Hydrodynamics of low-dimensional interacting systems: Advances, challenges and future directions
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On the dynamical origin of the vorticity alignment in homogeneous and isotropic turbulence

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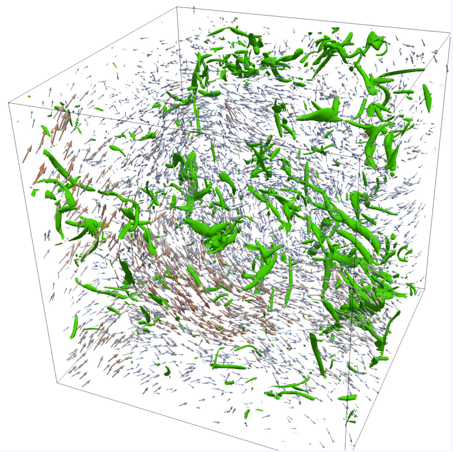
Homogeneous and isotropic turbulence



Velocity field and iso-surface of $|\omega|^2$ $(Re_\lambda = 210)$

- Simulation of forced Navier–Stokes equations in a periodic cube

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{f},$$
$$\nabla \cdot \boldsymbol{u} = 0$$

- Statistically steady, homogeneous and isotropic turbulence

Orientational order in turbulence

- Scaling laws in turbulence
  Moments of the velocity increments exhibit universal scaling laws such as Kolmogorov $2/3$ law

$$\left\langle \left\{ [\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},t)-\boldsymbol{u}(\boldsymbol{x},t)]\cdot\frac{\boldsymbol{r}}{|\boldsymbol{r}|} \right\}^2 \right\rangle = C\epsilon^{2/3}r^{2/3}$$

- Vorticity alignment (Ashurst, Kernstein, Kerr and Gibson 1987)
  The vorticity vector $\boldsymbol{\omega}=\nabla\times\boldsymbol{u}$ tends to align with a certain direction.

Vorticity alignment: rate-of-strain tensor

- The equations of the vorticity $\boldsymbol{\omega}(\boldsymbol{x},t) = \nabla \times \boldsymbol{u}(\boldsymbol{x},t)$

$$\partial_t \boldsymbol{\omega} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} + \nu \nabla^2 \boldsymbol{\omega} + \nabla \times \boldsymbol{f}.$$

- Rate-of-strain tensor S

$$S_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right)$$

- With S, the vorticity equations become

$$\partial_t \omega_i + u_\ell \partial_\ell \omega_i = S_{i\ell}\omega_\ell + \nu \partial_\ell^2 \omega_i + (\nabla \times \boldsymbol{f})_i.$$

- S can be diagonalized at each $\boldsymbol{x}$ and t

$$S = P^{-1} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} P.$$

Here $\lambda_1 + \lambda_2 + \lambda_3 = 0$ due to $\nabla \cdot \boldsymbol{u} = 0$.

Vorticity alignment

- Rate-of-strain tensor $S_{ij} = (\partial_i u_j + \partial_j u_i)/2$ can be diagonalized at each $\boldsymbol{x}$ and t

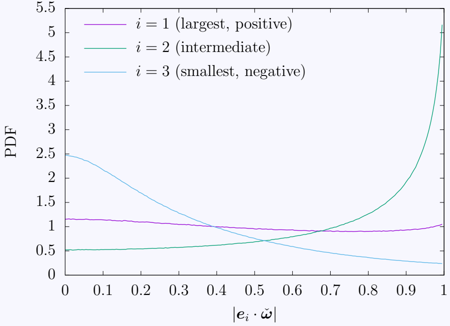
$$S = P^{-1} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} P.$$

Here $\lambda_1 + \lambda_2 + \lambda_3 = 0$ due to $\nabla \cdot \boldsymbol{u} = 0$.

- Let us set $\lambda_1 \geq \lambda_2 \geq \lambda_3$ ($\lambda_1 \geq 0$ and $\lambda_3 \leq 0$).
  Let $\boldsymbol{e}_1, \boldsymbol{e}_2$ and $\boldsymbol{e}_3$ be the associated eigenvectors (principal axes of S).

- Vorticity alignment (Ashurst *et al.* 1987)
  - The vorticity $\boldsymbol{\omega}(\boldsymbol{x}, t)$ tends to align with $\boldsymbol{e}_2(\boldsymbol{x}, t)$ in turbulence.
  - $\boldsymbol{e}_2$: the eigenvector of the intermediate eigenvalue λ_2 of S

Vorticity alignment: discovery in simulation



$Re_\lambda = 130$

- Probability density function of

$$\cos\theta_i = \boldsymbol{e}_i(\boldsymbol{x}, t) \cdot \frac{\boldsymbol{\omega}(\boldsymbol{x}, t)}{|\boldsymbol{\omega}(\boldsymbol{x}, t)|}$$

$\boldsymbol{e}_1$: eigenvector of the largest eigenvalue $\lambda_1 \geq 0$

$\boldsymbol{e}_2$: eigenvector of the intermediate eigenvalue λ_2

$\boldsymbol{e}_3$: eigenvector of the smallest eigenvalue $\lambda_3 \leq 0$

- Most probably $\cos\theta_2 = 1$ $(\theta_2 = 0)$
  $\boldsymbol{\omega}$ prefers to be aligned with $\boldsymbol{e}_2$.
- The PDFs are same for higher Reynolds numbers
  (Buaria, Bodenschatz & Pumir 2020).

Vorticity alignment: remarks

- The vorticity alignment with $\boldsymbol{e}_2$
  - Spontaneous orientational order in turbulence

    “This result was unexpected by the statistical physics community, but had been anticipated by vortex models such as Tennekes (1968), Lundgren (1982) and Vieillefosse (1982, 1984).“

    quoted from Schumacher, Kerr & Horiuti (2013).
  - The vorticity magnitude is not maximally amplified

$$\partial_t \frac{|\boldsymbol{\omega}|^2}{2} + (\boldsymbol{u} \cdot \nabla) \frac{|\boldsymbol{\omega}|^2}{2} = \boldsymbol{\omega} S \boldsymbol{\omega} + \nu \boldsymbol{\omega} \cdot \nabla^2 \boldsymbol{\omega} + \boldsymbol{\omega} \cdot (\nabla \times \boldsymbol{f}).$$

    Or, the nonlinearity is depleted (e.g., Constantin 1994).
- Why the alignment with $\boldsymbol{e}_2$ dominates has not been answered.

Outline

- Why the alignment of ω with e_2 dominates has not been answered (Schumacher, Kerr & Horiuti 2013).

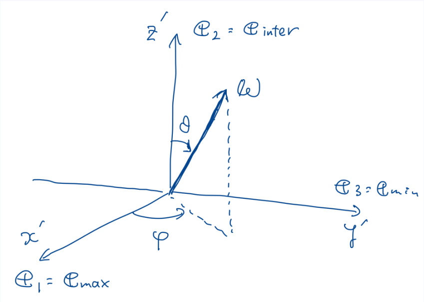
- Outline of this talk

  We study

  (1) the alignment with local spherical coordinates spanned by e_i's,

  (2) a dynamical model for the alignment
  (evolution of the angle between ω and e_2),

Local strain coordinates



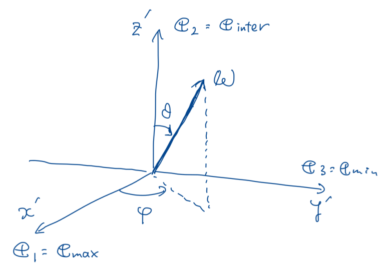
Local strain coordinates

- Eigenvalues of the rate-of-strain tensor $S(\boldsymbol{x}, t)$
  $\lambda_1 \geq \lambda_2 \geq \lambda_3$
  Here $\mathrm{tr} S = \lambda_1 + \lambda_2 + \lambda_3 = 0$ ($\lambda_1 \geq 0$ and $\lambda_3 \leq 0$)
- The corresponding eigenvectors $\boldsymbol{e}_1, \boldsymbol{e}_2$ and $\boldsymbol{e}_3$.
  $\boldsymbol{e}_1 \Rightarrow x'$ axis, $\boldsymbol{e}_2 \Rightarrow z'$ axis, $\boldsymbol{e}_3 \Rightarrow y'$ axis
- The vorticity is then written as

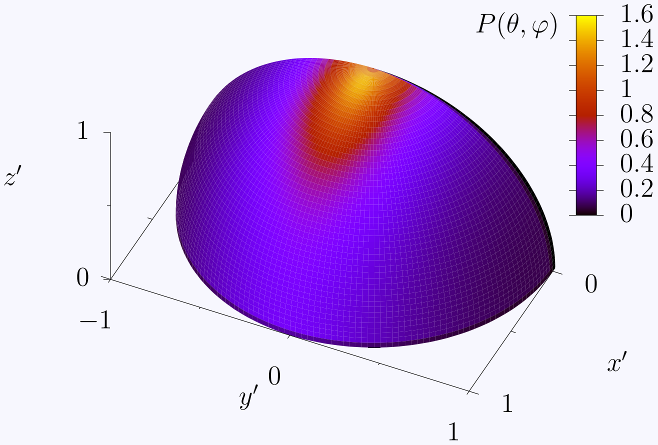
$$\boldsymbol{\omega} = \begin{pmatrix} \omega \sin\theta \cos\varphi \\ \omega \sin\theta \sin\varphi \\ \omega \cos\theta \end{pmatrix}$$

  $0 \leq \theta \leq \pi/2$ and $-\pi/2 \leq \varphi \leq \pi/2$.
- Calculate PDF $P(\theta, \varphi)$ from simulation data!

PDF $P(\theta, \varphi)$ from simulation

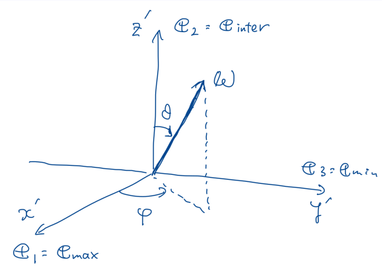


Local strain coordinates

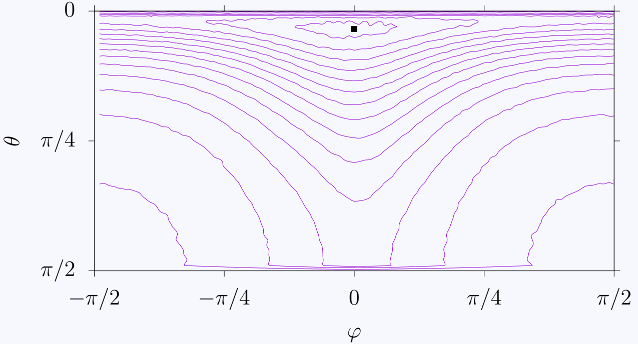


Probability to find the direction of $\boldsymbol{\omega}$: $P(\theta, \varphi) \sin\theta \mathrm{d}\theta \mathrm{d}\varphi$
($Re_\lambda = 210$)

PDF $P(\theta, \varphi)$ from simulation



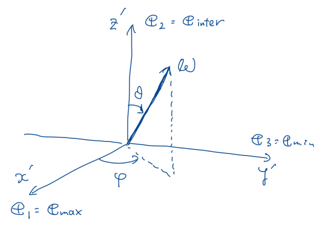
Local strain coordinates

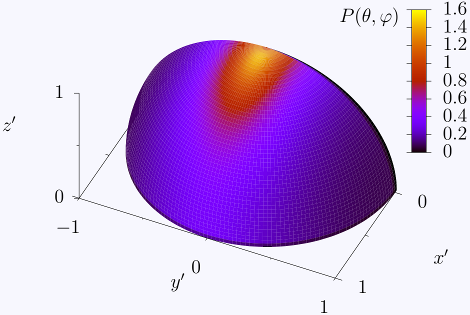


Contours of $P(\theta, \varphi)$ with peak at $(\theta_*, \varphi_*) \simeq (0.035\pi, 0)$

$(Re_\lambda = 210)$

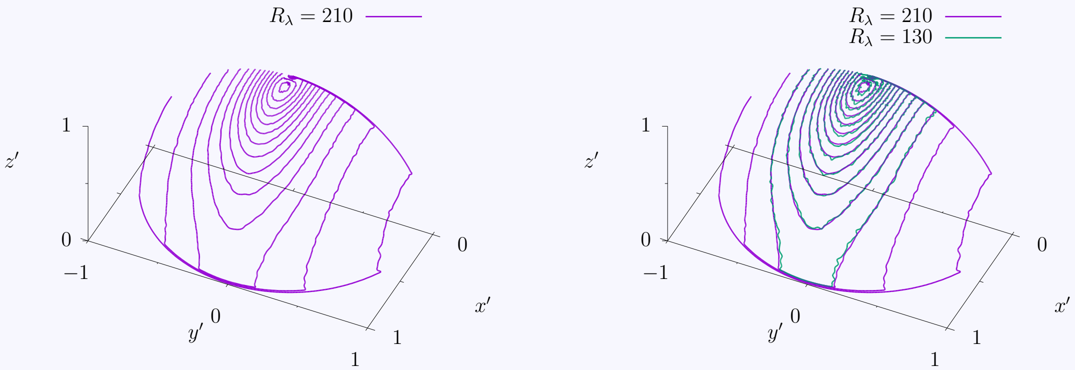
Observation on PDF $P(\theta, \varphi)$





- PDF $P(\theta, \varphi)$
  - Symmetric with respect to the y' axis
  - The peak is at $(\theta_*, \varphi_*) \simeq (0.035\pi, 0) = (6.3^\circ, 0^\circ)$
- Implications
  - The vorticity does not have the e_3-component on average.
  - θ_* may not be zero.

Re-number dependence of PDF $P(\theta, \varphi)$



- $P(\theta, \varphi)$ does not depend on the Reynolds number at least in $100 \lesssim R_\lambda \lesssim 200$.
- The peak $(\theta_*, \varphi_*) \simeq (0.035\pi, 0) = (6.3^\circ, 0^\circ)$ does not either.

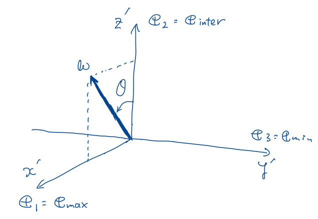
- Outline of this talk

  We study

  (1) the alignment with local spherical coordinates spanned by e_i's,

  (2) a dynamical model for the alignment
  (evolution of the angle between ω and e_2),

Evolution of polar angle $\theta(t)$



- Can one model evolution of the polar angle $\theta(t)$ in the local strain coordinates?
- Let us assume $\varphi = 0$ for simplicity.
- Then, the vorticity is written as

$$\boldsymbol{\omega}(t) = \begin{pmatrix} \omega(t)\sin\theta(t) \\ 0 \\ \omega(t)\cos\theta(t) \end{pmatrix}$$

- Observe that

$$\tan\theta(t) = \frac{\boldsymbol{\omega}(t)\cdot \boldsymbol{e}_1(t)}{\boldsymbol{\omega}(t)\cdot \boldsymbol{e}_2(t)}.$$

- Can we model evolution of $\boldsymbol{\omega}(t), \boldsymbol{e}_1(t)$ and $\boldsymbol{e}_2(t)$?

Model of evolution of polar angle $\theta(t)$

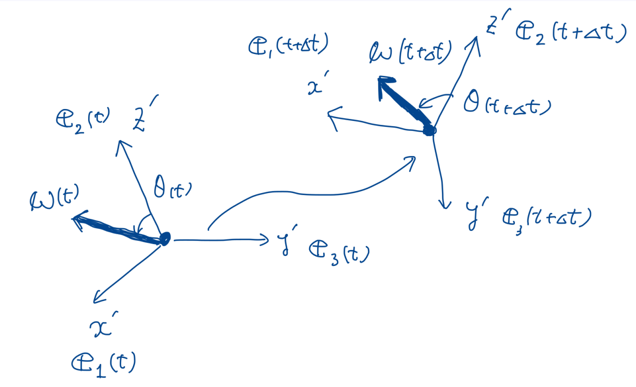
- The model we develop is

$$
\begin{aligned}
\frac{\mathrm{d}\theta}{\mathrm{d}t} &= \frac{[2(\lambda_1-\lambda_2)+\omega][2(\lambda_1-\lambda_2)-\omega]}{8(\lambda_1-\lambda_2)}\sin 2\theta,\\
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= \omega(\lambda_1\sin^2\theta+\lambda_2\cos^2\theta),\\
\frac{\mathrm{d}\lambda_1}{\mathrm{d}t} &= \frac{1}{3}(-2\lambda_1^2+\lambda_2^2+\lambda_3^2)+\frac{\omega^2}{12}(3\cos^2\theta-2),\\
\frac{\mathrm{d}\lambda_2}{\mathrm{d}t} &= \frac{1}{3}(\lambda_1^2-2\lambda_2^2+\lambda_3^2)+\frac{\omega^2}{12}(3\sin^2\theta-2),\\
\frac{\mathrm{d}\lambda_3}{\mathrm{d}t} &= \frac{1}{3}(\lambda_1^2+\lambda_2^2-2\lambda_3^2)+\frac{\omega^2}{12}.
\end{aligned}
$$

(cf. Viellefosse 1982; Majda 1991).

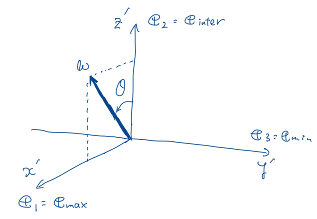
- We assume $\varphi = 0$ and ignore the viscosity and the forcing.

Sketch of derivation of the model



$$\begin{cases} \partial_t \boldsymbol{\omega} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} &= S\boldsymbol{\omega} + \cancel{\nu\nabla^2\boldsymbol{\omega}}, \\ \partial_t S + (\boldsymbol{u}\cdot\nabla)S &= -S^2 - \Omega^2 - (\nabla\otimes\nabla)p + \cancel{\nu\nabla^2 S}, \\ \nabla^2 p &= -S_{ij}S_{ij} + \dfrac{|\boldsymbol{\omega}|^2}{2} \end{cases} \Rightarrow \begin{cases} \dot{\boldsymbol{\omega}} &= S\boldsymbol{\omega}, \\ \dot{S} &= -S^2 - \Omega^2 - (\nabla\otimes\nabla)p, \\ \nabla^2 p &= -S_{ij}S_{ij} + \dfrac{|\boldsymbol{\omega}|^2}{2} \end{cases}$$

Evolution of the vorticity



- Lagrangian evolution of the vorticity in the local strain coordinates

$$\dot{\boldsymbol{\omega}} \simeq S\boldsymbol{\omega} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_3 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \omega \sin\theta \\ 0 \\ \omega\cos\theta \end{pmatrix} = \begin{pmatrix} \lambda_1 \omega \sin\theta \\ 0 \\ \lambda_2 \omega \cos\theta \end{pmatrix}$$

Therefore

$$\boldsymbol{\omega}(t+\Delta t) = \begin{pmatrix} (1+\Delta t \lambda_1)\omega\sin\theta \\ 0 \\ (1+\Delta t\lambda_2)\omega\cos\theta \end{pmatrix} + O(\Delta t^2),$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \omega(\lambda_1 \sin^2\theta + \lambda_2\cos^2\theta)$$

$\omega(t) = |\boldsymbol{\omega}(t)|$

Evolution of the rate-of-strain tensor

- Lagrangian evolution of the rate-of-strain tensor S in the local strain coordinates

$$\dot{S} \simeq -S^2 - \Omega^2 - (\nabla \otimes \nabla)p$$

- Ω (anti-symmetric part of the velocity gradient tensor) is

$$\Omega = \frac{1}{2}\begin{pmatrix} 0 & \omega_{z'} & -\omega_{y'} \\ -\omega_{z'} & 0 & \omega_{x'} \\ \omega_{y'} & -\omega_{x'} & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & \omega\cos\theta & 0 \\ -\omega\cos\theta & 0 & \omega\sin\theta \\ 0 & -\omega\sin\theta & 0 \end{pmatrix}$$

- The pressure Hessian is here modeled as an identity-matrix form using $\nabla^2 p = -S_{ij}S_{ij} + |\boldsymbol{\omega}|^2/2 = \mathrm{tr}[(\nabla \otimes \nabla)p]$

$$(\nabla \otimes \nabla)p \sim \frac{1}{3}\left(-\lambda_1^2 - \lambda_2^2 - \lambda_3^2 + \frac{\omega^2}{2}\right)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(Vieillefosse 1982).

Evolution of the rate-of-strain tensor

- The rate-of-strain $S(t + \Delta t)$ is given as

$$S(t+\Delta t) \simeq \begin{pmatrix} \lambda_1 + \Delta t \left[\frac{1}{3}(-2\lambda_1^2 + \lambda^2 + \lambda_3^2) + \frac{\omega^2}{12}(3\cos^2\theta - 2)\right] & 0 & -\Delta t \frac{\omega^2}{4}\cos\theta\sin\theta \\ 0 & \lambda_3 + \Delta t \left[\frac{1}{3}(\lambda_1^2 + \lambda^2 - 2\lambda_3^2) + \frac{\omega^2}{12}\right] & 0 \\ -\Delta t \frac{\omega^2}{4}\cos\theta\sin\theta & 0 & \lambda_2 + \Delta t \left[\frac{1}{3}(\lambda_1^2 - 2\lambda^2 + \lambda_3^2) + \frac{\omega^2}{12}(3\sin^2\theta - 2)\right] \end{pmatrix} + O(\Delta t^2).$$

- We then solve the eigenvalue problem of $S(t + \Delta t)$.
  The largest eigenvalue and eigenvector are

$$\lambda_1(t+\Delta t) = \frac{1}{2}\left\{\lambda_1(t) + \lambda_2(t) + \Delta t(a+c) + \sqrt{[\lambda_1(t) - \lambda_2(t) + \Delta t(a-c)]^2 + 4\Delta t^2 b^2}\right\},$$

$$\boldsymbol{e}_1(t+\Delta t) \parallel \begin{pmatrix} \lambda_1(t+\Delta t) - \lambda_2(t) - \Delta t c \\ 0 \\ -\Delta t b \end{pmatrix},$$

where $a = \frac{1}{3}(-2\lambda_1^2 + \lambda_2^2 + \lambda_3^2) + \frac{\omega^2}{12}(3\cos^2\theta - 2)$, $b = -\frac{\omega^2}{4}\cos\theta\sin\theta$,
$c = \frac{1}{3}(\lambda_1^2 - 2\lambda_2^2 + \lambda_3^2) + \frac{\omega^2}{12}(3\sin^2\theta - 2)$

Evolution of the rate-of-strain tensor

- The intermediate eigenvalue and eigenvector are

$$\lambda_2(t+\Delta t) = \frac{1}{2}\left\{\lambda_1(t)+\lambda_2(t)+\Delta t(a+c) - \sqrt{[\lambda_1(t)-\lambda_2(t)+\Delta t(a-c)]^2 + 4\Delta t^2 b^2}\right\},$$

$$\boldsymbol{e}_2(t+\Delta t) \parallel \begin{pmatrix} -\Delta t b \\ 0 \\ \lambda_1(t)+\Delta t a - \lambda_2(t+\Delta t). \end{pmatrix}$$

- The smallest eigenvalue $\lambda_3(t+\Delta t)$ and eigenvector $\boldsymbol{e}_3(t+\Delta t)$ are

$$\lambda_3(t+\Delta t) = \lambda_3 + \Delta t\left[\frac{1}{3}(\lambda_1^2+\lambda_2^2-2\lambda_3^2)+\frac{\omega^2}{12}\right] + O(\Delta t^2),$$

$$\boldsymbol{e}_3(t+\Delta t) \parallel \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

where $a = \frac{1}{3}(-2\lambda_1^2+\lambda_2^2+\lambda_3^2)+\frac{\omega^2}{12}(3\cos^2\theta-2)$, $b = -\frac{\omega^2}{4}\cos\theta\sin\theta$, $c = \frac{1}{3}(\lambda_1^2-2\lambda_2^2+\lambda_3^2)+\frac{\omega^2}{12}(3\sin^2\theta-2)$

Evolution of the eigenvalues and the polar angle

- The expressions of $\lambda_j(t+\Delta t)$ yield

$$\frac{\mathrm{d}\lambda_1}{\mathrm{d}t} = \frac{1}{3}(-2\lambda_1^2+\lambda_2^2+\lambda_3^2) + \frac{\omega^2}{12}(3\cos^2\theta-2),$$
$$\frac{\mathrm{d}\lambda_2}{\mathrm{d}t} = \frac{1}{3}(\lambda_1^2-2\lambda_2^2+\lambda_3^2) + \frac{\omega^2}{12}(3\sin^2\theta-2),$$
$$\frac{\mathrm{d}\lambda_3}{\mathrm{d}t} = \frac{1}{3}(\lambda_1^2+\lambda_2^2-2\lambda_3^2) + \frac{\omega^2}{12}.$$

- For the polar angle $\theta(t+\Delta t)$

$$\tan\theta(t+\Delta t) = \frac{\boldsymbol{\omega}(t+\Delta t)\cdot\boldsymbol{e}_1(t+\Delta t)}{\boldsymbol{\omega}(t+\Delta t)\cdot\boldsymbol{e}_2(t+\Delta t)}$$
$$= \tan\theta + \Delta t\,\frac{[2(\lambda_1-\lambda_2)+\omega][2(\lambda_1-\lambda_2)-\omega]}{4(\lambda_1-\lambda_2)}\tan\theta + O(\Delta t^2).$$

Therefore

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{[2(\lambda_1-\lambda_2)+\omega][2(\lambda_1-\lambda_2)-\omega]}{8(\lambda_1-\lambda_2)}\sin 2\theta.$$

Summary of the model

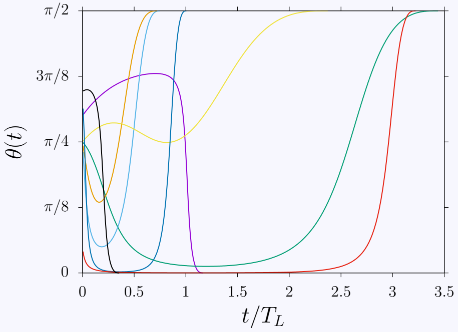
- The model (ODEs)

$$
\begin{aligned}
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= \omega(\lambda_1 \sin^2\theta + \lambda_2 \cos^2\theta), \\
\frac{\mathrm{d}\lambda_1}{\mathrm{d}t} &= \frac{1}{3}(-2\lambda_1^2 + \lambda_2^2 + \lambda_3^2) + \frac{\omega^2}{12}(3\cos^2\theta - 2), \\
\frac{\mathrm{d}\lambda_2}{\mathrm{d}t} &= \frac{1}{3}(\lambda_1^2 - 2\lambda_2^2 + \lambda_3^2) + \frac{\omega^2}{12}(3\sin^2\theta - 2), \\
\frac{\mathrm{d}\theta}{\mathrm{d}t} &= \frac{[2(\lambda_1 - \lambda_2) + \omega][2(\lambda_1 - \lambda_2) - \omega]}{8(\lambda_1 - \lambda_2)} \sin 2\theta.
\end{aligned}
$$

$\lambda_3 = -\lambda_1 - \lambda_2 \quad (\lambda_1 \geq \lambda_2 \geq \lambda_3)$

- Lagrangian evolution
- We assumed the azimuthal angle $\varphi(t) = 0$ (the PDF is symmetric).
- We ignored the viscosity and the forcing.
- Off-diagonal components of the pressure Hessian were ignored [Restricted Euler equation (Vieillefosse 1982)].

Numerical simulation of the model



8 trajectories

T_L : Large scale turnover time of DNS

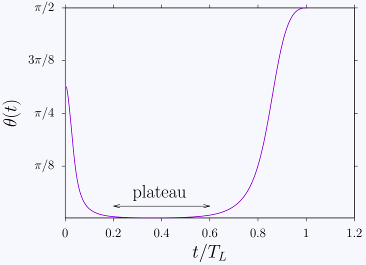
- The model (λ_3 is erased)

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \omega(\lambda_1 \sin^2\theta + \lambda_2 \cos^2\theta),$$
$$\frac{\mathrm{d}\lambda_1}{\mathrm{d}t} = \frac{1}{3}[-(\lambda_1-\lambda_2)^2 + 3\lambda_2^2] + \frac{\omega^2}{12}(3\cos^2\theta - 2),$$
$$\frac{\mathrm{d}\lambda_2}{\mathrm{d}t} = \frac{1}{3}[-(\lambda_1-\lambda_2)^2 + 3\lambda_1^2] + \frac{\omega^2}{12}(3\sin^2\theta - 2),$$
$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{[2(\lambda_1-\lambda_2)+\omega][2(\lambda_1-\lambda_2)-\omega]}{8(\lambda_1-\lambda_2)}\sin 2\theta.$$

- Initial values are taken from DNS (by imposing $|\varphi| \leq 0.01\pi$).

- For large t, $\theta(t) \to 0$ or $\pi/2$. $\omega(t)$ and $\lambda_1(t)$ seem diverge.
- Some trajectories show plateau of $\theta(t)$ close to $\theta = 0$.

Equilibrium point of the model



- The model (λ_3 is erased)

$$
\begin{aligned}
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= \omega(\lambda_1 \sin^2\theta + \lambda_2 \cos^2\theta), \\
\frac{\mathrm{d}\lambda_1}{\mathrm{d}t} &= \frac{1}{3}[-(\lambda_1-\lambda_2)^2 + 3\lambda_2^2] + \frac{\omega^2}{12}(3\cos^2\theta - 2), \\
\frac{\mathrm{d}\lambda_2}{\mathrm{d}t} &= \frac{1}{3}[-(\lambda_1-\lambda_2)^2 + 3\lambda_1^2] + \frac{\omega^2}{12}(3\sin^2\theta - 2), \\
\frac{\mathrm{d}\theta}{\mathrm{d}t} &= \frac{[2(\lambda_1-\lambda_2)+\omega][2(\lambda_1-\lambda_2)-\omega]}{8(\lambda_1-\lambda_2)}\sin 2\theta.
\end{aligned}
$$

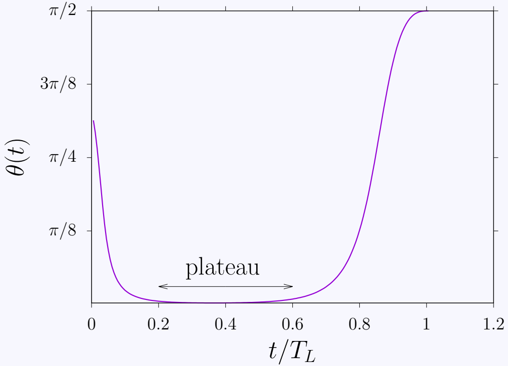
- The model has an equilibrium point,

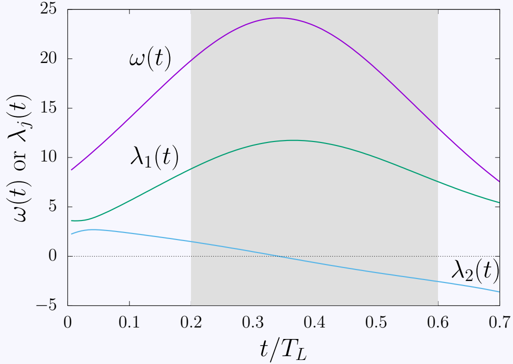
$$\theta = 0, \quad \omega = 2\lambda_1, \quad \lambda_2 = 0.$$

(No equilibrium point with $\theta = \pi/2$)

- Does the plateau correspond to meandering around this equilibrium point?

Plateau and the equilibrium of the model





- In the plateau, the equilibrium condition of the model

$$\theta = 0, \quad \omega = 2\lambda_1, \quad \lambda_2 = 0$$

  is roughly satisfied.
- The plateau is a long stay around the equilibrium point.

Insight from the model

- The model (ODEs)

$$
\begin{aligned}
\frac{\mathrm{d}\omega}{\mathrm{d}t} =& \omega(\lambda_1 \sin^2\theta + \lambda_2 \cos^2\theta), \\
\frac{\mathrm{d}\lambda_1}{\mathrm{d}t} =& \frac{1}{3}(-2\lambda_1^2 + \lambda_2^2 + \lambda_3^2) + \frac{\omega^2}{12}(3\cos^2\theta - 2), \\
\frac{\mathrm{d}\lambda_2}{\mathrm{d}t} =& \frac{1}{3}(\lambda_1^2 - 2\lambda_2^2 + \lambda_3^2) + \frac{\omega^2}{12}(3\sin^2\theta - 2), \\
\frac{\mathrm{d}\theta}{\mathrm{d}t} =& \frac{[2(\lambda_1 - \lambda_2) + \omega][2(\lambda_1 - \lambda_2) - \omega]}{8(\lambda_1 - \lambda_2)} \sin 2\theta \\
\lambda_3 =& -\lambda_1 - \lambda_2 \quad (\lambda_1 \geq \lambda_2 \geq \lambda_3)
\end{aligned}
$$

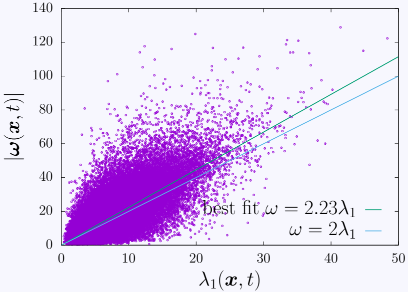
  - The polar angle $\theta = 0$ corresponds to the vorticity alignment with e_2.
  - There are long transient states around the equilibrium point,

$$\theta = 0, \quad \omega = 2\lambda_1, \quad \lambda_2 = 0 \tag{1}$$

- Insight from the model

  The vorticity alignment with e_2 corresponds to meandering around an equilibrium like Eq.(1).

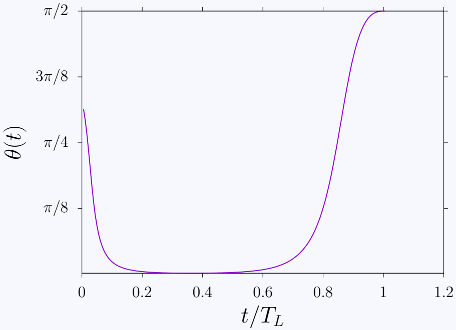
Check with Navier–Stokes simulation data



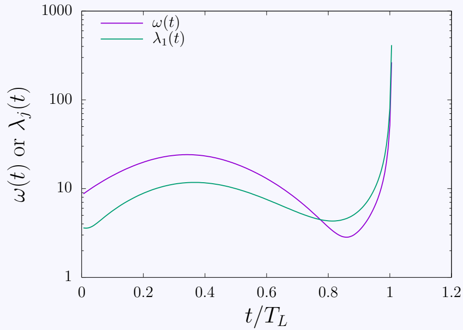
Data conditioned with $|\varphi| \le 0.01\pi$ and $\theta \le 0.05\pi$ ($Re_\lambda = 210$)

- Insight form the model : $\theta = 0, \omega = 2\lambda_1, \lambda_2 = 0$
- Scatter plot of DNS grid data satisfying the alignment $\boldsymbol{\omega}(\boldsymbol{x},t) \parallel \boldsymbol{e}_2(\boldsymbol{x},t)$
  - least square fit : $|\boldsymbol{\omega}(\boldsymbol{x},t)| = 2.23\lambda_1(\boldsymbol{x},t)$
  - Not always $|\lambda_2(\boldsymbol{x},t)| \ll \lambda_1(\boldsymbol{x},t)$
- For 3D scatter plot (λ_1, λ_2 and $|\boldsymbol{\omega}|$) least sq. fit: $|\boldsymbol{\omega}(\boldsymbol{x},t)| = 2.23\lambda_1(\boldsymbol{x},t) + 0.0473\lambda_2(\boldsymbol{x},t)$
- The insight carries over to turbulence data! $|\boldsymbol{\omega}(\boldsymbol{x},t)| \simeq 2\lambda_1(\boldsymbol{x},t)$

Problem of the model



$\theta(t) \to \pi/2$



$\omega(t)$ and $\lambda_1(t)$ seem to diverge

- For large t, $\omega(t)$ and $\lambda_1(t)$ becomes too large (blows up). (as expected, Veillefosse 1982, 1984)
- The model does not yield a statistically steady state.
- The fixed point ($\theta = 0, \omega = 2\lambda_1, \lambda_2 = 0$) is neutrally stable.

Summary

- The preferential alignment (PA) of the vorticity, $\boldsymbol{\omega} \parallel \boldsymbol{e}_2$.
  $\boldsymbol{e}_2$: the eigenvector of the intermediate eigenvalue λ_2 of the rate-of-strain tensor S.

  (1) Analyzed PA in the spherical local strain coordinates
  - the PDF peak is $(\theta, \varphi) \simeq (0.035\pi, 0)$.

  (2) A dynamical-system model of the angles
  - PA corresponds to the neutral fixed point of the model.

- The model does not yield a statistically steady state.
- We developed the two-angle model similiary, but leading to a similar blowup.
- Better modeling of the pressure Hessian and the viscous term is needed (many proposals are available including stochastic noises).
- In the Navier–Stokes turbulence, suppression of the growth of the vorticity due to the pressure is an active subject.