Macroscopic Particle Transport in Dissipative Bosonic Systems

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arXiv: 2503.13731

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Outline

- Speed limit of information propagation and particle transport
- Introduction to Optimal transport theory
- Bosonic transport in open quantum systems
- Conclusion and Outlook

Speed of information propagation

- In closed quantum systems, the information propagation has a speed limit due to unitary evolution
- The upper limit is Lieb-Robinson bound:

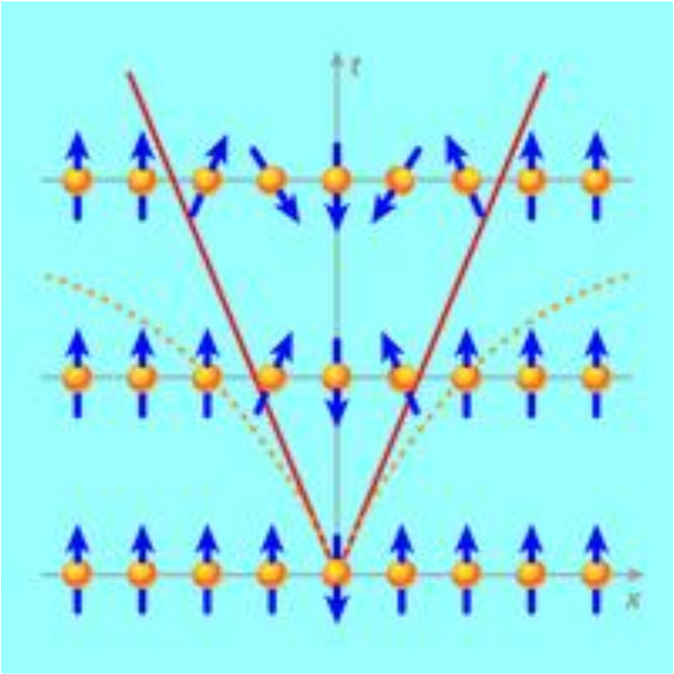
$$\|[A(t), B]\| \leqslant c \exp[-(d_{XY} - vt)/\xi]$$

Lieb and Robinson, Commun. Math. Phys. (1972)

A, B: local operators on different supports X and Y

The Lieb-Robinson bound indicates the locality even in the absence of relativistic assumptions.

T. Kuwahara, et al., PRX **10**, 031010 (2020)



$A \longleftrightarrow B$

d_{XY}

effective light cone

Speed of information propagation

- Spin and fermion systems (bounded local operators):

1. Short-range systems: $\|J_{ij}\| \leqslant e^{-|i-j|/\xi}$

M. B. Hastings, et al., Comm. Math. Phys. 265, 781 (2006)

$$\|[A(t), B]\| \leqslant c \exp[-(d_{XY} - vt)/\xi] \quad \text{linear light cone}$$

2. Long-range systems: $\|J_{ij}\| \leqslant \dfrac{1}{|i-j|^{\alpha}}$

M. Foss-Feig, et al., PRL **114**, 157201(2015)
T. Kuwahara, et al., PRX 10, 031010 (2020)

$$\tau \gtrsim d_{XY}^{\min(1, \alpha - 2D)} \quad \text{linear light cone if } \alpha \geq 2D + 1.$$

- Bosonic case:

Only short-range hopping with low-density initial states can be evaluated:

C. Yin, et al., PRX **12**, 021039 (2022), T. Kuwahara, et al., Nat. Commun. **15**, 2520 (2024).

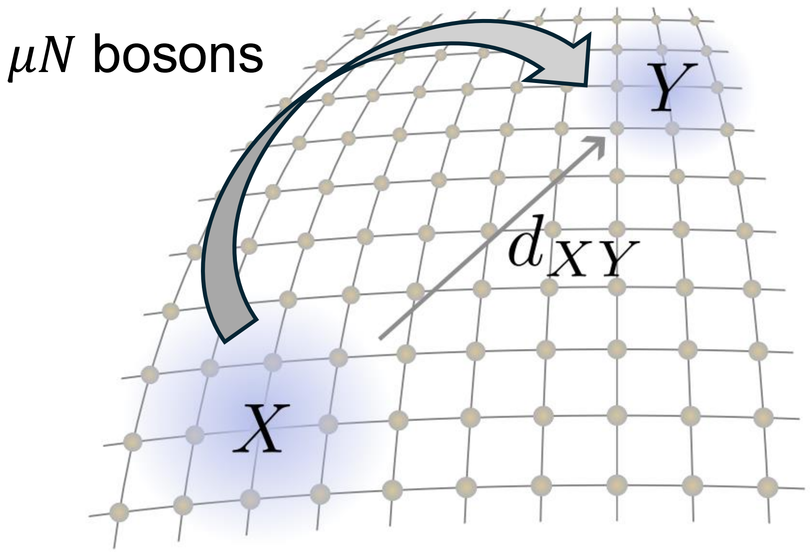
Speed of particle transport

- On the other hand, we also focus on macroscopic particle transport, which also represents a type of speed limit for information propagation

- Setups: bosons on the lattices:
  1. long-range hopping and interactions
  2. time-dependent Hamiltonians
- Criterion:

$$n_Y(\tau) \geqslant n_{X^c}(0) + \mu N$$

X^c: the complementary region of X on the lattice



$\mu \in (0,1]$: ratio of transported particles
N: total number of bosons

T. Vu, et al., Quantum 8, 1483 (2024)

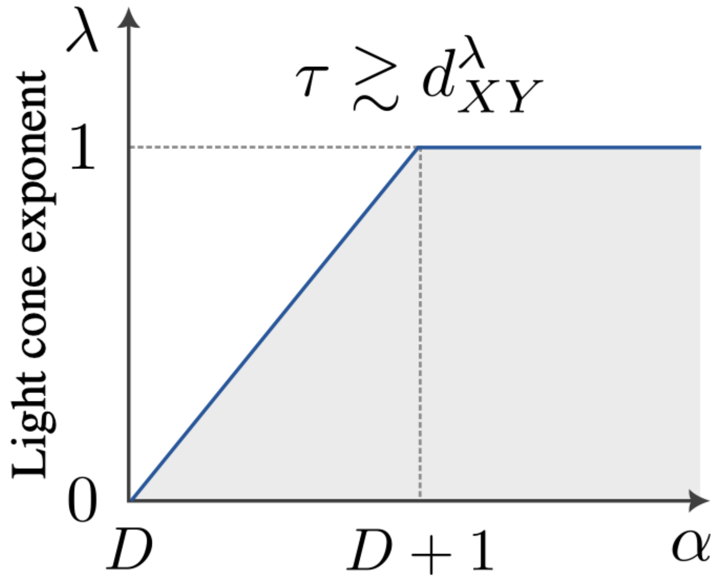
Results in closed quantum systems

- Then how fast can the particle be transported?
- Bosonic Hamiltonian:

$$H = \sum_{i \neq j} J_{ij}(t) b_i^\dagger b_j + \sum_{Z \subset \Lambda} h_Z(n_i, t)$$

$|J_{ij}| \leqslant J/|i-j|^\alpha$:symmetric long-range hopping with $\alpha > D$ (spatial dimension)

$h_Z(t)$: an arbitrary function of density operators n_i



- Given a transport distance d_{XY}, the transport time of μN bosons satisfies:

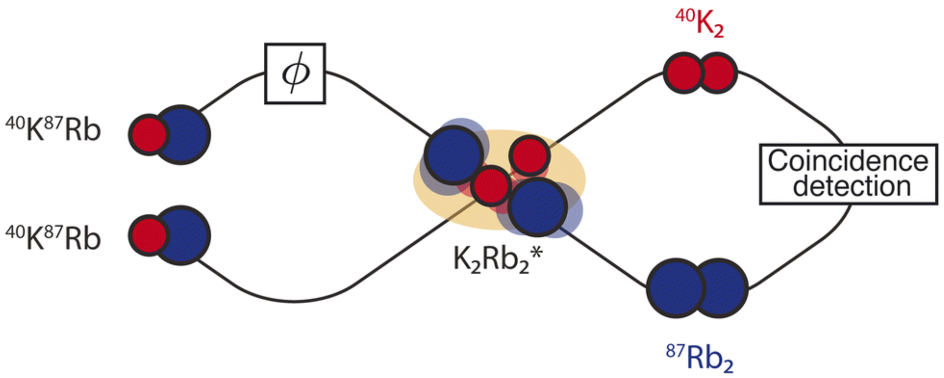
$$\tau \geqslant \frac{\mu}{J\varphi} d_{XY}^{\min(1, \alpha - D)}$$

φ: O(1) constant

T. Vu, et al., Quantum 8, 1483 (2024)

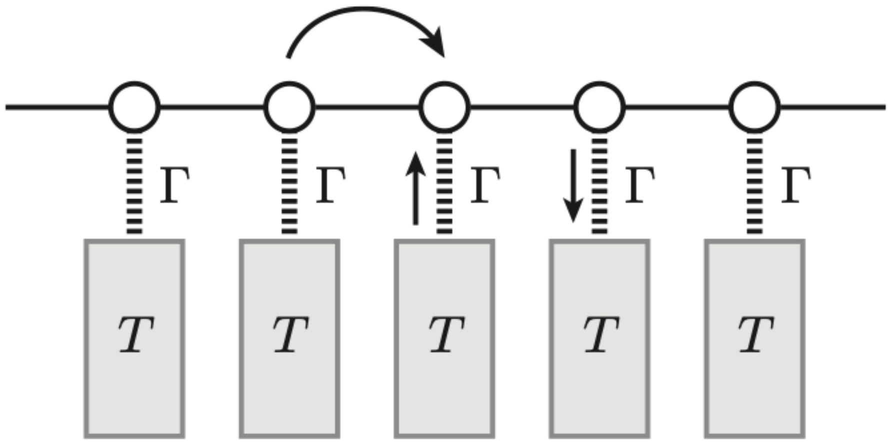
Speed of particle transport

- Up to now, the studies are restricted in closed quantum systems.
- However, the systems inevitably suffer from loss in ultracold atomic systems: inelastic collision, chemical reaction, coupling to bath, …



chemical reactions of KRb molecules

Science **322**, 231 (2008)
Science **327**, 853 (2010)
PRL. **104**, 030402 (2010)



boson/fermion chains coupled to baths

Phys. Rev. B **32**, 1846 (1985)
Rev. Mod. Phys. **86**, 779 (2014)

Speed of particle transport

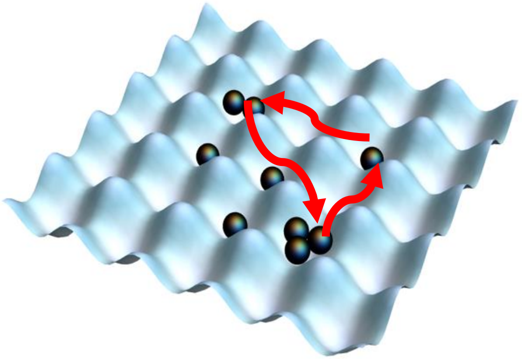
- When the dissipation rate is small, we can describe the dynamics of density matrix via Lindblad equation:

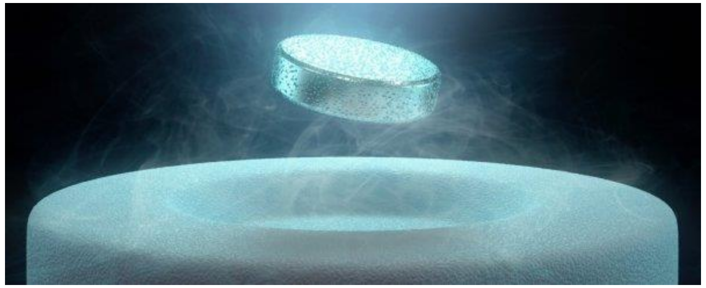
$$\frac{d\rho}{dt} = -i[H,\rho] + \frac{\gamma}{2}\sum_i \left(2L_i\rho L_i^\dagger - L_i^\dagger L_i\rho - \rho L_i^\dagger L_i\right)$$

i: lattice sites, L_i: local Lindblad operators, γ: dissipation rate

H. Li, et al., arXiv: 2406.08868 K. Yamamoto, etal., PRL. **127**, 055301 (2021)

- Previous studies show the superfluidity and superconductivity under the two-body loss:
- Question: What is the speed limit for macroscopic particle transport in open quantum systems?





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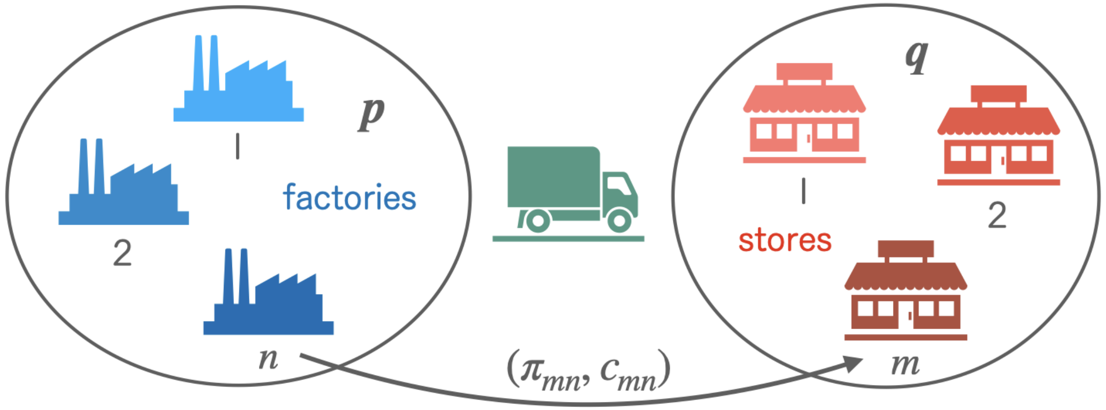
Optimal transport theory

- Motivation: transport cost of goods



- Total cost:

$$\sum_{m,n} \pi_{mn} c_{mn}$$

π_{mn}: fraction of goods from factory n to store m
c_{mn}: cost per unit of goods from factory n to store m

- L^1- Wasserstein distance:

$$W(p,q) = \min_{\pi} \sum_{m,n} \pi_{mn} c_{mn}$$

Transport plan π_{mn} (joint distribution):

$$\sum_m \pi_{mn} = p_n, \sum_n \pi_{mn} = q_m$$

cost matrix c_{mn} satisfies: $c_{mn} + c_{nk} \geqslant c_{mk}$

(triangular inequality)

Kantorovich-Rubinstein duality

- If the cost matrix is symmetric: $c_{mn} = c_{nm}$:

$$W(p,q) = \max_{\phi} \phi^T (p-q)$$

the maximum is taken for the vectors ϕ satisfying $|\phi_m - \phi_n| \leq c_{mn}$.

- For a general cost matrix:

$$W(p,q) \leqslant \max_{\phi} \phi^T (p-q)$$

the maximum is taken for the vectors ϕ satisfying $\phi_m - \phi_n \leq c_{mn}$.

(applied in open quantum systems)

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Criterion for Transport

- For one-body loss, we can see

$$\frac{d}{dt}\langle n_i\rangle = -\gamma\langle n_i\rangle \Rightarrow \langle n_i\rangle(t) = e^{-\gamma t}\langle n_i\rangle(0)$$

if there is no Hamiltonian. Hence, the 1-norm of distribution $p_i = n_i/N$ is decreasing with time.

$$||p||_1 = \sum_n p_n$$

- Criterion for transport in open quantum systems:

$$n_Y(\tau) = n'_{X^c}(\tau) + \mu N$$

$n_Y(\tau)$: particle number of Y by switching on tunneling from X to X^c

$n'_{X^c}(\tau)$: particle number of X^c by switching off tunneling from X to X^c

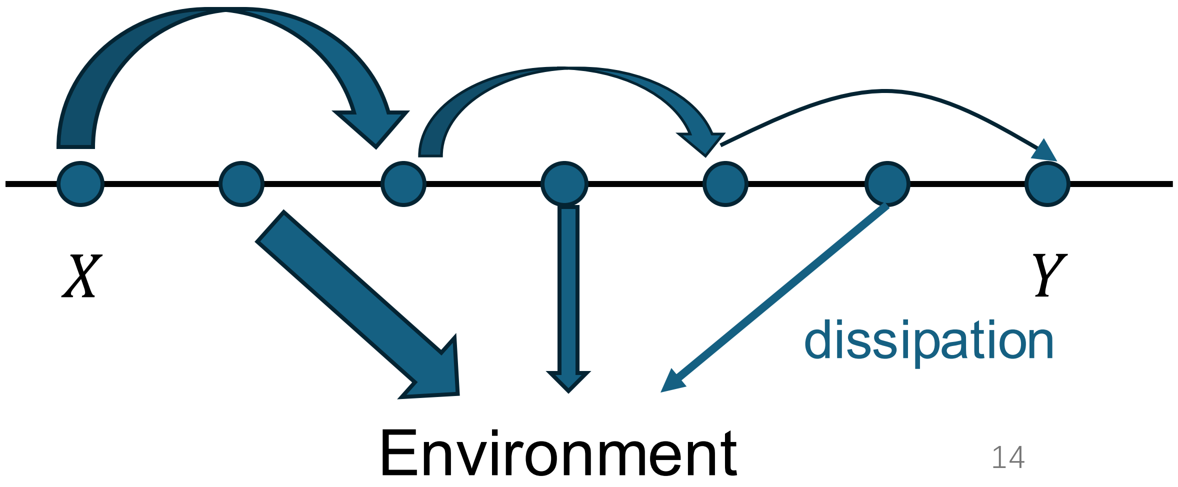
Bosonic transport in open quantum systems

- For one-body loss: $L_i = b_i$, the transport time satisfies

$$\tau e^{-\gamma\tau} \geqslant \frac{\mu}{J\varphi} d_{XY}^{\min(1,\alpha-D)}$$

- The exponential decay is due to the loss effect: $N_t = N_0 e^{-\gamma t}$

- Therefore, the current also decays due to the loss, leading to the deceleration of the transport process.



One-body loss

- Corollaries from the result: $$\tau e^{-\gamma\tau} \geqslant \frac{\mu}{J\varphi} d_{XY}^{\min(1,\alpha-D)}$$

1. There is an upper bound for the transportable particle numbers:

$$\mu \leq \min\left(\frac{J\varphi\zeta(\alpha-\alpha_\varepsilon-D+1)}{e\gamma d_{XY}^{\alpha_\varepsilon}}, 1\right)$$

2. There exists a transport distance limit for bosons:

$$d_l = \left(\frac{J\varphi}{\mu e\gamma}\right)^{1/\min(1,\alpha-D)}$$

If we take $\mu = 1/N$, the limit represents the size of bosons (the farthest distance that the bosons can reach):

$$d_l = \left(\frac{NJ\varphi}{e\gamma}\right)^{1/\min(1,\alpha-D)}$$

Many-body loss

- However, for many-body loss: $L_i = b_i^n$, the transport time satisfies:

$$\tau \geqslant \frac{\mu}{J\varphi} d_{XY}^{\min(1,\alpha-D)}$$

This exactly meets the result in closed quantum systems.

- Reason: decoherence-free subspaces $\mathcal{H}$ D. A. Lidar, et al., PRL **81**, 2594 (1998)

$$\rho \in \mathcal{H} \qquad 2L_i\rho L_i^\dagger - L_i^\dagger L_i\rho - \rho L_i^\dagger L_i = 0$$

$$\Updownarrow$$

for an arbitrary site i

$$L_i|n\rangle = 0, \forall |n\rangle \in \mathcal{H}$$

Many-body loss

- One-body loss: does not exist nontrivial states.
- Many-body loss: exists!

$$L_i|n\rangle = 0, \forall |n\rangle \in \mathcal{H}$$

- Examples: the ground state of Bose-Hubbard model with $U \to \infty$:

$$H = \sum_{i \neq j} J_{ij}(b_i^\dagger b_j + b_j^\dagger b_i) + U \sum_i \hat{n}_i(\hat{n}_i - 1)$$

- Since the double-occupancy leads to an infinite increase of energy, the ground state consists of single-occupied states.

- The existence of decoherence-free subspaces can enhance the transport process.

Sketch of proof

- The proof relies on the generalized Wasserstein distance.

Traditional Wasserstein distance: between balanced distributions.

$$\sum_m \pi_{mn} = p_n, \sum_n \pi_{mn} = q_m$$

Closed quantum systems: compare the distance between the initial and the final distributions with the same norms: $W(p,q), ||p||_1 = ||q||_1$.

- However, open quantum system leads to imbalancedness of norms

$$\|p_t\|_1 \leqslant \|p_0\|_1$$

Generalized Wasserstein distance?

Sketch of Proof

- Generalized Wasserstein distance:
- One-body loss:

$$\tilde{W}(p_\tau, p_0) = W(p_\tau, p_0 e^{-\gamma\tau})$$

- General cases:

$$\widetilde{W}(\boldsymbol{x}, \boldsymbol{y}) := \min_{\boldsymbol{x} \succeq \boldsymbol{x}' \succeq \boldsymbol{0}, \boldsymbol{y}' \succeq \boldsymbol{y}, \|\boldsymbol{x}'\|_1 = \|\boldsymbol{y}'\|_1} W(\boldsymbol{x}', \boldsymbol{y}')$$

$$||x||_1 \geq ||x'||_1 = ||y'||_1 \geq ||y||_1$$

- The dynamics of particle number distribution: $x_i(t) := \mathrm{tr}(b_i^\dagger b_i \rho_t)/N$

One-body loss:

$$\dot{x}_i(t) = \underbrace{\frac{1}{N} \sum_{j, j \neq i} 2J_{ij}(t)\, \mathrm{Im}[\mathrm{tr}(b_j^\dagger b_i \rho_t)]}_{\text{current}} - \underbrace{\gamma x_i(t)}_{\text{dissipation}}$$

Sketch of Proof

- The Wasserstein distance can be bounded by

$$W(e^{-\gamma\tau}\boldsymbol{x}_0,\boldsymbol{x}_\tau) \geq \min_{i\in Y,j\in X} c_{ij} \sum_{i\in Y,j\in X} \pi_{ij} \geq \mu d_{XY}^{\alpha_\varepsilon}$$

$$\alpha_\varepsilon = \min(1, \alpha - D)$$

$$c_{ij} = \|i - j\|^{\alpha_\varepsilon}$$

$$W(e^{-\gamma\tau}\boldsymbol{x}_0,\boldsymbol{x}_\tau) \leq \tau e^{-\gamma\tau} J\varphi$$

$$\Longrightarrow \quad \tau e^{-\gamma\tau} \geqslant \frac{\mu}{J\varphi} d_{XY}^{\min(1,\alpha-D)}$$

- For many-body-loss case:

ϕ_{ij}: current flow from j to i

$$\dot{x}_i(t) = -d_i(t) + \sum_{j(\neq i)} \phi_{ij}(t)$$

$$d_i(t) := -(\gamma/2N)\,\mathrm{tr}(\hat{d}_i\rho_t)$$ dissipation

$$\hat{d}_i := 2(b_i^\dagger)^n \hat{n}_i b_i^n - \hat{n}_i (b_i^\dagger)^n b_i^n - (b_i^\dagger)^n b_i^n \hat{n}_i$$

Sketch of Proof

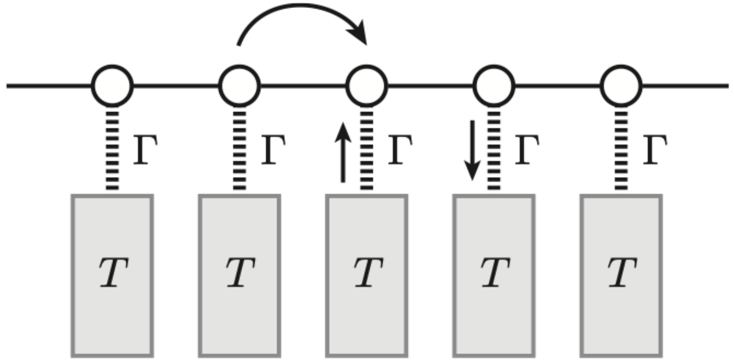
- Similarly,

$$\begin{aligned}\widetilde{W}(\boldsymbol{x}_0,\boldsymbol{x}_\tau) &= \min_{\boldsymbol{x}_0\succeq\boldsymbol{x}'\succeq\boldsymbol{0},\boldsymbol{x}''\succeq\boldsymbol{x}_\tau,\|\boldsymbol{x}'\|_1=\|\boldsymbol{x}''\|_1}\min_{\pi}\sum_{i,j}\pi_{ij}c_{ij}\\ &\geq \mu d_{XY}^{\alpha_\varepsilon},\\ \widetilde{W}(\boldsymbol{x}_0,\boldsymbol{x}_\tau) &\leq \frac{1}{2}\int_0^\tau dt\sum_{i\neq j}c_{ij}|\phi_{ij}(t)|\\ &\leq \tau J\varphi\zeta(\alpha-\alpha_\varepsilon-D+1)\end{aligned}$$

$$\Longrightarrow \qquad \tau \geqslant \frac{\mu}{J\varphi}d_{XY}^{\min(1,\alpha-D)}$$

- The equality holds when there exist decoherence-free subspaces.

Gain and loss



- So far, we have only considered the lossy case. In condensed matter physics, the coupling to baths includes both the gain effects and loss effects:
- The master equation is given by

$$\frac{d\rho_t}{dt} = i[\rho_t, H] + \sum_{i,j}^{\alpha=1,2} \frac{\gamma_\alpha}{2}(2L_i^\alpha \rho_t L_j^{\alpha\dagger} - L_j^{\alpha\dagger} L_i^\alpha \rho_t - \rho_t L_j^{\alpha\dagger} L_i^\alpha)$$

$$L_i^1 := b_i \qquad\qquad L_i^2 := b_i^\dagger$$

- The coefficients $\gamma_1(\gamma_2)$ are loss(gain) rates. Here we only focus on the case with $\gamma_1 > \gamma_2$ (loss dominant).

Gain and loss

- The dynamics of the particle number distribution:

$$\dot{x}_i(t) = \frac{1}{N}\sum_{j,j\neq i} 2J_{ij}(t)\,\mathrm{Im}[\mathrm{tr}(b_j^\dagger b_i \rho_t)] - \underbrace{\gamma_1 x_i(t)}_{\text{loss}} + \underbrace{\gamma_2\left[x_i(t) + \frac{1}{N}\right]}_{\text{gain}}$$

- Since it contains both loss and gain, the dynamics of number distribution without Hamiltonian is given by

$$y_i(\tau) = y_i(0)e^{-\Delta\gamma\tau} + \frac{\gamma_2}{N\Delta\gamma}(1 - e^{-\Delta\gamma\tau})$$

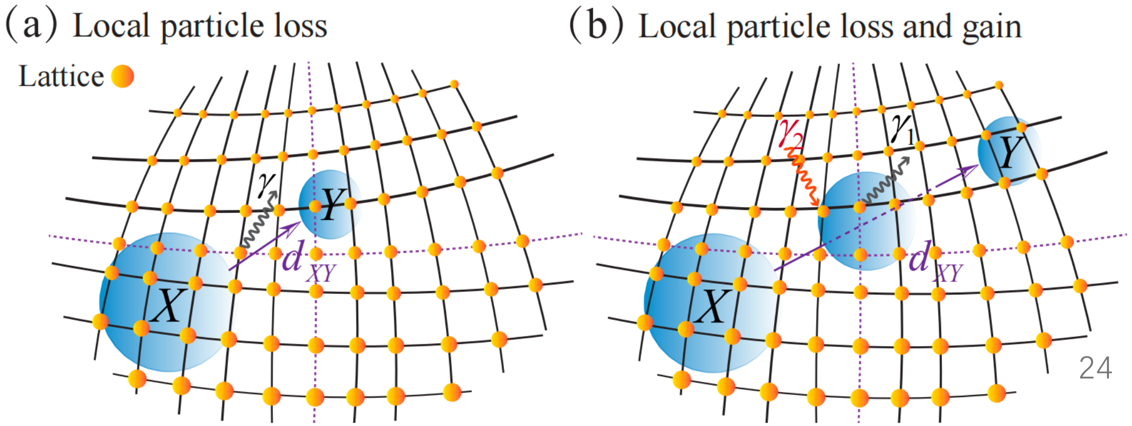
- Therefore, the criterion becomes: $x_Y(\tau) - y_{X^c}(\tau) \geq \mu$.

Gain and loss

- By introducing a similar Wasserstein distance, we have

$$\tau e^{-\Delta\gamma\tau} + \frac{\gamma_2}{\Delta\gamma}\mathcal{N}^{-1}\left(\frac{1-e^{-\Delta\gamma\tau}}{\Delta\gamma} - \tau e^{-\Delta\gamma\tau}\right) \geq \kappa_1^{\varepsilon} d_{XY}^{\alpha_\varepsilon} \qquad \Delta\gamma := \gamma_1 - \gamma_2$$

- Since the left-hand side is larger than $\tau e^{-\gamma\tau}$, the transport time is decreased by the gain effect.
- In the balanced limit: $\Delta\gamma \to 0$, the left-hand side becomes τ, which returns back to closed quantum systems.



Gain and loss

$$\tau_c := (\Delta\gamma - \gamma_2/\mathcal{N})^{-1}$$

- Corollaries from the result:

1. The upper bound for transportable particle number:

$$\mathcal{N} := N/|\Lambda|$$

$$\mu \leq \min\left(\frac{J\varphi\zeta(\alpha-\alpha_\varepsilon-D+1)}{d_{XY}^{\alpha_\varepsilon}}B(\tau_c), 1\right) \quad B(t) := \begin{cases} \frac{e^{-\Delta\gamma t}}{(\Delta\gamma)^2 t} + \frac{\gamma_2\mathcal{N}^{-1}}{(\Delta\gamma)^2} & \text{if } \mathcal{N} > \frac{\gamma_2}{\Delta\gamma}, \\ \frac{\gamma_2\mathcal{N}^{-1}}{(\Delta\gamma)^2} & \text{if } \mathcal{N} \leq \frac{\gamma_2}{\Delta\gamma}, \end{cases}$$

The dense limit is nothing different from the one-body-loss case.

However, for the dilute limit we have

$$B(\tau_c) = \gamma_2\mathcal{N}^{-1}/(\Delta\gamma)^2 \to \infty$$

, indicating there is no upper bound by enlarging the lattice.

Gain and loss

2. Size of bosons:

$$d_l = (NJ\varphi\zeta(\alpha - \alpha_\varepsilon - D + 1)B(\tau_c))^{1/\alpha_\varepsilon}$$

In the dilute limit, the size becomes

$$d_l = \left(\frac{\gamma_2|\Lambda|J\varphi\zeta(\alpha - \alpha_\varepsilon - D + 1)}{(\Delta\gamma)^2}\right)^{1/\alpha_\varepsilon}$$

The farthest transport distance can be also determined by lattice sites.

- Reason: decoherence-free subspace:

$$\rho = \bigotimes_i \left[\frac{1}{1 - \gamma_2/\gamma_1}\sum_{n=0}^{\infty}\left(\frac{\gamma_2}{\gamma_1}\right)^n |n\rangle_i\langle n|\right]$$

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Conclusion and Outlook

- Construct optimal transport theory in open quantum systems
- Derive the lower bound for transport time τ for dissipative cases
- Show the upper bound for transportable particle number and transport distance limit
- Show that the decoherence-free subspace enhances the transport

  (no transport distance or particle number limit, enhance the transport)

Conclusion and Outlook

Possible topics in the future:

- probability of observing particles beyond d_{XY} when the time is less than the lower bound
- microscopic transport theory
- Lieb-Robinson bound for long-range systems
- …

Thank you!