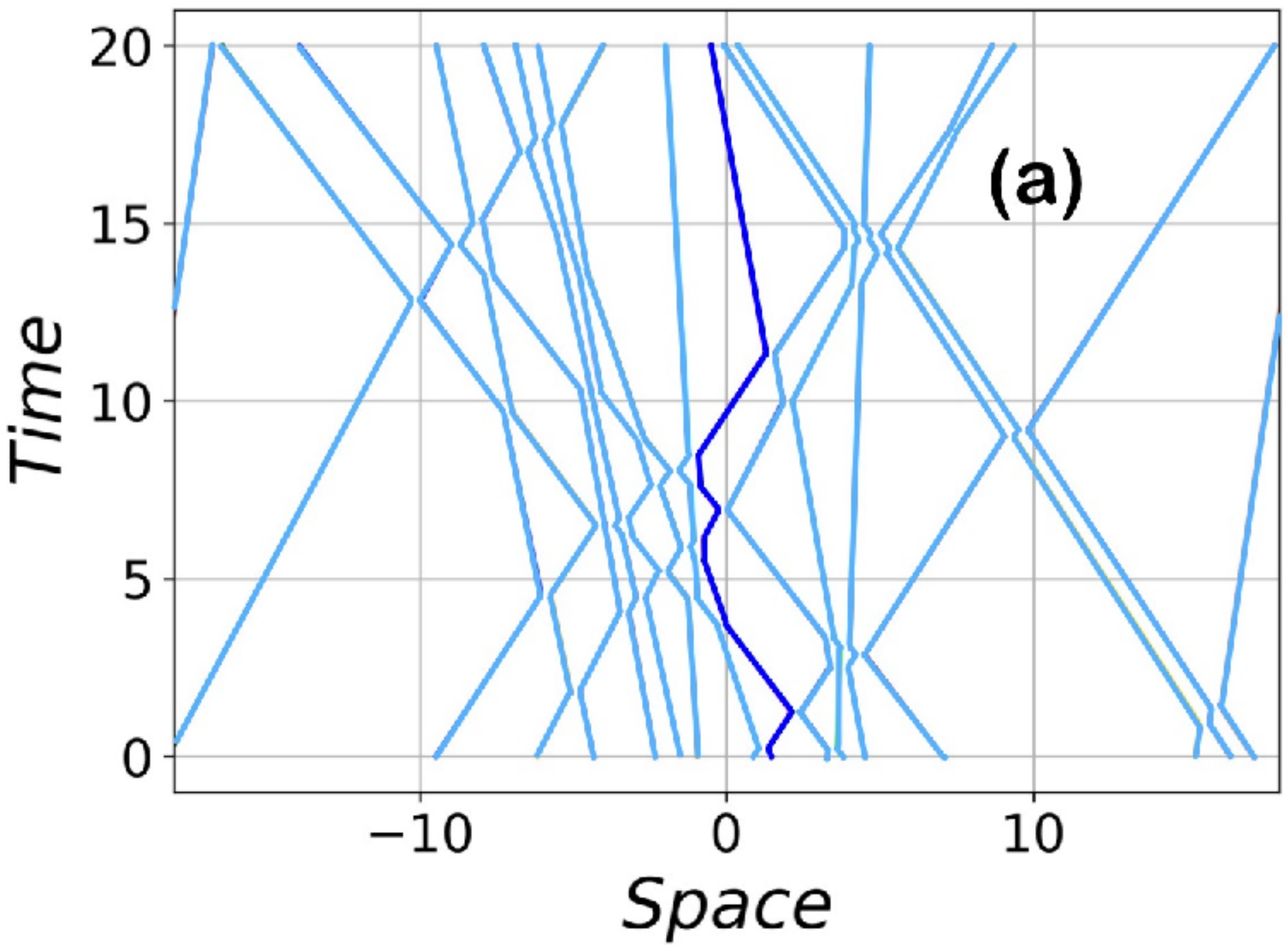
Exchange labels:
Does not become
Non-interacting !!

We get interacting Quasiparticles.

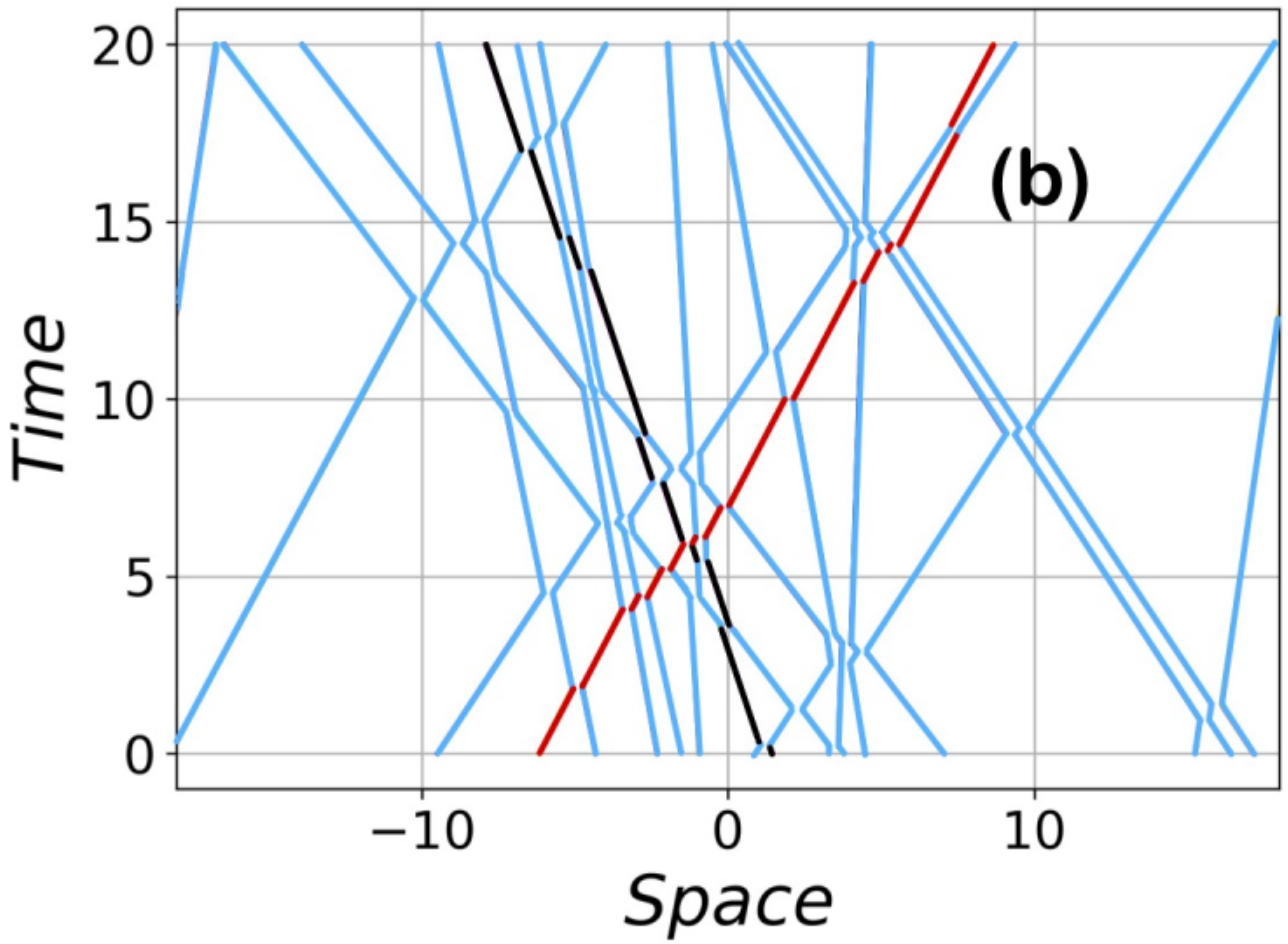
Quasiparticles



Actual rod trajectories



Quasiparticle trajectories



Gas of hard rods ≡ Gas of interacting quasiparticles

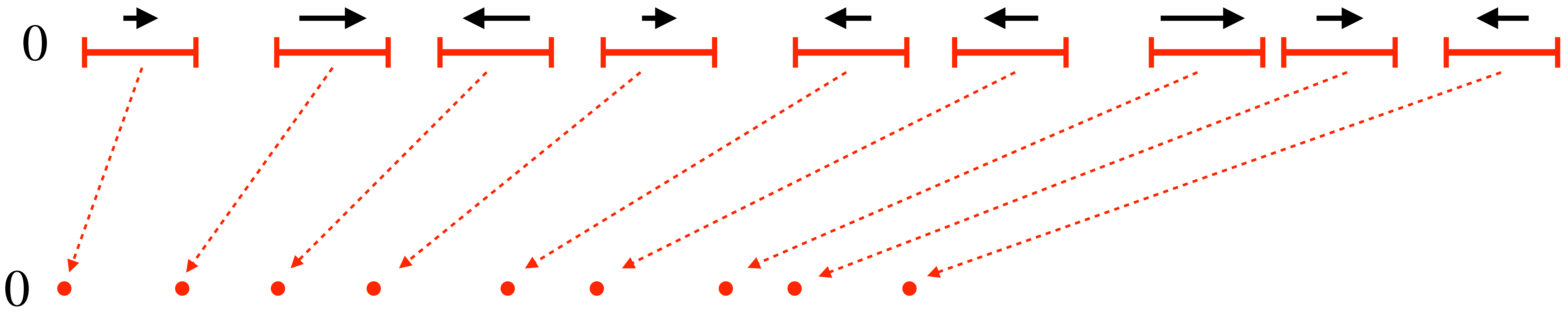
The quasiparticle picture will be useful for hydrodynamic description

Solving microscopic dynamics

Solving microscopic dynamics

One can map the dynamics of hard rod gas (HRG) to the dynamics of hard point gas (HPG)

$$\underbrace{\{x_i, v_i\}}_{\text{HRG config}^{\text{n.}}} \rightarrow \underbrace{\{x'_i, v_i\}}_{\text{HPG config}^{\text{n.}}}$$



point particle coordinates

$$x'_i = x_i - (i - 1/2)a = x_i - a \sum_{j \neq i} \Theta(x_i - x_j)$$

$$v'_i = v_i$$

Lebowitz, Percus, Sykes Phys. Rev. (1969)

Trajectories of hard rods and mapping to hard point gas

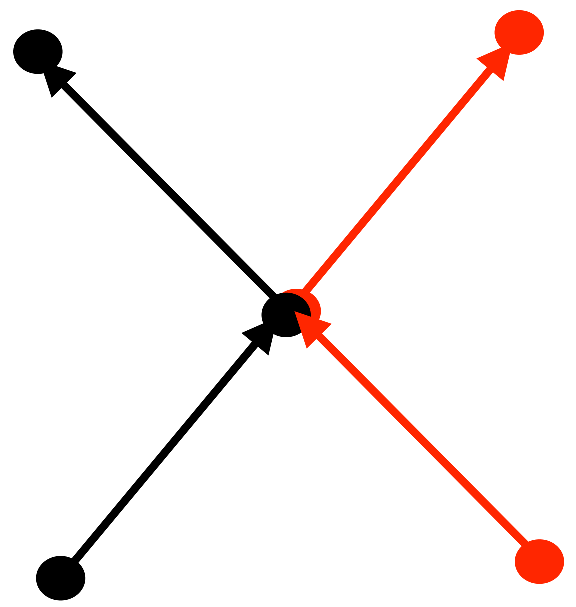
Initial HR coordinates	Transform	Initial HP coordinates	Evolve Point particles	Final HP coordinates	Transform Back	Final HR coordinates

$$\{x_i(0), v_i(0)\} \xrightarrow[x_i'=x_i-(i-1)a]{} \{x_i'(0), v_i\} \xrightarrow[x_i'(t)=x_i'(0)+v_i t]{} \{x_i(t), v_i(t)\} \xrightarrow[x_i=x_i'+(i-1)a]{} \{x_i'(t), v_i\}$$

Hydrodynamic evolution of hard rod gas?

(Euler) Generalised hydrodynamics (GHD)

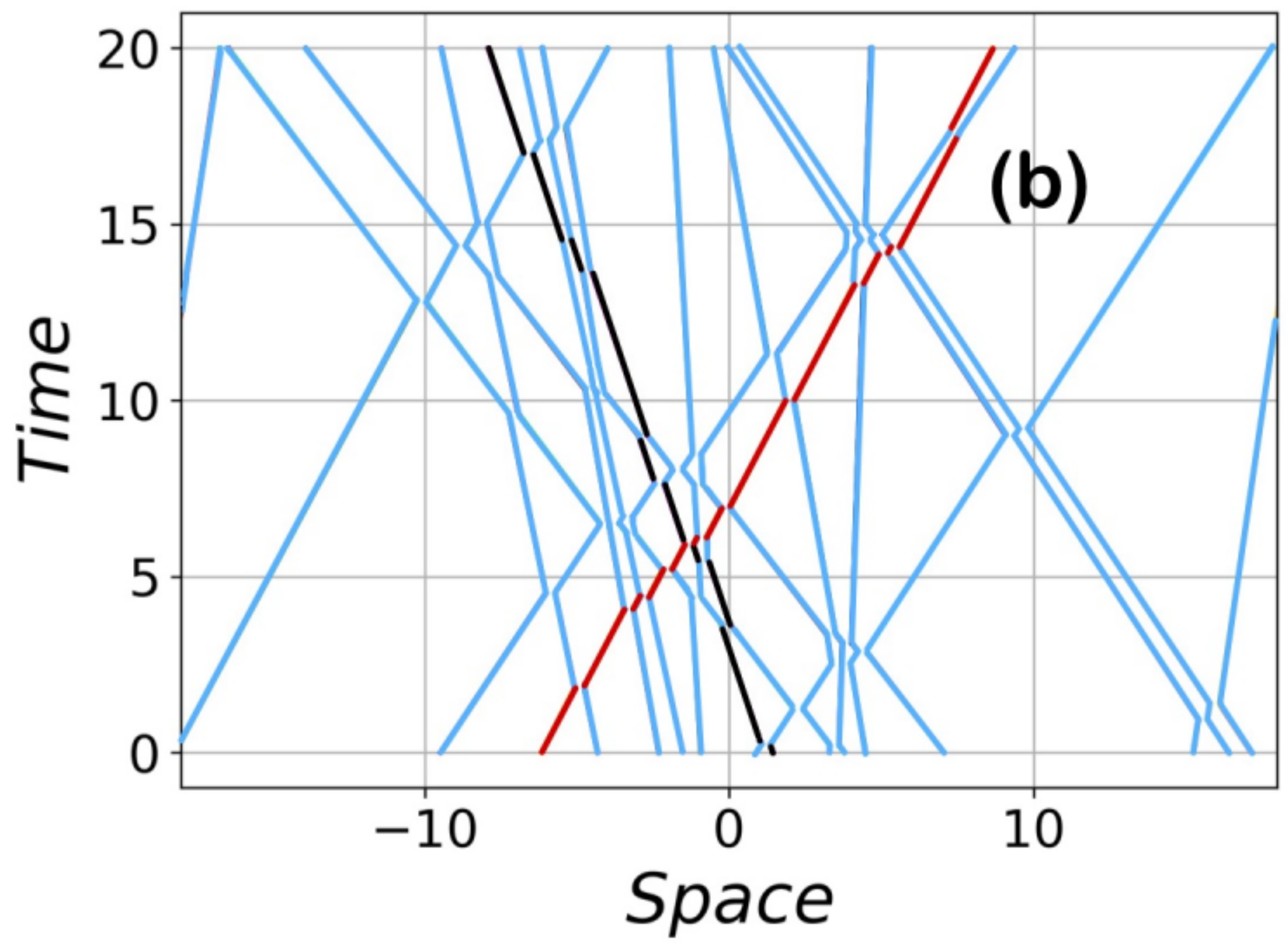
Hard point gas (a=0)



$f^0(v, x', t) =$ Phase space density of point particles

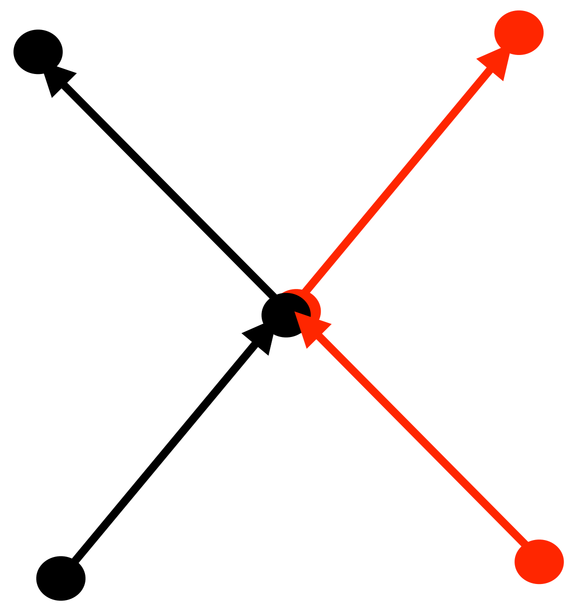
$$\partial_t f^0(v, x', t) + \partial_{x'} \left(v \; f^0(v, x', t) \right) = 0$$

For hard rods the GHD can be written in terms of phase space density of quasiparticles.



(Euler) Generalised hydrodynamics

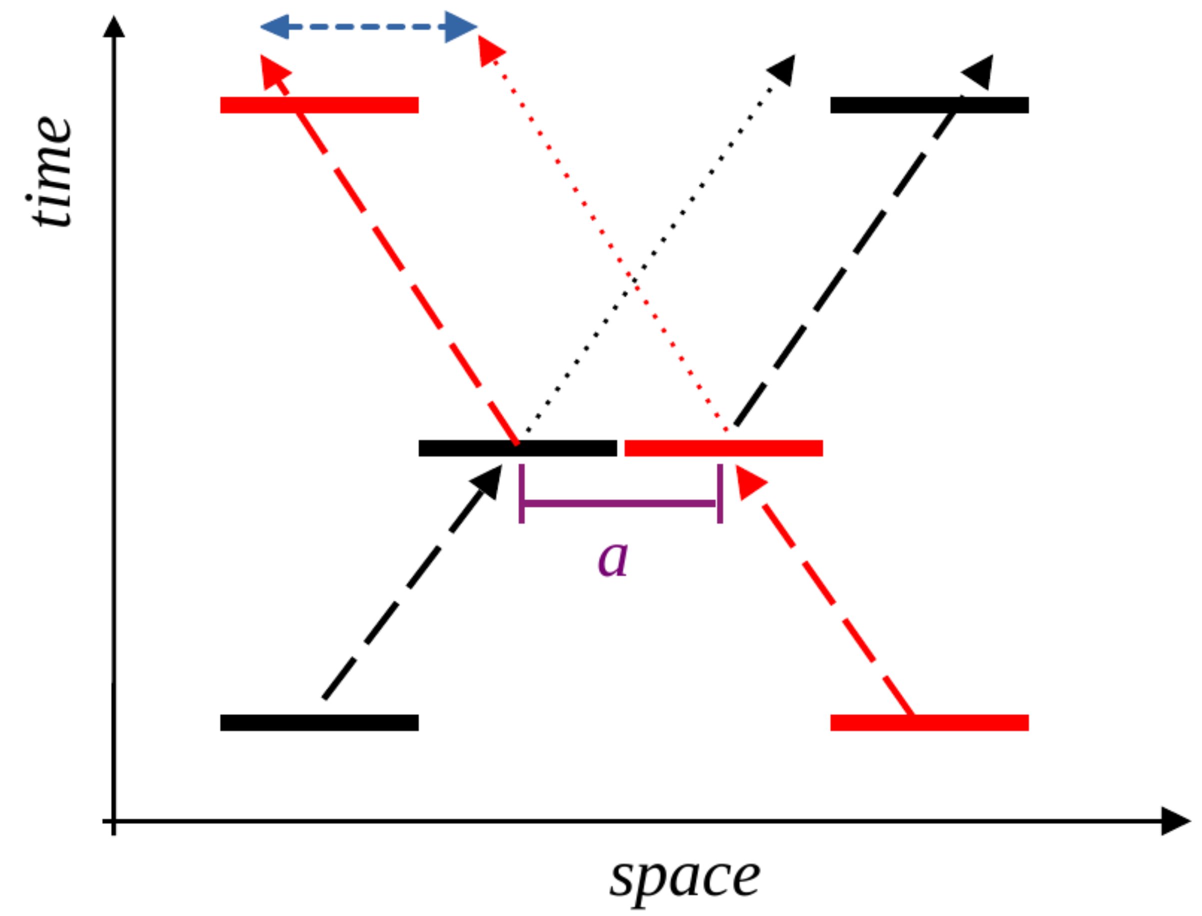
Hard point gas



$$f^0(v,x',t) = \text{Phase space density of point particles}$$

$$\partial_t f^0(v,x',t) + \partial_{x'}\left(v \; f^0(v,x',t)\right) = 0$$

Hard rod gas



$$f(x,v,t) = \text{Phase space density of quasi-particles}$$

$$\partial_t f(v,x,t) + \partial_x \left(v_{\text{eff}}(v) \; f(v,x,t)\right) = 0$$

$$v_{\text{eff}}(v) = \frac{v - a\rho u}{1 - a\rho}, \;\; \text{where}$$

Mass density: $\rho(x,t) = \int f(x,v,t)dv,$

Flow velocity: $u(x,t) = \frac{1}{\rho}\int v \; f(x,v,t)dv.$

Doyon, Lecture notes (2019)

Doyon & Spohn, JStat Mech (2017)

How to solve the generalised hydrodynamic equation?

Mapping to HPG in continuous variables

Recall: $x_i = x'_i + a \sum_{j \neq i} \Theta(x'_i - x'_j)$

$$x = x' + aF^0(x', t)$$

$$f(x, v, t) = \frac{f^0(x', v, t)}{1 + a\rho^0(x', t)},$$

$$\rho^0(x', t) = \int dv \; f^0(x', v, t),$$

$$F^0(x', t) = \int_0^x dx' \; \rho^0(x', t),$$

Recall microscopic solution:

Initial HR coordinates	Transform	Initial HP coordinates	Evolve Point particles	Final HP coordinates	Transform Back	Final HR coordinates

$$\{x_i(0), v_i(0)\} \xrightarrow[x'_i = x_i - (i-1)a]{} \{x'_i(0), v_i\} \xrightarrow[x'_i(t) = x'_i(0) + v_i t]{} \{x_i(t), v_i(t)\} \xrightarrow[x_i = x'_i + (i-1)a]{} \{x'_i(t), v_i\}$$

Hard rod hydrodynamics → hard point particle Hydrodynamics

$$x = x' + aF^0(x', t)$$

$$f^0(v, x', t) = \text{Phase space density of point particles}$$
$$f(v, x, t) = \text{Phase space density of hard rods}$$

$$f(x, v, t) = \frac{f^0(x', v, t)}{1 + a\rho^0(x', t)},$$

$$\rho^0(x', t) = \int dv \; f^0(x', v, t),$$

$$F^0(x', t) = \int_0^x dx' \; \rho^0(x', t),$$

Euler GHD of hard rods:

$$\partial_t f(v, x, t) + \partial_x \left(v_{\text{eff}}(v) \; f(v, x, t)\right) = 0$$

Euler GHD of hard point particles:

$$\partial_t f^0(v, x', t) + \partial_{x'} \left(\; v \; f^0(v, x', t)\right) = 0$$

Solve

$$f^0(x', v, t)$$

Transform back to Hard rods

$$f(x, v, t)$$

Solution of the Euler GHD equations of hard rod gas

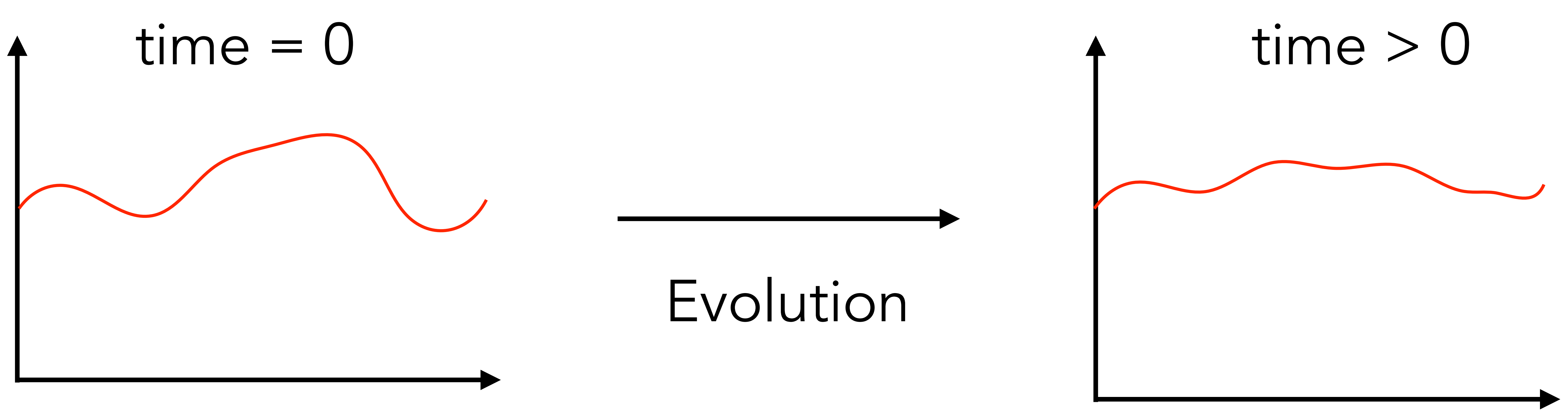
$$f_{\text{euler}}(x(x'), v, t) = \frac{f^0(x', v, t)}{1 + a\rho^0(x', t)}$$

$$x(x') = x + aF^0(x', t)$$

Thus the mapping to HPG allows one to solve the E-GHD equations

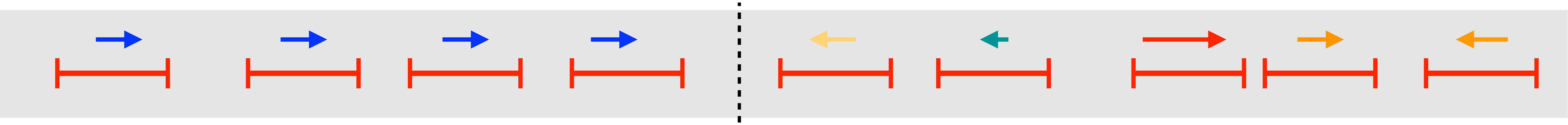
and

study non-equilibrium evolution



An example

Uniformly filled box with inhomogeneous velocity distribution



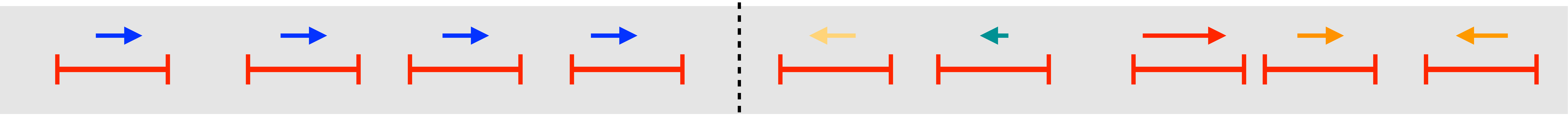
Rods have velocity 1

(Special component)

Velocities are chosen from Maxwell distribution

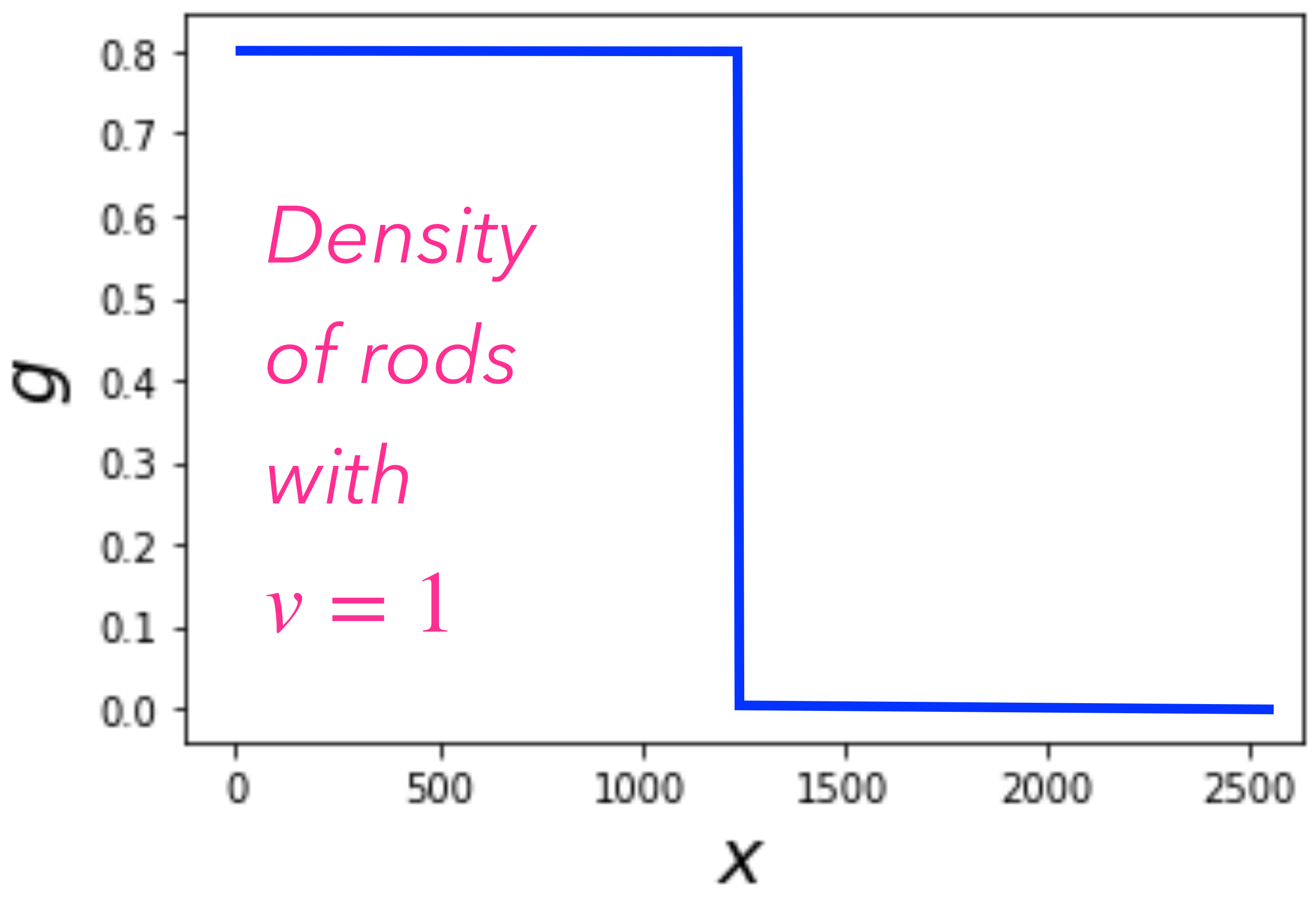
(Background component)

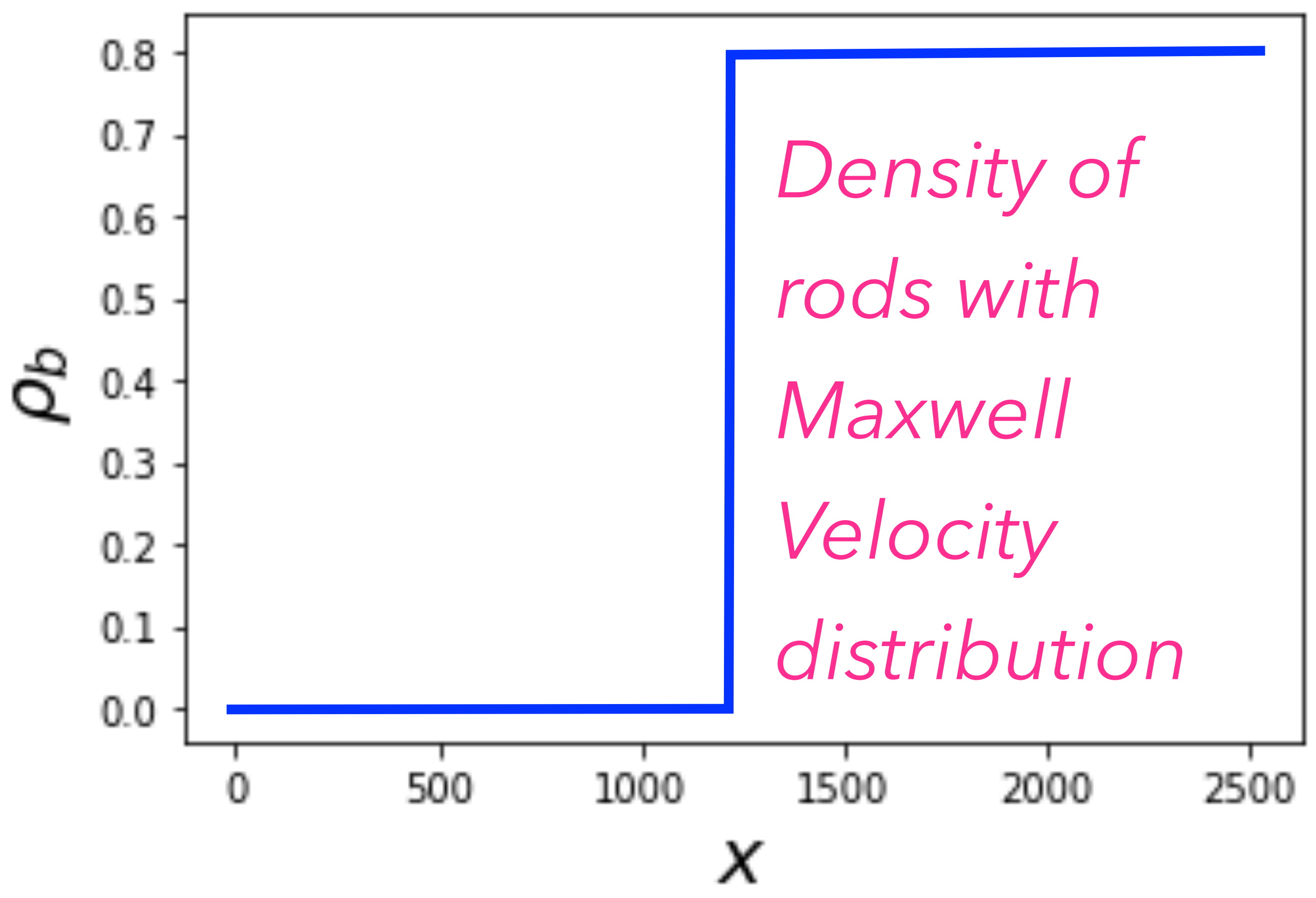
Uniformly filled box with inhomogeneous velocity distribution



Rods have velocity 1

Velocities are chosen from Maxwell distribution





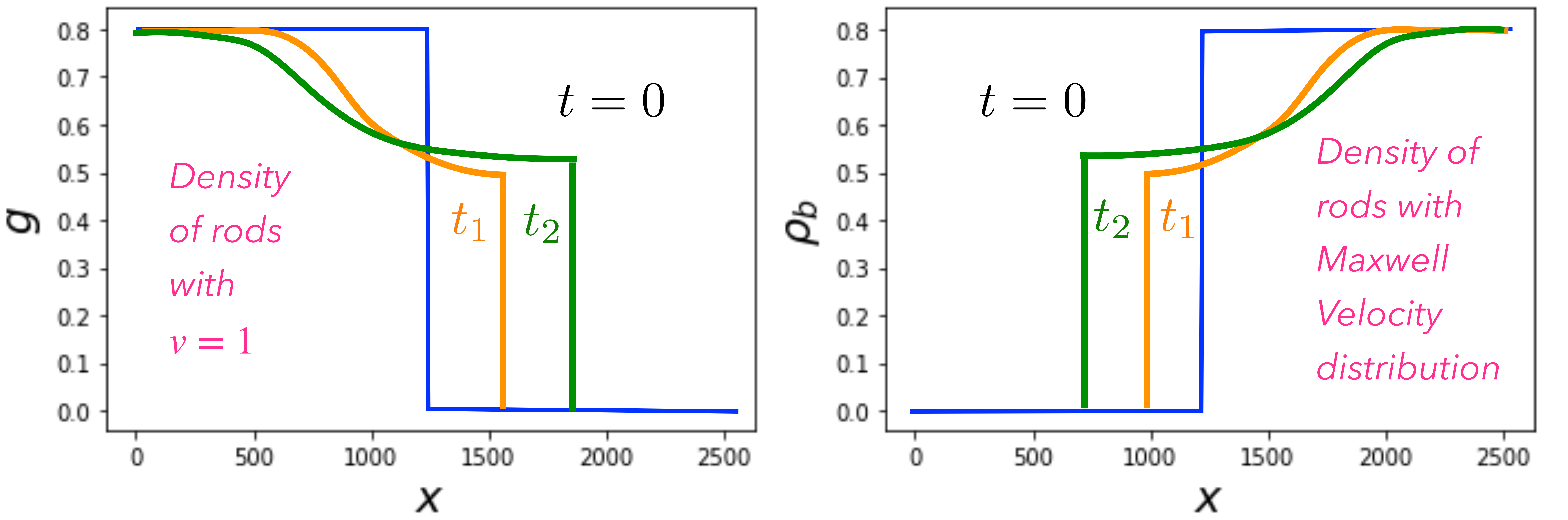
$$f(x,v,0) = \underbrace{\rho\ \delta(v-1)\ \theta(L/2-x)}_{\text{special rods}} + \underbrace{\rho\ p_{\rm mx}(v)\ \theta(x-L/2)}_{\text{background rods}}$$

How do the density profiles of different components evolve?

$$f(x,v,0) = \underbrace{\rho\ \delta(v-1)\ \theta(L/2-x)}_{\text{special rods}} + \underbrace{\rho\ p_{\mathrm{mx}}(v)\ \theta(x-L/2)}_{\text{background rods}}$$

$$\downarrow$$

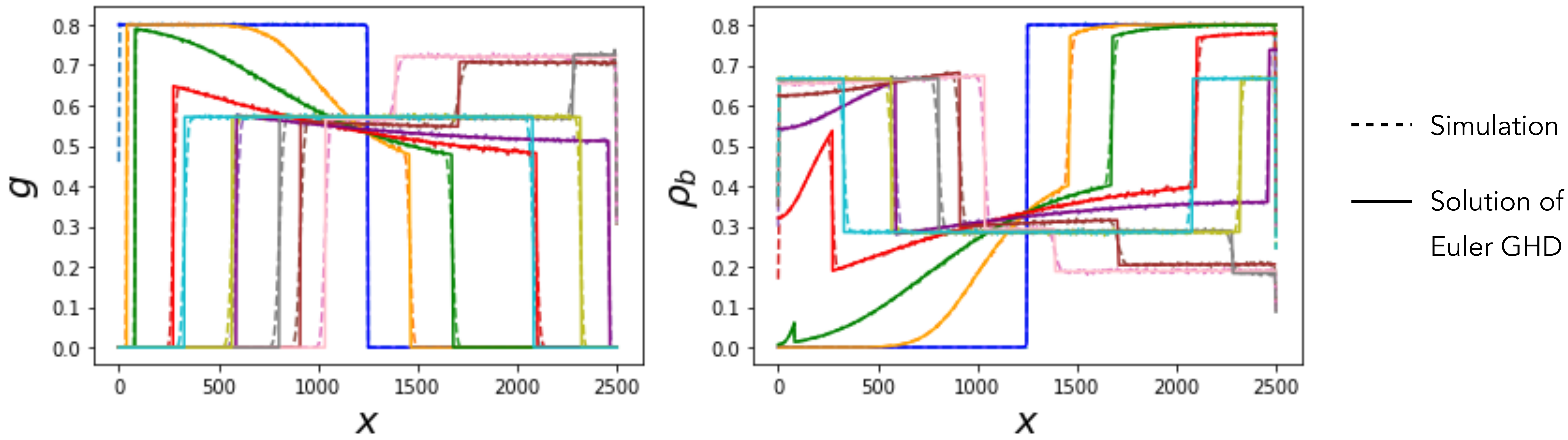
$$f(x,v,t) = \underbrace{g(x,t)\ \delta(v-1)}_{\text{special rods}} + \underbrace{f_b(x,v,t)}_{\text{background rods}}$$



Singh, Dhar, Spohn, **A.K.** Jstat. Phys. (2024) 191:66

Comparison of simulation result with GHD solution

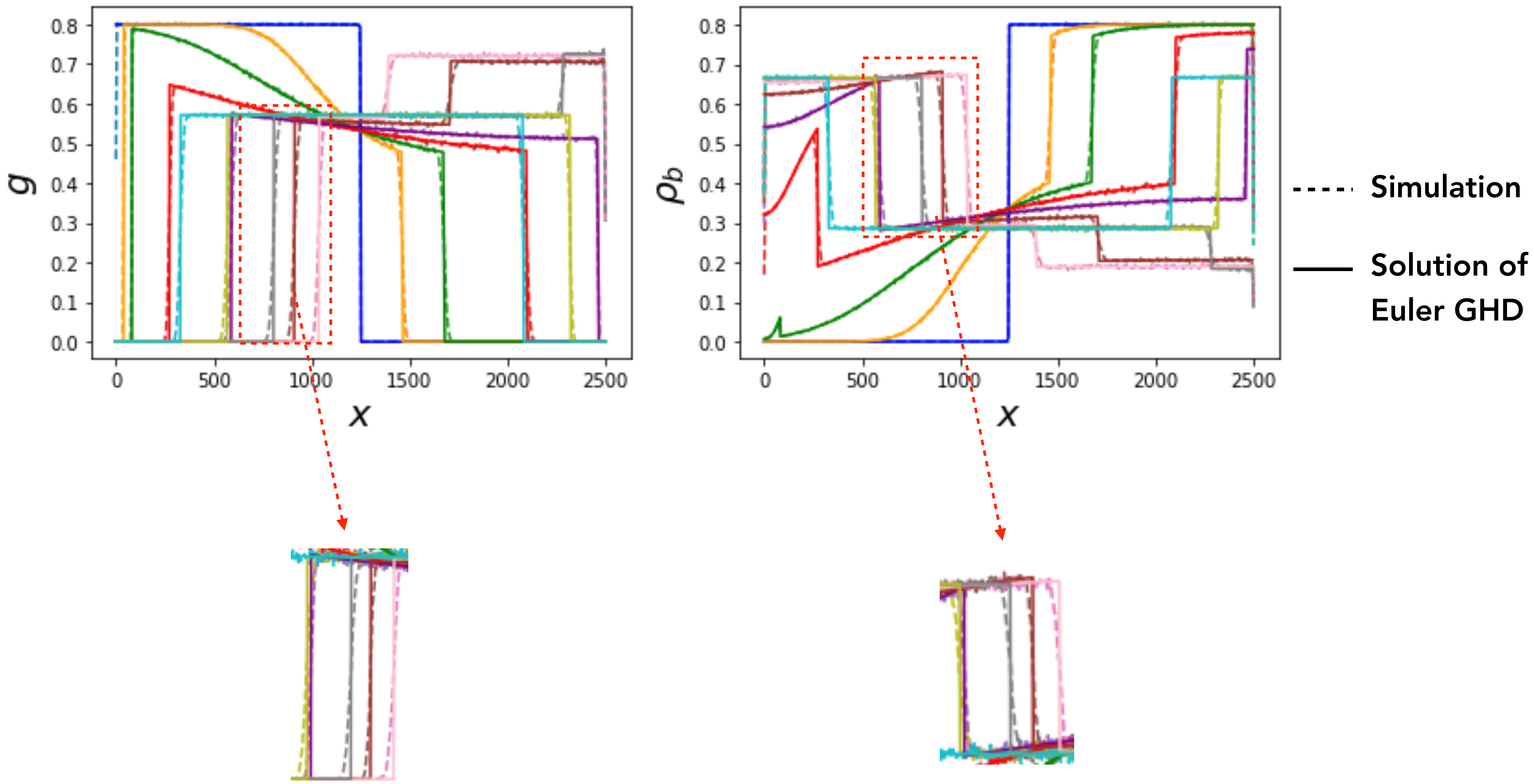
$$f(x,v,t) = \underbrace{g(x,t)\ \delta(v-1)}_{\text{special rods}} + \underbrace{f_b(x,v,t)}_{\text{background rods}}$$



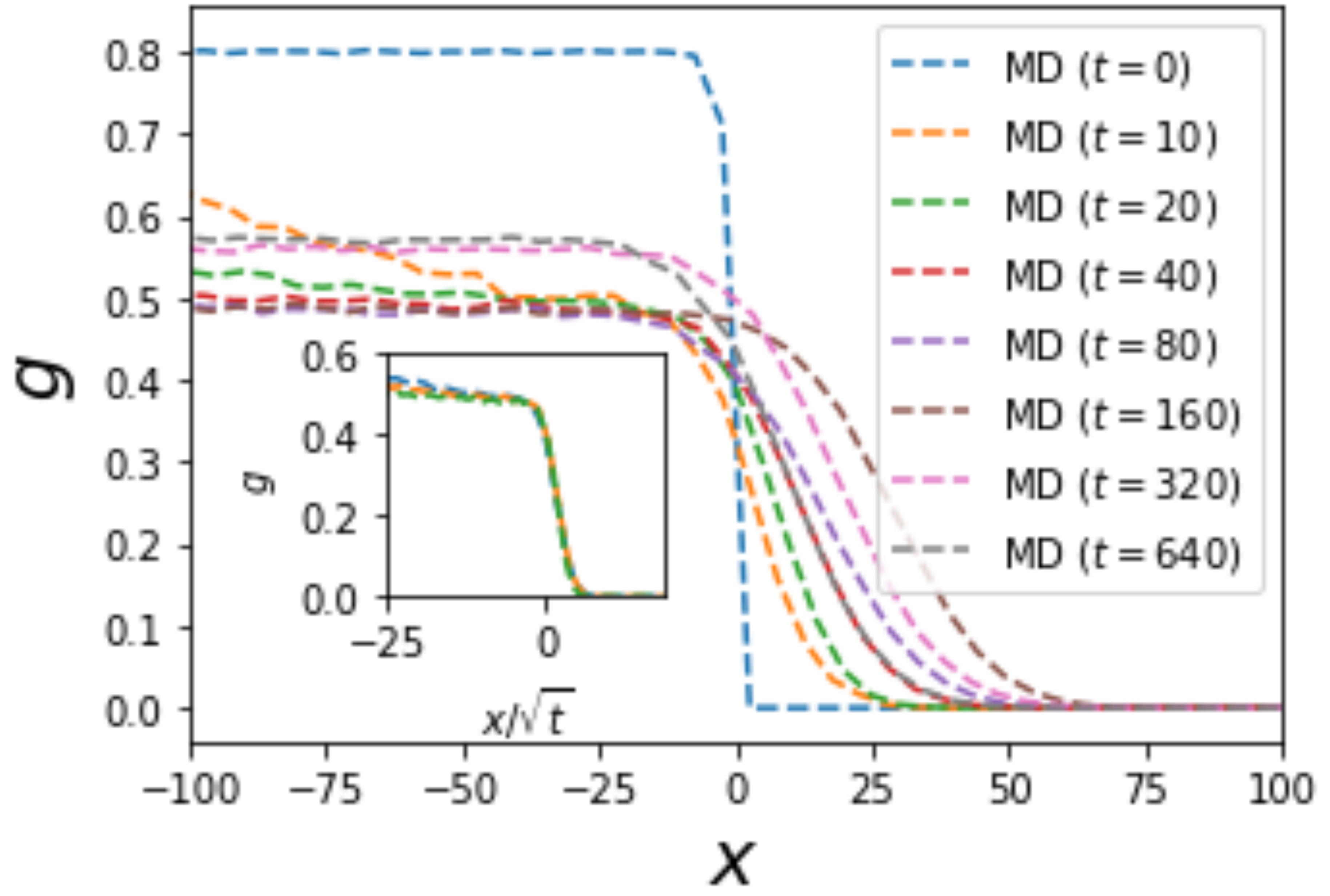
Solution of the Euler GHD equation matches quite well.

Singh, Dhar, Spohn, **A.K.** Jstat. Phys. (2024) 191:66

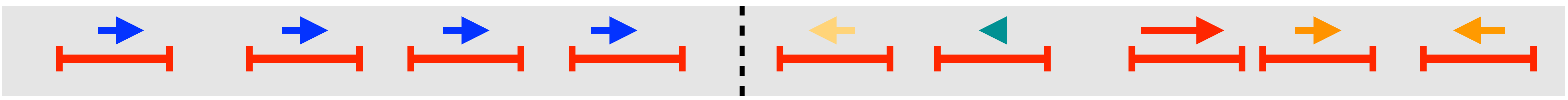
Disagreement with the Euler solution at the discontinuities

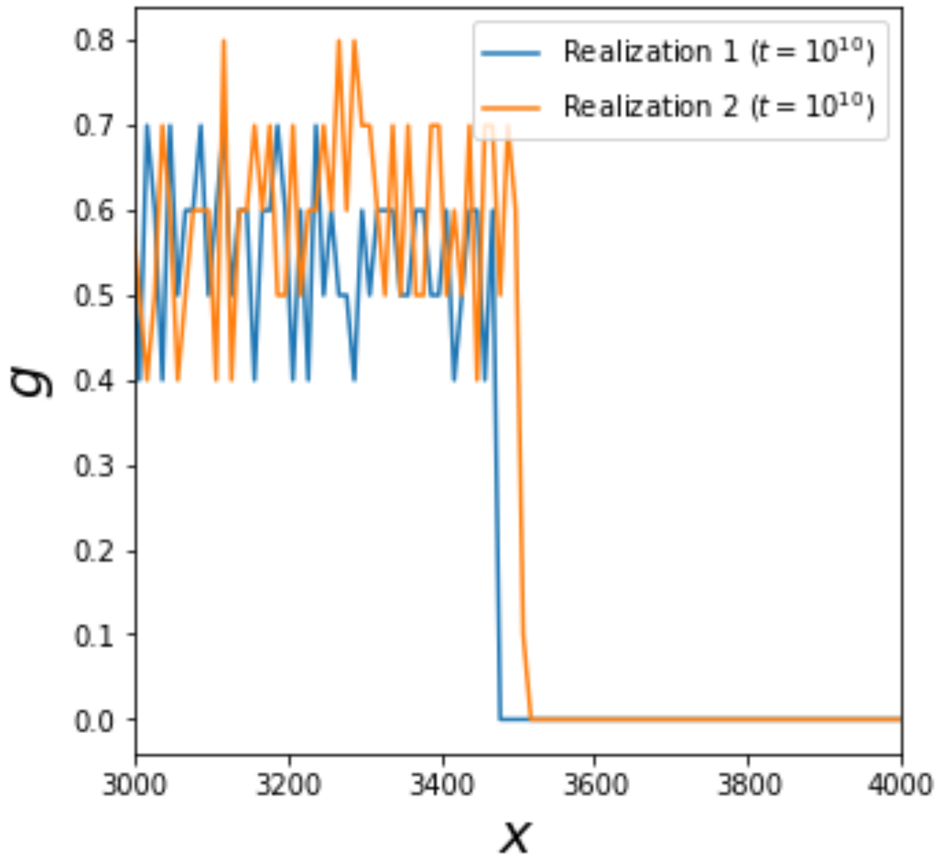


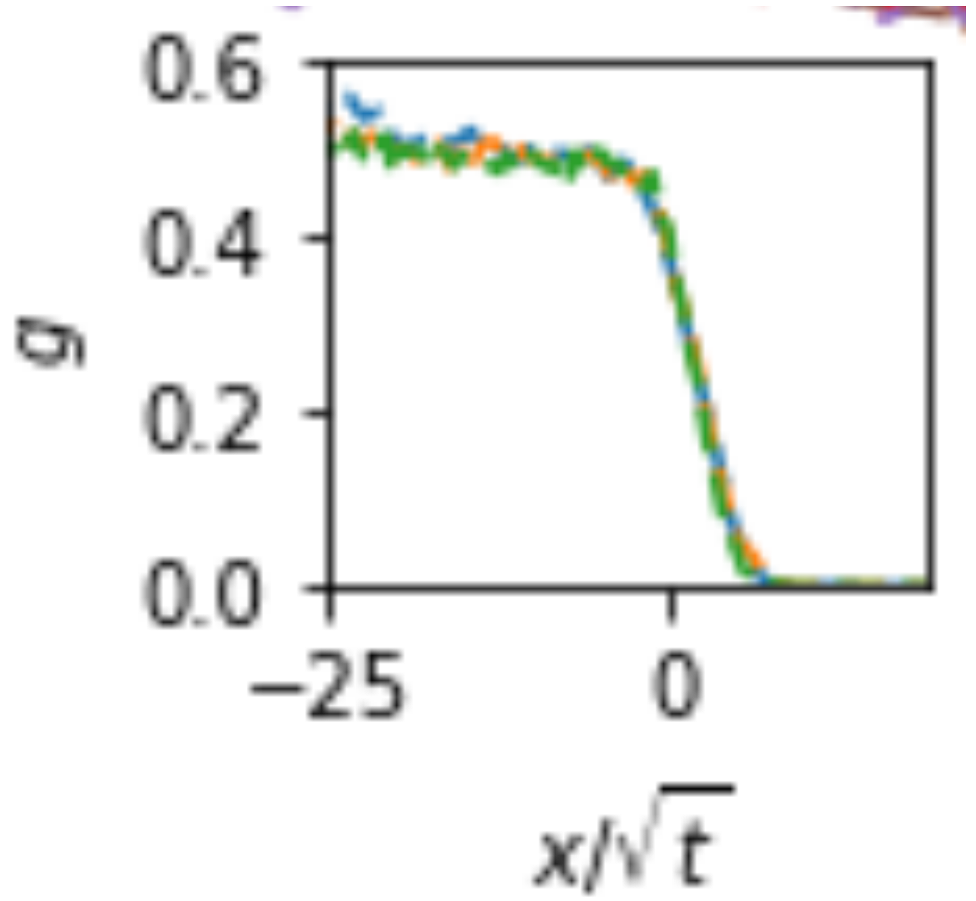
Initial discontinuity spreads with time



Recall initial condition:







$\xi(t)$ width of the shock $\propto$ fluctuation in the # of background particles to the left of the shock

$\propto \sqrt{\text{Mean \# of such particles}}$

$\propto \sqrt{t}$

How to obtain the spread theoretically

– Navier-Stokes correction to E-GHD

– Microscopic solution

Microscopic approach

For certain type of initial conditions one can perform exact microscopic calculations.

in which,

we choose the initial condition for hard point particles
and then convert to hard rod configurations

Initial condition

– *We first choose N locations randomly and independently from $p(\hat{x}')$.*

Initial condition

– *We first choose* N *locations randomly and independently from* $p(\hat{x}')$*.*

Initial condition

– *We first choose* N *locations randomly and independently from* $p(\hat{x}')$.

Initial condition

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Initial condition

– *We first choose N locations randomly and independently from $p(\hat{x}')$.*

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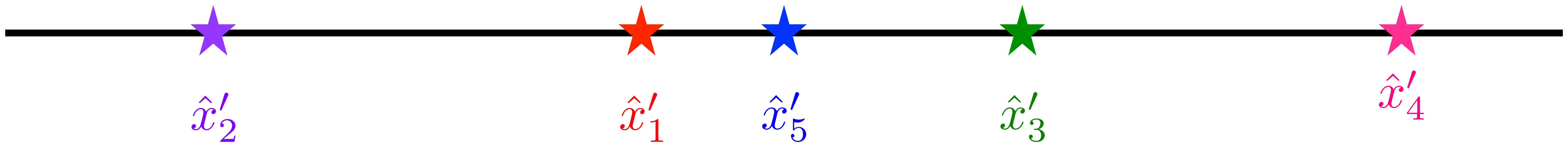
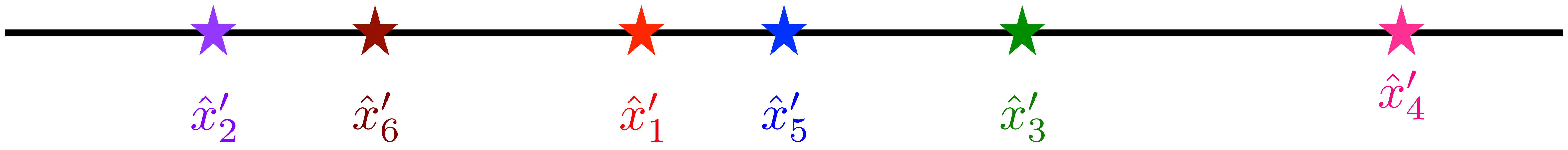


Initial condition

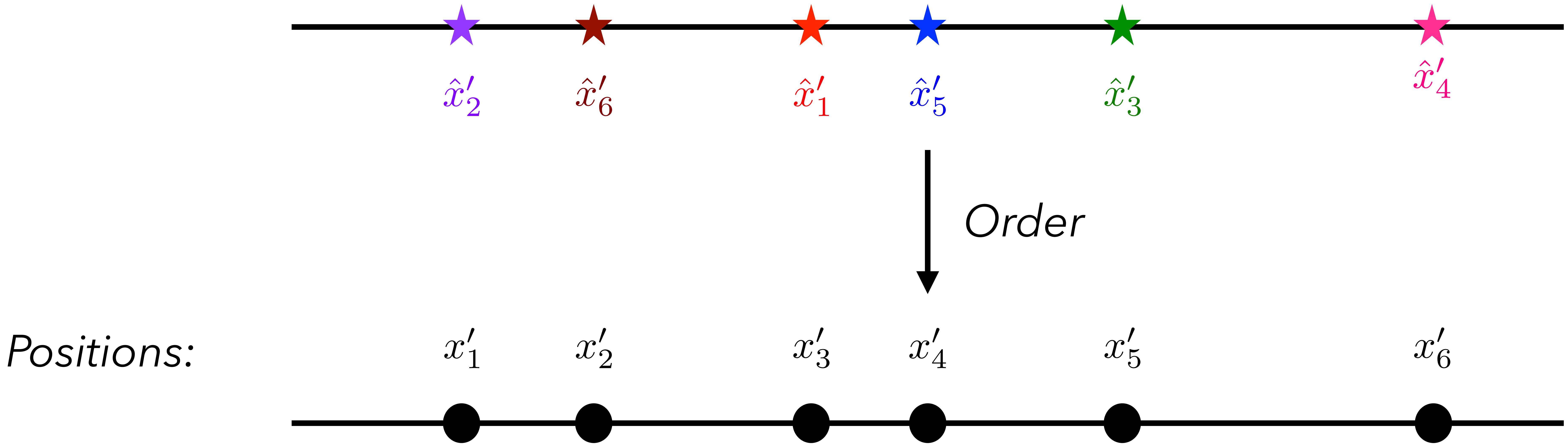
– *We first choose N locations randomly and independently from $p(\hat{x}')$.*



Initial condition

– *We first choose N locations randomly and independently from $p(\hat{x}')$.*

– *We arrange these locations in increasing order and place point particles there.*

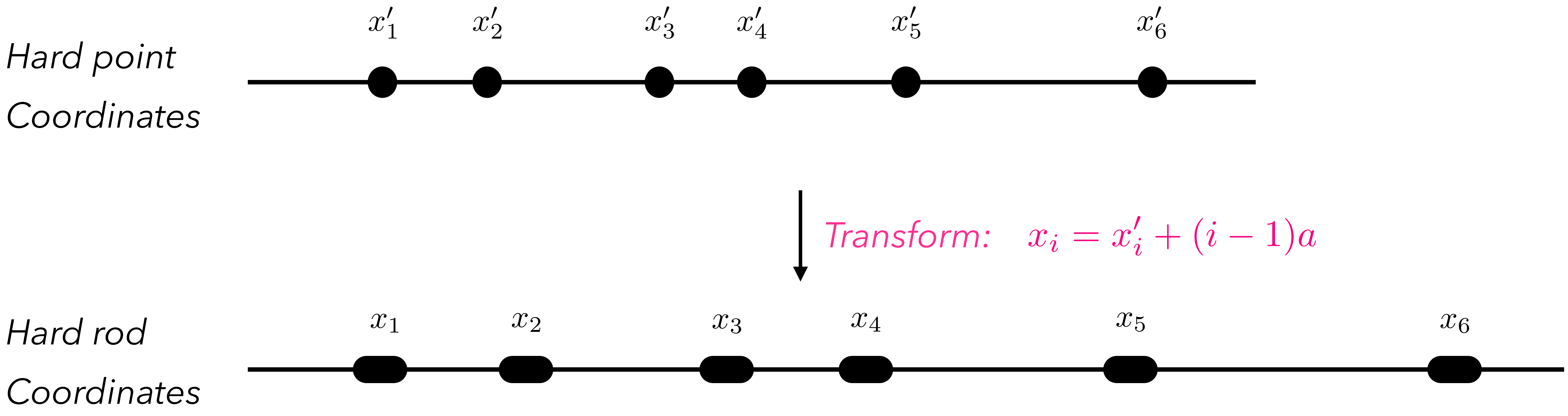


Initial condition

– *We first choose* N *locations randomly and independently from* $p(\hat{x}')$.

– *We arrange these locations in increasing order and place point particles there.*

– *Hard rod coordinates are obtained using the transformation* $x_i = x_i' + a \sum_{j \neq i} \Theta(x_i' - x_j')$



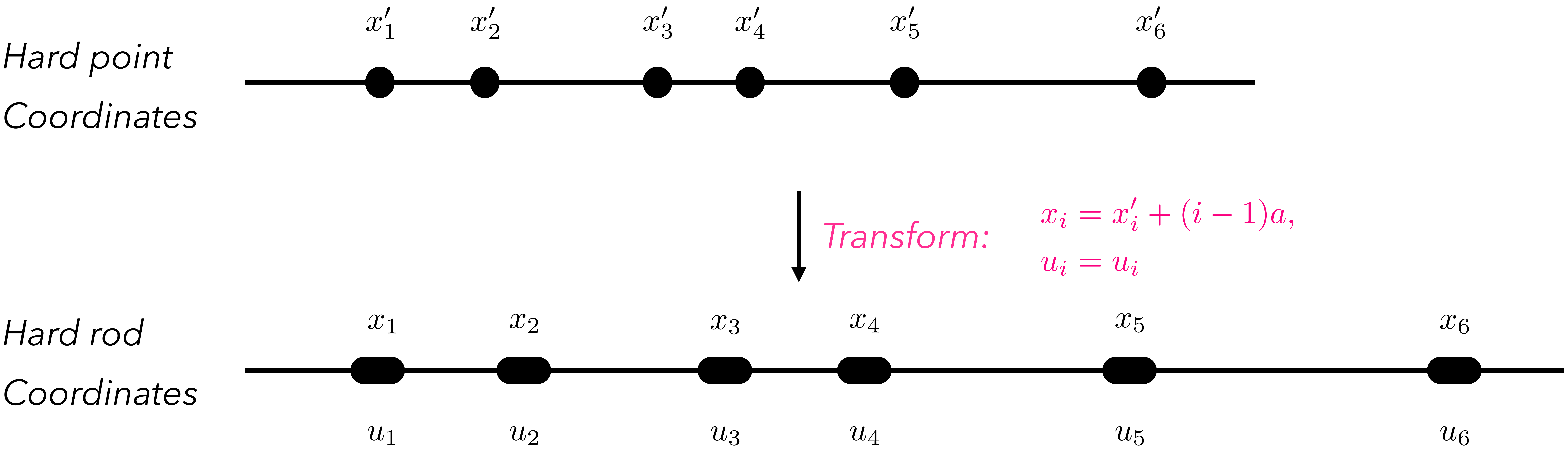
Initial condition

– *We first choose* N *locations randomly and independently from* $p(\hat{x}')$.

– *We arrange these locations in increasing order and place point particles there.*

– *Hard rod coordinates are obtained using the transformation* $x_i = x_i' + a \sum_{j \neq i} \Theta(x_i' - x_j')$

– *Choose the velocities of the rods independently from a distribution* $h(u)$.



Exact microscopic solution

$$\rho_{\text{rod}}(x,t) = \langle \hat{\rho}_{\text{rod}}(x,t) \rangle = \left\langle \sum_i \delta(x - x_i(t)) \right\rangle$$

Using the mapping to hard point particles

$$\rho_{\text{rod}}(x,t) = \sum_{i=1}^{N} \mathsf{P}_i^{\text{point}}(x_i')$$

where $x_i' = [x - (i-1)a]$.

$$\mathsf{P}_i^{\text{point}}(z)dz = \text{Prob.} \left[\begin{array}{c} \text{that } i^{th} \text{ point particle is} \\ \text{within } [z, z+dz] \text{ at time } t \end{array} \right]$$

For i.i.d. initial conditions on point particles

$$\mathsf{P}_i^{\text{point}}(z) = N \binom{N-1}{i-1} q(z,t)^{i-1} \; p(z,t) \; (1 - q(z,t))^{N-i}, \quad 0 < z < L',$$

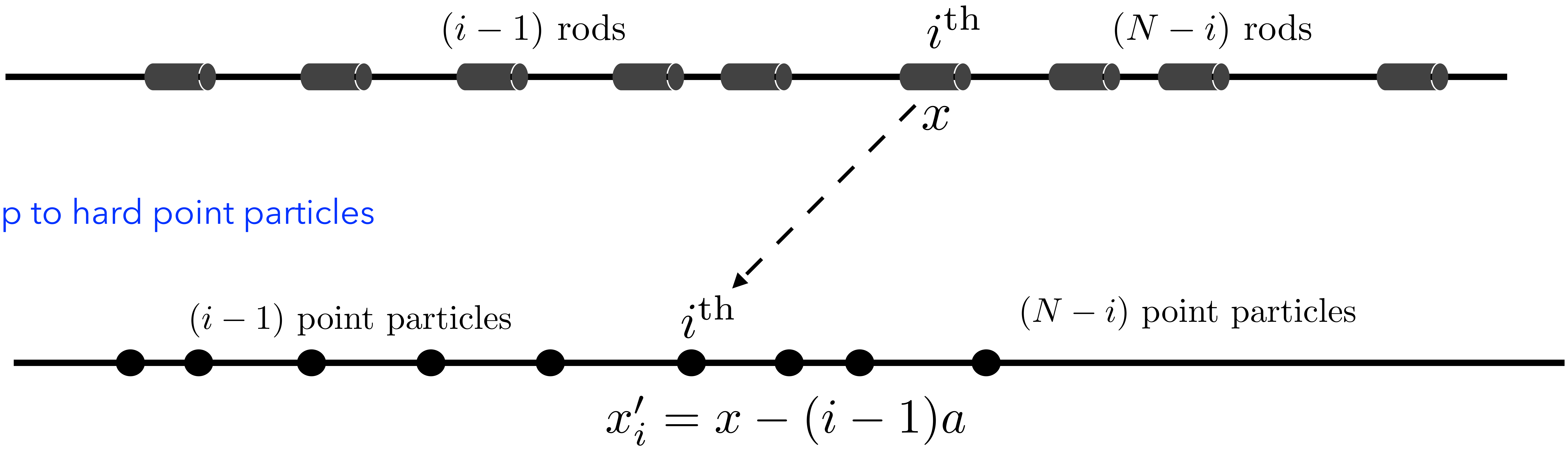
$$p(z,t)dz = \text{Prob. [that a point particle is within } [z, z+dz] \text{ at time } t]$$

$$q(y,t) = \int^{y} dz \; p(z,t).$$

Exact microscopic solution

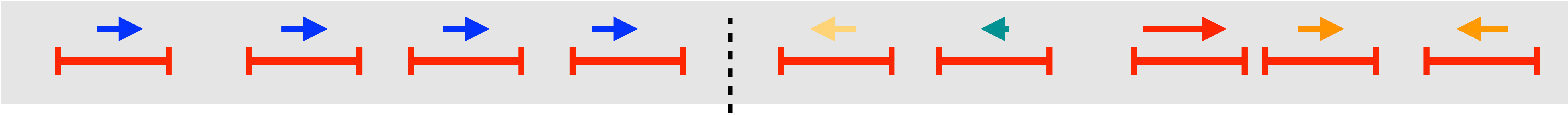
$$\rho_{\text{rod}}(x,t) = \langle \hat{\rho}_{\text{rod}}(x,t) \rangle = \left\langle \sum_i \delta(x - x_i(t)) \right\rangle$$

Map to hard point particles



$$\rho_{\text{rod}}(x,t) = \sum_{i=1}^{N} \mathrm{P}_i^{\text{point}}(x'_i) \qquad \mathrm{P}_i^{\text{point}}(z)dz = \text{Prob.} \left[\begin{array}{c} \text{that } i^{th} \text{ point particle is} \\ \text{within } [z, z+dz] \text{ at time } t \end{array} \right]$$

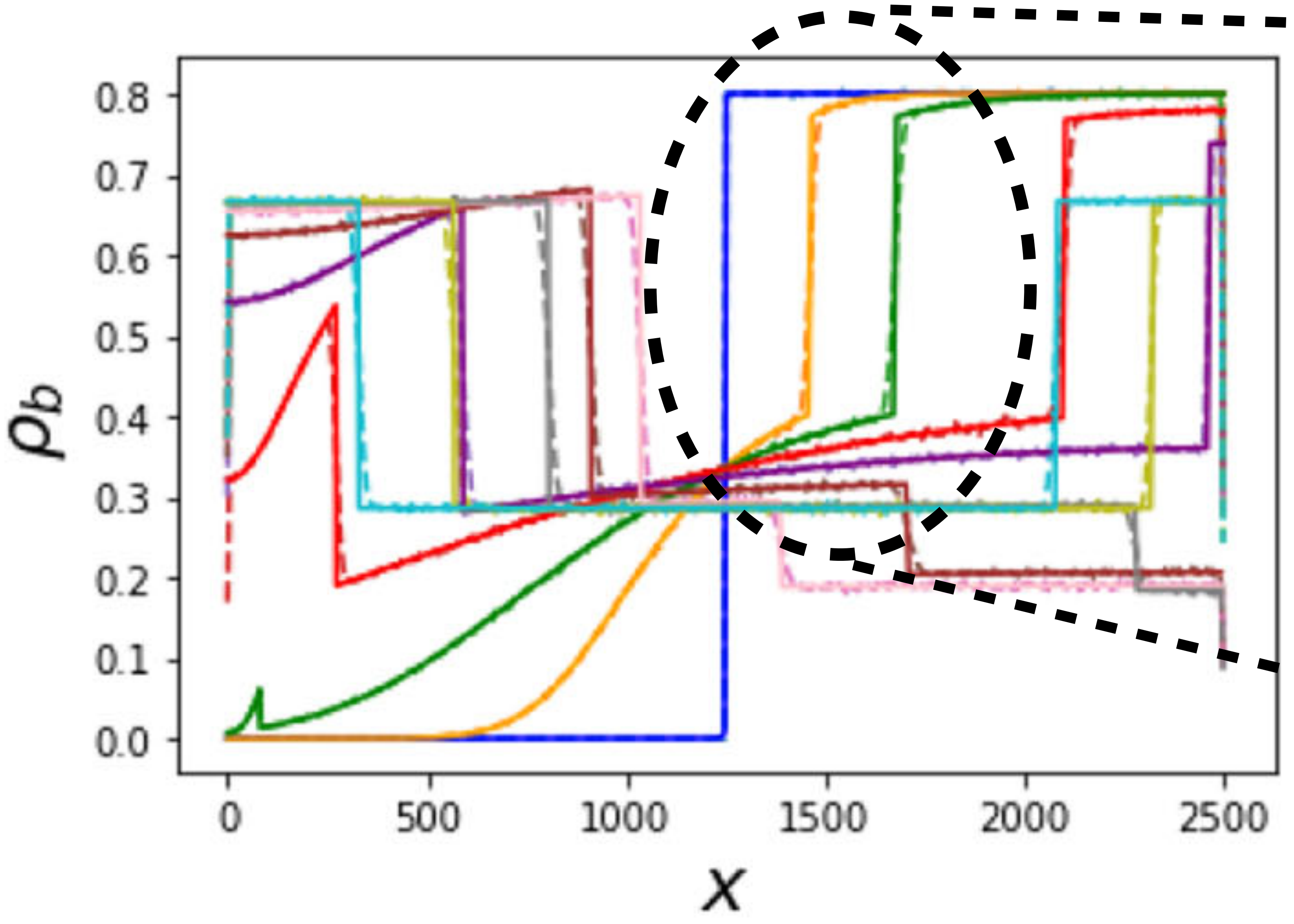
Microscopic solution to the velocity domain wall problem



Rods in the left half has velocity 1

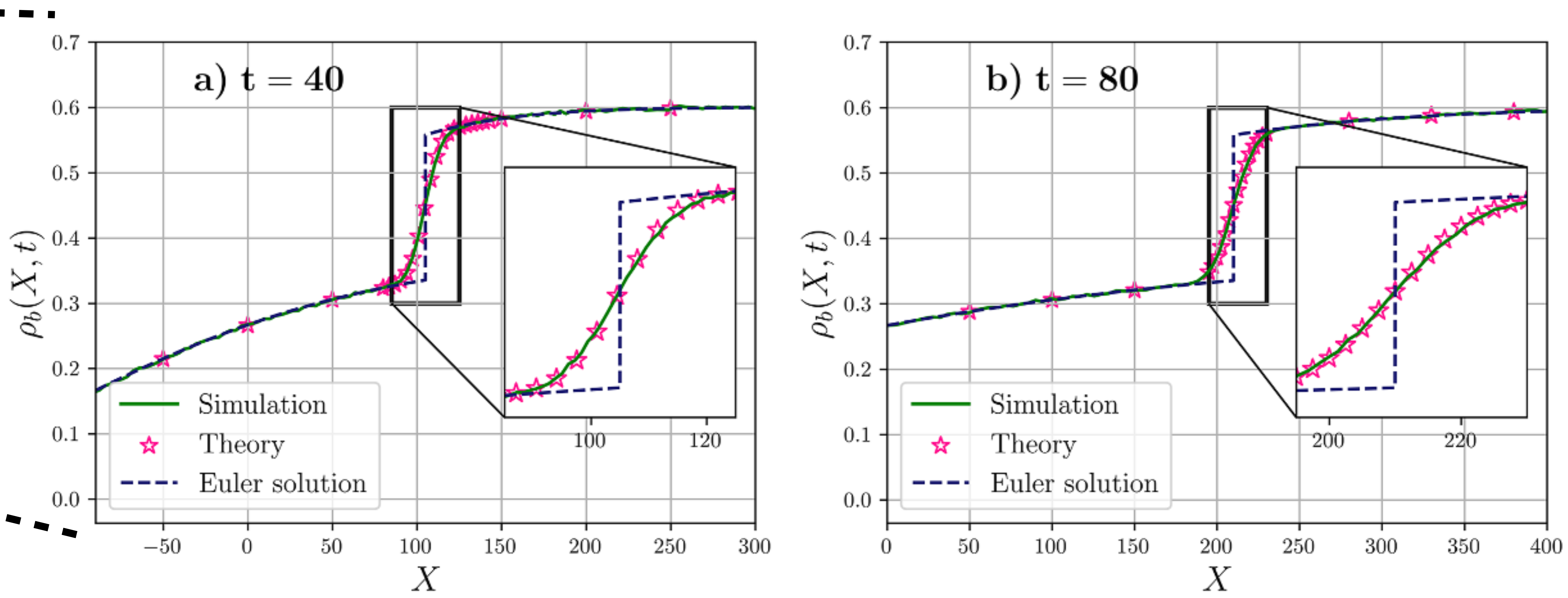
Velocities of the rods in the right half are chosen from Maxwell distribution

Recall: Comparison only with Euler soln.



Singh, Dhar, Spohn, **A.K.** Jstat. Phys. (2024) 191:66

Comparison with microscopic solution



M. J. Powdel & **A. K.**, J. Stat. Mech. (2024) 123205

Large N asymptotic of the microscopic solution $\rightarrow$ correction to Euler solution

In the large N limit with small a keeping Na finite,

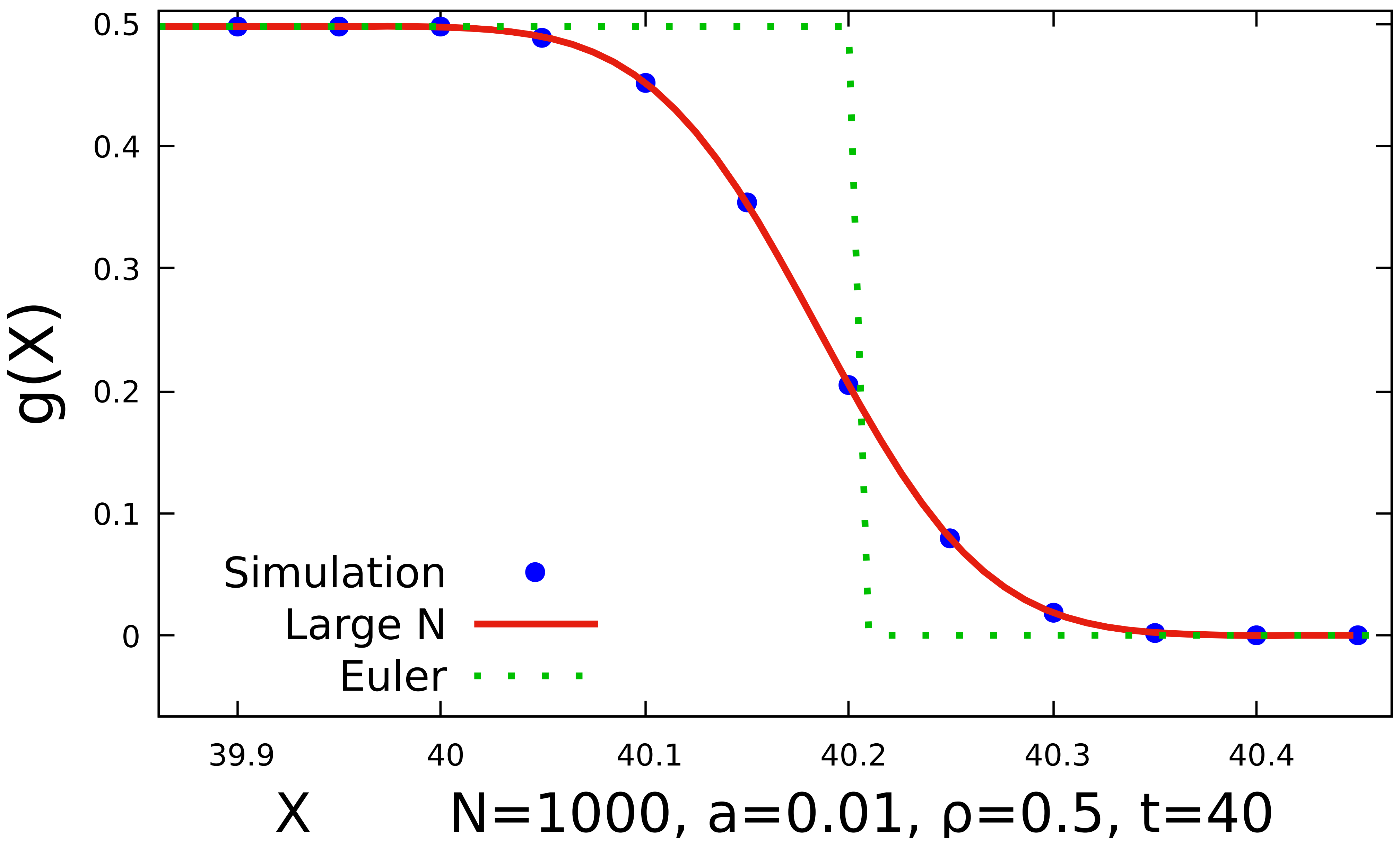
$$\rho(x,t) \approx \int_{-\infty}^{\infty} dz' \; \frac{\sqrt{N}\exp\left(-\frac{N(x-z'-aF^0(z',t))^2}{2a^2\Sigma_a^2(z',t)}\right)}{\sqrt{2\pi a^2 \Sigma_a^2(z',t)}}\rho^0(z',t)$$

where,

$$\Sigma_a^2(z,t) = F^0(z,t)(N - F^0(z,t))$$
$$\rho^0(z,t) = Np(z,t)$$
$$F^0(z,t) = Nq(z,t).$$

$$\Big\downarrow \; N \to \infty$$

$$\rho_{\text{euler}}(x,t) \approx \int_{-\infty}^{\infty} dz' \; \delta(x - z' - aF^0(z',t))\rho^0(z',t)$$
$$= \frac{\rho^0(x'(x),t)}{1 + a\rho^0(x'(x),t)}$$



(Numerical) solution of the Navier-Stokes equation

GHD with Navier-Stokes correction

$$\partial_t f + \partial_x (v_{\text{eff}} f) = \partial_x \mathcal{N}, \quad \text{Navier-Stokes term}$$

$$\mathcal{N} = \frac{a^2}{2(1 - a\rho)} \int dw |v - w| (f(w) \partial_x f(v) - f(v) \partial_x f(w))$$

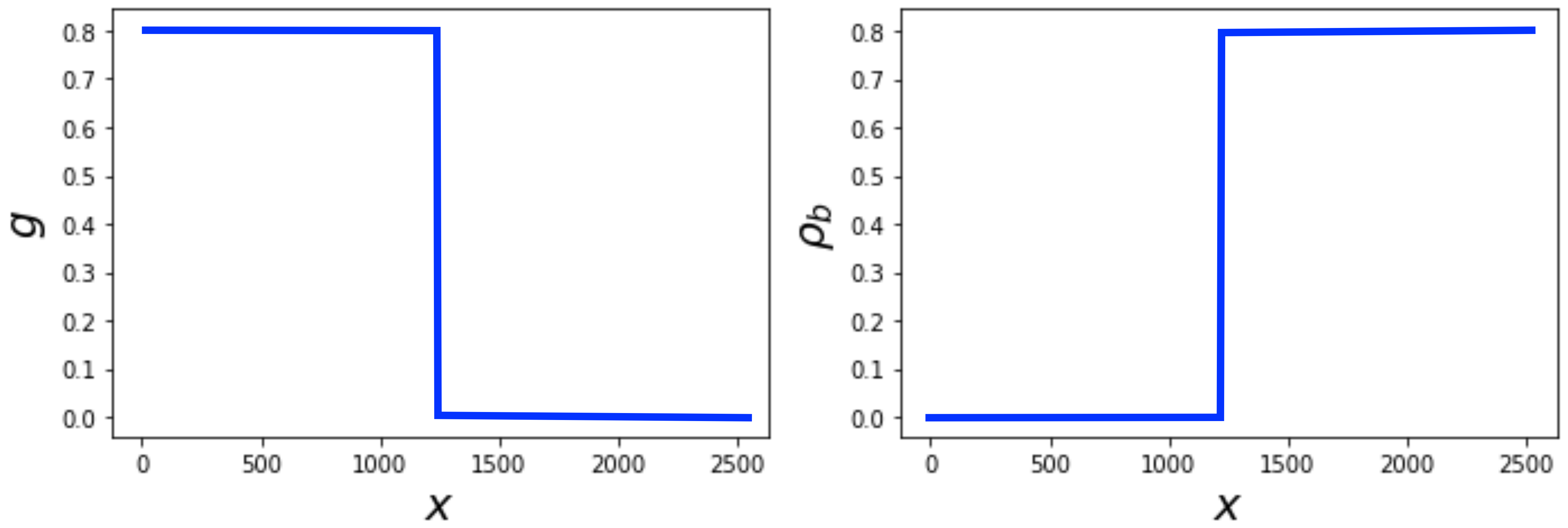
Doyon & Spohn, (2017)

Boldirghini & Suhov (1997)

We would like to solve this equation numerically

Ideally for this domain wall initial condition

$$f(x, v, 0) = \underbrace{\rho\ \delta(v - 1)\ \theta(L/2 - x)}_{\text{special rods}} + \underbrace{\rho\ p_{\text{mx}}(v)\ \theta(x - L/2)}_{\text{background rods}}$$



GHD with Navier-Stokes correction

$$\partial_t f + \partial_x (v_{\rm eff} f) = \partial_x \mathcal{N},$$ Navier-Stokes term

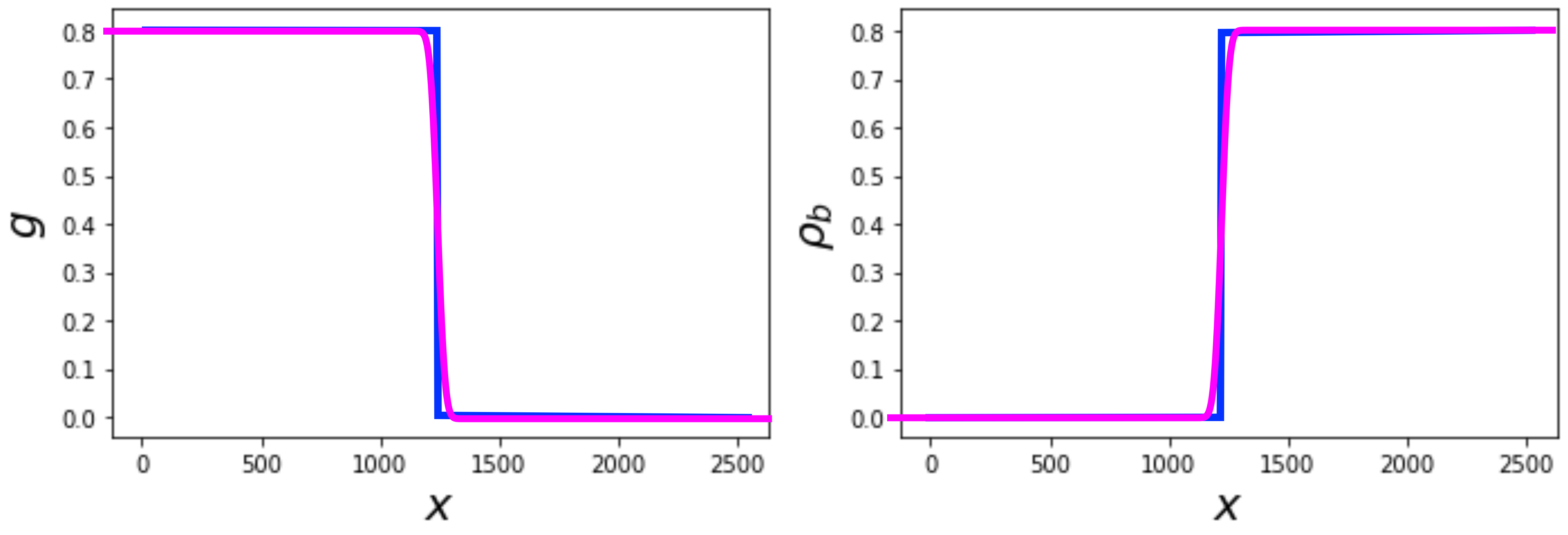
$$\mathcal{N} = \frac{a^2}{2(1-a\rho)} \int dw |v-w| (f(w)\partial_x f(v) - f(v)\partial_x f(w))$$

Doyon & Spohn, (2017)
Boldirghini & Suhov (1997)

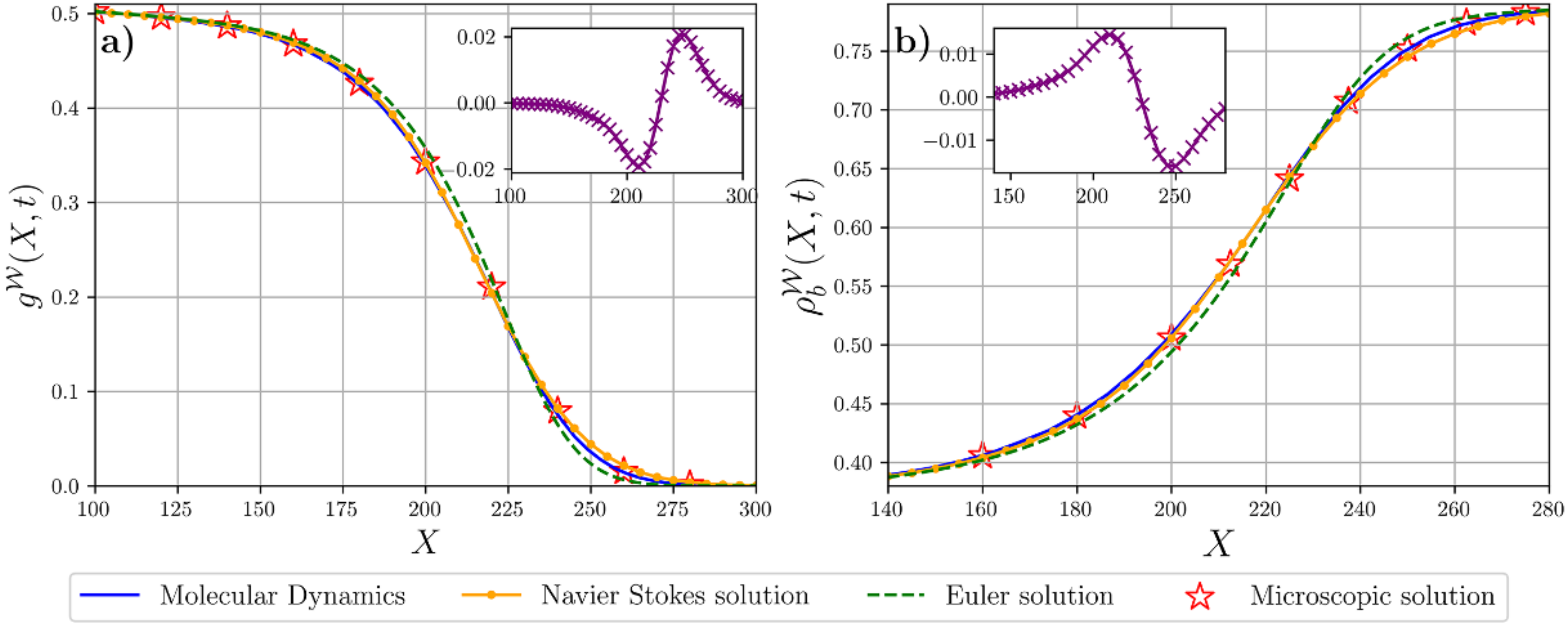
We would like to solve this equation numerically
Ideally for this domain wall initial condition

$$f(x,v,0) = \underbrace{\rho\ \delta(v-1)\ \theta(L/2-x)}_{\text{special rods}} + \underbrace{\rho\ p_{\rm mx}(v)\ \theta(x-L/2)}_{\text{background rods}}$$

Instead we solved for a slightly smoother version



Comparison with the solution of Navier-Stokes equation

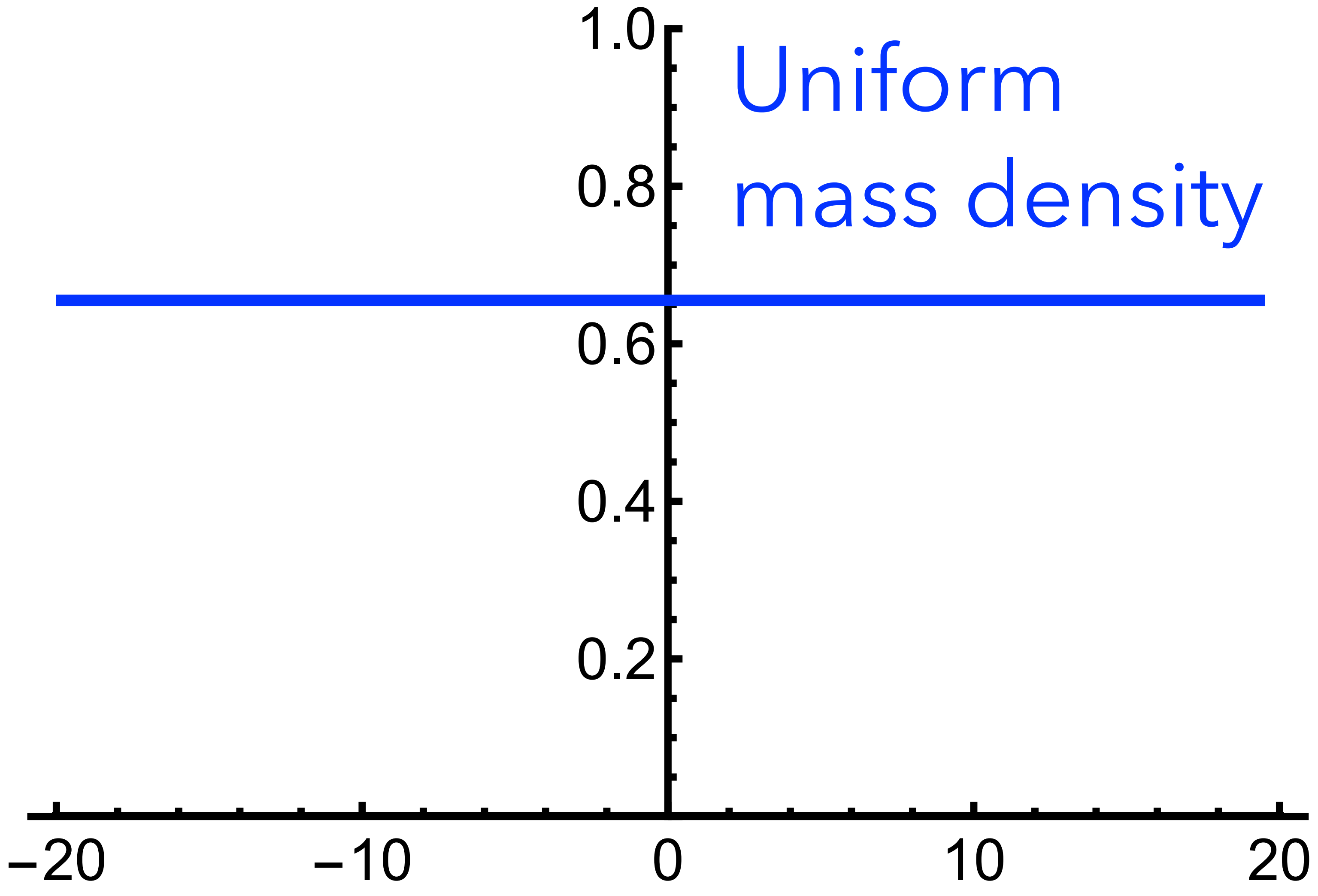


The effect of the Navier-Stokes correction can be seen more prominently in the tagged quasi-particle problem

The tagged quasi-particle problem (homogeneous background)

Lebowitz, Percus, Sykes Phys. Rev. (1969)

Homogeneously filled box with velocity chosen from Maxwell distribution, additionally there is one particle at the origin with velocity v_0



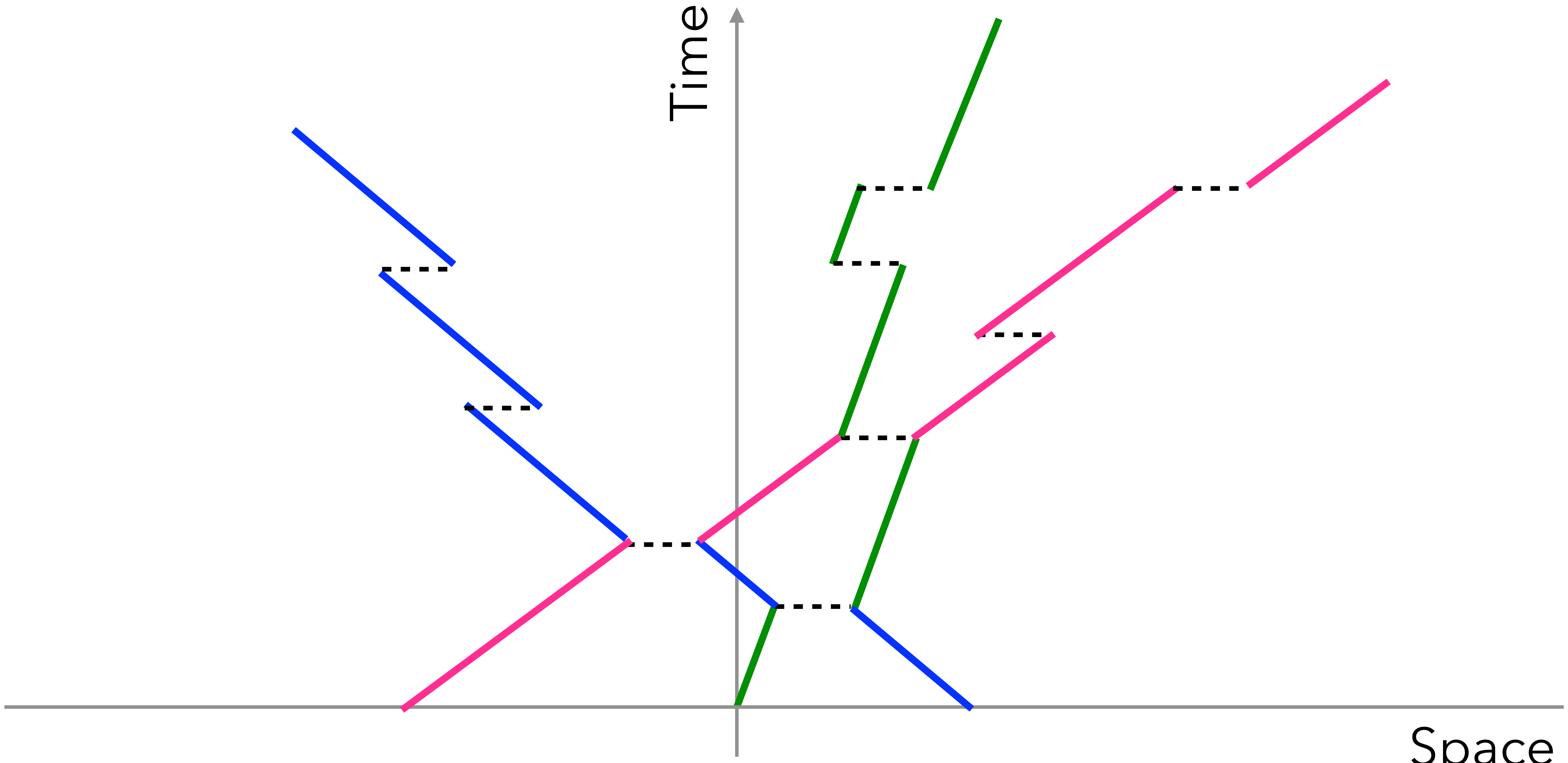
How will the tagged quasiparticle move ?

Case of hard point particles (a=0)

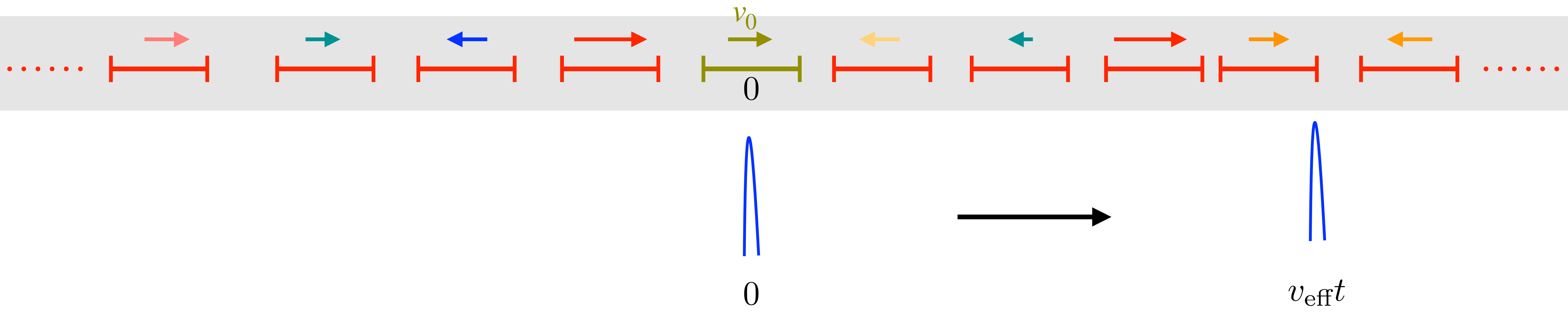
$$\rho_{v_0}(x,0) = \delta(x) \longrightarrow \rho_{v_0}(x,t) = \delta(x - v_0 t)$$

This will not be true for hard rods $(a \neq 0)$

For hard rod gas, a quasiparticle moves randomly



On an average it will move by amount $v_{\text{eff}}t$



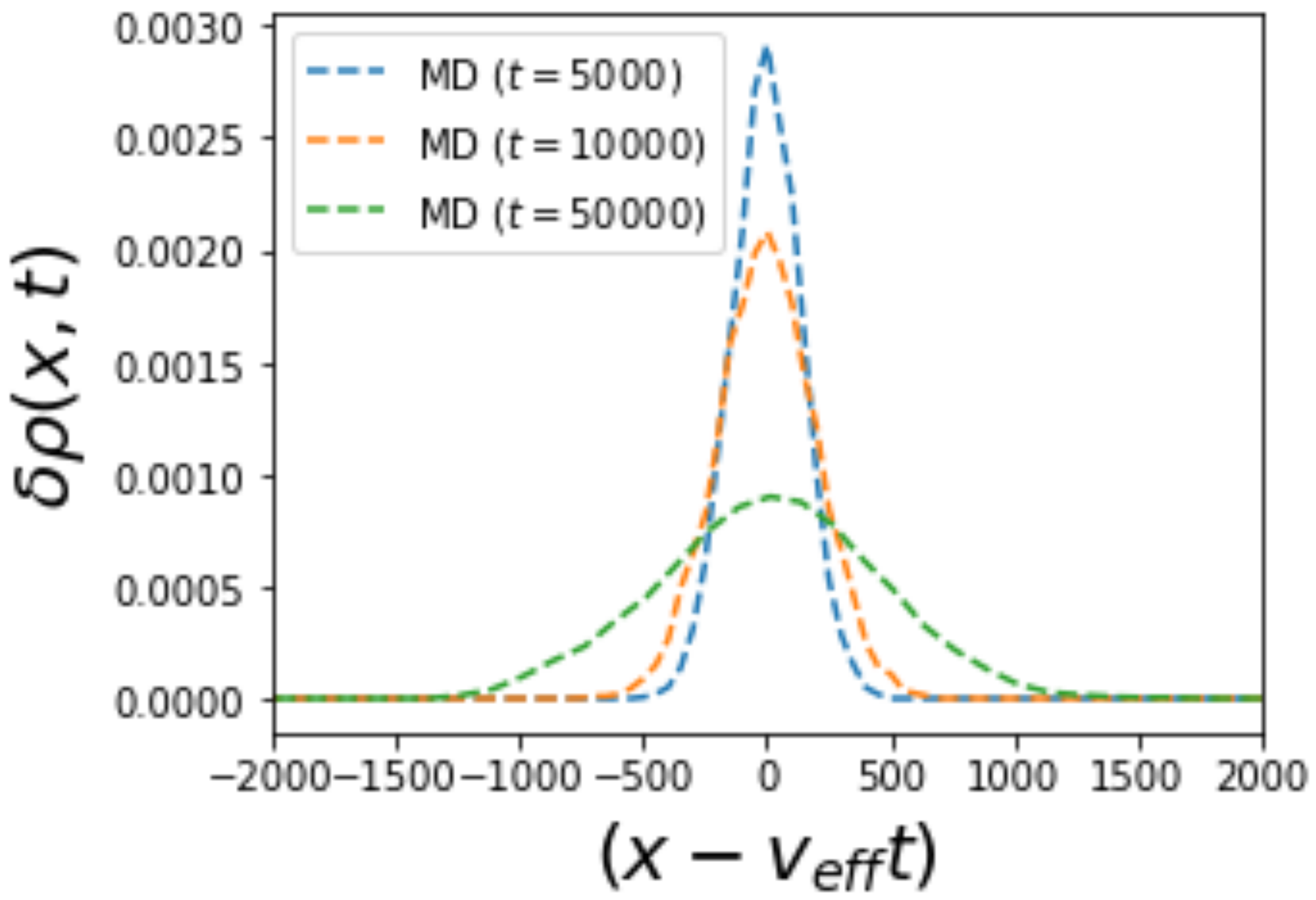
Euler GHD =>

$$\rho_{v_0}(x,0) = \delta(x) \longrightarrow \rho_{v_0}(x,t) = \delta(x - v_{\text{eff}}(v_0)t)$$

$$v_{\text{eff}} = \frac{v_0}{1 - a\rho_0}$$

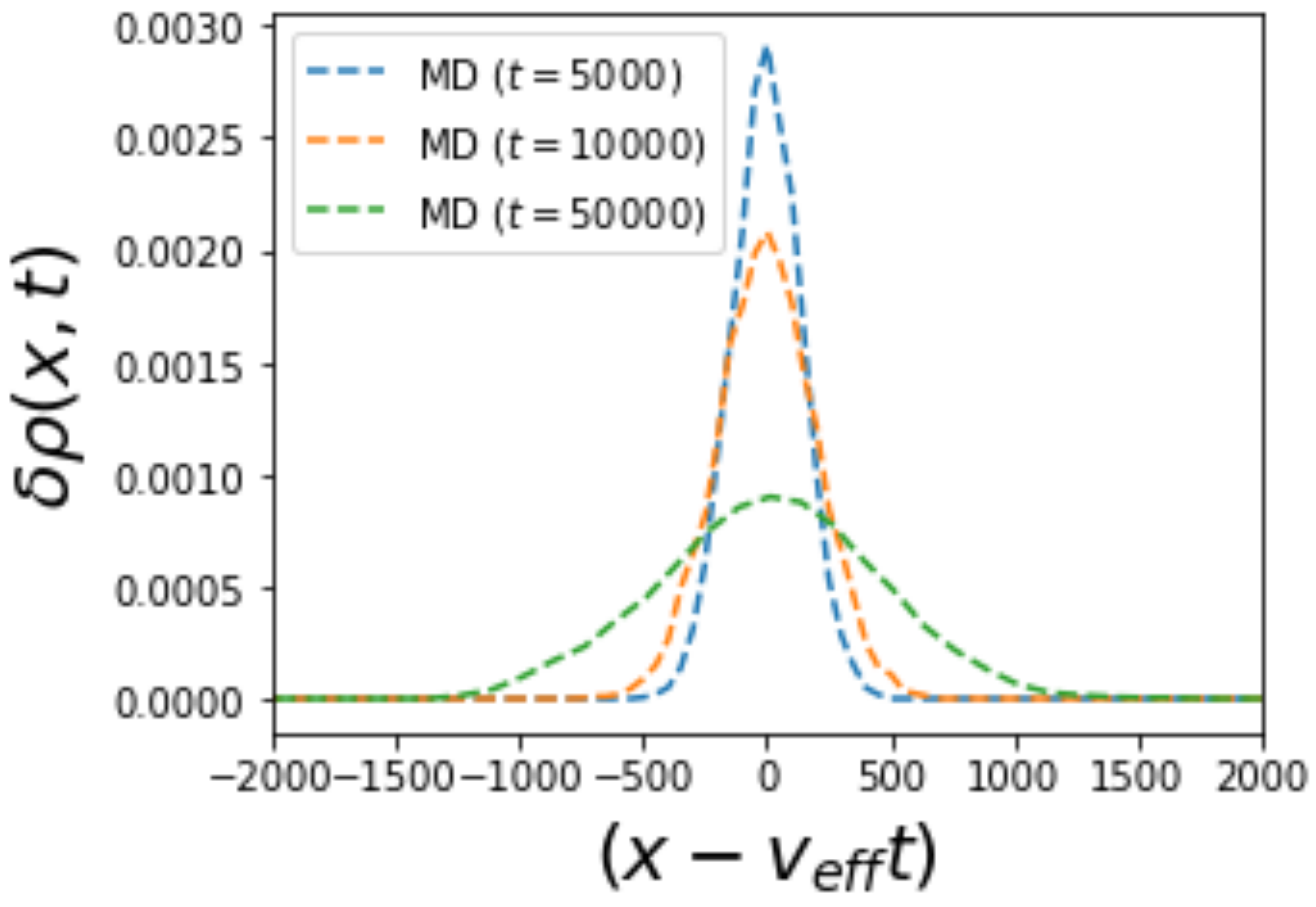
What is the probability to find the quasiparticle at x at time t?

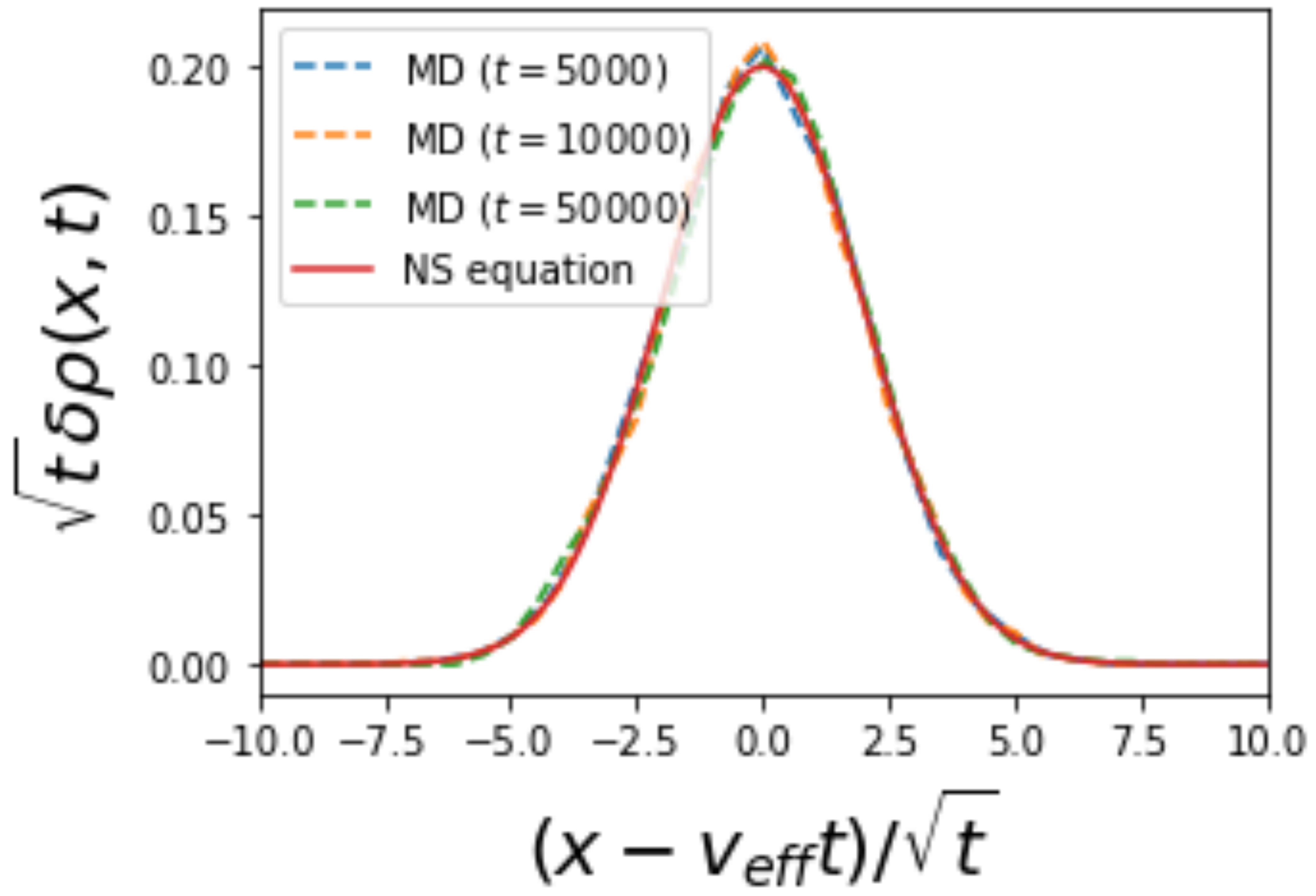
From numerical simulation we observe spreading of the distribution



Singh, Dhar, Spohn, **A.K.** JSP, (2023)

Numerical distribution has diffusive scaling





Singh, Dhar, Spohn, **A.K.** JSP, (2023)

Diffusive solution from GHD with NS correction

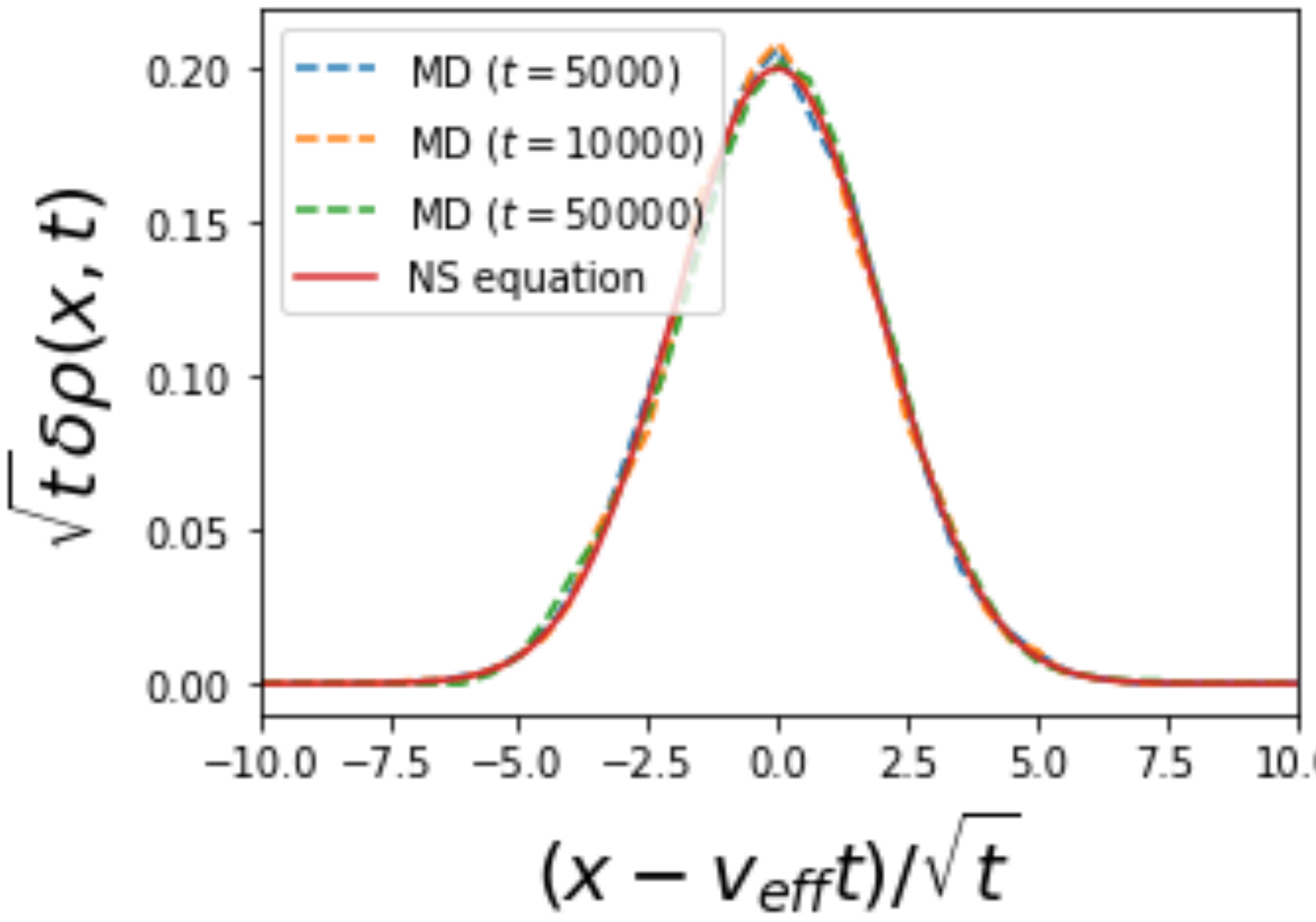
$$\partial_t f + \partial_x(v_{\text{eff}} f) = \partial_x \mathcal{N},$$
$$\mathcal{N} = \frac{a^2}{2(1-a\rho)} \int dw |v-w| (f(w)\partial_x f(v) - f(v)\partial_x f(w)).$$

Solution of the linearised NS GHD:

$$\delta\rho(x,t) = \frac{1}{\sqrt{2\pi n a^2 \mu(v_0) t}} e^{-\frac{(x-v_{\text{eff}}t)^2}{2a^2 n\mu(v_0)t}}$$

Singh, Dhar, Spohn, Kundu, JSP, (2023)

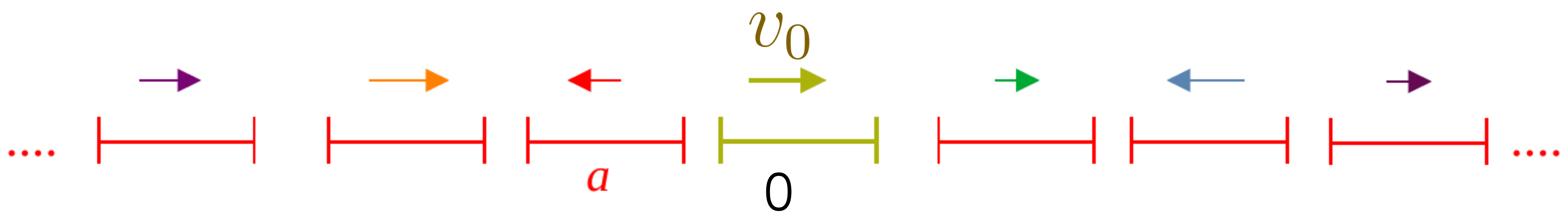
$$\mu(v_0) = \int dv |v-v_0| h(v),$$
$$v_{\text{eff}} = \frac{v_0}{1-a\rho_0},\ n = \frac{\rho_0}{1-a\rho_0},$$



Agrees with

Lebowitz, Percus, Sykes, Phys. Rev. (1969)

Gaussian distribution of single tagged quasiparticle



- JL Lebowitz, JK Percus, and J Sykes, *Physical review*, 171(1):224, 1968 (Solving kinetic equations)

- S. K. Singh, A. Dhar, H. Spohn, and A. Kundu, *Journal of Statistical Physics,* 191(6):66, 2024 (Solving HD equations)

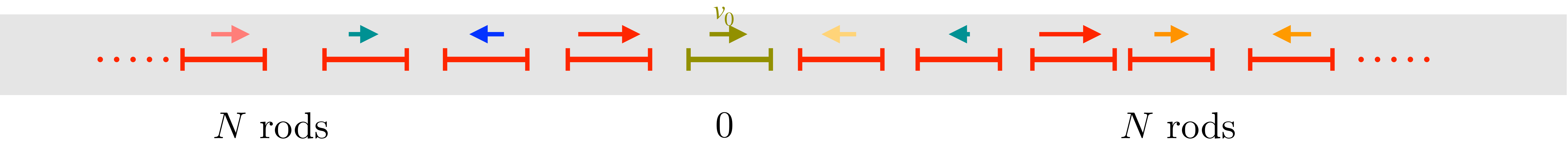
- M. J. Powdel, and A. Kundu, Journal of Statistical Mechanics, 2024 (12), 123205 (Using a microscopic approach)

$$\mathbb{P}_t(X) = \frac{1}{\sqrt{2\pi\Sigma_a^2(t)}} \exp\left(-\frac{(X-\langle X\rangle)^2}{2\Sigma_a^2(t)}\right)$$

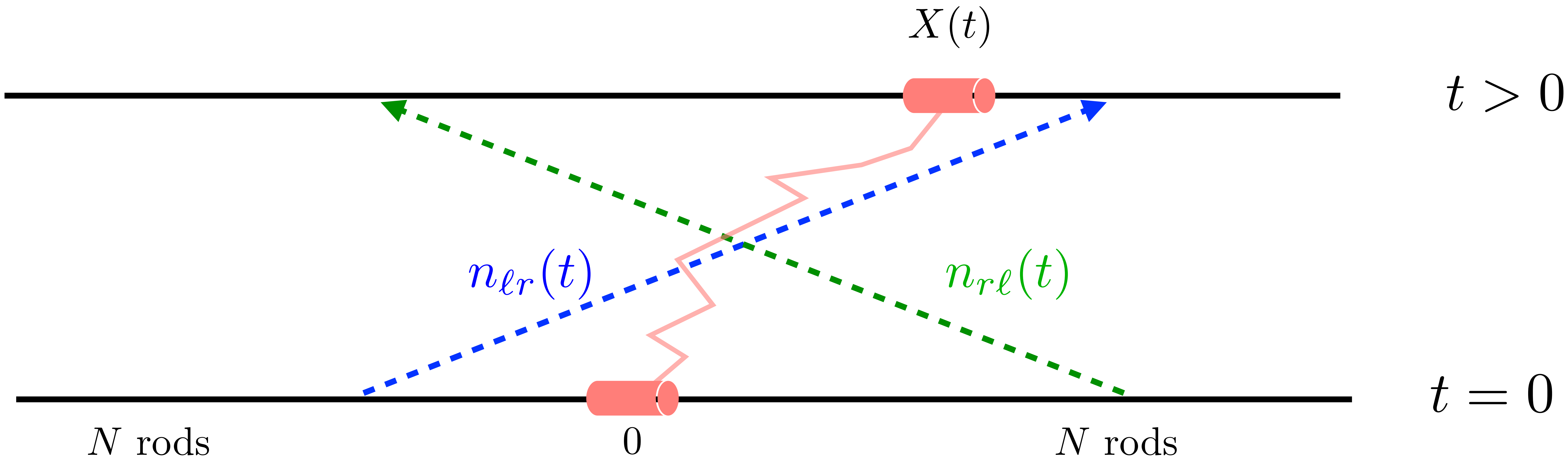
$$\Sigma_a^2(t) = \mathscr{D}(v)\ t,$$

$$\mathscr{D}(v) = \varphi_0 a^2 \int_{-\infty}^{\infty} dw\ |w-v| \hbar(w)$$

Basic steps of the computation



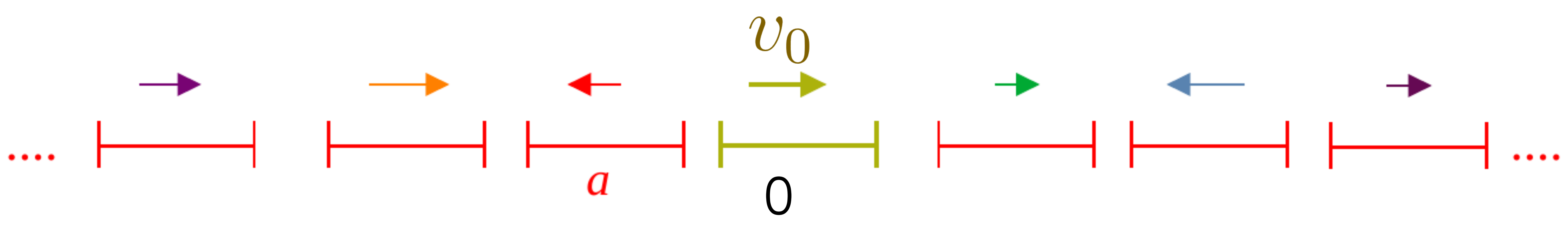
$$X(t) = v_0 t + a[n_{r\ell}(t) - n_{\ell r}(t)]$$



$$\text{Joint pdf.:} \;\; \mathscr{P}(n_{r\ell} = n,\; n_{r\ell} = m) = \mathscr{P}(n)\;\mathscr{P}(m)$$

$$= \text{Independent Poisson distribution for large } N$$

Recall: Single tagged quasiparticle



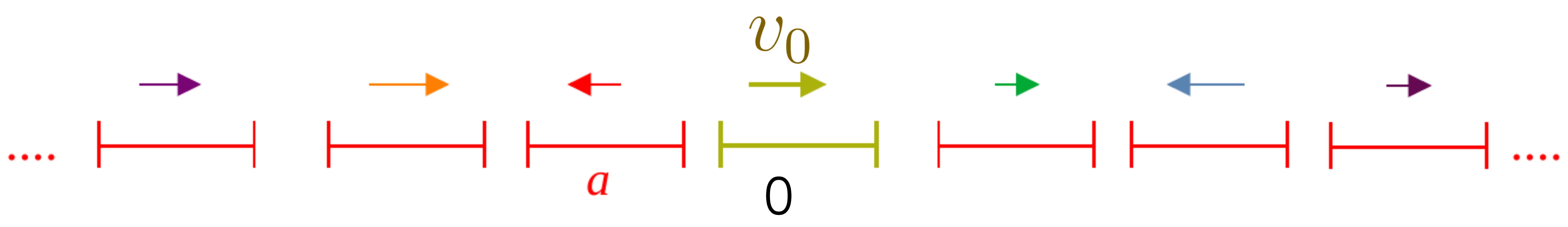
$$\mathbb{P}_t(X) = \frac{1}{\sqrt{2\pi \Sigma_a^2(t)}} \exp\left(-\frac{(X - \langle X \rangle)^2}{2\Sigma_a^2(t)} \right)$$

$$\Sigma_a^2(t) = \mathscr{D}(v)\ t,$$

$$\mathscr{D}(v) = \varphi_0 a^2 \int_{-\infty}^{\infty} dw\ |w - v| \hbar(w)$$

Tracer rod seems to move like a Brownian particle

Recall: Single tagged quasiparticle



$$\mathbb{P}_t(X) = \frac{1}{\sqrt{2\pi\Sigma_a^2(t)}} \exp\left(-\frac{(X - \langle X\rangle)^2}{2\Sigma_a^2(t)}\right)$$

$$\Sigma_a^2(t) = \mathscr{D}(v)\ t,$$

$$\mathscr{D}(v) = \varphi_0 a^2 \int_{-\infty}^{\infty} dw\ \big|w - v\big| \hbar(w)$$

Tracer rod seems to move like a Brownian particle

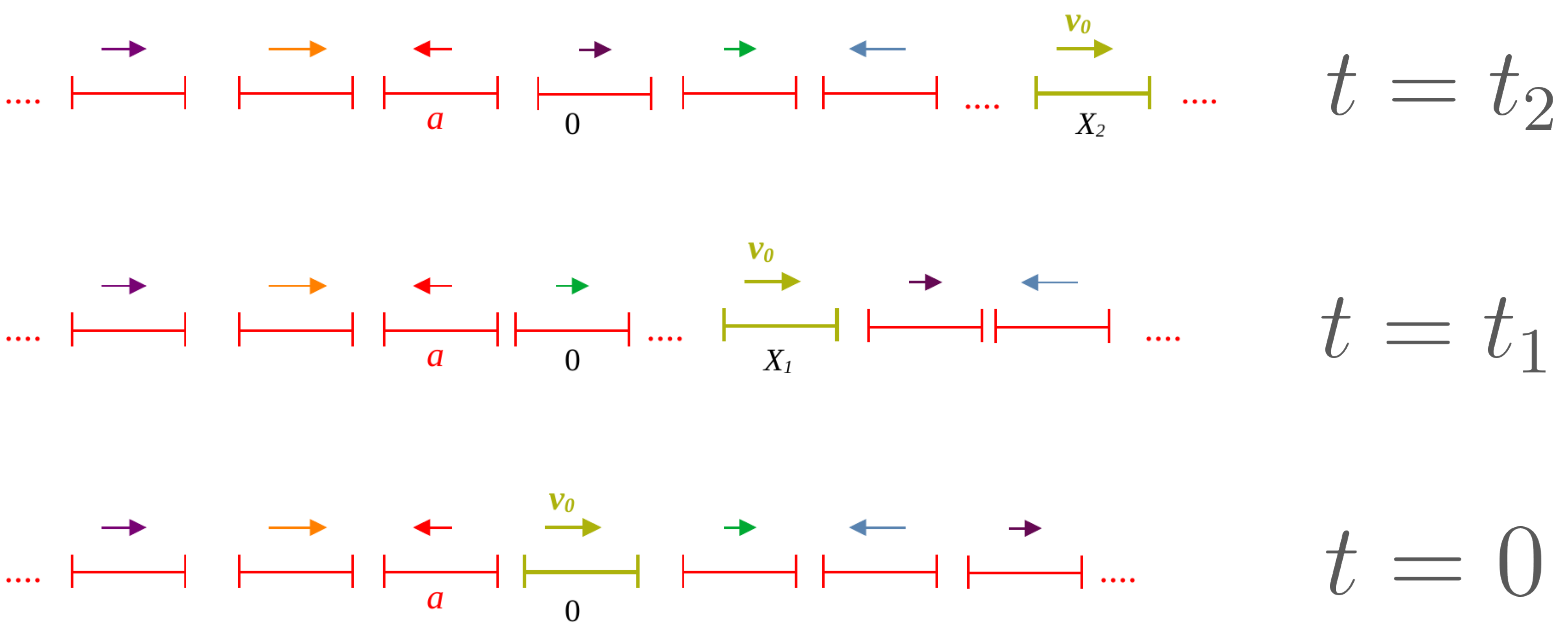
Further evidence?

Autocorrelation

To get more evidence we compute…

correlation of single tracer rod positions at two different times

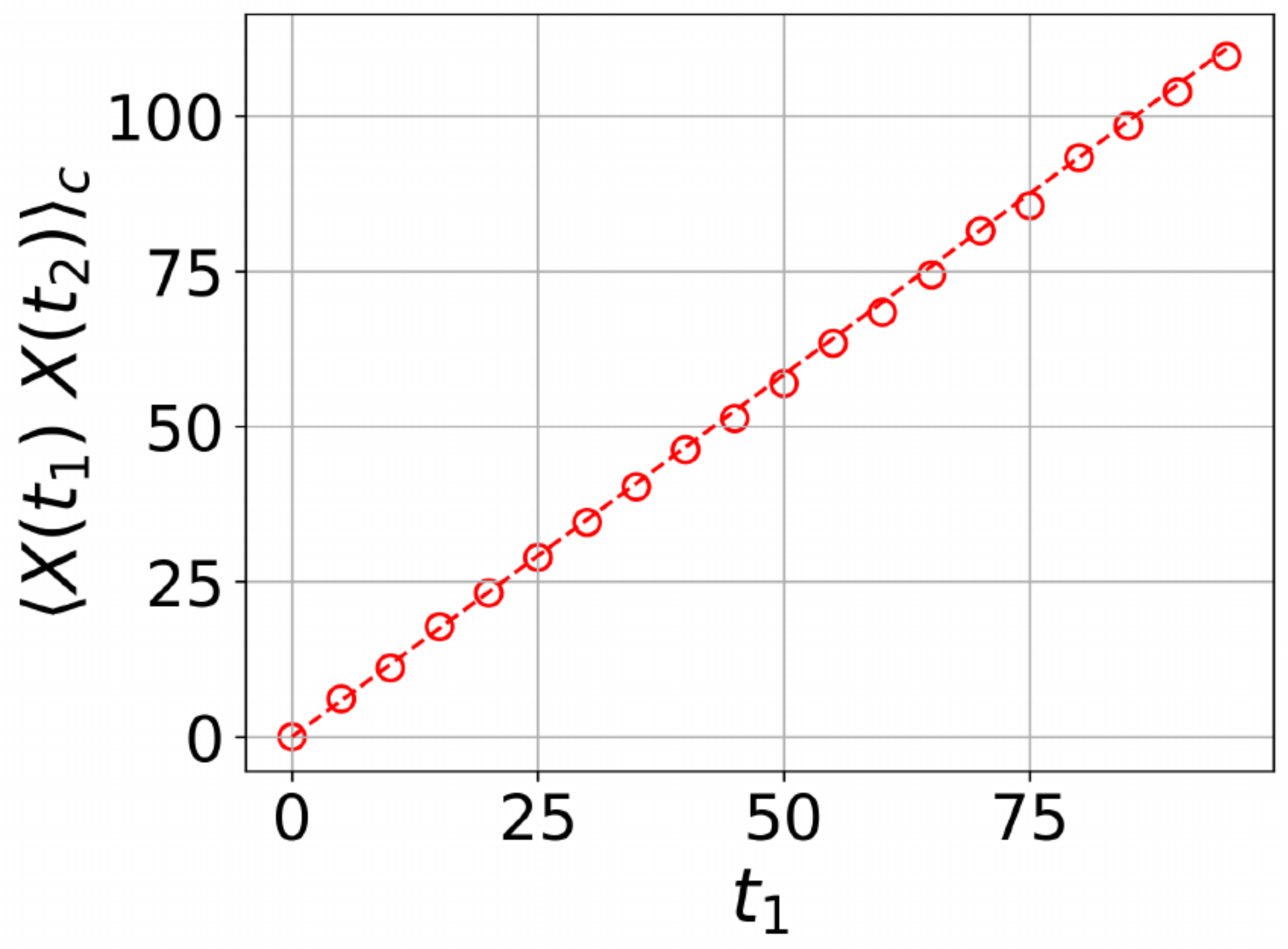
$$\langle X_{v_0}(t_1) X_{v_0}(t_2) \rangle_c$$



Single tracer auto-correlation (homogeneous background)

$$\langle X_{v_0}(t_1) X_{v_0}(t_2)\rangle_c \sim \mathscr{D}(v_0)\ \min(t_1, t_2)$$

Seema, Mukherjee, Dhar, Spohn. Kundu (2025)



- Similar to the two time correlation for uniformly drifted Brownian particle

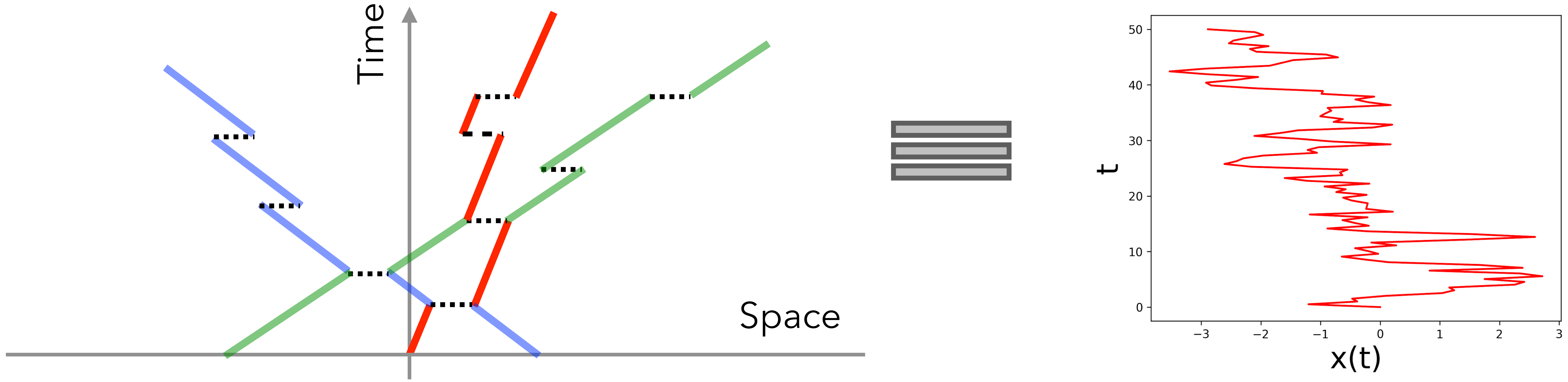
$$\frac{dx}{dt} = v_0 + \eta(t)$$

$$\langle \eta(t_1)\eta(t_2)\rangle = 2\mathscr{D}\delta(t_1 - t_2)$$

Quasiparticles perform drifted Brownian motion

Quasiparticles in homogeneous background move like drifted Brownian particles

$$\frac{dX_v}{dt} = \frac{v}{1 - a\varrho_0} + \xi_v(t),$$



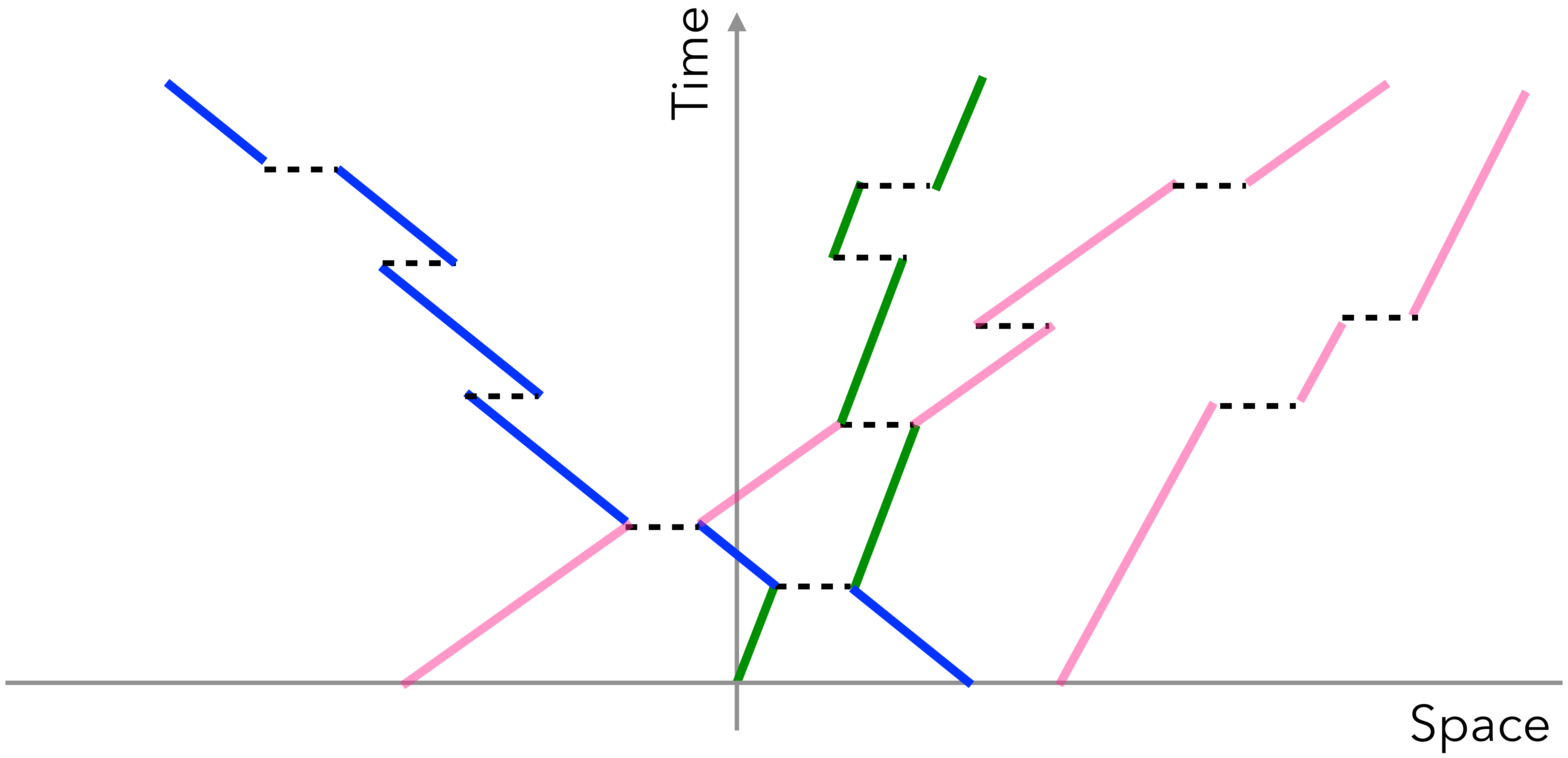
This fact was proved rigorously using a probabilistic approach.

Ferrari and Olla, The Annals of Applied Probability, 35(2):1125, 2025

For Toda quasiparticle, a similar stochastic motion has recently been predicted.

A. Aggarwal, arXiv:2503.11407

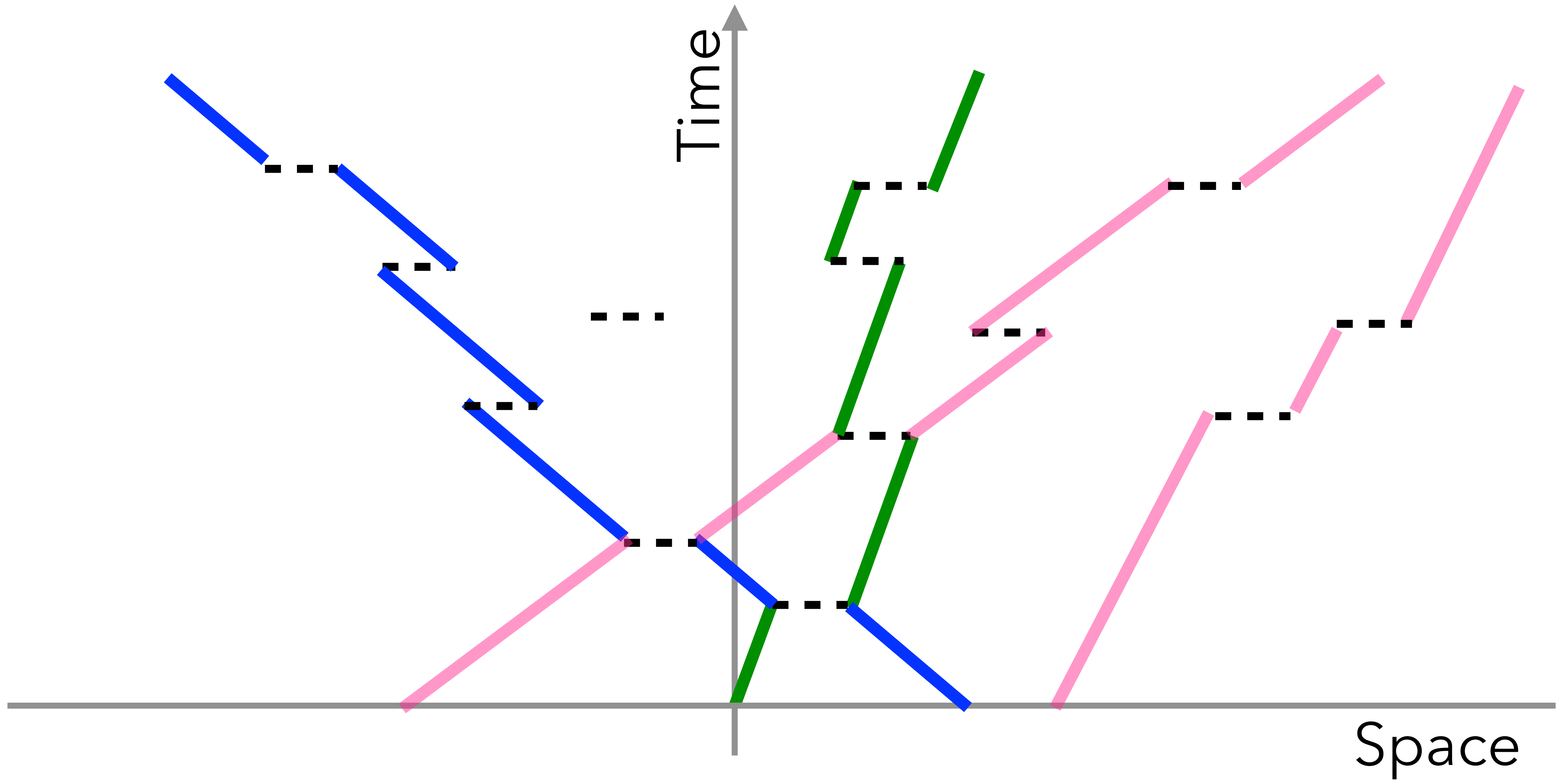
Two quasiparticles



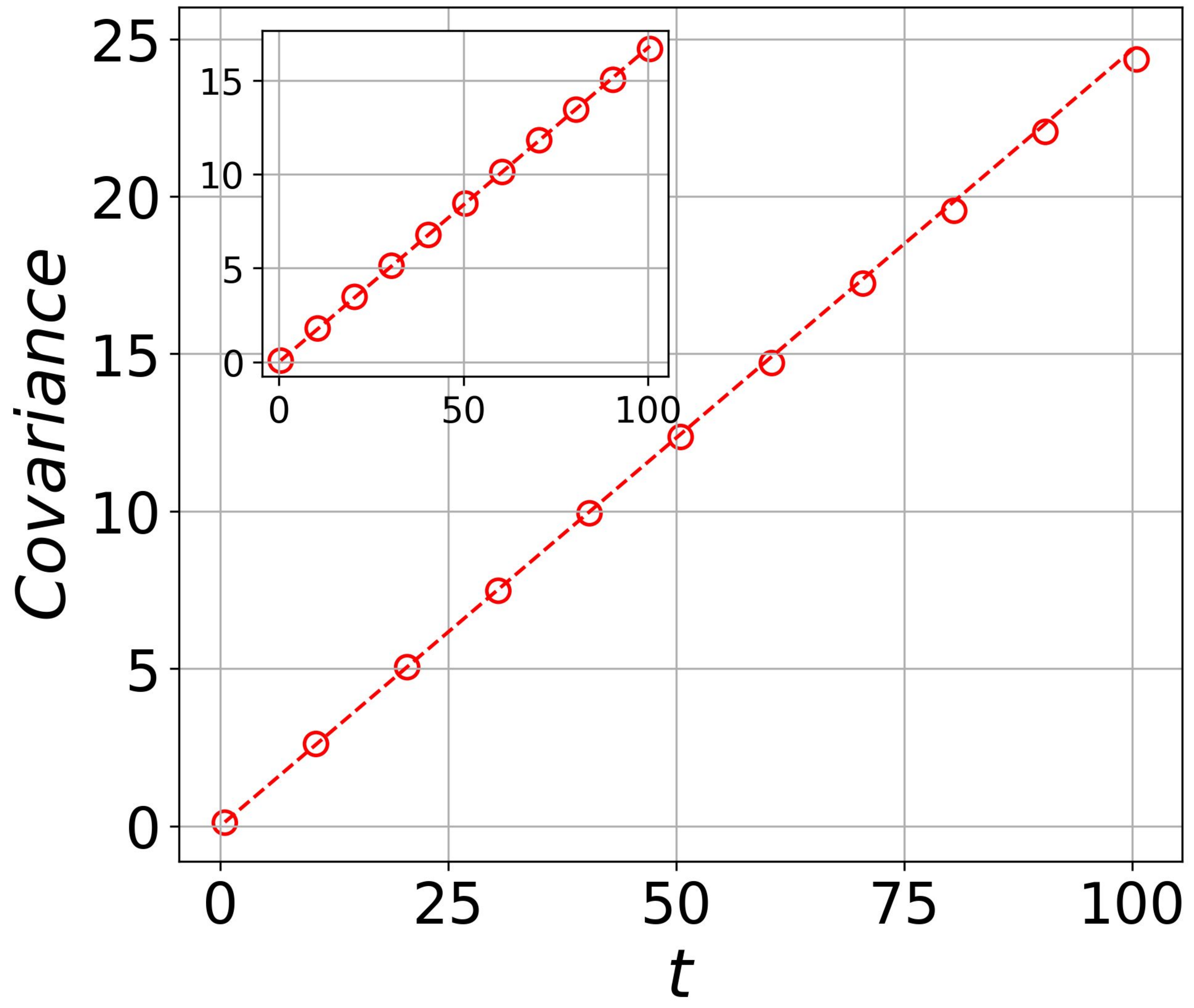
Are they correlated?

We compute co-variance: $\langle X_v(t) X_u(t) \rangle_c = \langle X_v(t) X_u(t) \rangle - \langle X_v(t) \rangle \langle X_u(t) \rangle$

Co-variance (homogeneous background)



$$\langle X_{v0}(t) Y_{u0}(t) \rangle_c \sim (\mathscr{D}(v_0) + \mathscr{D}(u_0) - |v_0 - u_0|)\, t$$



$$N = 5000, \;\; a = 0.5, v_0 = 0.5, \;\; u_0 = 1.0$$

Recall:

$$\mathscr{D}(v) = \varphi_0 a^2 \int_{-\infty}^{\infty} dw \; |w - v| \hbar(w)$$

Quasiparticles are correlated

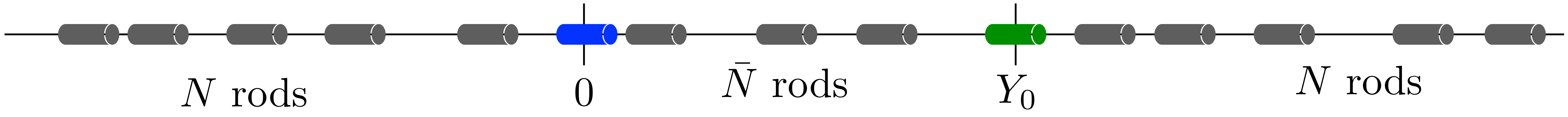
– – –

in an interesting way !

Variance of the separation $\langle (Y-X)^2\rangle_c$ (Homogeneous background)

Tracer rods moving in a homogeneous background with the same velocity $v_0 = u_0$

Density: $\rho_0, \quad \varphi_0 = \frac{\rho_0}{1-a\rho_0}$



Separation: $Y(t) - X(t)$

$$\langle (Y-X)^2\rangle_c = ?$$

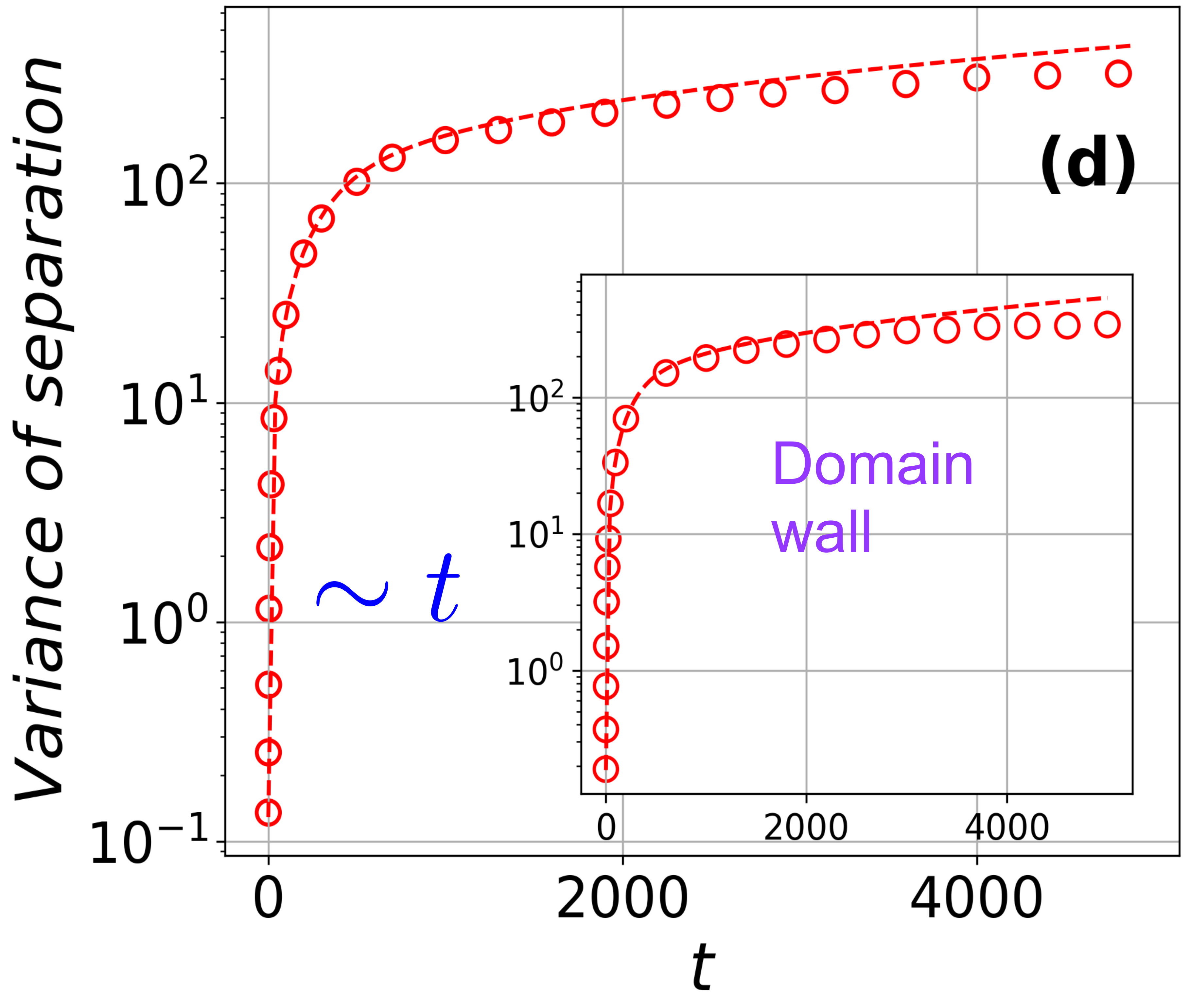
Mean: $\langle Y(t) - X(t)\rangle \sim Y_0$

Seema, Mukherjee, Dhar, Spohn. Kundu (2025)

Variance of the separation: large t asymptotic

Variance:

$$\langle (Y(t) - X(t))^2 \rangle_c \sim \begin{cases} t & \text{for small } t \\ \text{const.} & \text{for large } t \end{cases}$$



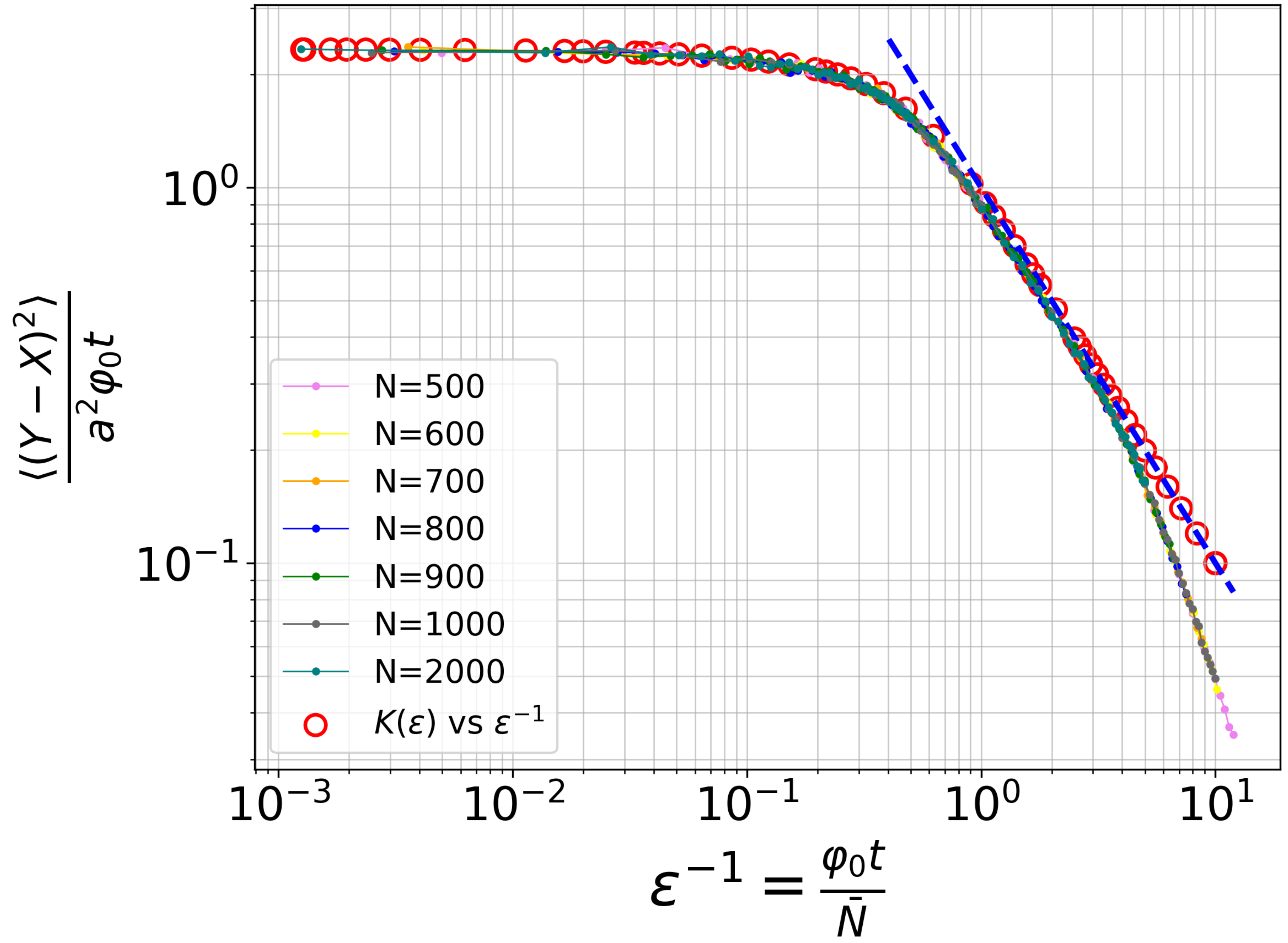
Seema, Mukherjee, Dhar, Spohn. Kundu (2025)

Scaling of the variance of the separation $\langle (Y-X)^2\rangle_c$

$$\frac{\langle (Y-X)^2\rangle_c}{t} = a^2\varphi_0\,\mathcal{K}\left(\frac{\bar N}{t\varphi_0}\right),\ \text{where } y_0 = \frac{\bar N}{\varphi_0}, \text{and}$$

Density: $\rho_0, \quad \varphi_0 = \frac{\rho_0}{1-a\rho_0}$

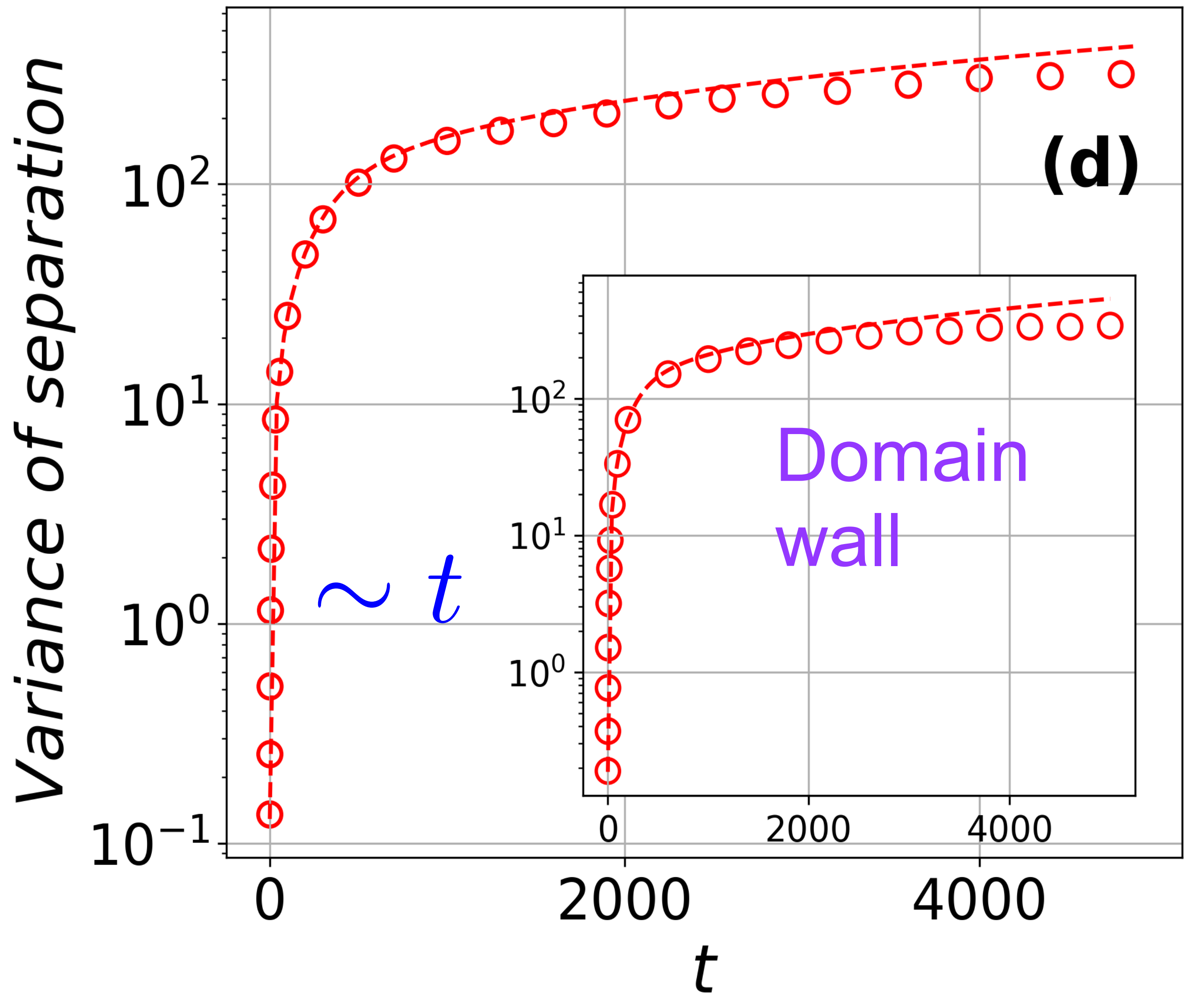
$$\mathcal{K}(\epsilon) = 2[\mathcal{D}(v_0) - \mathcal{F}(v_0-\epsilon) - \bar{\mathcal{F}}(v_0+\epsilon)] - \frac{1}{\epsilon}[\mathcal{F}(v_0) - \mathcal{F}(v_0-\epsilon) + \bar{\mathcal{F}}(v_0) - \bar{\mathcal{F}}(v_0+\epsilon)]^2.$$

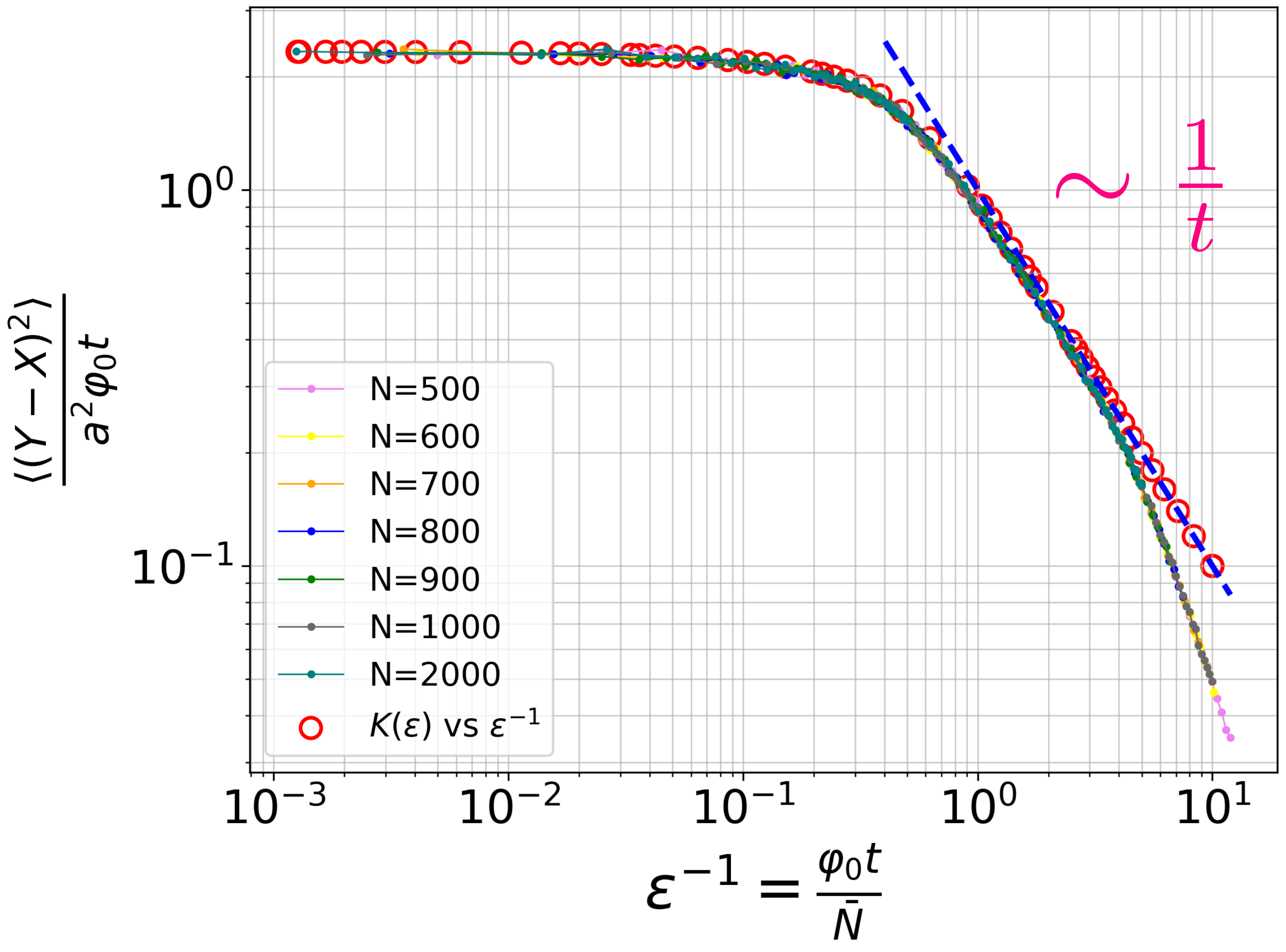


Seema, Mukherjee, Dhar, Spohn. Kundu (2025)

Variance of the separation: large t asymptotic

$$\langle (Y(t)-X(t))^2\rangle_c \sim \begin{cases} t & \text{for small } t \\ \text{const.} & \text{for large } t \end{cases}$$





Diffusion coefficient of the separation:

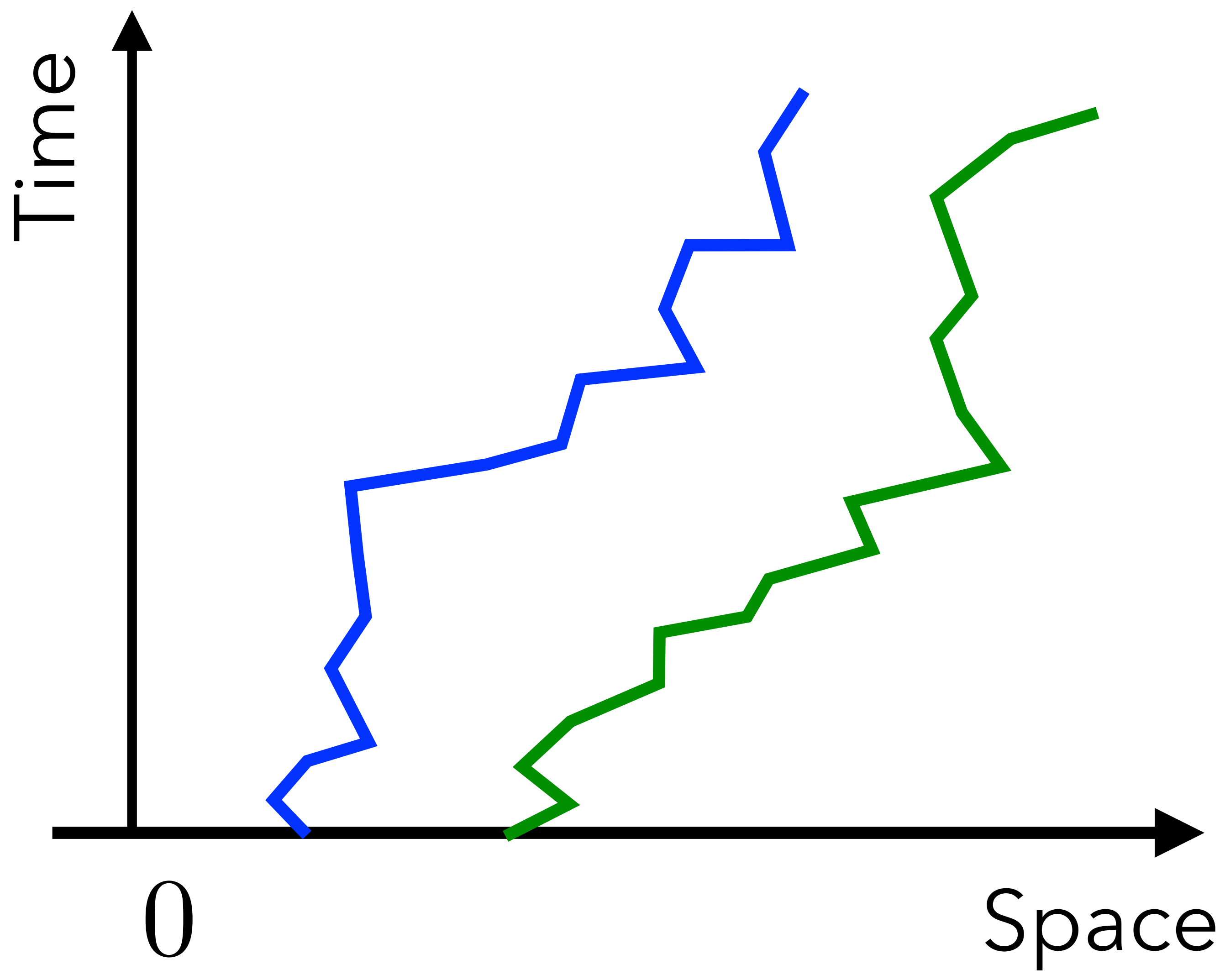
$$\frac{\langle (Y(t)-X(t))^2\rangle_c}{t} \to 0, \quad \text{as } t \to \infty$$

Seema, Mukherjee, Dhar, Spohn. Kundu (2025)

Rigid body motion

Variance of separation: $$\frac{\langle (Y(t)-X(t))^2\rangle_c}{t} \to 0, \quad \text{as } t\to\infty$$ whereas $$\langle X(t)^2\rangle_c \sim \mathscr{D}(v_0)\ t$$ $$\langle Y(t)^2\rangle_c \sim \mathscr{D}(u_0)\ t$$

Two tagged quasiparticles seems to move like a rigid body with respect to the motion of their centre of mass !!



P. Ferrari and S. Olla, Ann. Appl. Probab. 35(2): 1125-1142 (2025)

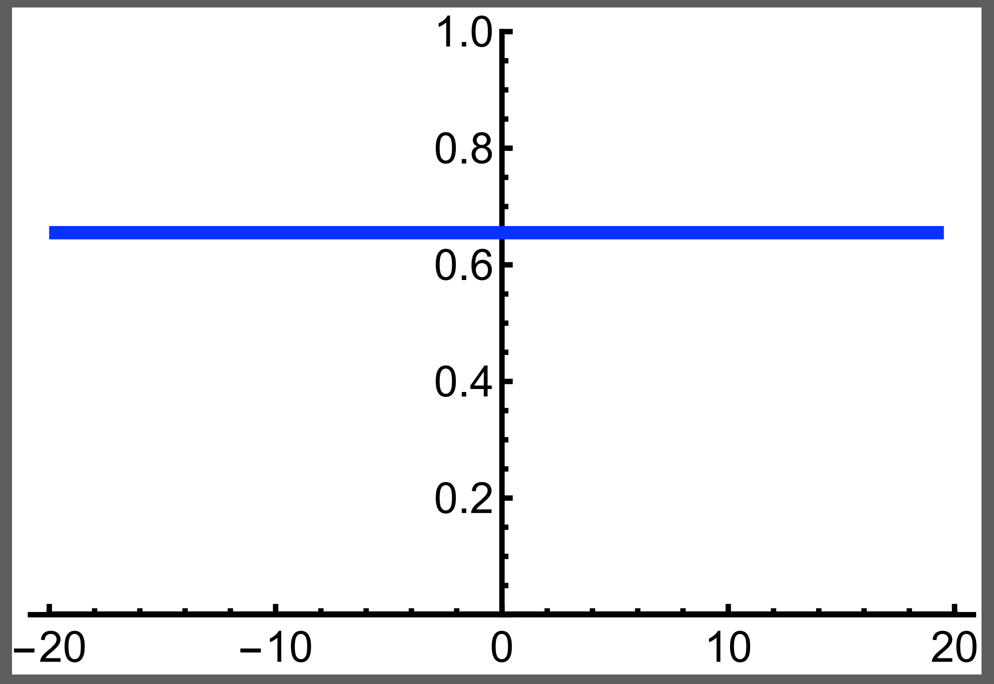
Seema, Mukherjee, Dhar, Spohn. Kundu (2025)

Summary of tagged quasiparticle statistics

1. Homogeneous background

Variance 1:	$\langle X_{v_0}(t)^2\rangle_c$	$\sim$	$\mathscr{D}(v_0)\; t$
Variance 2:	$\langle Y_{u_0}(t)^2\rangle_c$	$\sim$	$\mathscr{D}(u_0)\; t$
Auto-correlation:	$\langle X_{v_0}(t_1)X_{v_0}(t_2)\rangle_c$	$\sim$	$\mathscr{D}(v_0)\;\; \min(t_1,t_2)$
Co-variance:	$\langle X_{v_0}(t)Y_{u_0}(t)\rangle_c$	$\sim$	$(\mathscr{D}(u_0)+\mathscr{D}(v_0)-\lvert v_0-u_0\rvert)\; t$

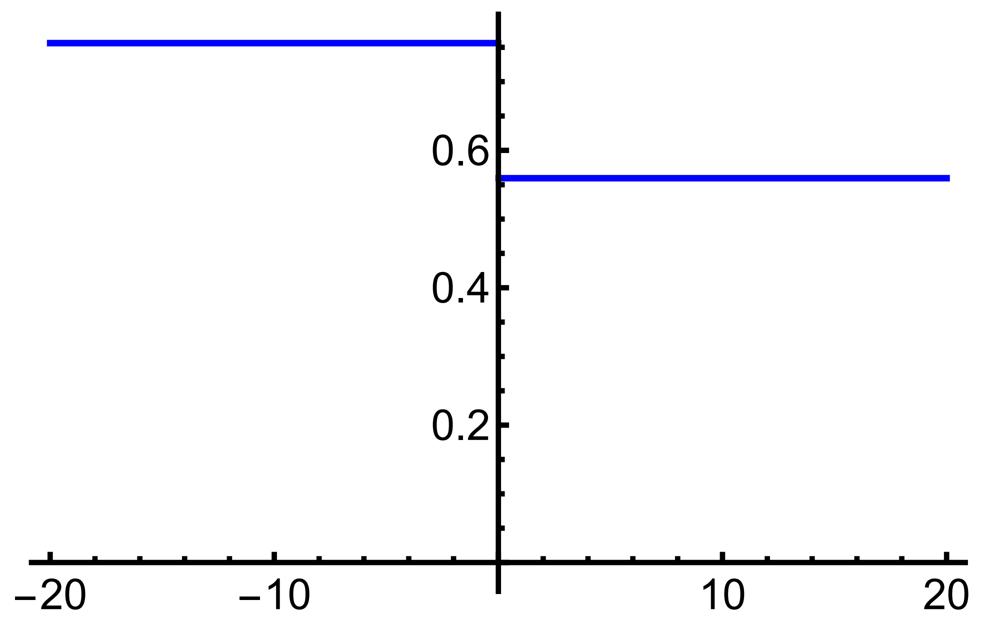
$$\mathscr{D}(v) = \varphi_0 a^2 \int_{-\infty}^{\infty} dw\; \lvert w-v\rvert \hbar(w)$$

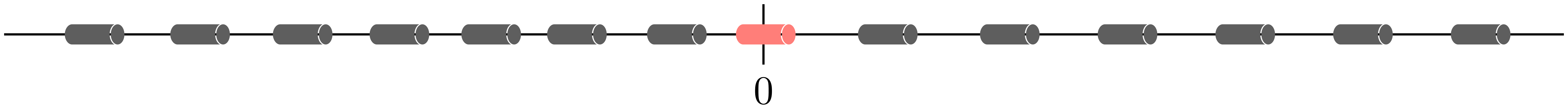


2. Also have expressions for inhomogeneous background.

Seema, Mukherjee, Dhar, Spohn. Kundu (2025)

Inhomogeneous background: Step profile





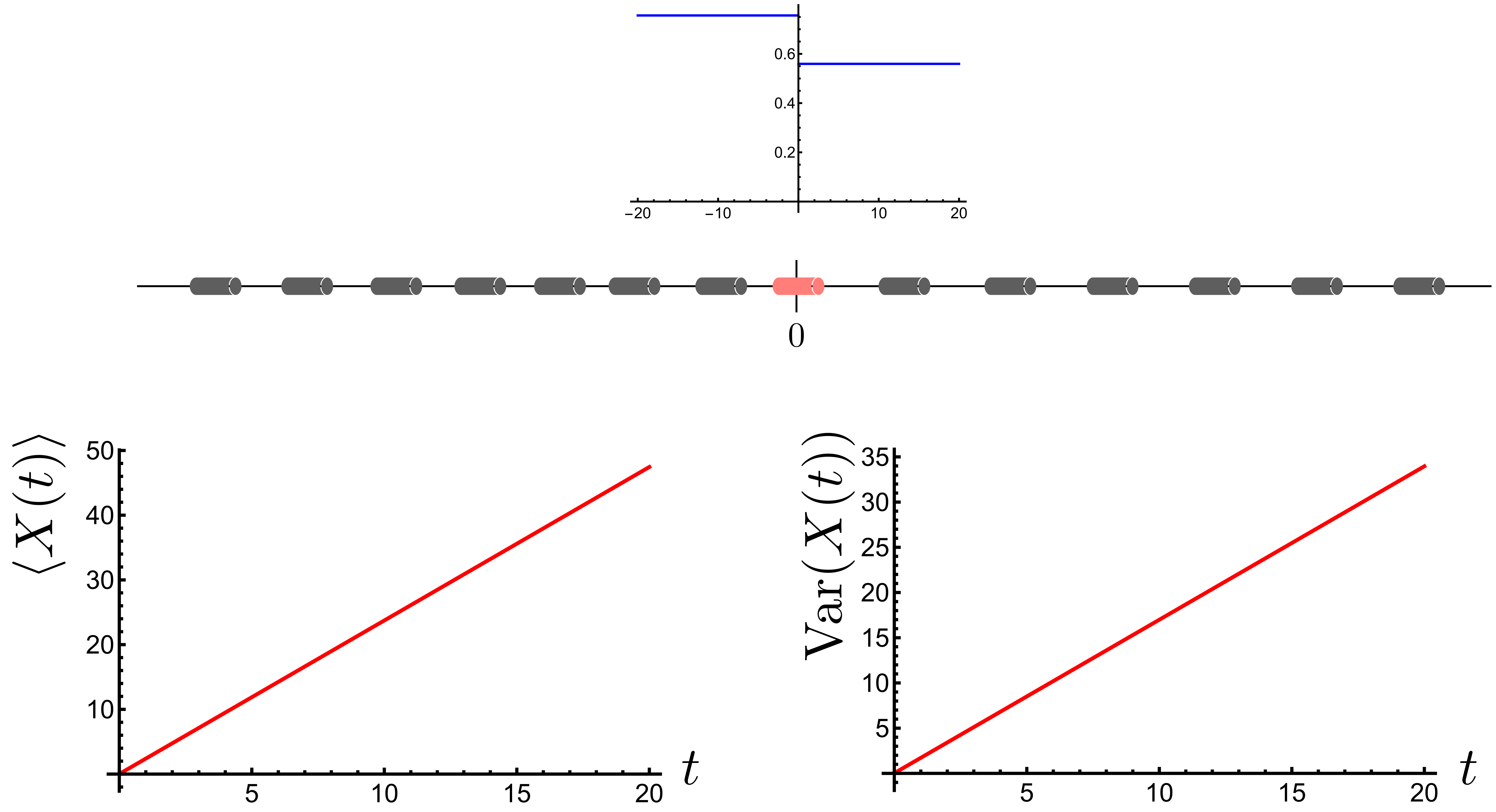
Domain wall background:

$$\begin{aligned} \langle X_{v_0}^2(t)\rangle_c &= a^2\ \mathcal{D}_{\rm dw}(v_0,\rho_r,\rho_\ell)\ t \\ \langle X_{v_0}(t_1)X_{v_0}(t_2)\rangle_c &= a^2\ \mathcal{D}_{\rm dw}(v_0,\rho_r,\rho_\ell)\ \min(t_1,t_2) \\ \mathcal{D}_{\rm dw}(v_0) &= \varphi_r\int_{-\infty}^{v_0} du\ (v_0-u)\hbar(u) + \varphi_\ell\int_{v_0}^{\infty} du\ (u-v_0)\hbar(u) \end{aligned}$$

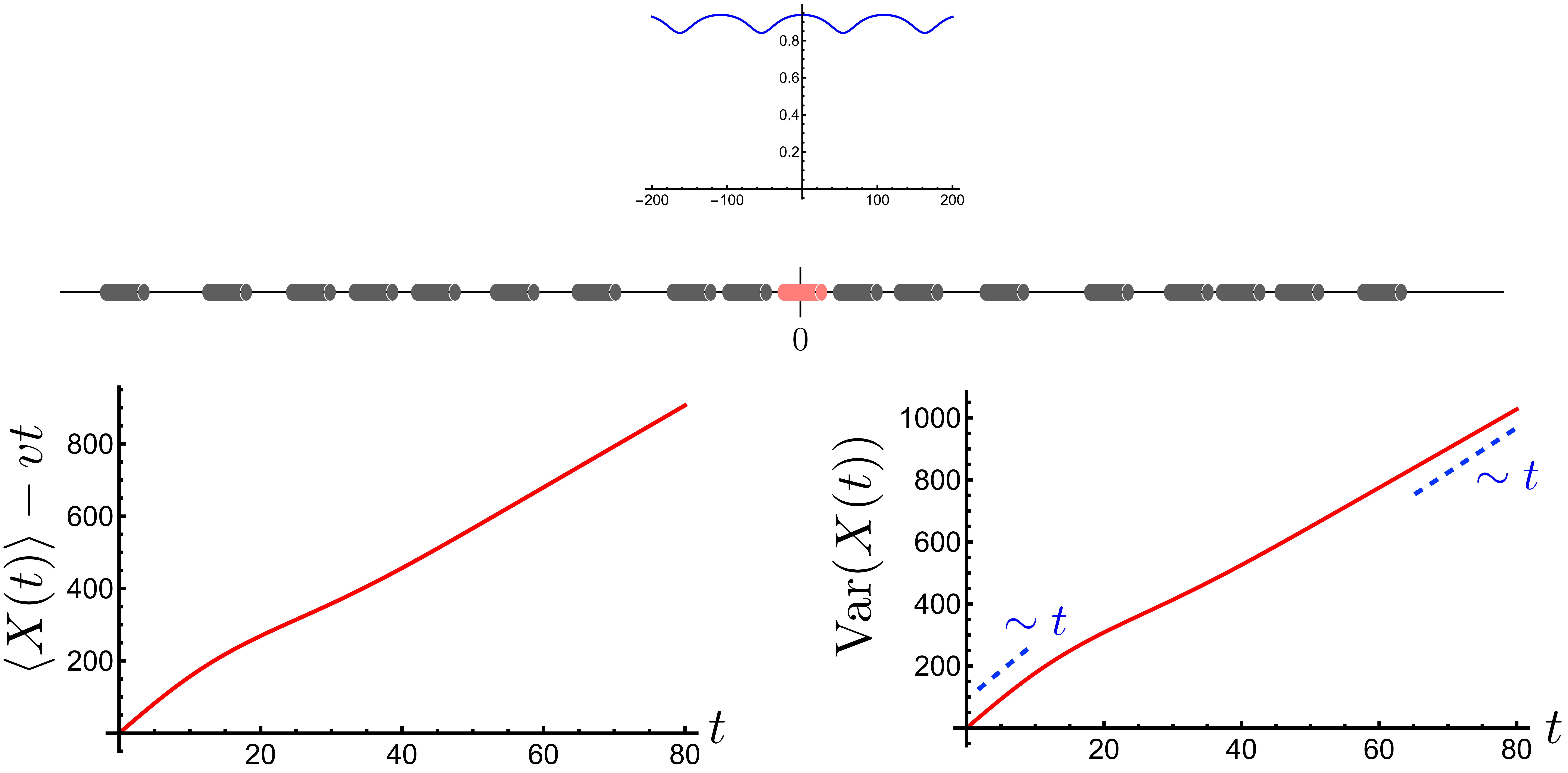
Homogeneous background:

$$\begin{aligned} \langle X_{v_0}^2(t)\rangle_c &= a^2\ \varphi_0\ \mathcal{D}_{\rm h}(v_0)\ t \\ \langle X_{v_0}(t_1)X_{v_0}(t_2)\rangle_c &= a^2\ \varphi_0\ \mathcal{D}_{\rm h}(v_0)\ \min(t_1,t_2) \\ \mathcal{D}_{\rm h}(v_0) &= \varphi_0\int_{-\infty}^{\infty} du\ |v_0-u|\hbar(u) \end{aligned}$$

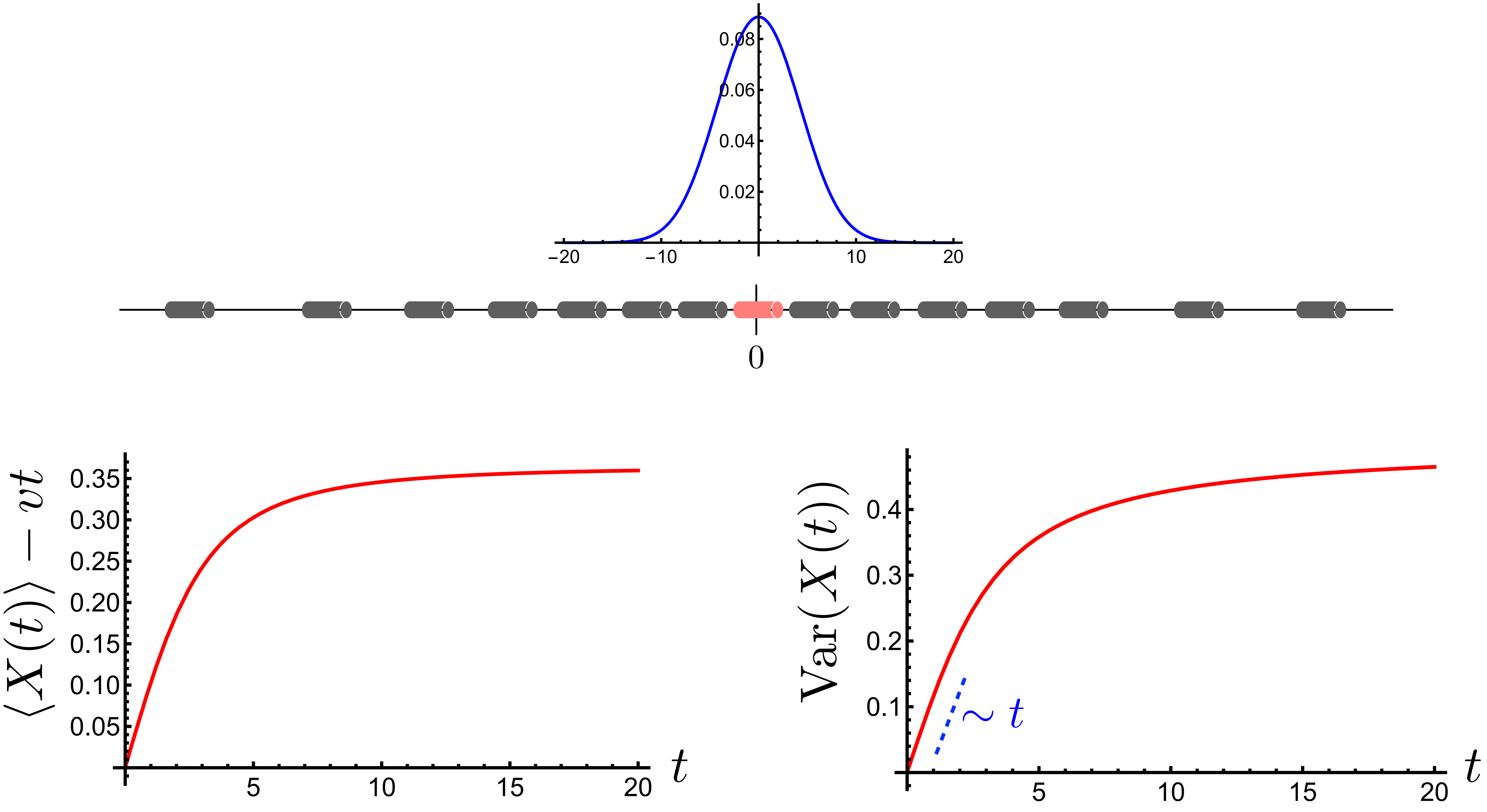
Inhomogeneous background: Step profile



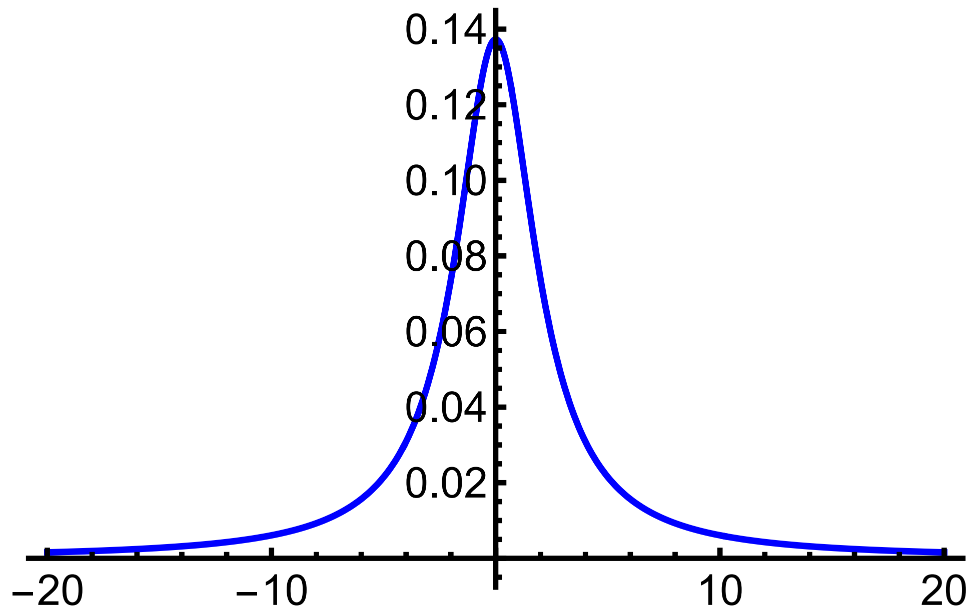
Inhomogeneous background: Sinusoidal profile

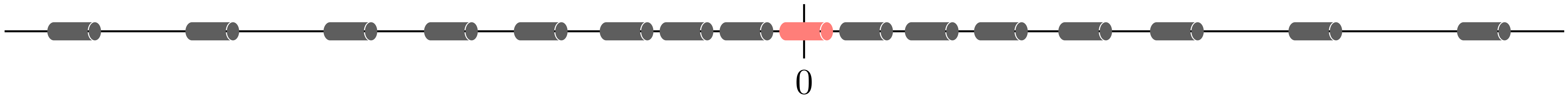


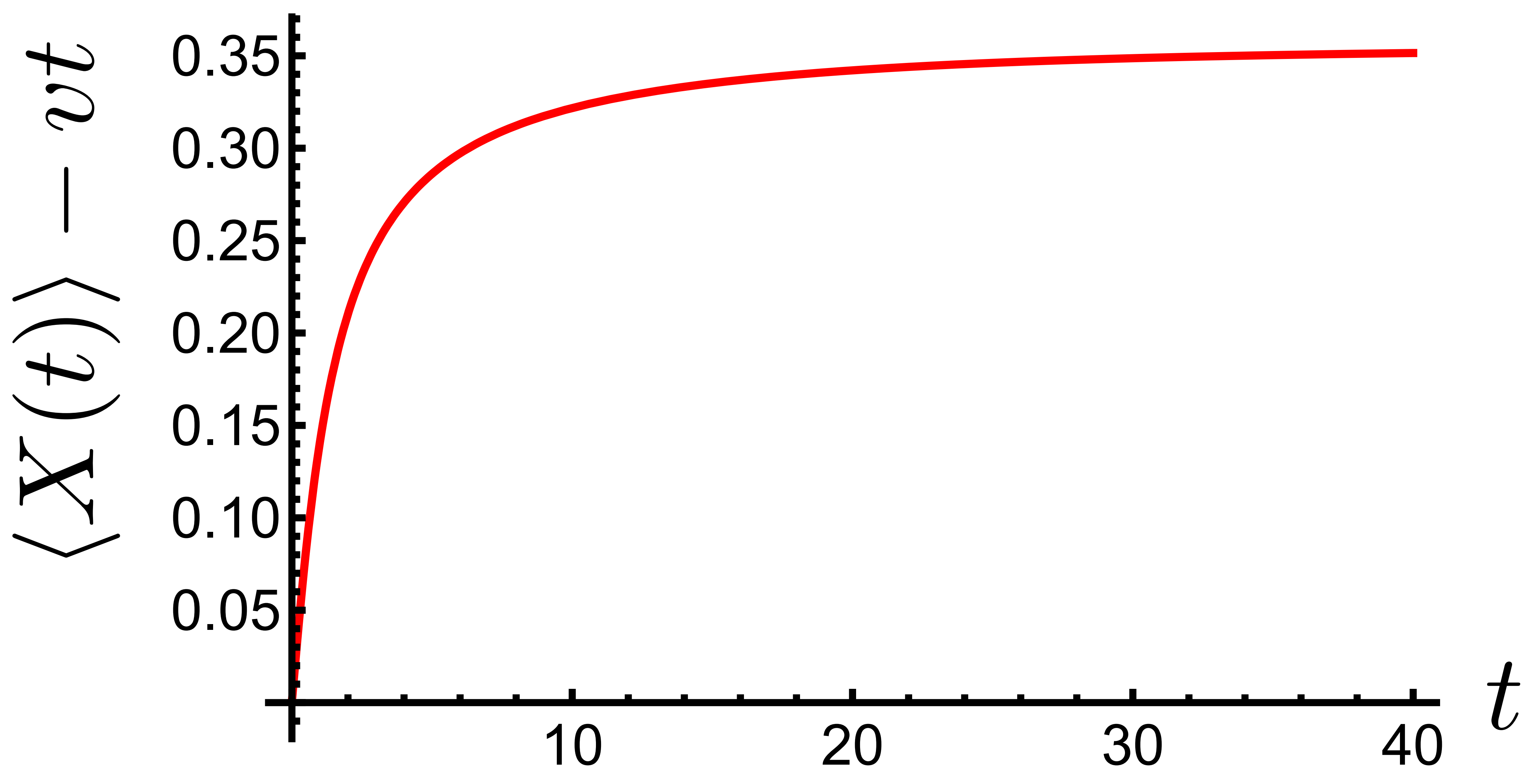
Inhomogeneous background: Gaussian profile

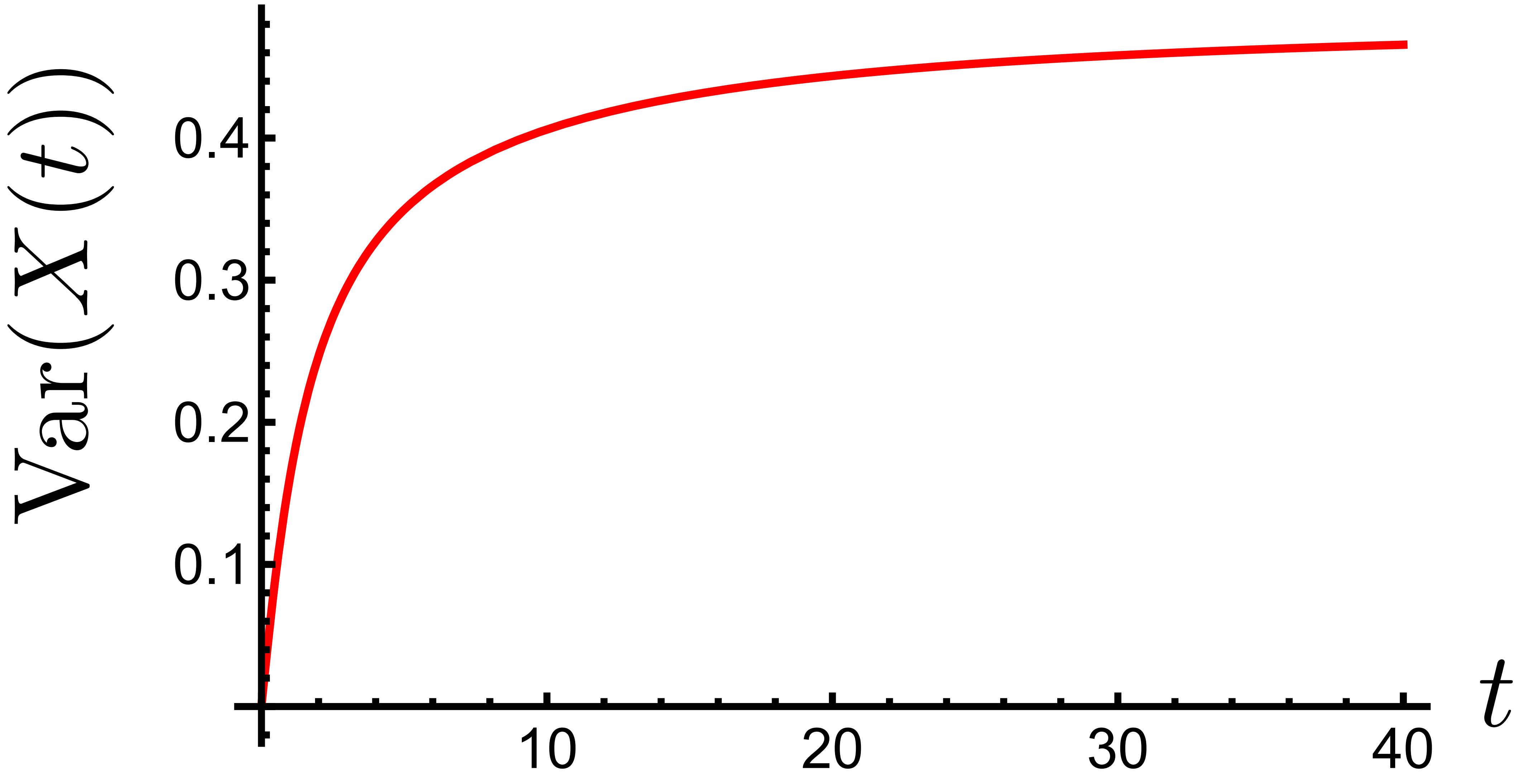


Inhomogeneous background: Power law profile

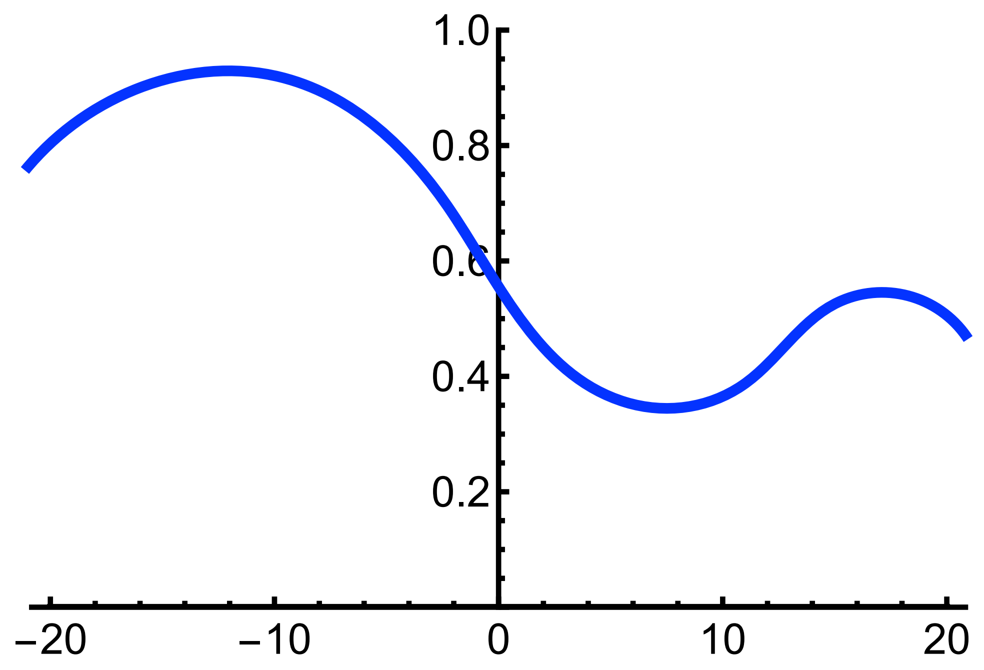


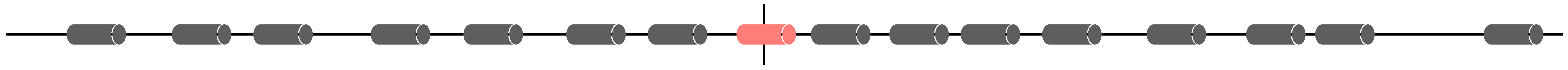






Inhomogeneous background: Generic inhomogeneous profile





Homogeneous background:

$$\langle X_{v_0}^2(t)\rangle_c = a^2\ \varphi_0\ \mathcal{D}_{\rm h}(v_0)\ t$$
$$\langle X_{v_0}(t_1)X_{v_0}(t_2)\rangle_c = a^2\ \varphi_0\ \mathcal{D}_{\rm h}(v_0)\ \min(t_1,t_2)$$

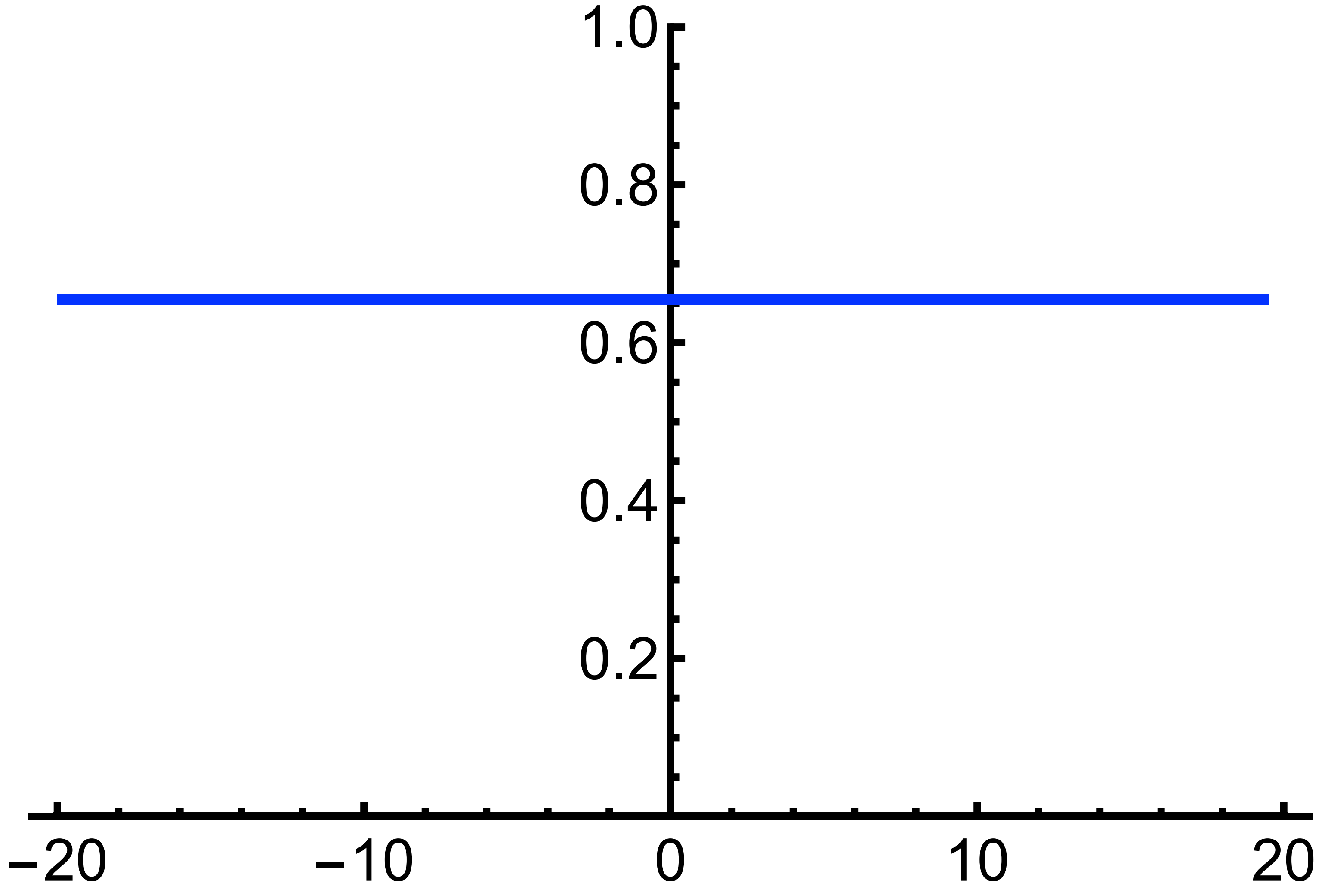
Domain wall background:

$$\langle X_{v_0}^2(t)\rangle_c = a^2\ \mathcal{D}_{\rm dw}(v_0,\rho_r,\rho_\ell)\ t$$
$$\langle X_{v_0}(t_1)X_{v_0}(t_2)\rangle_c = a^2\ \mathcal{D}_{\rm dw}(v_0,\rho_r,\rho_\ell)\ \min(t_1,t_2)$$

Generic inhomogeneous background:

$$\langle X_{v_0}^2(t)\rangle_c = a^2\ \mathcal{D}_{\rm ih}[\rho(X);v_0,t]$$
$$\langle X_{v_0}(t_1)X_{v_0}(t_2)\rangle_c = a^2\ \mathcal{D}_{\rm ih}[\rho(X);v_0,\min(t_1,t_2)]$$

Coming back to homogeneous background case



Recall: Quasiparticle statistics (Homogeneous Background)

$$
\begin{aligned}
\text{Variance 1:} \quad & \langle X_{v_0}(t)^2\rangle_c & \sim \quad & \mathscr{D}(v_0)\ t \\
\text{Variance 2:} \quad & \langle Y_{u_0}(t)^2\rangle_c & \sim \quad & \mathscr{D}(u_0)\ t \\
\text{Auto-correlation:} \quad & \langle X_{v_0}(t_1) X_{v_0}(t_2)\rangle_c & \sim \quad & \mathscr{D}(v_0)\ \min(t_1, t_2) \\
\text{Co-variance:} \quad & \langle X_{v_0}(t) Y_{u_0}(t)\rangle_c & \sim \quad & \mathscr{G}(u, v)\ t
\end{aligned}
$$

$$\mathscr{D}(v) = \varphi_0 a^2 \int_{-\infty}^{\infty} dw\ |w - v| \hbar(w)$$

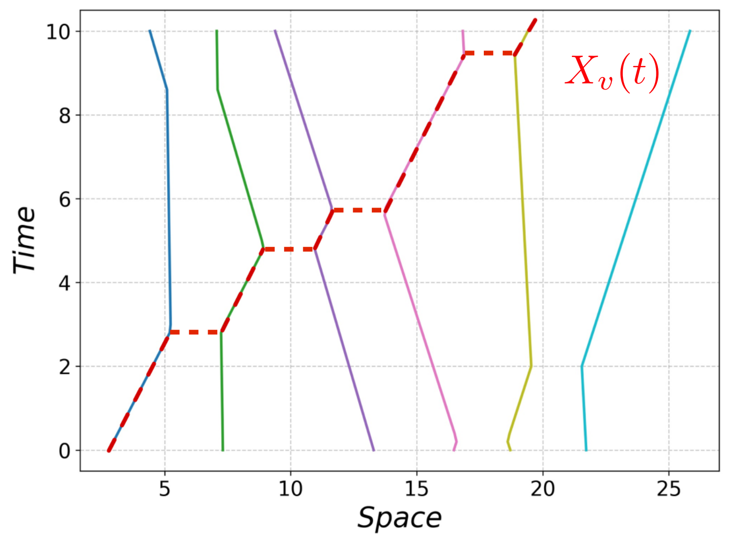
$$\mathscr{G}(u, v) = \mathscr{D}(u) + \mathscr{D}(v) - |u - v|$$

Summary ⇒ suggests

Quasiparticles perform correlated drifted Brownian motion

Quasiparticles in homogeneous background move like drifted Brownian particles

$$\frac{dX_v}{dt} = \frac{v}{1 - a\varrho_0} + \xi_v(t),$$



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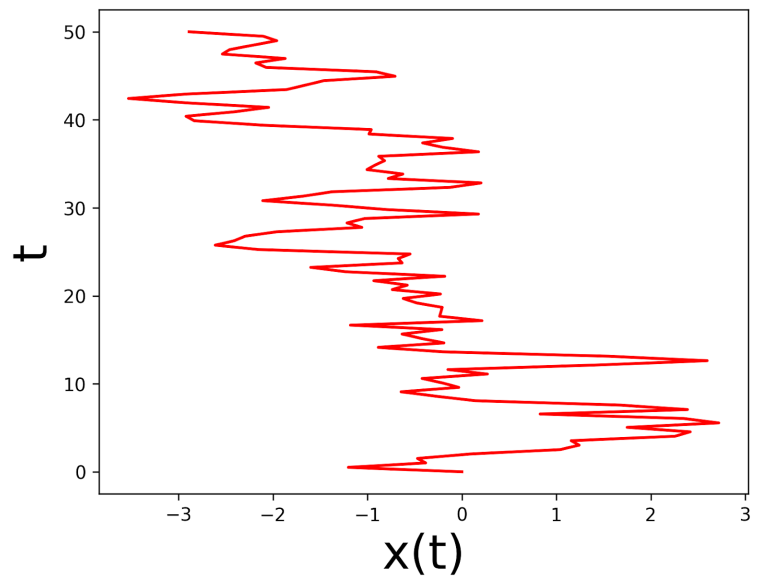

they are also correlated with other quasiparticles.

$$\frac{dX_i}{dt} = \bar{v}_i + \xi_i(t), \quad \text{for} \quad i = 1, 2, ..., \mathcal{N}, \quad \text{with} \quad \bar{v}_i = \frac{v_i}{1 - a\varrho_0},$$

$$\langle \xi_i(t)\xi_j(t') \rangle = \frac{\mathcal{G}(v_i, v_j)}{\sqrt{\mathcal{D}(v_i)}\sqrt{\mathcal{D}(v_j)}} \; \delta(t - t'), \qquad \text{for} \quad i, j = 1, 2, ..., \mathcal{N},$$

$$\langle \xi_i(t)\xi_i(t') \rangle = \mathcal{D}(v_i) \; \delta(t - t'),$$

Gas of Brownian particles

⇒ A homogeneous gas of hard rods can be thought of a gas of non-interacting but correlated (drifted) brownian particles.

$$\frac{dX_i}{dt} = \bar{v}_i + \xi_i(t), \quad \text{for} \quad i = 1, 2, ..., \mathcal{N}, \quad \text{with} \quad \bar{v}_i = \frac{v_i}{1 - a\varrho_0},$$

$$\langle \xi_i(t)\xi_j(t') \rangle = \frac{\mathcal{G}(v_i, v_j)}{\sqrt{\mathcal{D}(v_i)}\sqrt{\mathcal{D}(v_j)}} \, \delta(t - t'), \quad \text{for} \quad i, j = 1, 2, ..., \mathcal{N},$$

$$\langle \xi_i(t)\xi_i(t') \rangle = \mathcal{D}(v_i) \, \delta(t - t'),$$

⇒ Phenomenological derivation of fluctuating hydrodynamics

Phenomenological derivation of the fluctuating hydrodynamics for homogeneous gas

Empirical density:

$$\rho(X, v, t) = \sum_{i=1}^{\mathcal{N}} \delta(X - X_i)\ \delta(v - v_i).$$

Fourier transform:

$$\hat{\rho}(k, v, t) = \int_{-\infty}^{\infty} dX\ e^{\iota k X}\ \rho(X, v, t) = \sum_j e^{\iota k X_j}\ \delta(v - v_j).$$

$$\hat{\rho}(k, v, t + dt) = \sum_j \left[1 + \iota k\ \bar{v}_j dt + \iota k\ \delta\xi_j(dt) - \frac{k^2}{2}(\delta\xi_j)^2 + O(dt^2)\right]\ e^{\iota k X_j(t)} \delta(v - v_j)$$

$$\partial_t \hat{\rho}(k, v, t) = \iota k \bar{v} \hat{\rho}(k, v, t) - \frac{k^2}{2}\ \mathcal{D}(v) \hat{\rho}(k, v, t) + \iota k\ \zeta_k(v, t)$$

$$\zeta_k(v, t) = \frac{1}{dt} \sum_j e^{\iota k X_j(t)}\ \delta(v - v_j) \delta\xi_j(dt)$$

Seema, Mukherjee, Dhar, Spohn. Kundu (2025)

Phenomenological derivation of the fluctuating hydrodynamics for homogeneous gas

Inverse Fourier transform:

$$\partial_t \not{f}(X,v,t) = \frac{v}{1-a\varrho_0}\partial_X \not{f}(X,v,t) + \frac{\mathscr{D}(v)}{2}\partial_X^2 \not{f}(X,v,t) + \partial_X \not{f}(X,v,t)\sqrt{\mathscr{D}(v)}\dot{W}_t(v),$$

$$\langle \dot{W}_t(v)\dot{W}_s(v)\rangle = \delta(t-s)\mathscr{D}(v), \quad \text{and}, \quad \langle \dot{W}_t(u)\dot{W}_s(v)\rangle = \delta(t-s)\frac{\mathscr{G}(u,v)}{\sqrt{\mathscr{D}(v)}\sqrt{\mathscr{D}(u)}}.$$

$$\mathscr{D}(v) = \varphi_0 a^2 \int_{-\infty}^{\infty} dw\ \left|w-v\right|\hbar(w)$$

Seema, Mukherjee, Dhar, Spohn. Kundu (2025)

Phenomenological derivation of the fluctuating hydrodynamics for homogeneous gas

Inverse Fourier transform:

$$\partial_t \mathscr{f}(X,v,t) = \frac{v}{1-a\varrho_0}\partial_X \mathscr{f}(X,v,t) + \frac{\mathscr{D}(v)}{2}\partial_X^2 \mathscr{f}(X,v,t) + \partial_X \mathscr{f}(X,v,t)\sqrt{\mathscr{D}(v)}\dot{W}_t(v),$$

$$\langle \dot{W}_t(v)\dot{W}_s(v)\rangle = \delta(t-s)\mathscr{D}(v), \quad \text{and}, \quad \langle \dot{W}_t(u)\dot{W}_s(v)\rangle = \delta(t-s)\frac{\mathscr{G}(u,v)}{\sqrt{\mathscr{D}(v)}\sqrt{\mathscr{D}(u)}}.$$

$$\mathscr{D}(v) = \varphi_0 a^2 \int_{-\infty}^{\infty} dw\ |w-v| \mathscr{h}(w)$$

Proved rigorously using a probabilistic method

Macroscopic diffusive fluctuations for generalized hard rods dynamics
P. Ferrari and S. Olla, Ann. Appl. Probab. 35(2): 1125-1142 (2025)

Seema, Mukherjee, Dhar, Spohn. Kundu (2025)

—- Euler GHD (even with NS terms) can only provide mean behaviour of the conserved densities, not their fluctuations.

—- Fluctuating HD can provide statistics beyond mean

Density correlations

—- Ballistic macroscopic fluctuation theory
Doyon B, Perfetto G, Sasamoto T and Yoshimura T,
SciPost Phys. 15 136 (2023)

These correlations essentially originate from the randomness in the initial configurations.

These correlations essentially originate from the randomness in the initial configurations.

Quasiparticle statistical properties can be obtained from Euler scale correlation of the phase space density

Recall: for homogeneous background

Variance 1:	$\langle X_{v_0}(t)^2\rangle_c$	$\sim$	$\mathscr{D}(v_0)\ t$
Variance 2:	$\langle Y_{u_0}(t)^2\rangle_c$	$\sim$	$\mathscr{D}(u_0)\ t$
Auto-correlation:	$\langle X_{v_0}(t_1)X_{v_0}(t_2)\rangle_c$	$\sim$	$\mathscr{D}(v_0)\ \min(t_1,t_2)$
Co-variance:	$\langle X_{v_0}(t)Y_{u_0}(t)\rangle_c$	$\sim$	$(\mathscr{D}(u_0)+\mathscr{D}(v_0)-\lvert v_0-u_0\rvert)\ t$

$$\mathscr{D}(v) = \varphi_0 a^2 \int_{-\infty}^{\infty} dw\ \lvert w-v\rvert \hbar(w)$$

Tagged particle fluctuation from Euler scale correlation in a homogeneous gas

Displacement of the tracer rod can be written as

$$X(t) = v_0 t + a \left[\underbrace{n_{r\ell}}_{\substack{\text{Number of rods} \\ \text{crossed from } r \to \ell}} - \underbrace{n_{\ell r}}_{\substack{\text{Number of rods} \\ \text{crossed from } \ell \to r}} \right],$$

This can be expressed in terms of the mass density $\rho(Y,t)$ of hard rods.

$$\underbrace{\int_{-\infty}^{X(t)} dY \ \rho(Y,t)}_{\substack{\text{Number of rods on the left} \\ \text{of the quasiparticle at } t>0}} = \underbrace{\int_{-\infty}^{0} dY \ \rho(Y,0)}_{\substack{\text{Number of rods on the left} \\ \text{of the quasiparticle at } t=0}} + \underbrace{(n_{r\ell} - n_{\ell r})}_{\substack{\text{Net number of rods that crossed the trajectory} \\ \text{of the tagged quasiparticle from } r \to \ell \text{ till } t>0}},$$

Hence we have:

$$X(t) = v_0 t + a \left[\int_{-\infty}^{X(t)} dY \ \rho(Y,t) - \int_{-\infty}^{0} dY \ \rho(Y,0) \right],$$

Tagged particle fluctuation from Euler scale correlation

$$X(t) = v_0 t + a\left[\int_{-\infty}^{X(t)} dY\ \rho(Y,t) - \int_{-\infty}^{0} dY\ \rho(Y,0)\right],$$

This can be further expressed in terms of the point particle mass density as

$$X(t) = v_0 t + a\left[\int_{-\infty}^{v_0 t} dy \underbrace{\varphi(y,t)}_{\substack{\text{Hard point}\\ \text{mass density at } t}} - \int_{-\infty}^{0} dy\ \varphi(y,0)\right],$$

Tagged particle fluctuation from Euler scale correlation

$$X(t) = v_0 t + a\left[\int_{-\infty}^{v_0 t} dy \underbrace{\varphi(y,t)}_{\substack{\text{Hard point} \\ \text{mass density at } t}} - \int_{-\infty}^{0} dy\, \varphi(y,0)\right],$$

Variance: $\Sigma_X^2(t) = \langle \mathscr{A}(0)^2 \rangle_c + \langle \mathscr{A}(t)^2 \rangle_c - 2\langle \mathscr{A}(t)\mathscr{A}(0)\rangle_c,$

where

$$\langle \mathscr{A}(t)\mathscr{A}(t')\rangle_c = a^2 \int_{-\infty}^{0} dy \int_{-\infty}^{0} dy' \, \langle \varphi(y,t)\varphi(y',t')\rangle_c.$$

Mass density correlation of the hard point particles at Euler space time scale

Euler scale correlation of the phase space density

We assume the hard rod gas starts in a homogeneous state with mass density ϱ_0 and velocities chosen from a symmetric distribution $\hbar(v) = \hbar(-v)$.

For this case, the initial correlation of the corresponding hard point density is

$$\langle f(x, v, 0) f(y, u, 0)\rangle_c = \varphi_0 \; \delta(x - y)\delta(v - u) \; \hbar(v).$$

This initial correlation gets evolved in time.

Euler scale correlation of the phase space density

We assume the hard rod gas starts in a homogeneous state with mass density ϱ_0 and velocities chosen from a symmetric distribution $\hbar(v) = \hbar(-v)$.

For this case, the initial correlation of the corresponding hard point density is

$$\langle f(x,v,0) f(y,u,0)\rangle_c = \varphi_0 \, \delta(x-y)\delta(v-u) \, \hbar(v).$$

This $\Rightarrow$ the following (point particle) mass-density correlation

$$\begin{aligned} \langle \varphi(y,t)\varphi(y',t')\rangle_c &= \int dv \int du \, \langle f(y,v,t) f(y',u,t')\rangle_c \\ &= \int dv \int du \, \langle f(y-vt,v,0) f(y'-ut',u,0)\rangle_c . \\ &= \varphi_0 \frac{1}{|t-t'|} \, \hbar\left(\frac{y-y'}{|t-t'|}\right), \end{aligned}$$

One can use this correlation to study the tagged particle fluctuations.

Tagged particle fluctuation from Euler scale correlation

Variance: $\Sigma_X^2(t) = \langle \mathscr{A}(0)^2 \rangle_c + \langle \mathscr{A}(t)^2 \rangle_c - 2\langle \mathscr{A}(t)\mathscr{A}(0) \rangle_c,$

where

$$\langle \mathscr{A}(t)\mathscr{A}(t') \rangle_c = a^2 \int_{-\infty}^{0} dy \int_{-\infty}^{0} dy' \, \langle \varphi(y,t)\varphi(y',t') \rangle_c.$$

Recall: The (point particle) mass-density correlation

$$\langle \varphi(y,t)\varphi(y',t') \rangle_c = \varphi_0 \frac{1}{|t-t'|} \, \hbar \left(\frac{y-y'}{|t-t'|} \right),$$

One gets the same result obtained previously using microscopic approach

$$\Sigma_X^2(t) = t \; a^2 \varphi_0 \int_{-\infty}^{\infty} dw \, |w - v_0| \, \hbar(w) = t \; a^2 \varphi_0 \; \mathcal{D}(v_0),$$

Tagged particle fluctuation from Euler scale correlation

Following similar method

$$\text{Auto-correlation:} \quad \langle X_{v_0}(t_1) X_{v_0}(t_2) \rangle_c \sim \mathscr{D}(v_0) \; \min(t_1, t_2)$$

$$\text{Co-variance:} \quad \langle X_{v_0}(t) Y_{u_0}(t) \rangle_c \sim (\mathscr{D}(u_0) + \mathscr{D}(v_0) - |v_0 - u_0|) \; t$$

Seema, Mukherjee, Dhar, Spohn. Kundu (2025)

Equilibrium space-time correlation of hard rod gas

Recall: The (point particle) mass-density correlation

$$\langle \varphi(y,t)\varphi(y',t')\rangle_c = \varphi_0 \frac{1}{|t-t'|}\, \hbar\left(\frac{y-y'}{|t-t'|}\right),$$

$$\varrho_0 = \frac{\varphi_0}{1+a\varphi_0}$$

For hard rod gas this gets translated to

$$\begin{aligned}\mathcal{C}^{\mathrm{eq}}_{0,0}(X,t;X',t') &= \langle \varrho(X,t)\varrho(X',t')\rangle \\ &= \varrho_0(1-a\varrho_0)^3 \frac{1}{|t-t'|}\hbar\left(\frac{(1-a\varrho_0)\,(x_a-x_b)}{|t-t'|}\right),\end{aligned}$$

This result is true for homogeneous gas.

What about inhomogeneous case?

Ballistic macroscopic fluctuation theory (BMFT)

Doyon, Perfetto, Sasamoto, Yoshimura, SciPost 15, 136 (2023).

Physical Review Letters, 131(2):027101, 2023.

Long-range correlation

Phase space density correlation

$$\mathscr{C}(\mathtt{x}_a, u_a, \mathtt{t}_a; \mathtt{x}_b, u_b; \mathtt{t}_b) := \langle \not{f}(\mathtt{x}_a, u_a, \mathtt{t}_a)\not{f}(\mathtt{x}_b, u_b, \mathtt{t}_b)\rangle_c$$

where the empirical phase space density is

$$\not{f}(\mathtt{x}, u, \mathtt{t}) = \sum_{i=1}^{N} \delta(\mathtt{x} - \mathtt{x}_i)\delta(u - v_i)$$

$$\mathscr{C}(\mathtt{x}_a, u_a, \mathtt{t}_a; \mathtt{x}_b, u_b; \mathtt{t}_b) = -\left[\frac{\partial}{\partial\lambda} \frac{\langle \not{f}(\mathtt{x}_b, u_b, \mathtt{t}_b) e^{-\lambda \not{f}(\mathtt{x}_a, u_a, \mathtt{t}_a)}\rangle_{\mathbb{P}_\mathrm{r}}}{\langle e^{-\lambda \not{f}}(\mathtt{x}_a, u_a, \mathtt{t}_a)\rangle_{\mathbb{P}_\mathrm{r}}}\right]_{\lambda=0},$$

where $\mathbb{P}_\mathrm{r}(\mathtt{x}_i, v_i) =$ Joint pdf of $\{\mathtt{x}_i, v_i\}$

Hard rod gas initial condition

The joint probability distribution from which the initial positions and velocities of the rods are chosen:

$$\mathbb{P}_{\mathrm{r}}(\{\mathtt{x}_i, v_i\}) = \frac{1}{Z_r} \prod_{i=1}^{N} \Psi(\mathtt{x}_i, v_i) \prod_{i=1}^{N-1} \Theta(\mathtt{x}_{i+1} - \mathtt{x}_i - a),$$

with $\Psi(\mathtt{x}, v) = e^{-\psi(\mathtt{x}/\ell, v)}$.

A kundu, Arxiv: 2504.09201

Hard rod gas initial condition

The joint probability distribution from which the initial positions and velocities of the rods are chosen:

$$\mathbb{P}_{\mathrm{r}}(\{\mathtt{x}_i, v_i\}) = \frac{1}{Z_r} \prod_{i=1}^{N} \Psi(\mathtt{x}_i, v_i) \prod_{i=1}^{N-1} \Theta(\mathtt{x}_{i+1} - \mathtt{x}_i - a),$$

with $\Psi(\mathtt{x}, v) = e^{-\psi(\mathtt{x}/\ell, v)}$.

A kundu, Arxiv: 2504.09201

Recall the other initial condition: Point particle positions are first chosen from i.i.d. distribution and then ordered and transformed to hard rod coordinates.

$$\mathbb{P}^{\mathrm{point}}(\{x_i, v_i\}, 0) = N! \prod_{i=1}^{N} p_a(x_i) h(v_i) \prod_{i=1}^{N-1} \Theta(x_{i+1} - x_i),$$

$$\mathtt{x}_i = x_i + a \sum_{j \neq i} \Theta(x_i - x_j).$$

Generic initial condition

The joint probability distribution from which the initial positions and velocities of the rods are chosen:

$$\mathbb{P}_{\mathrm{r}}(\{\mathtt{x}_i, v_i\}) = \frac{1}{Z_r} \prod_{i=1}^{N} \Psi(\mathtt{x}_i, v_i) \prod_{i=1}^{N-1} \Theta(\mathtt{x}_{i+1} - \mathtt{x}_i - a),$$

$$\text{with} \quad \Psi(\mathtt{x}, v) = e^{-\psi(\mathtt{x}/\ell, v)}.$$

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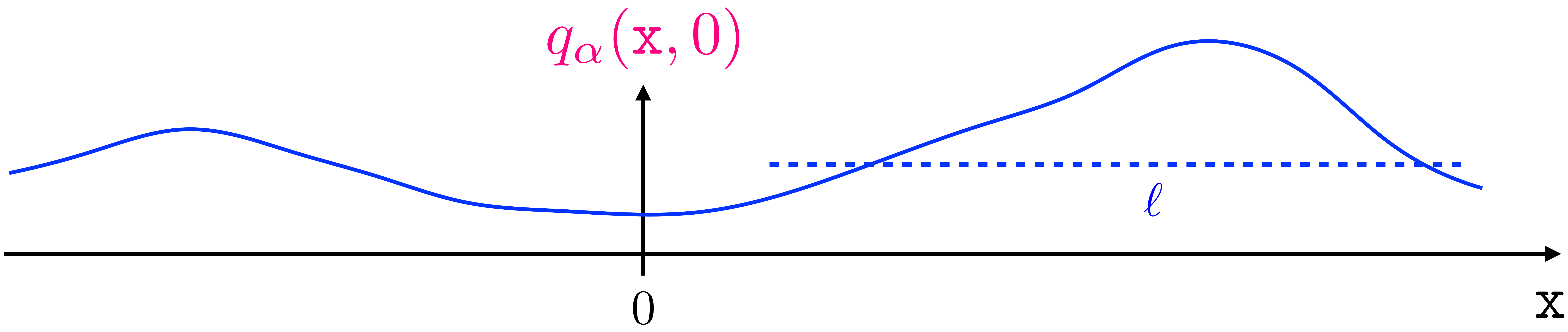


Ballistic macroscopic fluctuation theory of systems with ballistic transport

Principles:

1. Hydrodynamic principle: The large scale motion of the system can be described by mesoscopic means of local observables (coarse grained) which are functions of locally conserved densities. This means the currents are functions of the conserved densities → Euler Equations.

2. Principle of local relaxation: Mesoscopic means of local observables do not fluctuate independently from the conserved densities, but are fixed functions of conserved densities

3.

"Euler- scale fluctuations of time-evolved observables are obtained by deterministically transporting fluctuations of conserved quantities in the initial state via the Euler hydrodynamic equations of the model. "

$$\mathbb{O} = \int dx \int dv \ \mathfrak{b}(x,v)\mathfrak{f}(x,v,t)$$

Doyon, Perfetto, Sasamoto, Yoshimura, SciPost 15, 136 (2023).

Ballistic Macroscopic Fluctuation Theory

$$\mathbb{O} = \int dx \int dv\ \mathfrak{b}(x,v)\mathfrak{f}(x,v,t)$$

$$\langle e^{-\lambda \mathbb{O}} \rangle_{\mathbb{P}_\mathrm{r}} \approx \int \mathscr{D}[\mathfrak{f}(x,v,t)]\ \mathscr{P}[\mathfrak{f}(x,v,t)]\ e^{-\lambda \int dx \int dv\ \ \mathfrak{b}(x,v)\ \ \mathfrak{f}(x,v,t)}$$

$$\mathscr{P}[\mathfrak{f}(x,v,t)] = \int \mathscr{D}[\mathfrak{f}(x,v,0)]\ \underbrace{\mathscr{P}_{\mathrm{ini}}[\mathfrak{f}(x,v,0)]}_{\sim e^{-\ell \mathscr{F}[\mathfrak{f}(x,v,0)]}}\ \delta\left\{\partial_t \mathfrak{f}(x,v,t) + \partial_x \left(v_{\mathrm{eff}}(x,v,t)\mathfrak{f}(x,v,t)\right)\right\}$$

$$\langle \ldots \rangle_{\mathbb{P}_\mathrm{r}} \approx \langle \ldots \rangle_{\mathscr{P}[\mathfrak{f}]}$$

$$\mathscr{C}(\mathtt{x}_a, u_a, \mathtt{t}_a; \mathtt{x}_b, u_b; \mathtt{t}_b) = -\left[\frac{\partial}{\partial \lambda} \frac{\langle \ell(\mathtt{x}_b, u_b, \mathtt{t}_b) e^{-\lambda \ell(\mathtt{x}_a, u_a, \mathtt{t}_a)} \rangle_{\mathbb{P}_\mathrm{r}}}{\langle e^{-\lambda \ell}(\mathtt{x}_a, u_a, \mathtt{t}_a) \rangle_{\mathbb{P}_\mathrm{r}}}\right]_{\lambda=0},$$

where $\mathbb{P}_\mathrm{r}(\mathtt{x}_i, v_i) =$ Joint pdf of $\{\mathtt{x}_i, v_i\}$

Doyon, Perfetto, Sasamoto, Yoshimura, SciPost 15, 136 (2023).

Ballistic Macroscopic Fluctuation Theory

$$\mathbb{O} = \int dx \int dv \; \mathfrak{b}(x,v)\mathfrak{f}(x,v,t)$$

Benjamin's Talk

$$\langle e^{-\lambda \mathbb{O}} \rangle_{\mathbb{P}_{\mathrm{r}}} \approx \int \mathscr{D}[\mathfrak{f}(x,v,t)] \; \mathscr{P}[\mathfrak{f}(x,v,t)] \; e^{-\lambda \int dx \int dv \; \mathfrak{b}(x,v) \; \mathfrak{f}(x,v,t)}$$

$$\mathscr{P}[\mathfrak{f}(x,v,t)] = \int \mathscr{D}[\mathfrak{f}(x,v,0)] \; \underbrace{\mathscr{P}_{\mathrm{ini}}[\mathfrak{f}(x,v,0)]}_{\sim e^{-\ell \mathscr{F}[\mathfrak{f}(x,v,0)]}} \; \delta\left\{\partial_t \mathfrak{f}(x,v,t) + \partial_x \left(v_{\mathrm{eff}}(x,v,t)\mathfrak{f}(x,v,t)\right)\right\}$$

$$\left\langle \ldots \right\rangle_{\mathbb{P}_{\mathrm{r}}} \approx \left\langle \ldots \right\rangle_{\mathscr{P}[\mathfrak{f}]}$$

$$\mathscr{C}(\mathtt{x}_a, u_a, \mathtt{t}_a; \mathtt{x}_b, u_b; \mathtt{t}_b) = -\left[\frac{\partial}{\partial \lambda} \frac{\langle \mathfrak{f}(\mathtt{x}_b, u_b, \mathtt{t}_b) e^{-\lambda \mathfrak{f}(\mathtt{x}_a, u_a, \mathtt{t}_a)} \rangle_{\mathscr{P}[\mathfrak{f}]}}{\langle e^{-\lambda \mathfrak{f}}(\mathtt{x}_a, u_a, \mathtt{t}_a) \rangle_{\mathscr{P}[\mathfrak{f}]}}\right]_{\lambda=0},$$

Doyon, Perfetto, Sasamoto, Yoshimura, SciPost 15, 136 (2023).

$\langle \dots \rangle_{\mathbb{P}_{\mathrm{r}}} \approx \langle \dots \rangle_{\mathscr{P}[\mathfrak{f}]}$

Phase space density correlation

$$\mathscr{C}(\mathtt{x}_a, u_a, \mathtt{t}_a; \mathtt{x}_b, u_b; \mathtt{t}_b) = -\left[\frac{\partial}{\partial\lambda} \frac{\langle \mathfrak{f}(\mathtt{x}_b, u_b, \mathtt{t}_b) e^{-\lambda \mathfrak{f}(\mathtt{x}_a, u_a, \mathtt{t}_a)} \rangle_{\mathscr{P}[\mathfrak{f}]}}{\langle e^{-\lambda \mathfrak{f}}(\mathtt{x}_a, u_a, \mathtt{t}_a) \rangle_{\mathscr{P}[\mathfrak{f}]}}\right]_{\lambda=0},$$

Saddle point method for large ℓ:

$$\mathscr{C}(\mathtt{x}_a, u_a, \mathtt{t}_a; \mathtt{x}_b, u_b; \mathtt{t}_b) = \frac{1}{\ell} \mathcal{C}\left(\frac{\mathtt{x}_a}{\ell}, u_a, \frac{\mathtt{t}_a}{\ell}; \frac{\mathtt{x}_b}{\ell}, u_b, \frac{\mathtt{t}_b}{\ell}\right),$$

$$\mathcal{C}(x_a, u_a, t_a; x_b, u_b, t_b) = -\left[\frac{\partial}{\partial\lambda} \mathfrak{f}_\lambda^{\mathrm{sd}}(x_b, u_b, t_b)\right]_{\lambda=0}$$

Doyon, Perfetto, Sasamoto, Yoshimura, SciPost 15, 136 (2023).

Phase space density correlation

BMFT in terms of hard rod phase space densities

$$\langle e^{-\lambda \mathbb{O}} \rangle_{\mathbb{P}_r} \approx \int \mathscr{D}[\mathfrak{f}(x,v,t)]\ \mathscr{P}[\mathfrak{f}(x,v,t)]\ e^{-\lambda \int dx \int dv\ \mathfrak{b}(x,v)\ \mathfrak{f}(x,v,t)}$$

$$\mathscr{P}[\mathfrak{f}(x,v,t)] = \int \mathscr{D}[\mathfrak{f}(x,v,0)]\ \underbrace{\mathscr{P}_{\text{ini}}[\mathfrak{f}(x,v,0)]}_{\sim e^{-\ell \mathscr{F}[\mathfrak{f}(x,v,0)]}}\ \delta\left\{\partial_t \mathfrak{f}(x,v,t) + \partial_x \left(v_{\text{eff}}(x,v,t)\mathfrak{f}(x,v,t)\right)\right\}$$

$$\mathbb{O} = \int dx \int dv\ \mathfrak{b}(x,v)\mathfrak{f}(x,v,t)$$

BMFT in terms of point particle phase space densities

$$\mathscr{P}_{\text{free}}[f(x',v,t)] = \int \mathscr{D}[f(x',v,0)]\ \underbrace{\mathscr{P}^{\text{free}}_{\text{ini}}[f(x',v,0)]}_{\sim e^{-\ell \mathscr{F}_{\text{free}}[f(x',v,0)]}}\ \delta\left\{\partial_t f(x',v,t) + \partial_{x'} \left(v f(x',v,t)\right)\right\}$$

A kundu, Arxiv: 2504.09201

Phase space density correlation

$$\mathfrak{O} = \int dx \int dv \; \mathfrak{b}(x,v)\mathfrak{f}(x,v,t) \longrightarrow \mathfrak{O} = \int dx' \int dv \; \mathfrak{b}(x(x'),v) f(x',v,t)$$

We compute the saddle point density $\mathfrak{f}_{\mathrm{sd}}(x,v,t)$ of the hard rods.

Scaled correlation function:

$$\mathscr{C}(\mathtt{x}_a, u_a, \mathtt{t}_a; \mathtt{x}_b, u_b; \mathtt{t}_b) = \frac{1}{\ell}\mathcal{C}\left(\frac{\mathtt{x}_a}{\ell}, u_a, \frac{\mathtt{t}_a}{\ell}; \frac{\mathtt{x}_b}{\ell}, u_b, \frac{\mathtt{t}_b}{\ell}\right),$$

$$\mathcal{C}(x_a, u_a, t_a; x_b, u_b, t_b) = -\left[\frac{\partial}{\partial\lambda}\mathfrak{f}_\lambda^{\mathrm{sd}}(x_b, u_b, t_b)\right]_{\lambda=0}$$

Long-range non-equilibrium correlation

For large ℓ the correlation has scaling form

$$\mathscr{C}(\mathtt{x}_a, u_a, \mathtt{t}_a; \mathtt{x}_b, u_b; \mathtt{t}_b) = \frac{1}{\ell}\mathcal{C}\left(\frac{\mathtt{x}_a}{\ell}, u_a, \frac{\mathtt{t}_a}{\ell}; \frac{\mathtt{x}_b}{\ell}, u_b, \frac{\mathtt{t}_b}{\ell}\right),$$

$$\mathcal{C}_{0,0}(x_a, t_a; x_b; t_b) = \partial_{x_a}\partial_{x_b}\big[(1 - a\bar{\varrho}(x_a, t_a))(1 - a\bar{\varrho}(x_b, t_b))\, \mathscr{H}(x_a, t_a; x_b, t_b)\big],$$

where

$$\begin{aligned}
\mathscr{H}(x_a, t_a; x_b, t_b) = & \int dy' \int du\ \Theta(x'_a - ut_a - y')\Theta(x'_b - ut_b - y')\bar{\mathfrak{f}}(y', u, 0) \\
& - a \int dy' \int du \int dw\ (1 - a\bar{\varrho}(y,0))\Theta(x'_a - wt_a - y')\Theta(x'_b - ut_b - y') \\
& \times \ \left\{ \frac{2 - a\bar{\varrho}(y,0)}{(1 - a\bar{\varrho}(y,0))^2}\bar{\mathfrak{f}}(y,u,0)\bar{\mathfrak{f}}(y,w,0) - \bar{\mathfrak{f}}(y,u,0)\bar{\mathscr{f}}(x'_a, w, t_a) \right. \\
& \left. \qquad - \bar{\mathfrak{f}}(y,w,0)\bar{\mathscr{f}}(x'_b, u, t_b) - a\bar{\varrho}(y,0)\bar{\mathscr{f}}(x'_a, w, t_a)\bar{\mathscr{f}}(x'_b, u, t_b) \right\}.
\end{aligned}$$

A kundu, Arxiv: 2504.09201

Long-range non-equilibrium correlation

$$\mathcal{C}_{0,0}(x_a, t_a; x_b; t_b) = \partial_{x_a}\partial_{x_b}\big[(1 - a\bar{\varrho}(x_a, t_a))(1 - a\bar{\varrho}(x_b, t_b))\ \mathscr{H}(x_a, t_a; x_b, t_b)\big],$$

where

$$\begin{aligned}\mathscr{H}(x_a, t_a; x_b, t_b) = &\int dy' \int du\ \Theta(x_a' - ut_a - y')\Theta(x_b' - ut_b - y')\bar{\ell}(y', u, 0) \\ &- a \int dy' \int du \int dw\ (1 - a\bar{\varrho}(y, 0))\Theta(x_a' - wt_a - y')\Theta(x_b' - ut_b - y') \\ &\times \Big\{\frac{2 - a\bar{\varrho}(y, 0)}{(1 - a\bar{\varrho}(y, 0))^2}\bar{\mathfrak{f}}(y, u, 0)\bar{\mathfrak{f}}(y, w, 0) - \bar{\mathfrak{f}}(y, u, 0)\bar{\ell}(x_a', w, t_a) \\ &\quad - \bar{\mathfrak{f}}(y, w, 0)\bar{\ell}(x_b', u, t_b) - a\bar{\varrho}(y, 0)\bar{\ell}(x_a', w, t_a)\bar{\ell}(x_b', u, t_b)\Big\}.\end{aligned}$$

A kundu, Arxiv: 2504.09201

This was previously computed by

Doyon, Perfetto, Sasamoto, Yoshimura, SciPost 15, 136 (2023).
Physical Review Letters, 131(2):027101, 2023.
Hubner, Biagetti, De Nardis, Doyon. arXiv:2503.07794

Expression of $\mathcal{C}(x_a, u_a, t_a; x_b, u_b, t_b)$

$$\mathcal{C}_{0,0}(x_a, t_a; x_b; t_b) = \partial_{x_a}\partial_{x_b}\left[(1 - a\bar{\varrho}(x_a, t_a))(1 - a\bar{\varrho}(x_b, t_b))\ \mathscr{H}(x_a, t_a; x_b, t_b)\right],$$

where

$$\begin{aligned}\mathscr{H}(x_a, t_a; x_b, t_b) = &\int dy' \int du\ \Theta(x'_a - ut_a - y')\Theta(x'_b - ut_b - y')\bar{\mathscr{f}}(y', u, 0) \\ &- a\int dy' \int du \int dw\ (1 - a\bar{\varrho}(y, 0))\Theta(x'_a - wt_a - y')\Theta(x'_b - ut_b - y') \\ &\times \left\{\frac{2 - a\bar{\varrho}(y,0)}{(1 - a\bar{\varrho}(y,0))^2}\bar{\mathfrak{f}}(y, u, 0)\bar{\mathfrak{f}}(y, w, 0) - \bar{\mathfrak{f}}(y, u, 0)\bar{\mathscr{f}}(x'_a, w, t_a)\right. \\ &\left. - \bar{\mathfrak{f}}(y, w, 0)\bar{\mathscr{f}}(x'_b, u, t_b) - a\bar{\varrho}(y,0)\bar{\mathscr{f}}(x'_a, w, t_a)\bar{\mathscr{f}}(x'_b, u, t_b)\right\}.\end{aligned}$$

A kundu, Arxiv: 2504.09201

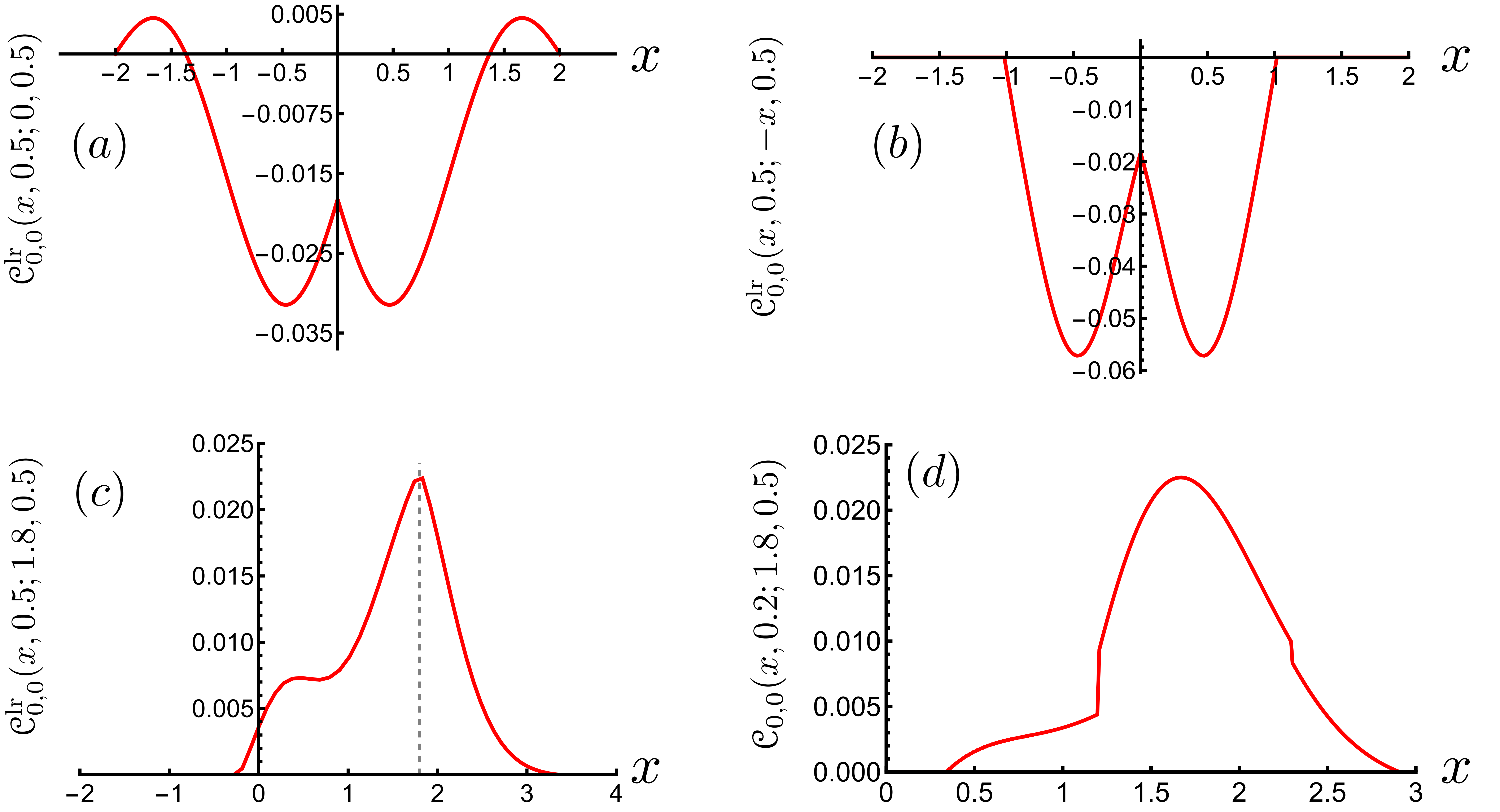
Satisfies three conditions of having non-zero long-range part:

1. In-homogeneous initial state.
2. Presence of interaction ($a \neq 0$)
3. Presence of more than one hydrodynamic modes with different velocities.

Doyon, Perfetto, Sasamoto, Yoshimura, SciPost 15, 136 (2023).

Physical Review Letters, 131(2):027101, 2023.

Plots of $\mathcal{C}(x_a, u_a, t_a; x_b, u_b, t_b)$



Conclusions

1. GHD has emerged as a fruitful description for studying large scale motion of integrable systems.
2. Hard rod system provides a simple yet non-trivial platform to understand various features of GHD, because, one can study this system both microscopically as well as hydrodynamically.
3. For certain class of initial conditions, one can perform **microscopic computations** which allow one to see **deviations of Euler solutions.**
4. Such deviations can also be described by Navier-Stokes (NS) corrections.
5. The effect of the NS terms can be exhibited prominently in **tagged quasiparticle problem.**
6. We find a tagged quasiparticle spreads diffusively —- in fact **moves effectively as a drifted Brownian motion.**
8. An effective description of hard rod gas as a gas of drifted and correlated Brownian particles provides a **phenomenological derivation of the FHD obtained by Ferrari and Olla.**
9. We showed that the variance and co-variance of tagged particles are **originated from the fluctuations and correlations in the initial state** that got transported ballistically by the Euler equations.

References:

Thermalization and hydrodynamics in an interacting integrable system: the case of hard rods
Singh, Dhar, Spohn, Kundu. Jstat. Phys. 191 (6), 66, (2024)

Conserved densities of hard rods: microscopic to hydrodynamic solutions
MJ Powdel, A Kundu, Jstat. Mech. 2024 (12), 123205

Dynamics of tagged quasiparticles in hard rod gas,
S. Chahal, I. Mukherjee, A. Dhar, H. Spohn, A. Kundu, Arxiv: 2506.xxxx

Macroscopic fluctuation theory of correlations in hard rod gas
A kundu, Arxiv: 2504.09201

Thank you