Signatures of integrability/non-integrability in two-dimensional quantum spin systems

local conserved quantities, operator growth and complex-time evolution

Hal Tasaki

N. Shiraishi and H. Tasaki, "The $S = 1/2$ XY and XYZ models on the two or higher dimensional hypercubic lattice do not possess nontrivial local conserved quantities" (2412.18504), H. Tasaki, unpublished

Hydrodynamics of low-dimensional interacting systems (YITP, Kyoto), June 11, 2025

motivation

mathematical studies of quantum many-body systems

exact solutions free fermion, Bethe ansatz, Yang-Baxter relation ....

only cover integrable models

rigorous, general theorems

cover models in a certain class,
both integrable and non-integrable models

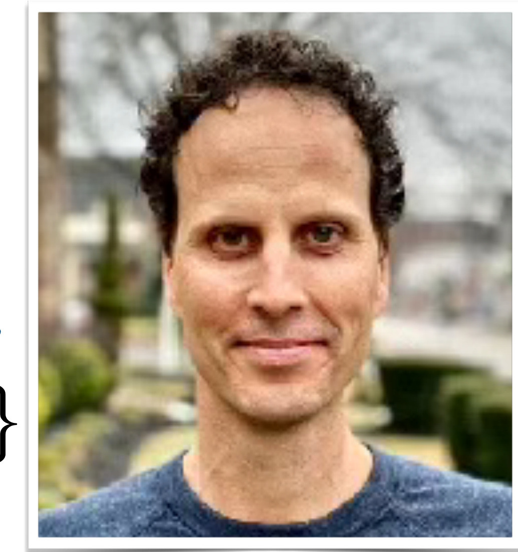
there are properties/phenomena (quantum chaos, thermalization, ETH = energy eigenstate thermalization hypothesis, standard hydrodynamics, ...) that are expected to take place only in non-integrable systems

mathematical results that exclusively apply to non-integrable systems??

early rigorous results that showed a concrete quantum model (with short-range interactions) exhibits a behavior that is never observed in integrable models

Bouch (2015) singularity in the imaginary-time evolution in a two-dimensional quantum spin system

$$\hat{H}_{\mathrm{B}} := \sum_{j,k\in\mathbb{Z}}\{\hat{X}_{(j,k)}\hat{Z}_{(j+1,k)} + \hat{Z}_{(j,k)}\hat{X}_{(j,k+1)}\}$$



Gabriel Bouch

Naoto Shiraishi

Shiraishi (2019) absence of nontrivial local conserved quantities in the $S = 1/2$ XYZ-h spin chain

Cao (2021) "quantum chaotic" behavior of the moment in the one and two-dimensional quantum Ising models

Xiangyu Cao

part 1

background
main results
idea of the proof

part 2

setting
operator growth
complex-time evolution
idea of the proof

summary and discussion

Shiraishi's work in 2019

$S=\frac{1}{2}$ **XYZ-h spin chain with Hamiltonian**

$$\hat{H}_{\text{XYZ-h}}=-\sum_{j=1}^{L}\{J_{\text{X}}\,\hat{X}_j\hat{X}_{j+1}+J_{\text{Y}}\,\hat{Y}_j\hat{Y}_{j+1}+J_{\text{Z}}\,\hat{Z}_j\hat{Z}_{j+1}+h\,\hat{Z}_j\}$$

integrable (can be mapped to a free fermion) if $J_{\text{Z}}=0$

series of local conserved quantities $[\hat{H}_{\text{XY-h}},\hat{Q}^{\pm}_{k_{\max}}]=0$

$$\hat{Q}_3^{+}=\sum_{j=1}^{L}\{J_{\text{X}}\,\hat{X}_j\hat{Z}_{j+1}\hat{X}_{j+2}+J_{\text{Y}}\,\hat{Y}_j\hat{Z}_{j+1}\hat{Y}_{j+2}-h(\hat{X}_j\hat{X}_{j+1}+\hat{Y}_j\hat{Y}_{j+1})\}$$

$$k_{\max}=3,4,\ldots$$

$$\hat{Q}_3^{-}=\sum_{j=1}^{L}\{\hat{X}_j\hat{Z}_{j+1}\hat{Y}_{j+2}-\hat{Y}_j\hat{Z}_{j+1}\hat{X}_{j+2}\}$$

$$\hat{Q}_4^{+}=\sum_{j=1}^{L}\{J_{\text{X}}(\hat{X}_j\hat{Z}_{j+1}\hat{Z}_{j+2}\hat{X}_{j+3}+\hat{Y}_j\hat{Y}_{j+1})+J_{\text{Y}}(\hat{Y}_j\hat{Z}_{j+1}\hat{Z}_{j+2}\hat{Y}_{j+3}+\hat{X}_j\hat{X}_{j+1})-h(\hat{X}_j\hat{Z}_{j+1}\hat{X}_{j+2}+\hat{Y}_j\hat{Z}_{j+1}\hat{Y}_{j+2})\}$$

$$\hat{Q}_4^{-}=\sum_{j=1}^{L}\{\hat{X}_j\hat{Z}_{j+1}\hat{Z}_{j+2}\hat{Y}_{j+3}-\hat{Y}_j\hat{Z}_{j+1}\hat{Z}_{j+2}\hat{X}_{j+3}\}$$

if $J_{\text{X}}\neq J_{\text{Y}}$, $J_{\text{Z}}\neq 0$, and $h\neq 0$, the model has no local conserved quantities with support size $3\leq k_{\max}\leq L/2$

Naoto Shiraishi, "Proof of the absence of local conserved quantities in the XYZ chain with a magnetic field", 2019

the first rigorous result that applies exclusively to a standard "non-solvable" model!

integrability and conserved quantities

Liouville integrability of a classical Hamiltonian system with $2n$ dimensional phase space

there exist n independent conserved quantities

quantum system with D dimensional Hilbert space and Hamiltonian $\hat{H}$

there always exist D "conserved quantities" $|\Psi_j\rangle\langle\Psi_j|$

integrable quantum many-body systems

one finds (via transfer matrices, boost operator, ...) a series of conserved quantities that are the sum of strictly local operators (some models possess quasi local conserved quantities)

the relationship between "integrability" and the presence of conserved quantities is subtle

absence of nontrivial local conserved quantities in one-dimensional quantum spin systems

quantum Ising model Chiba 2024

PXP model Park and Lee 2024

$S = 1/2$ chain with next-nearest neighbor interactions Shiraishi 2024

$S = 1$ chain with bilinear biquadratic interactions Park and Lee 2024

$S = 1$ chain with anisotropic bilinear biquadratic interaction Hokkyo, Yamaguchi, Chiba 2024

$S = 1/2$ chain with symmetric nearest neighbor interaction Yamaguchi, Chiba, Shiraishi 2024

$S = 1/2$ chain with symmetric next-nearest neighbor interaction Shiraishi 2025

complete classifications!

empirical rule: a simple quantum spin model is either integrable or does not possess local conserved quantities

recent general results for one-dimensional quantum spin systems

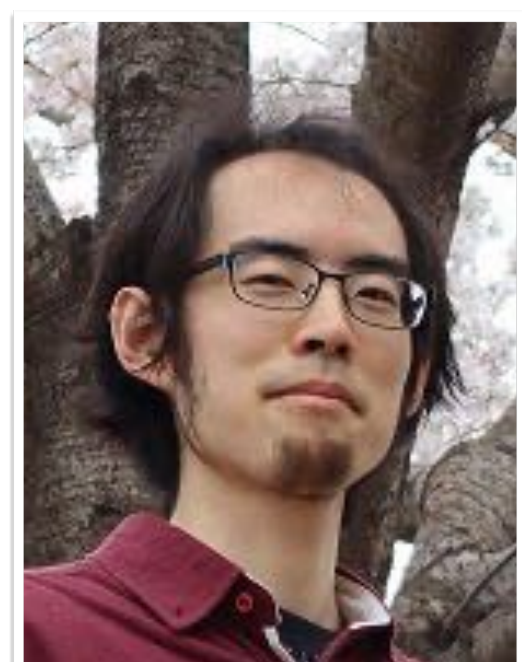
A. Hokkyo, “Rigorous Test for Quantum Integrability and Nonintegrability”, 2025

an efficient, rigorous scheme for establishing the absence of local conserved quantities in a general class of quantum spin systems (mostly in one dimension)

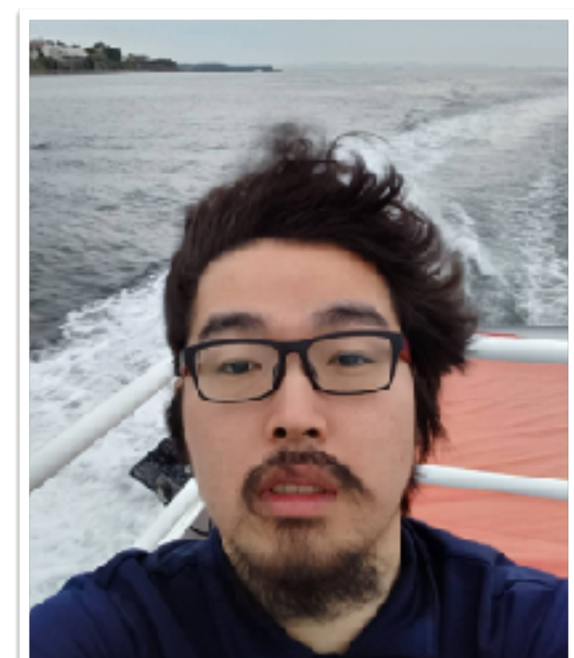
Akihiro Hokkyo

N. Shiraishi and M. Yamaguchi, “Dichotomy theorem distinguishing non-integrability and the lowest-order Yang-Baxter equation for isotropic spin chains”, 2025

a simple, concrete criterion that determines whether a model satisfies the Reshetikin condition or lacks local conserved quantities



Naoto Shiraishi



Mizuki Yamaguchi

the dichotomy theorem

N. Shiraishi and M. Yamaguchi, "Dichotomy theorem distinguishing non-integrability and the lowest-order Yang-Baxter equation for isotropic spin chains", 2025

a simple, concrete criterion that determines whether a model satisfies the Reshetikin condition or lacks local conserved quantities

the most general $SU(2)$ invariant n.n. spin S chain

$$\hat{H} = \sum_{j=1}^{L} \hat{h}_j \qquad \hat{h}_j = \sum_{n=1}^{2S} J_n (\hat{\boldsymbol{S}}_j \cdot \hat{\boldsymbol{S}}_{j+1})^n \qquad J_n \in \mathbb{R}$$

theorem: if $[[\hat{h}_j, \hat{h}_{j+1}], (\hat{h}_j + \hat{h}_{j+1})]$ contains a nonzero 3-support product, then the model has no nontrivial local-conserved quantities; else, the model satisfies the lowest order Yang-Baxter equation (the Reshetikin condition) and has conserved quantities $Q_3 = \sum_j [\hat{h}_j, \hat{h}_{j+1}]$ and

$$Q_4 = \sum_j \left([[\hat{h}_j, \hat{h}_{j+1}], \hat{h}_{j+1}] + 2[[\hat{h}_j, \hat{h}_{j+1}], \hat{h}_{j+2}] \right)$$

absence of nontrivial local conserved quantities in two or higher dimensional quantum spin systems

it is likely that a spin model is "less integrable" in higher dimensions

Yuuya Chiba

$S = 1/2$ Ising model with a magnetic field (not in the Z-direction) Chiba 2024

$S = 1/2$ XY and XYZ model with or without a magnetic field Shiraishi and Tasaki 2024

on the d-dimensional hypercubic lattice with $d \geq 2$, all "standard" $S = 1/2$ models (except for the classical Ising model) have no nontrivial local conserved quantities and are very likely to be "nonintegrable"

a close cousin of the Kitaev honeycomb model

$S = 1/2$ quantum compass model on the square lattice

Futami and Tasaki 2025

part 1

background
main results
idea of the proof

part 2

setting
operator growth
complex-time evolution
idea of the proof

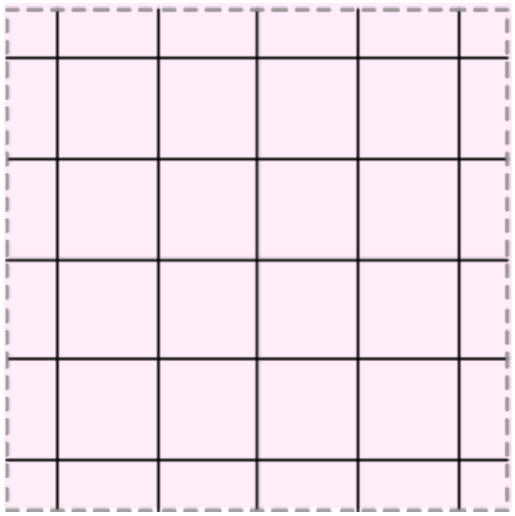
summary and discussion

$S = \frac{1}{2}$ model in two dimensions

operators of a single spin is spanned by

$$\hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{Y} = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \hat{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\Lambda = \{1, \ldots, L\}^2$ $L \times L$ **square lattice with periodic boundary conditions**



$\hat{X}_u, \hat{Y}_u, \hat{Z}_u$ **copies of** $\hat{X}, \hat{Y}, \hat{Z}$ **at site** $u \in \Lambda$

Hamiltonian of the XYZ model

$$\hat{H} = -\frac{1}{2} \sum_{\substack{u,v \in \Lambda \\ (|u-v|=1)}} \left\{ J_{\mathrm{X}}\, \hat{X}_u \hat{X}_v + J_{\mathrm{Y}}\, \hat{Y}_u \hat{Y}_v + J_{\mathrm{Z}}\, \hat{Z}_u \hat{Z}_v \right\}$$

$$- \sum_{u \in \Lambda} \left\{ h_{\mathrm{X}}\, \hat{X}_u + h_{\mathrm{Y}}\, \hat{Y}_u + h_{\mathrm{Z}}\, \hat{Z}_u \right\}$$

$$J_{\mathrm{X}}, J_{\mathrm{Y}}, J_{\mathrm{Z}}, h_{\mathrm{X}}, h_{\mathrm{Y}}, h_{\mathrm{Z}} \in \mathbb{R}, \qquad J_{\mathrm{X}} \neq 0, J_{\mathrm{Y}} \neq 0$$

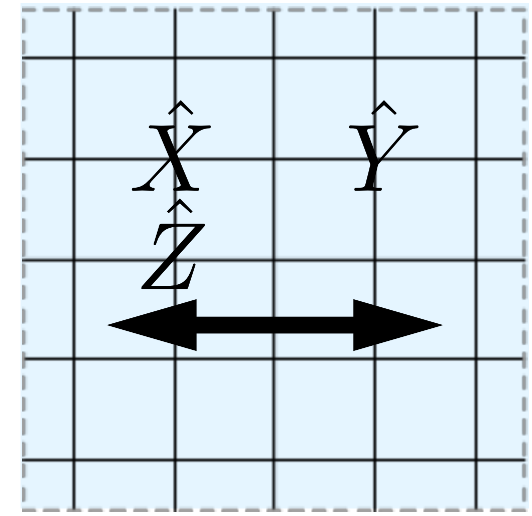
local conserved quantities

conserved quantities that are linear combinations of strictly local products

$\boldsymbol{A} = \bigotimes_{u \in S} \hat{A}_u$ product of Pauli matrices

$S \subset \Lambda \qquad \hat{A}_u = \hat{X}_u, \hat{Y}_u, \hat{Z}_u$

$\boldsymbol{A} =$



$\mathrm{Wid}\,\boldsymbol{A} = 3$

$\mathcal{P}_\Lambda$ the set of all products

$\mathrm{Wid}\,\boldsymbol{A}$ the horizontal width of the support $S \subset \Lambda$

candidate of a local conserved quantity with width k_{max} such that $2 \le k_{\mathrm{max}} \le \frac{L}{2}$

$$\hat{Q} = \sum_{\substack{\boldsymbol{A} \in \mathcal{P}_\Lambda \\ (\mathrm{Wid}\,\boldsymbol{A} \le k_{\mathrm{max}})}} q_{\boldsymbol{A}}\, \boldsymbol{A} \qquad q_{\boldsymbol{A}} \in \mathbb{C}$$

$q_{\boldsymbol{A}} \neq 0$ for at least one $\boldsymbol{A}$ with $\mathrm{Wid}\,\boldsymbol{A} = k_{\mathrm{max}}$

$\hat{Q}$ is a local conserved quantity iff $[\hat{H}, \hat{Q}] = 0$

main theorems

Shiraishi and Tasaki 2024

$$\hat{H}=-\frac{1}{2}\sum_{\substack{u,v\in\Lambda\\(|u-v|=1)}}\{J_{\mathrm{X}}\,\hat{X}_u\hat{X}_v+J_{\mathrm{Y}}\,\hat{Y}_u\hat{Y}_v+J_{\mathrm{Z}}\,\hat{Z}_u\hat{Z}_v\}-\sum_{u\in\Lambda}\{h_{\mathrm{X}}\,\hat{X}_u+h_{\mathrm{Y}}\,\hat{Y}_u+h_{\mathrm{Z}}\,\hat{Z}_u\}$$

$$J_{\mathrm{X}},J_{\mathrm{Y}},J_{\mathrm{Z}},h_{\mathrm{X}},h_{\mathrm{Y}},h_{\mathrm{Z}}\in\mathbb{R},\quad J_{\mathrm{X}}\neq0,J_{\mathrm{Y}}\neq0$$

$$\hat{Q}=\sum_{\substack{\boldsymbol{A}\in\mathcal{P}_\Lambda\\(\mathrm{Wid}\,\boldsymbol{A}\le k_{\mathrm{max}})}}q_{\boldsymbol{A}}\,\boldsymbol{A}\qquad q_{\boldsymbol{A}}\in\mathbb{C}$$

$\hat{Q}$ is a local conserved quantity iff $[\hat{H},\hat{Q}]=0$

Theorem: there are no local conserved quantities $\hat{Q}$ with width k_{max} such that $3\le k_{\mathrm{max}}\le\frac{L}{2}$

Hamiltonian is a local conserved quantity with $k_{\mathrm{max}}=2$

Theorem: any local conserved quantity with $k_{\mathrm{max}}=2$ is written as $\hat{Q}=\eta\,\hat{H}+\hat{Q}_1$ with $\eta\neq0$, where $\hat{Q}_1$ is a linear combination of single-site Pauli matrices

part 1

background
main results
idea of the proof

part 2

setting
operator growth
complex-time evolution
idea of the proof

summary and discussion

basic strategy of the proof

Shiraishi 2019, 2024

$$\hat{H} = -\frac{1}{2} \sum_{\substack{u,v\in\Lambda \\ (|u-v|=1)}} \{J_{\mathrm{X}}\,\hat{X}_u\hat{X}_v + J_{\mathrm{Y}}\,\hat{Y}_u\hat{Y}_v + J_{\mathrm{Z}}\,\hat{Z}_u\hat{Z}_v\} - \sum_{u\in\Lambda}\{h_{\mathrm{X}}\,\hat{X}_u + h_{\mathrm{Y}}\,\hat{Y}_u + h_{\mathrm{Z}}\,\hat{Z}_u\}$$

$$\boldsymbol{A} = \bigotimes_{u\in S} \hat{A}_u \qquad [\hat{H}, \boldsymbol{A}] = \sum_{\boldsymbol{B}\in\mathcal{P}_\Lambda} \lambda_{\boldsymbol{A},\boldsymbol{B}}\,\boldsymbol{B}$$

written in terms of $J_{\mathrm{X}}, J_{\mathrm{Y}}, J_{\mathrm{Z}}, h_{\mathrm{X}}, h_{\mathrm{Y}}, h_{\mathrm{Z}}$

$$\hat{X}^2 = \hat{Y}^2 = \hat{Z}^2 = \hat{I}$$
$$\hat{X}\hat{Y} = -\hat{Y}\hat{X} = \mathrm{i}\hat{Z}$$
$$\hat{Y}\hat{Z} = -\hat{Z}\hat{Y} = \mathrm{i}\hat{X}$$
$$\hat{Z}\hat{X} = -\hat{X}\hat{Z} = \mathrm{i}\hat{Y}$$

$$\hat{Q} = \sum_{\substack{\boldsymbol{A}\in\mathcal{P}_\Lambda \\ (\mathrm{Wid}\boldsymbol{A}\le k_{\max})}} q_{\boldsymbol{A}}\,\boldsymbol{A}$$

$$[\hat{H}, \hat{Q}] = \sum_{\boldsymbol{B}\in\mathcal{P}_\Lambda} \left(\sum_{\substack{\boldsymbol{A}\in\mathcal{P}_\Lambda \\ (\mathrm{Wid}\boldsymbol{A}\le k_{\max})}} \lambda_{\boldsymbol{A},\boldsymbol{B}}\,q_{\boldsymbol{A}} \right) \boldsymbol{B}$$

basic relation for $\boldsymbol{B}$

$[\hat{H}, \hat{Q}] = 0 \iff \sum_{\boldsymbol{A}\in\mathcal{P}_\Lambda} \lambda_{\boldsymbol{A},\boldsymbol{B}}\,q_{\boldsymbol{A}} = 0$ for all $\boldsymbol{B}\in\mathcal{P}_\Lambda$

basic strategy of the proof

Shiraishi 2019, 2024

$$\hat{H} = -\frac{1}{2} \sum_{\substack{u,v\in\Lambda \\ (|u-v|=1)}} \{J_{\mathrm{X}}\,\hat{X}_u\hat{X}_v + J_{\mathrm{Y}}\,\hat{Y}_u\hat{Y}_v + J_{\mathrm{Z}}\,\hat{Z}_u\hat{Z}_v\} - \sum_{u\in\Lambda}\{h_{\mathrm{X}}\,\hat{X}_u + h_{\mathrm{Y}}\,\hat{Y}_u + h_{\mathrm{Z}}\,\hat{Z}_u\}$$

$$\hat{Q} = \sum_{\substack{\boldsymbol{A}\in\mathcal{P}_\Lambda \\ (\mathrm{Wid}\,\boldsymbol{A}\le k_{\max})}} q_{\boldsymbol{A}}\,\boldsymbol{A}$$

basic relation for B

$$\sum_{\boldsymbol{A}\in\mathcal{P}_\Lambda} \lambda_{\boldsymbol{A},\boldsymbol{B}}\, q_{\boldsymbol{A}} = 0$$ for all $\boldsymbol{B}\in\mathcal{P}_\Lambda$

$\boxed{\text{coupled linear equations for } q_A}$

we shall prove $q_A = 0$ whenever $\mathrm{Wid}A = k_{\max}$ for $3 \le k_{\max} \le \frac{L}{2}$

contradicts the assumption that $q_A \neq 0$ for at least one A with $\mathrm{Wid}A = k_{\max}$

no local conserved quantities with $3 \le k_{\max} \le \frac{L}{2}$

1st step of the proof: Shiraishi shift 1

$k_{\max} = 3$

use basic relation for $\boldsymbol{B}$ with $\mathrm{Wid}\boldsymbol{B} = k_{\max} + 1$

$\boldsymbol{A}$
X
Y Z Y

$\boldsymbol{B}$
X
Y Z Z X

$\boldsymbol{A}'$
X
X Z X

$\boldsymbol{B}'$
X
X Z Y Z

$[\hat{X}_u \hat{X}_{u'}, \boldsymbol{A}] = 2\mathrm{i}\boldsymbol{B}$ $\quad [\hat{Y}_v \hat{Y}_{v'}, \boldsymbol{A}'] = -2\mathrm{i}\boldsymbol{B}$ $\quad [\hat{Z}_{u'} \hat{Z}_{u''}, \boldsymbol{A}'] = 2\mathrm{i}\boldsymbol{B}'$

basic relation for $\boldsymbol{B}$

$$2\mathrm{i}J_{\mathrm{X}}\, q_{\boldsymbol{A}} - 2\mathrm{i}J_{\mathrm{Y}}\, q_{\boldsymbol{A}'} = 0$$

basic relation for $\boldsymbol{B}'$

$$2\mathrm{i}J_{\mathrm{Z}}\, q_{\boldsymbol{A}'} = 0$$

$$q_{\boldsymbol{A}} = \frac{J_{\mathrm{Y}}}{J_{\mathrm{X}}}\, q_{\boldsymbol{A}'} \qquad \boldsymbol{A}' = \mathcal{S}(\boldsymbol{A})$$

Shiraishi shift

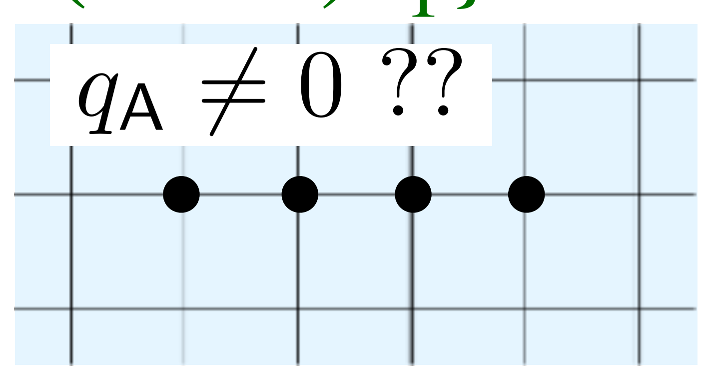
$$q_{\boldsymbol{A}} = 0$$

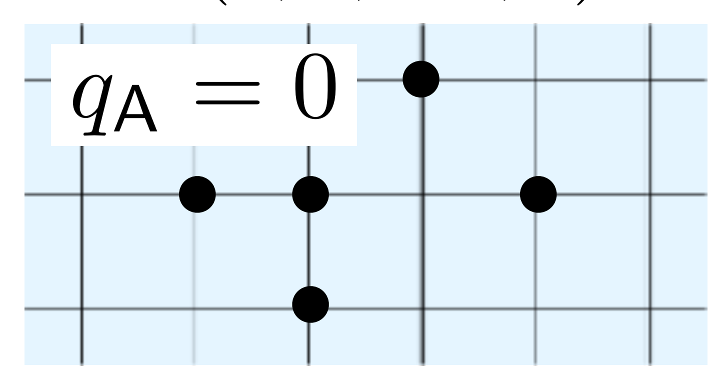
when $\mathrm{Wid}A = k_{\max}$ we have $q_A = 0$ unless

$\mathrm{Supp}A = \{u, u + e_1, \ldots, u + (k-1)e_1\}$

$e_1 = (1, 0, \ldots, 0)$

reduced to an essentially one-dimensional problem





1st step of the proof: Shiraishi shift 2

$k_{\max} = 4$ A **X Z Z Z**

X Z Z X Y

$\mathcal{S}(A)$ **Y Z X Y**

Y Z X Z X

$\mathcal{S}^2(A)$ **X X Z X**

B **X X Z Z Y**

$$q_{\boldsymbol{A}} = \frac{J_{\mathrm{X}}}{J_{\mathrm{Y}}} q_{\mathcal{S}(\boldsymbol{A})}$$

$$q_{\mathcal{S}(\boldsymbol{A})} = \frac{J_{\mathrm{Y}}}{J_{\mathrm{X}}} q_{\mathcal{S}^2(\boldsymbol{A})}$$

basic relation for B

$$-2\mathrm{i} J_{\mathrm{Y}}\, q_{\mathcal{S}^2(\boldsymbol{A})} = 0$$

$$q_{\boldsymbol{A}} = 0$$

1st step of the proof: Shiraishi shift 2

$k_{\max} = 4$

A	X Z Z Z
	X Z Z X Y
$\mathcal{S}(A)$	Y Z X Y
	Y Z X Z X
$\mathcal{S}^2(A)$	X X Z X
B	X X Z Z Y

basic relation for B

$$-2\mathrm{i}J_{\mathrm{Y}}\, q_{\mathcal{S}^2(\boldsymbol{A})} = 0$$

$$q_{\boldsymbol{A}} = 0$$

A	X Y Y X
	X Y Y Z Y
$\mathcal{S}(A)$	Z Y Z Y
	Z Y Z Z X
$\mathcal{S}^2(A)$	X Z Z X
	X Z Z Z Y
$\mathcal{S}^3(A)$	Y Z Z Y
	Y Z Z Z X
$\mathcal{S}^4(A)$	X Z Z X

$$q_{\boldsymbol{A}} = (\mathrm{const})\, q_{\hat{X}\hat{Z}\hat{Z}\hat{X}}$$

lemma: for any A with $\mathrm{Wid}A = k_{\max}$, we have either

$$q_{\boldsymbol{A}} = 0, \quad q_{\boldsymbol{A}} = \lambda' q_{\boldsymbol{C}_{\mathrm{XX}}}, \quad q_{\boldsymbol{A}} = \lambda'' q_{\boldsymbol{C}_{\mathrm{YX}}} \qquad (\lambda', \lambda'' \neq 0)$$

with

$$\boldsymbol{C}_{\mathrm{XX}} = \hat{X}_{x_0+e_1}\hat{Z}_{x_0+2e_1}\cdots\hat{Z}_{x_0+(k_{\max}-1)e_1}\hat{X}_{x_0+k_{\max}e_1}$$

$$\boldsymbol{C}_{\mathrm{YX}} = \hat{Y}_{x_0+e_1}\hat{Z}_{x_0+2e_1}\cdots\hat{Z}_{x_0+(k_{\max}-1)e_1}\hat{X}_{x_0+k_{\max}e_1} \qquad e_1 = (1,0,\ldots,0)$$

2nd step of the proof

we only need to control the coefficients of

$$C_{\mathrm{XX}} = \hat{X}_{x_0+e_1}\hat{Z}_{x_0+2e_1}\cdots\hat{Z}_{x_0+(k_{\max}-1)e_1}\hat{X}_{x_0+k_{\max}e_1}$$
$$C_{\mathrm{YX}} = \hat{Y}_{x_0+e_1}\hat{Z}_{x_0+2e_1}\cdots\hat{Z}_{x_0+(k_{\max}-1)e_1}\hat{X}_{x_0+k_{\max}e_1}$$

use basic relation for B with $\mathrm{Wid}B = k_{\max}$

	odd j		even j
$\hat{C}_j$	YZ --------------- ZZX	$\hat{C}_j$	XZ --------------- ZZY
$\hat{D}_j$	YZ ---- Z$\overset{\mathsf{Y}}{\mathsf{X}}$Z ---- ZZX	$\hat{D}_j$	XZ ---- Z$\overset{\mathsf{Y}}{\mathsf{X}}$Z ---- ZZY
$\hat{E}_j$	YZ ---- Z$\overset{\mathsf{Y}}{\mathsf{X}}$Z ---- ZY	$\hat{E}_j$	XZ ---- Z$\overset{\mathsf{Y}}{\mathsf{X}}$Z ---- ZX

only for odd k:

- $\hat{D}_{k-1}$ $\overset{\mathsf{Y}}{\mathsf{Z}}$Z ------------- ZZY
- $\hat{E}_{k-1}$ $\overset{\mathsf{Y}}{\mathsf{Z}}$Z ------------- ZX

$$\mathrm{Wid}\hat{\mathsf{C}}_j = \mathrm{Wid}\hat{\mathsf{D}}_j = k_{\max}$$
$$\mathrm{Wid}\hat{\mathsf{E}}_j = k_{\max} - 1$$

basic relation for D_j

$\downarrow$

$$q_{C_{\mathrm{XX}}} = q_{C_{\mathrm{YX}}} = 0$$

$q_A = 0$ **whenever** $\mathrm{Wid}A = k_{\max}$ **if** $3 \le k_{\max} \le \frac{L}{2}$

summary of part 1

☑ we proved that the XY and XYZ models on the d-dimensional hypercubic lattice with $d \geq 2$ possess no local conserved quantities

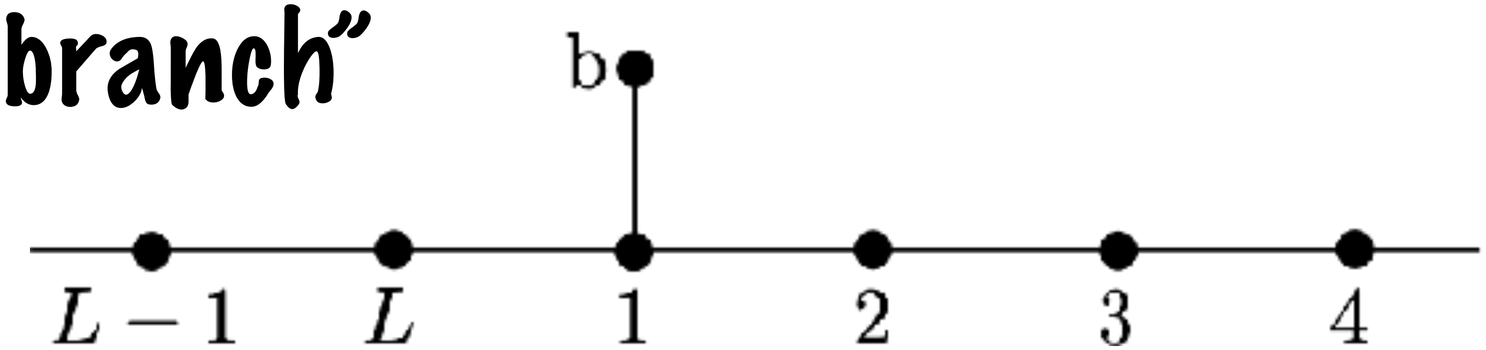
☑ the theorem applies to the simplest XX model

$$\hat{H} = -\frac{1}{2} \sum_{\substack{u,v\in\Lambda \\ (|u-v|=1)}} \{\hat{X}_u \hat{X}_v + \hat{Y}_u \hat{Y}_v\}$$

easily solved in 1D

quantum many-body models becomes "less solvable" in higher dimensions

☑ the same proof works for the system on a ladder or even a chain with a "branch"



summary of part 1

☑ various quantum many-body models were proved to possess no nontrivial local conserved quantities and hence are very likely to be "nonintegrable"

☑ for quantum spin chains, there seems to be a deep relationship between integrability and the absence/presence of nontrivial conserved quantities

all these results are interesting by themselves
but do we learn anything about, say, time-evolution?

☑ the absence of nontrivial local conserved quantities means any local operators change in time

→ operator growth (part 2)

part 1

background
main results
idea of the proof

part 2

setting
operator growth
complex-time evolution
idea of the proof

summary and discussion

$S=\frac{1}{2}$ model on $\mathbb{Z}^2$

$\hat{X}_u, \hat{Y}_u, \hat{Z}_u$ **copies of** $\hat{X}, \hat{Y}, \hat{Z}$ **at site** $u \in \mathbb{Z}^2$

algebra of local operators

$$\mathfrak{A}_{\mathrm{loc}} = \{\text{polynomials of } \hat{X}_u,\ \hat{Y}_u,\ \hat{Z}_u \text{ with } u \in \mathbb{Z}^2\}$$

remark: the notion of locality is different from that in part 1

$\mathcal{P}$ **set of all products** $\boldsymbol{A} = \bigotimes_{u\in S} \hat{A}_x$

$$\hat{A}_u = \hat{X}_u, \hat{Y}_u, \hat{Z}_u \quad S \subset \mathbb{Z}^2 \quad |S| < \infty$$

normalized Hilbert-Schmidt inner product

$\hat{A}, \hat{B} \in \mathfrak{A}_{\mathrm{loc}}$

$$\langle \hat{A}, \hat{B} \rangle_{\mathrm{NHS}} := \rho_\infty(\hat{A}^\dagger \hat{B})$$

$$\|\hat{A}\|_{\mathrm{NHS}} := \sqrt{\langle \hat{A}, \hat{A} \rangle_{\mathrm{NHS}}}$$

the infinite-temperature Gibbs state

$$\rho_\infty(\hat{A}) := \frac{\mathrm{Tr}_{\mathcal{H}_S}[\hat{A}]}{\mathrm{Tr}_{\mathcal{H}_S}[\hat{1}]}$$

$$\langle \boldsymbol{A}, \boldsymbol{B} \rangle_{\mathrm{NHS}} = \delta_{\boldsymbol{A},\boldsymbol{B}} \qquad \boldsymbol{A}, \boldsymbol{B} \in \mathcal{P}$$

Hamiltonian and time-evolution

(formal) Hamiltonian of the XYZ model

$$\hat{H} = -\frac{1}{2} \sum_{\substack{u,v\in\mathbb{Z}^2 \\ (|u-v|=1)}} \{J_\mathrm{X}\,\hat{X}_u\hat{X}_v + J_\mathrm{Y}\,\hat{Y}_u\hat{Y}_v + J_\mathrm{Z}\,\hat{Z}_u\hat{Z}_v\}$$

$$- \sum_{u\in\mathbb{Z}^2} \{h_\mathrm{X}\,\hat{X}_u + h_\mathrm{Y}\,\hat{Y}_u + h_\mathrm{Z}\,\hat{Z}_u\}$$

$$J_\mathrm{X}, J_\mathrm{Y}, J_\mathrm{Z}, h_\mathrm{X}, h_\mathrm{Y}, h_\mathrm{Z} \in \mathbb{R}$$

the generator of time-evolution (a.k.a the Liouvillian)

$\hat{A} \in \mathfrak{A}_\mathrm{loc}$ $\qquad$ $\mathrm{i}[\hat{H}_S, A]$

$$\delta(\hat{A}) := \mathrm{i}\,[\hat{H}, \hat{A}] \qquad (\delta(\hat{A}))^\dagger = \delta(\hat{A}^\dagger)$$

may not converge aboslutely!

time-evolution

$$\hat{A}(t) := \lim_{S\uparrow\mathbb{Z}^2} e^{\mathrm{i}\hat{H}_S t}\hat{A}_0\, e^{-\mathrm{i}\hat{H}_S t} = e^{t\delta}\hat{A} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\,\delta^n(\hat{A}_0)$$

$\hat{A}(t) \in \mathfrak{A} := \overline{\mathfrak{A}_\mathrm{loc}}$ **for any** $t \in \mathbb{R}$

Lieb-Robinson bound

part 1

background
main results
idea of the proof

part 2

setting
operator growth
complex-time evolution
idea of the proof

summary and discussion

operator growth

$\hat{A}_0 \in \mathfrak{A}_{\rm loc} \quad \|\hat{A}_0\|_{\rm NHS} = 1$

$$\hat{A}(t) := e^{\mathrm{i}\hat{H}t}\hat{A}_0\, e^{-\mathrm{i}\hat{H}t} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\,\delta^n(\hat{A}_0) \qquad \delta(\hat{A}) := \mathrm{i}\,[\hat{H}, \hat{A}]$$

the sequence of operators should contain information about the time-evolution

$\hat{A}_0, \delta(\hat{A}_0), \delta^2(\hat{A}_0), \delta^3(\hat{A}_0), \ldots$

$$\hat{O}_0 = \hat{A}_0$$

Gram-Shmidt

$$\hat{O}_n = \delta^n(\hat{A}_0) + \sum_{j=1}^{n-1} \alpha_j^{(n)}\,\delta^j(\hat{A}_0)$$

with $\langle \hat{O}_n, \hat{O}_j \rangle_{\rm NHS} = 0$ for all $j = 1, \ldots, n-1$

$\hat{O}_0, \hat{O}_1, \hat{O}_2, \hat{O}_3, \ldots$

$\hat{O}_n$ the operator that appears “for the first time” in the n-th recursion

Lanczos coefficient

$$b_n := \|\hat{O}_n\|_{\rm NHS} / \|\hat{O}_{n-1}\|_{\rm NHS}$$

$$\|\hat{O}_n\|_{\rm NHS} = b_1 b_2 \cdots b_n$$

characterizes intrinsic operator growth (with respect to the infinite-temperature Gibbs state ρ_∞)

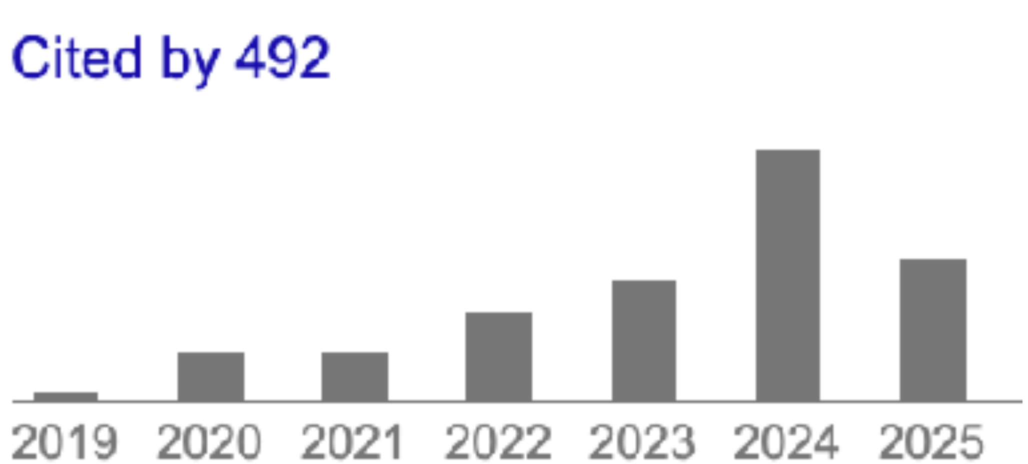
universal operator growth hypothesis

Parker, Cao, Avdoshkin, Scaffidi, Altman 2019

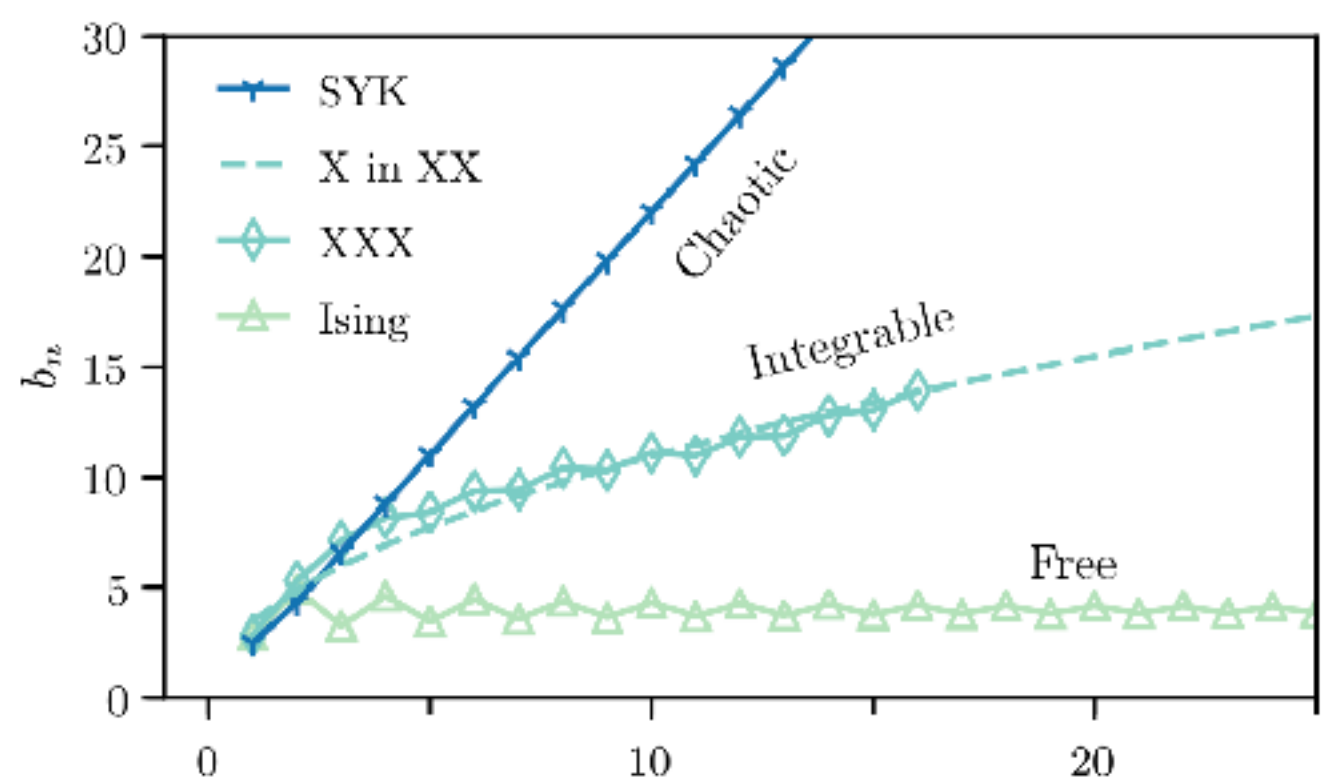
Nandy, Matsoukas-Roubeas, Martinez-Azcona, Dymarsky, del Campo 2024

the growth of the Lanczos coefficients b_n captures the essential feature of the quantum dynamics

$$b_n \sim \begin{cases} \text{const} & \text{free models} \\ \text{const}\, n^{\delta} \ \ (0 < \delta < 1) & \text{interacting integrable models} \\ \begin{cases} \text{const}\, n & d \geq 2 \\ \text{const}\, \dfrac{n}{\log n} & d = 1 \end{cases} & \text{chaotic models} \end{cases}$$



Parker, Cao, Avdoshkin, Scaffidi, Altman 2019



universal operator growth hypothesis

Parker, Cao, Avdoshkin, Scaffidi, Altman 2019

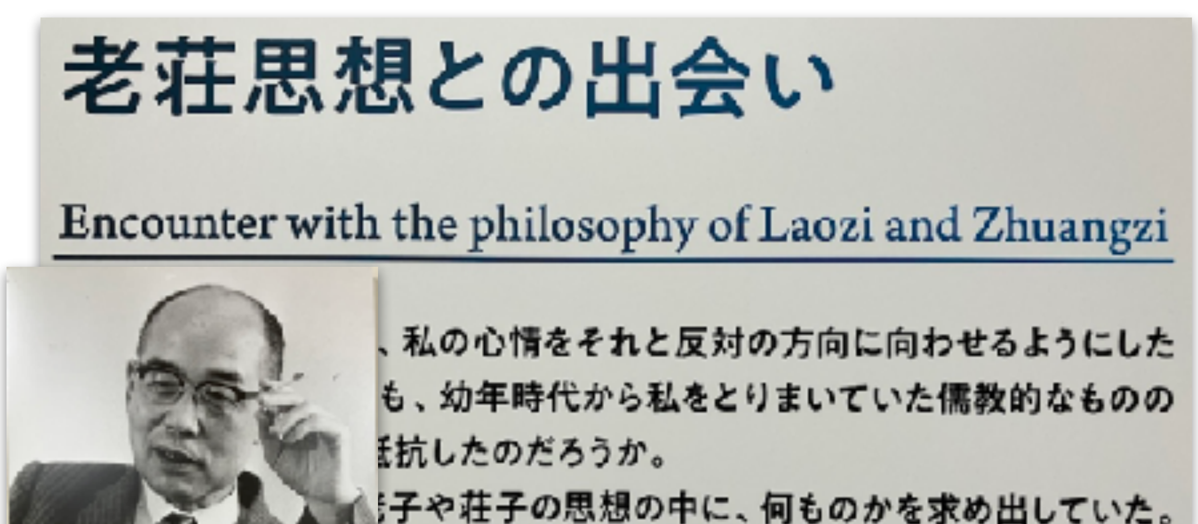
☑ well-defined notion (even from math point of view)

☑ applicable to essentially any quantum chaotic system, not only in the semi-classical (or any) limit

☑ seems to be an "almost" necessary and sufficient condition for quantum chaos

a subtle counterexample: a semi-classical integrable model wth saddle-dominated scrambling shows $b_n \sim \alpha\, n$

see, e.g., Nandy et al. 2024



"Zhuangzi", Fit for Emperors and Kings
The Death of Primal Chaos

南海之帝為儵 北海之帝為忽 中央之帝為渾沌
儵与忽 時相与遇於渾沌之地
渾沌待之甚善

儵与忽謀報渾沌之徳曰
人皆有七竅 以視聴食息
此独無有
嘗試鑿之

日鑿一竅 七日而渾沌死

『荘子』応帝王篇

main theorem

Shiraishi, Tasaki 2024, Tasaki unpublished

$$\hat{H} = -\frac{1}{2} \sum_{\substack{u,v \in \mathbb{Z}^2 \\ (|u-v|=1)}} \{J_{\mathrm{X}}\, \hat{X}_u \hat{X}_v + J_{\mathrm{Y}}\, \hat{Y}_u \hat{Y}_v + J_{\mathrm{Z}}\, \hat{Z}_u \hat{Z}_v\} - \sum_{u \in \mathbb{Z}^2} \{h_{\mathrm{X}}\, \hat{X}_u + h_{\mathrm{Y}}\, \hat{Y}_u + h_{\mathrm{Z}}\, \hat{Z}_u\}$$

Theorem: in all models except for the classical Ising model, there exist $\hat{A}_0$, $\alpha > 0$, and an infinite set $G \subset \mathbb{N}$ such that $b_1 b_2 \cdots b_n \ge \alpha^n n!$ for any $n \in G$

the proof for the quantum Ising model makes use of the idea due to Cao (2021)

by combining the result in Bouch (2015) with our method, the same statement for the Bouch model follows

$$\hat{H}_{\mathrm{B}} := \sum_{j,k \in \mathbb{Z}} \{\hat{X}_{(j,k)} \hat{Z}_{(j+1,k)} + \hat{Z}_{(j,k)} \hat{X}_{(j,k+1)}\}$$

main theorem

Shiraishi, Tasaki 2024, Tasaki unpublished

$$\hat{H} = -\frac{1}{2} \sum_{\substack{u,v\in\mathbb{Z}^2 \\ (|u-v|=1)}} \{J_{\mathrm{X}}\,\hat{X}_u\hat{X}_v + J_{\mathrm{Y}}\,\hat{Y}_u\hat{Y}_v + J_{\mathrm{Z}}\,\hat{Z}_u\hat{Z}_v\} - \sum_{u\in\mathbb{Z}^2} \{h_{\mathrm{X}}\,\hat{X}_u + h_{\mathrm{Y}}\,\hat{Y}_u + h_{\mathrm{Z}}\,\hat{Z}_u\}$$

Theorem: in all models except for the classical Ising model, there exist $\hat{A}_0$, $\alpha > 0$, and an infinite set $G \subset \mathbb{N}$ such that $b_1 b_2 \cdots b_n \geq \alpha^n n!$ for any $n \in G$

essentially shows $b_n \gtrsim \alpha\, n$, the behavior expected (almost) only in systems exhibiting quantum chaos!

any standard $S = 1/2$ quantum spin system in two or higher dimensions exhibit the signature of quantum chaos

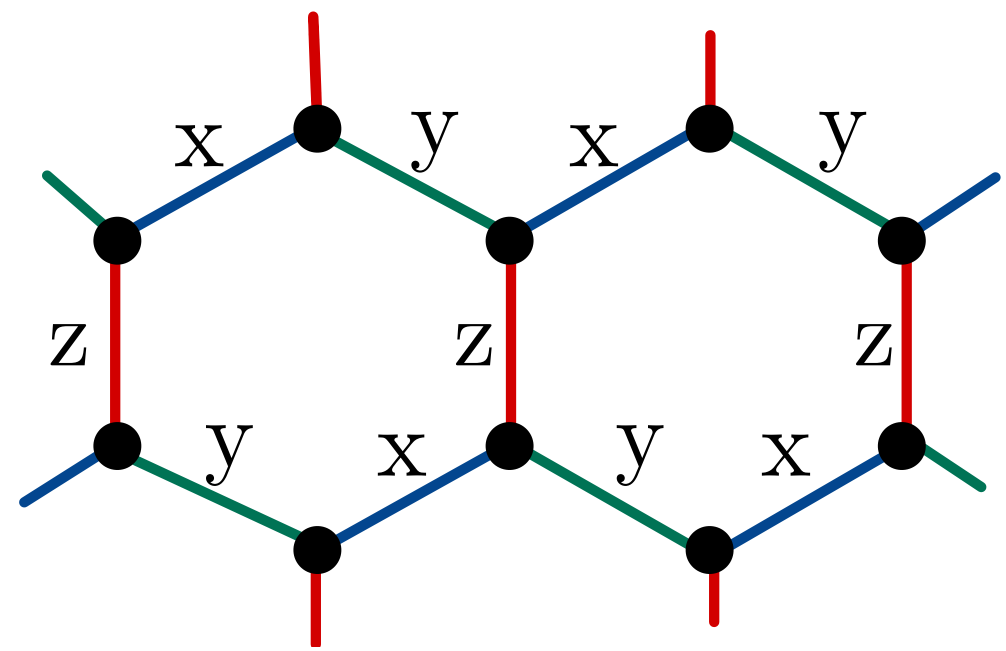
Cao (2021) proved $\max\{b_1, b_2, \ldots, b_n\} \gtrsim \alpha\, n/\log n$ for the one-dimensional Ising model with a slanted magnetic field $\hat{H}_{\mathrm{Ising}} = \sum_j \{\hat{Z}_j \hat{Z}_{k+1} + h\hat{X}_j + h'\hat{Z}_j\}$

Kitaev honeycomb model

two-dimensional $S = 1/2$ model with infinitely many local conserved quantities

$$\hat{H}_{\mathrm{Kitaev}} = \sum_{\{u,v\}\in\mathcal{B}_{\mathrm{x}}} \hat{X}_u\hat{X}_v + \sum_{\{u,v\}\in\mathcal{B}_{\mathrm{y}}} \hat{Y}_u\hat{Y}_v + \sum_{\{u,v\}\in\mathcal{B}_{\mathrm{z}}} \hat{Z}_u\hat{Z}_v$$

sets of bonds $\mathcal{B}_{\mathrm{x}} \cup \mathcal{B}_{\mathrm{y}} \cup \mathcal{B}_{\mathrm{z}}$



for the initial operator

$$\hat{A}_0 = \hat{Z}_u\hat{Z}_v \text{ for } \{u,v\} \in \mathcal{B}_{\mathrm{z}}$$

it is easy to prove

free model

$$2^{n/2} \le b_1 b_2 \cdots b_n \le 6^n \longrightarrow b_n \sim \text{const}$$ for all n

the behavior of b_n seems to be more complicated for a general initial operator

part 1

background
main results
idea of the proof

part 2

setting
operator growth
complex-time evolution
idea of the proof

summary and discussion

complex-time evolution

$$\hat{A}(t) := e^{\mathrm{i}\hat{H}t}\,\hat{A}(0)\,e^{-\mathrm{i}\hat{H}t} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\,\delta^n(\hat{A}(0))$$

$$\delta(\hat{A}) = \mathrm{i}\,[\hat{H},\hat{A}]$$

$$\mathfrak{A} = \overline{\mathfrak{A}_{\mathrm{loc}}}$$

Theorem (Araki, 1969) in a $d=1$ quantum spin system with a finite-ranged translation invariant Hamiltonian, $\hat{A}(t)$ with any $\hat{A}(0) \in \mathfrak{A}$ converges in $\mathfrak{A}$ (in the operator norm) for any $t \in \mathbb{C}$

what about systems in $d \geq 2$?

Proposition: in any quantum spin system with a finite-ranged uniformly bounded Hamiltonian, there exists $r_0 > 0$ such that $\hat{A}(t)$ with any $\hat{A}(0) \in \mathfrak{A}$ converges in $\mathfrak{A}$ (in the operator norm) for any $t \in \mathbb{C}$ with $|t| \leq r_0$

Zobov (2000) found a singularity at an imaginary time in the Heisenberg model in $d = \infty$

singularity at an imaginary time

$$\hat{A}(\mathrm{i}\beta) := e^{-\beta\hat{H}}\,\hat{A}(0)\,e^{\beta\hat{H}} = \sum_{n=0}^{\infty}\frac{(\mathrm{i}\beta)^n}{n!}\,\delta^n(\hat{A}(0))$$

$$\hat{H}_{\mathrm{B}} := \sum_{j,k\in\mathbb{Z}}\{\hat{X}_{(j,k)}\hat{Z}_{(j+1,k)} + \hat{Z}_{(j,k)}\hat{X}_{(j,k+1)}\}$$

Theorem (Bouch, 2015) $\hat{A}(\mathrm{i}\beta)$ **with** $\hat{A}(0) = \hat{X}_o$ **does not converge in** $\mathfrak{A}$ **(in the operator norm) for** $\beta \in \mathbb{R}$ **with sufficiently large** $|\beta|$**.**

$$\hat{H} = -\frac{1}{2}\sum_{\substack{u,v\in\mathbb{Z}^2\\(|u-v|=1)}}\{J_{\mathrm{X}}\,\hat{X}_u\hat{X}_v + J_{\mathrm{Y}}\,\hat{Y}_u\hat{Y}_v + J_{\mathrm{Z}}\,\hat{Z}_u\hat{Z}_v\} - \sum_{u\in\mathbb{Z}^2}\{h_{\mathrm{X}}\,\hat{X}_u + h_{\mathrm{Y}}\,\hat{Y}_u + h_{\mathrm{Z}}\,\hat{Z}_u\}$$

Theorem: in all models except for the classical Ising model, $\hat{A}(\mathrm{i}\beta)$ **with some** $\hat{A}(0)$ **does not converge in** $\mathfrak{A}$ **(in the operator norm) for** $\beta \in \mathbb{R}$ **with sufficiently large** $|\beta|$**.**

Shiraishi, Tasaki 2024, Tasaki unpublished

the operator grows rapidly and reaches infinity within a finite imaginary time!

no Lieb-Robinson!

characterization of quantum chaos

$$\hat{A}(\mathrm{i}\beta) := e^{-\beta\hat{H}}\,\hat{A}(0)\,e^{\beta\hat{H}} = \sum_{n=0}^{\infty}\frac{(\mathrm{i}\beta)^n}{n!}\,\delta^n(\hat{A}(0))$$

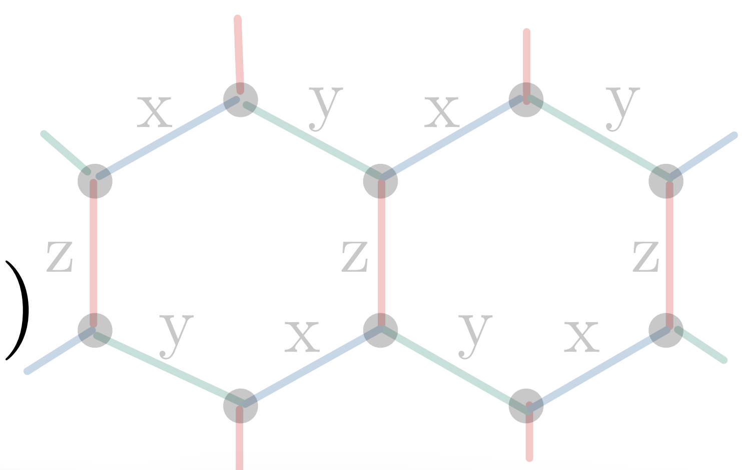
Avdoshkin, Dymarsky 2020

in a chaotic system, $\hat{A}(\mathrm{i}\beta)$ with $\beta\in\mathbb{R}$ grows rapidly

$d=1$ double exponential in $|\beta|$

$d\geq 2$ reaches infinity at a finite $|\beta|$

$$\hat{A}(t) := e^{\mathrm{i}\hat{H}t}\,\hat{A}(0)\,e^{-\mathrm{i}\hat{H}t} = \sum_{n=0}^{\infty}\frac{t^n}{n!}\,\delta^n(\hat{A}(0))$$



Theorem: in the Kitaev honeycomb model, $\hat{A}(t)$ with any $\hat{A}(0)\in\mathfrak{A}$ converges in $\mathfrak{A}$ (in the operator norm) for any $t\in\mathbb{C}$

Tasaki unpublished

part 1

background
main results
idea of the proof

part 2

setting
operator growth
complex-time evolution
idea of the proof

summary and discussion

$$\hat{H} = -\frac{1}{2}\sum_{\substack{u,v\in\mathbb{Z}^2\\(|u-v|=1)}} \{J_\mathrm{X}\,\hat{X}_u\hat{X}_v + J_\mathrm{Y}\,\hat{Y}_u\hat{Y}_v + J_\mathrm{Z}\,\hat{Z}_u\hat{Z}_v\} - \sum_{u\in\mathbb{Z}^2}\{h_\mathrm{X}\,\hat{X}_u + h_\mathrm{Y}\,\hat{Y}_u + h_\mathrm{Z}\,\hat{Z}_u\}$$

$$J_\mathrm{X} \neq 0,\ J_\mathrm{Y} \neq 0$$

operator growth from $\hat{X}_o$

$$\hat{X}_o, \delta(\hat{X}_o), \delta^2(\hat{X}_o), \delta^3(\hat{X}_o), \ldots \qquad \delta(\hat{A}) := \mathrm{i}\,[\hat{H}, \hat{A}]$$

Gram-Shmidt

$$\hat{O}_0, \hat{O}_1, \hat{O}_2, \hat{O}_3, \ldots$$

expand in products $\delta^n(\hat{X}_o) = \sum_{\boldsymbol{B}\in\mathcal{P}} c_{\boldsymbol{B}}\boldsymbol{B}$

if there is $\boldsymbol{B}$ that does not appear in $\delta^j(\hat{X}_o)$ with $j = 1,2,\ldots,n-1$, then we have

$$\hat{O}_n = c_{\boldsymbol{B}}\boldsymbol{B} + \sum_{\boldsymbol{B}'\neq\boldsymbol{B}} \tilde{c}_{\boldsymbol{B}'}\boldsymbol{B}'$$

$$b_1 b_2 \cdots b_n = \|\hat{O}_n\|_{\mathrm{NHS}} \geq |c_{\boldsymbol{B}}|$$

we shall look for such $\boldsymbol{B}$ with large $|c_{\boldsymbol{B}}|$

some examples of B and c_B

$\cdots\cdots\boxed{\cdot\cdot\quad} \longleftrightarrow [\hat{X}_u \hat{X}_v, \cdots\cdots]$ $\qquad$ $\cdots\cdots\boxed{\cdot\cdot\quad} \longleftrightarrow [\hat{Y}_u \hat{Y}_v, \cdots\cdots]$

the simplest construction

$\boxed{\mathbf{X}} \rightarrow \mathbf{Z}\boxed{\mathbf{Y}} \rightarrow \mathbf{Z}\,\mathbf{Z}\boxed{\mathbf{X}} \longrightarrow \overbrace{\mathbf{Z}\,\mathbf{Z}\,\mathbf{Z}\ldots\mathbf{Z}\,\mathbf{Z}}^{n}\mathbf{X} = B$

grown in a unique manner

$$c_B = (-4J_{\mathrm{X}}J_{\mathrm{Y}})^{n/2}$$

appears for the first time in $\delta^n(\hat{X}_o)$

a better strategy

$\overbrace{\mathbf{Z}\,\mathbf{Z}\,\mathbf{Z}\ldots\mathbf{Z}\,\mathbf{Z}}^{n/2}\mathbf{X} \rightarrow \mathbf{Z}\boxed{\mathbf{Z}}\mathbf{Z}\ldots\mathbf{Z}\boxed{\mathbf{Z}}\mathbf{X} \rightarrow \begin{matrix} & \mathbf{X} & & & & \mathbf{X} & \\ \mathbf{X} & \mathbf{Y} & \mathbf{X} & \ldots & \mathbf{X} & \mathbf{Y} & \mathbf{X} \\ \mathbf{Y} & & \mathbf{Y} & & \mathbf{Y} & & \end{matrix} = B$

grow this

take $n/2$ commutation relations in an arbitrary order

grown in at least $(n/2)!$ ways (in fact $(n-1)!!$ ways)

$$b_1 b_2 \cdots b_n \geq |c_B| \geq (4|J_{\mathrm{X}}J_{\mathrm{Y}}|)^{n/2}\left(\frac{n}{2}\right)! \qquad b_n \gtrsim \alpha\, n^{1/2}$$

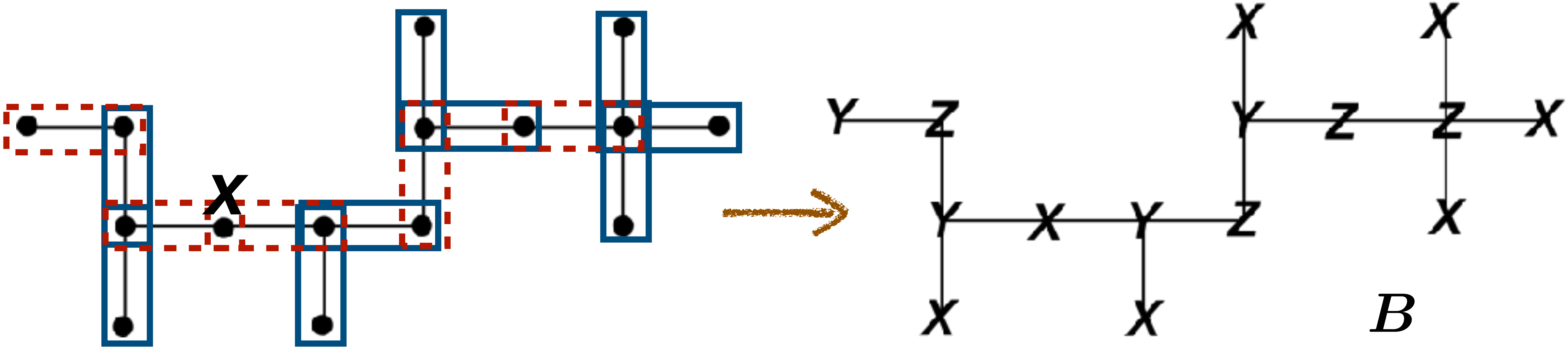
general construction on a rooted tree

$$T \subset \mathbb{Z}^2,\ |T| = n+1,\ T \ni o$$

$$\tilde{T} = \{\{u,v\} \mid u,v \in T,\ |u-v| = 1\}$$

assume $(T,\tilde{T})$ is connected and contains no loops

we start from $\hat{X}_o$ and grow B supported on T



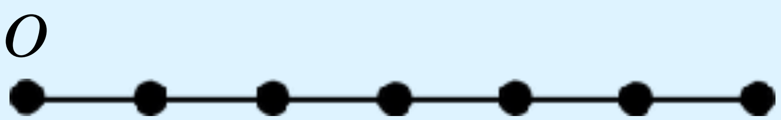
then $|c_{\boldsymbol{B}}| \ge (2\min\{|J_{\mathrm{X}}|, |J_{\mathrm{Y}}|\})^n\, N(T)$

$N(T)$ the number of ways to grow a rooted tree $(T,\tilde{T})$ starting from the root and adding edges one by one

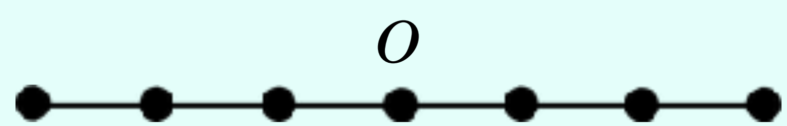
$\cdots\cdots \boxed{} \longleftrightarrow [\hat{X}_u \hat{X}_v, \cdots\cdots]$ $\quad \cdots\cdots \boxed{} \longleftrightarrow [\hat{Y}_u \hat{Y}_v, \cdots\cdots]$

the number of ways to grow a tree

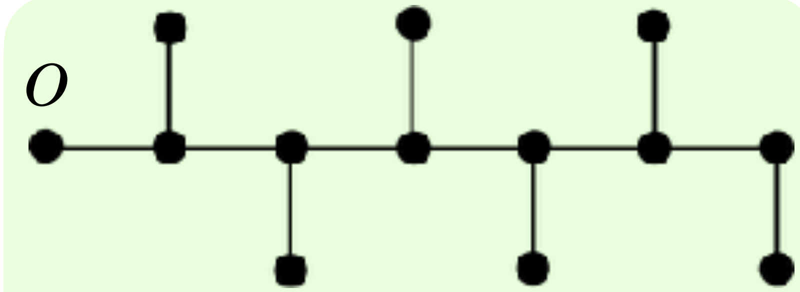
$N(T)$ the number of ways to grow a rooted tree $(T, \tilde{T})$ starting from the root and adding edges one by one



$N(T) = 1$



$N(T) = \binom{n}{n/2}$

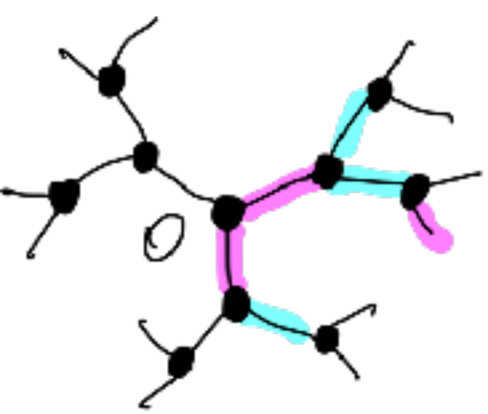


$N(T) = (n-1)!!$

Conjecture: there is $C > 0$ such that for any $n = 1,2,\ldots$ there exists a rooted tree $(T, \tilde{T})$ with $N(T) \geq n!/C^n$

expected in $\mathbb{Z}^d$ with $d \geq 2$

easily proved in the Bethe lattice



the total # of ways to grow trees with n edges $= 3 \times 4 \times \cdots \times (n+2)$

the number of distinct rooted trees $\leq 3^{2n}$

thus $\exists T$ s.t. $N(T) \geq \dfrac{3 \times 4 \times \cdots \times (n+2)}{3^{2n}} \geq \dfrac{n!}{9^n}$

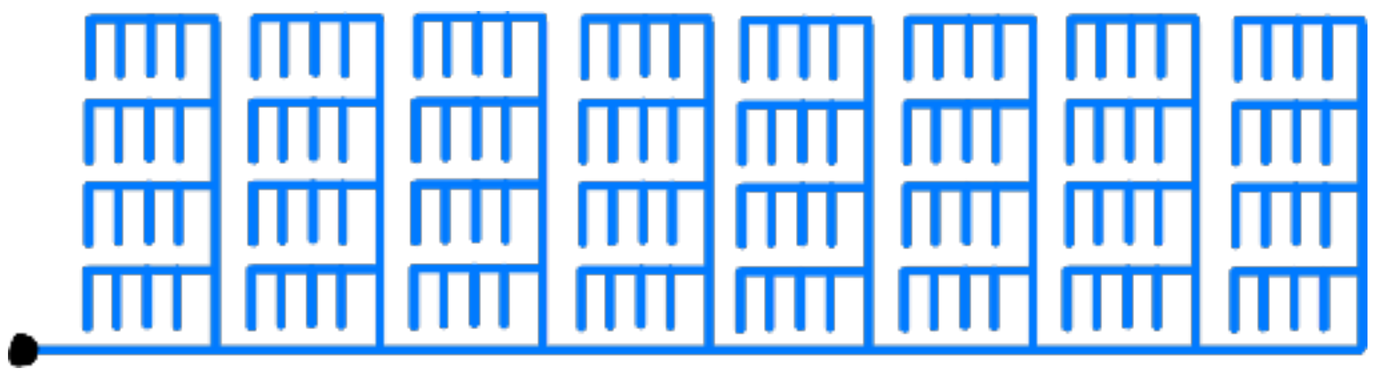
trees that can be grown in "too many" ways

$N(T)$ the number of ways to grow a rooted tree $(T, \tilde{T})$ starting from the root and adding edges one by one

Theorem (Bouch 2015) there are $C > 0$ and an infinite set $G \subset \mathbb{N}$ such that for any $n \in G$ there exists a rooted tree $(T, \tilde{T})$ with $|T| = n + 1$ and $N(T) \geq n!/C^n$

a tour de force hierarchical construction

see Tasaki (2024)



Bouch tree

Gabriel Bouch

this proves, for $n \in G$, the desired $b_1 b_2 \cdots b_n \geq \alpha^n n!$ for Lanczos coefficients in a large class of models in $d \geq 2$!!

the set G is infinite, but is extremely sparse...

imaginary-time evolution $\beta \in \mathbb{R}$

initial operator $\hat{A} \in \mathfrak{A}_{\mathrm{loc}},\ \hat{A}^\dagger = \hat{A},\ \|\hat{A}\|_{\mathrm{NHS}} = 1$

$$\hat{A}^{(N)}(\mathrm{i}\beta) := \sum_{m=0}^{2N} \frac{(\mathrm{i}\beta)^m}{m!}\,\delta^m(\hat{A}) \qquad \hat{A}(\mathrm{i}\beta) = \lim_{N\uparrow\infty} \hat{A}^{(N)}(\mathrm{i}\beta)$$

we shall prove $\|\hat{A}^{(N)}(\mathrm{i}\beta)\|$ diverges as $N \uparrow \infty$ if $|\beta|$ is large

not straightforward, as the coefficients usually have mixed signs

imaginary-time autocorrelation

$$\langle \hat{A}, \hat{A}^{(N)}(\mathrm{i}\beta)\rangle_{\mathrm{NHS}} = \sum_{m=0}^{2N} \frac{(\mathrm{i}\beta)^m}{m!} \langle \hat{A}, \delta^m(\hat{A})\rangle_{\mathrm{NHS}}$$

since $\langle \hat{A}, \delta(\hat{B})\rangle_{\mathrm{NHS}} = -\langle \delta(\hat{A}), \hat{B}\rangle_{\mathrm{NHS}}$, **we have**

$$\langle \hat{A}, \delta^m(\hat{A})\rangle_{\mathrm{NHS}} = \begin{cases} (-1)^{\frac{m}{2}} \langle \delta^{\frac{m}{2}}(\hat{A}), \delta^{\frac{m}{2}}(\hat{A})\rangle_{\mathrm{NHS}}, & m \text{ even;} \\ 0 & m \text{ odd.} \end{cases}$$

lower bound for $\|\hat{A}^{(N)}(\mathrm{i}\beta)\|$

$$\langle \hat{A}, \hat{A}^{(N)}(\mathrm{i}\beta)\rangle_{\text{NHS}} = \sum_{n=0}^{N} \frac{(\mathrm{i}\beta)^{2n}}{(2n)!}(-1)^n \langle \delta^n(\hat{A}), \delta^n(\hat{A})\rangle_{\text{NHS}}$$

$$= \sum_{n=0}^{N} \frac{\beta^{2n}}{(2n)!}\left(\|\delta^n(\hat{A})\|_{\text{NHS}}\right)^2$$

nonnegative for $\beta \in \mathbb{R}$

moment $\mu_{2n} = \left(\|\delta^n(\hat{A})\|_{\text{NHS}}\right)^2$

noting

$$\langle \hat{A}, \hat{A}^{(N)}(\mathrm{i}\beta)\rangle_{\text{NHS}} \leq \overbrace{\boxed{\|\hat{A}\|_{\text{NHS}}}}^{=1} \|\hat{A}^{(N)}(\mathrm{i}\beta)\|_{\text{NHS}} \leq \|\hat{A}^{(N)}(\mathrm{i}\beta)\|$$

we get $$\|\hat{A}^{(N)}(\mathrm{i}\beta)\| \geq \sum_{n=0}^{N} \frac{\beta^{2n}}{(2n)!}\mu_{2n}$$

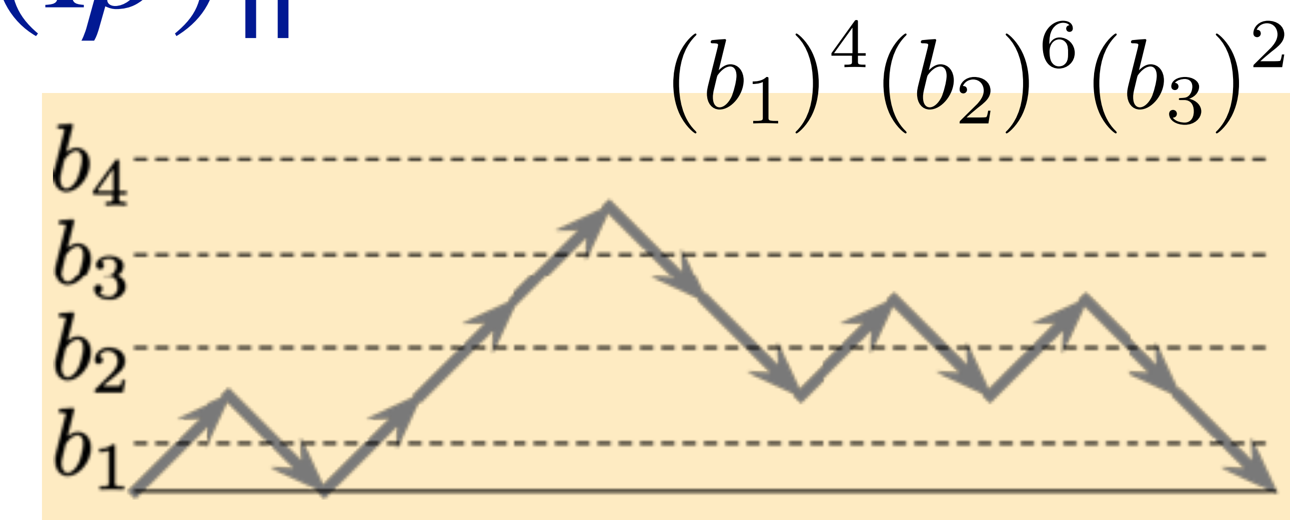
good news!

moments are related to Lanczos coefficients!

divergence of $\|\hat{A}^{(N)}(\mathrm{i}\beta)\|$

$$\mu_{2n} = \sum_{(p_1,\ldots,p_{2n})} b_{p_1} b_{p_2} \cdots b_{p_{2n}}$$

sum over Dyck paths



for example $\mu_2 = (b_1)^2 \qquad \mu_4 = (b_1 b_2)^2 + (b_1)^4$

in particular we see $\mu_{2n} \geq (b_1 b_2 \cdots b_n)^2$

$$\|\hat{A}^{(N)}(\mathrm{i}\beta)\| \geq \sum_{n=0}^{N} \frac{\beta^{2n}}{(2n)!} \mu_{2n} \geq \sum_{n=0}^{N} \frac{\beta^{2n}}{(2n)!} (b_1 b_2 \cdots b_n)^2$$

$$\geq \sum_{n \in G \cap [0,N]} \frac{\beta^{2n}}{(2n)!} (\alpha\, n!)^2 = \sum_{n \in G \cap [0,N]} (\alpha\beta)^{2n} \frac{(n!)^2}{(2n)!}$$

diverges as $N \uparrow \infty$ if $\alpha\,|\beta| > 2$

works for any model where we have $b_1 b_2 \cdots b_n \geq \alpha^n\, n!$

summary of part 2

☑ for a large class of $S = 1/2$ quantum spin systems, we established that the Lanczos coefficients exhibit the behavior expected for a quantum chaotic system

☑ for a large class of $S = 1/2$ quantum spin systems, we established that the imaginary-time evolution of a local operator exhibits a singularity at a finite "time"

- the same or similar results were proved, but only for restricted (fine-tuned) models
- the trees that can be grown in "too many" ways by Bouch play essential roles in the proof (thus the result is limited to n from the sparse set G)
- the proof requires a full infinite 2D lattice

part 1

background
main results
idea of the proof

part 2

setting
operator growth
complex-time evolution
idea of the proof

summary and discussion

summary and discussion

- recently, mathematical/theoretical physicists started proving that a concrete quantum spin system with short-range interactions exhibits a behavior that is never expected in an integrable model

Bouch 2015, Shiraishi 2019, Cao 2021, ...

- we proved such results for a class of standard $S = 1/2$ quantum spin systems in two or higher dimensions

this is a new trend – with plenty of room for further progress!

future issues

part 1

☑ extend the proof to quasi-local conserved quantities

☑ clarify the relationship between (non)integrability and the presence/absence of conserved charges

part 2

☑ prove the existence of trees that can be grown in "too many" ways for any number n of bonds

☑ show something about (observable) time-dependent quantities

overall

☑ justify other (physically interesting) properties of non-integrable or quantum chaotic systems

The emperor of the South Sea was called Shu [Brief], the emperor of the North Sea was called Hu [Sudden], and the emperor of the central region was called Hun-tun [Chaos]. Shu and Hu from time to time came together for a meeting in the territory of Hun-tun, and Hun-tun treated them very generously. Shu and Hu discussed how they could repay his kindness. "All men," they said, "have seven openings so they can see, hear, eat, and breathe. But Hun-tun alone doesn't have any. Let's trying boring him some!"

Every day they bored another hole, and on the seventh day Hun-tun died.

“Zhuangzi", Chapter 7 Fit for Emperors and Kings

https://terebess.hu/english/chuangtzu.html

so far, we have bored only three (not very critical) holes