GRADUATE SCHOOL OF / FACULTY OF SCIENCE
KYOTO UNIVERSITY

Extreme Universe
A New Paradigm for Spacetime and Matter from Quantum Information
Grant-in-Aid for Transformative Research Areas (A)

Sparse Sachdev-Ye-Kitaev-like models: spectral correlations and information scrambling

"Quantum Gravity and Information in Expanding Universe"

16:30-17:30, 17 February 2025

Masaki TEZUKA (Kyoto Univ.)

Contents and collaborators

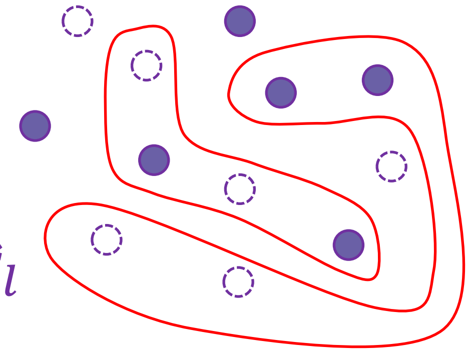
- **The SYK model: a maximally chaotic quantum mechanical model**
- **Binary-coupling sparse SYK**
  - Phys. Rev. B **107**, L081103 (2023) with Onur Oktay, Enrico Rinaldi, Masanori Hanada, and Franco Nori
- **Chaotic-integrable transition in SYK_{4+2}**
  - PRL **120**, 241603 (2018) with Antonio M. García-García, Bruno Loureiro, and Aurelio Romero-Bermúdez
  - **Many-body transition point and inverse participation ratio**: Phys. Rev. Research **3**, 013023 (2021) with Felipe Monteiro, Tobias Micklitz, and Alexander Altland
  - **Entanglement entropy**: Phys. Rev. Lett. **127**, 030601 (2021) with F. Monteiro, A. Altland, David A. Huse, and T. Micklitz
- **Quantum error correction in SYK-like models**
  - Phys. Rev. Research **6**, L022021 (2024) with Yoshifumi Nakata
- **Model of Pauli spins**
  - JHEP **05**(2024)280 with M. Hanada, Antal Jevicki, Xianlong Liu, and E. Rinaldi
- **Singular-value correlations in non-Hermitian sparse SYK**
  - Phys. Rev. B **111**, L060201 (2025) with Pratik Nandy and Tanay Pathak

The Sachdev-Ye-Kitaev (SYK) model

2N Majorana or N Dirac fermions randomly coupled to each other

Solvable in large-N limit, **maximally chaotic** at low T

[Maldacena, Shenker, and Stanford JHEP 2016]



[**Majorana** version]

$$\hat{H} = \sum_{1 \le a<b<c<d \le 2N} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d$$

[A. Kitaev: talks at KITP (**Feb 12**, Apr 7, and May 27, **2015**)]

[**Dirac** version]

$$\hat{H} = \sum_{ij;kl} J_{ij;kl} \hat{c}_i^{\dagger} \hat{c}_j^{\dagger} \hat{c}_k \hat{c}_l$$

[A. Kitaev's talks]
[S. Sachdev: PRX **5**, 041025 (2015)]

Gaussian random distribution

cf. SY model [Sachdev and Ye, PRL 1993]

>1300 citations after 2015

Studied for long time in the **nuclear theory** context

[French and Wong (1970)][Bohigas and Flores (1971)]

"Two-body Random Ensemble"

- Holographic correspondence to **black holes**
- Many variants of the model: bosonic, multiflavor, supersymmetric, nonhermitian, ...

Solvable in the $N \gg 1$ limit (after sample average $\langle \cdots \rangle_{\{J\}}$)

Non-perturbative Hamiltonian = 0,

$$\hat{H} = \frac{\sqrt{3!}}{(2N)^{3/2}} \sum_{1 \le a < b < c < d \le 2N} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d$$

as perturbation

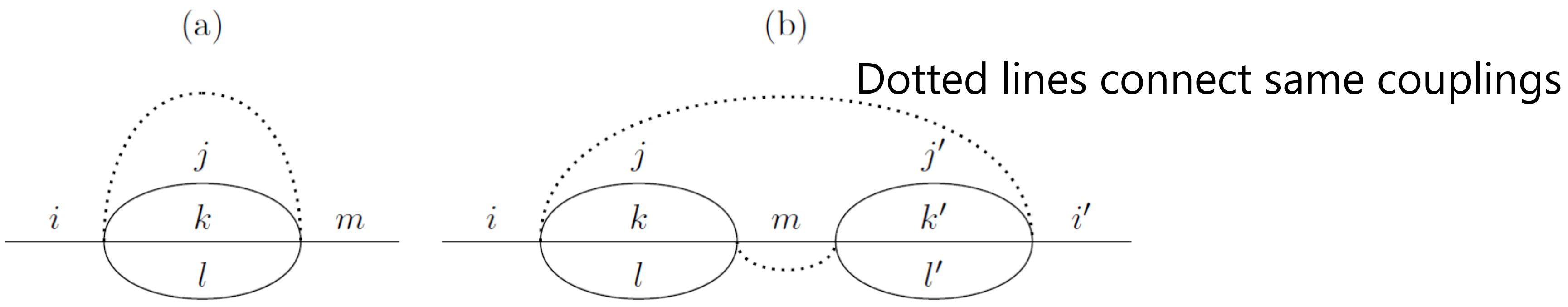
Free two-point function

$$G_{0,ij}(t) = -\langle \mathrm{T} \chi_i(t) \chi_j(0) \rangle = -\mathrm{sgn}(t) \delta_{ij}$$

$\langle J_{abcd}{}^2 \rangle_{\{J\}} = J^2$, Gaussian distribution

$\langle J_{abcd} J_{abce} \rangle_{\{J\}} = 0$ if $d \neq e$ ➔ Most diagrams average to zero

Only “melon-type” diagrams survive sample averaging



$O(1)$ melon

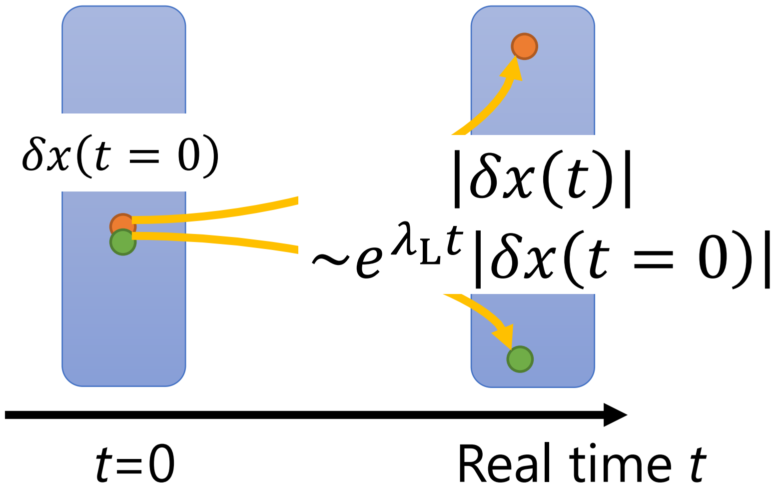
$O(N^{-2})$ not melon

Lyapunov exponent and out-of-time-order correlators (OTOC)

$$F(t) = \langle \hat{W}^\dagger(t)\hat{V}^\dagger(0)\hat{W}(t)\hat{V}(0)\rangle \quad W(t) = \mathrm{e}^{iHt} W \mathrm{e}^{-iHt}$$

Classical chaos:

Infinitesimally different initial coords



λ_{L}: Lyapunov exponent

$$\left(\frac{\partial x(t)}{\partial x(0)}\right)^2 = \{x(t), p(0)\}_{\mathrm{PB}}{}^2 \to e^{2\lambda_{\mathrm{L}} t}$$

Quantum dynamics:

$$C_T(t) = \langle [\hat{x}(t), \hat{p}(0)]^2 \rangle$$

For operators V and W, consider

$$C(t) = \langle |[W(t), V(t=0)]|^2 \rangle = \langle W^\dagger(t) V^\dagger(0) W(t) V(0) \rangle + \cdots$$

[Wiener 1938][Larkin & Ovchinnikov 1969]

OTOC $\sim e^{2\lambda_{\mathrm{L}} t}$ at long times, $\lambda_{\mathrm{L}} > 0$: chaotic

"Black holes are fastest quantum scramblers"
[P. Hayden and J. Preskill 2007] [Y. Sekino and L. Susskind 2008] [Shenker and Stanford 2014]

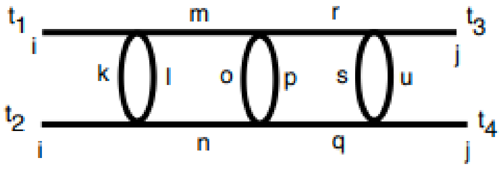
$\lambda_{\mathrm{L}} \leq 2\pi k_{\mathrm{B}} T/\hbar$ (chaos bound)

[J. Maldacena, S. H. Shenker, and D. Stanford, JHEP08(2016)106]

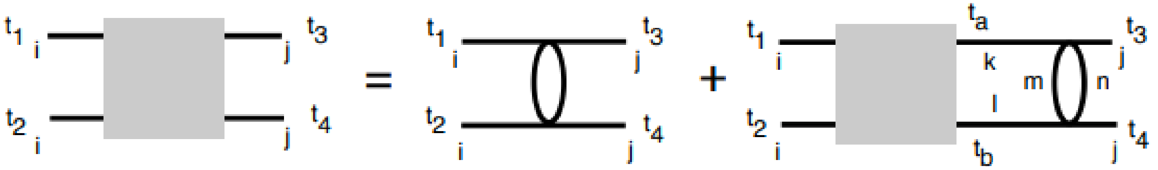
➔ SYK model can be solved in large-N; satisfies this bound at low T

Out-of-time-ordered correlators (OTOCs)

$$\langle \hat{\chi}_i(t_1)\hat{\chi}_i(t_2)\hat{\chi}_j(t_3)\hat{\chi}_j(t_4)\rangle$$

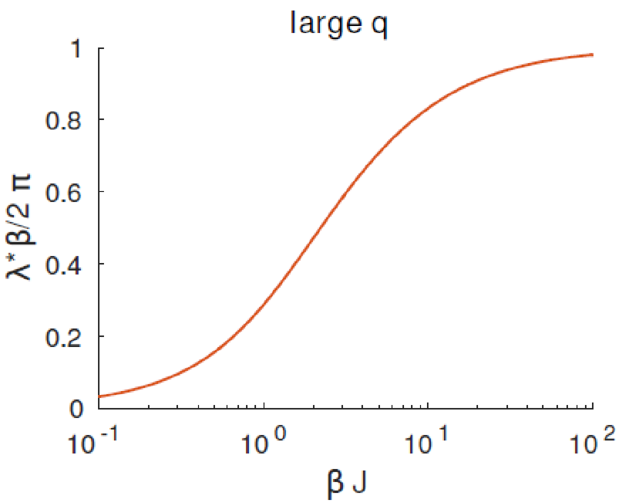


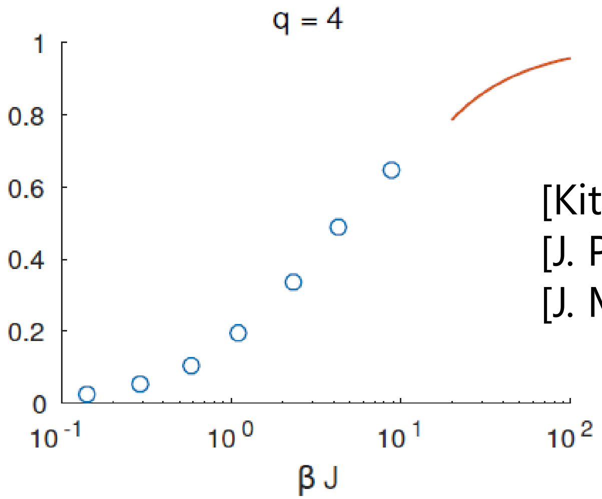
(a)



$$\Gamma(t_1,t_2,t_3,t_4) = \Gamma_0(t_1,t_2,t_3,t_4) + \int dt_a dt_b\, \Gamma(t_1,t_2,t_a,t_b)K(t_a,t_b,t_3,t_4)$$

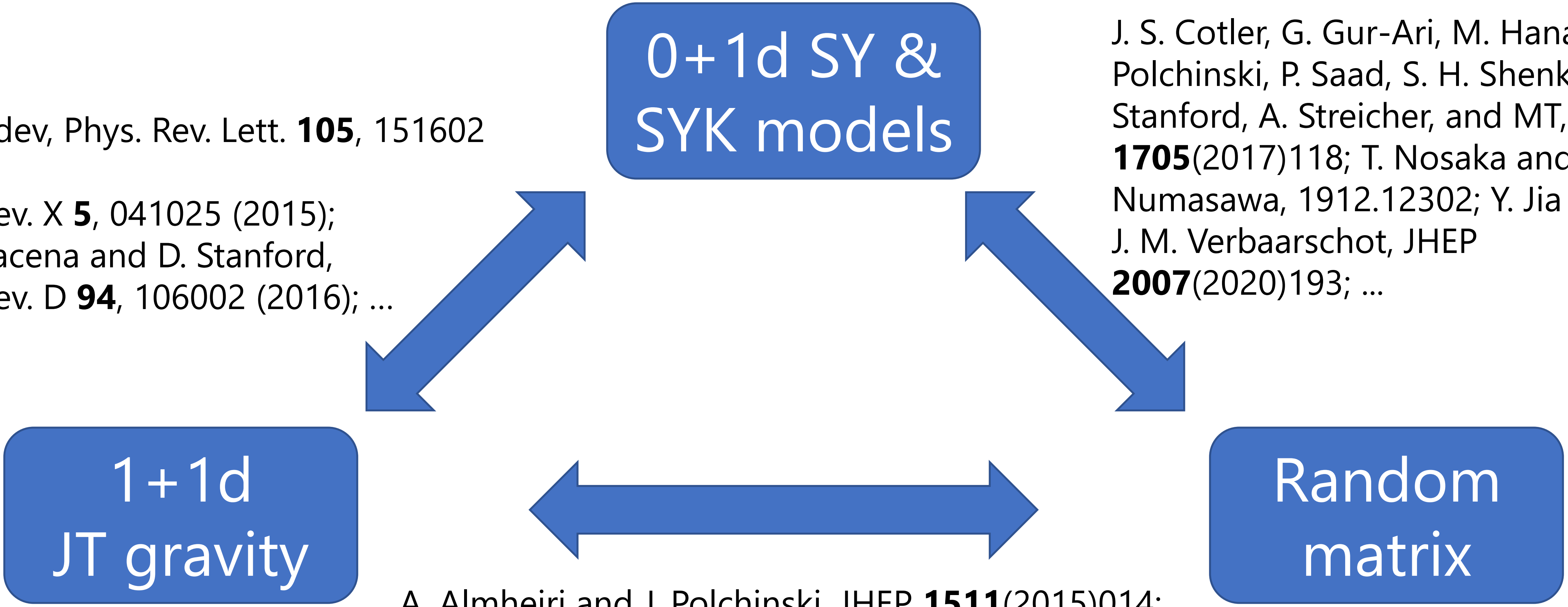
Regularized OTOC can be calculated for large-*N* SYK model, satisfies the chaos bound
$\lambda_{\mathrm{L}} = 2\pi k_{\mathrm{B}}T/\hbar$ at low *T* limit





[Kitaev's talks]
[J. Polchinski and V. Rosenhaus, JHEP 1604 (2016) 001]
[J. Maldacena and D. Stanford, Phys. Rev. D **94**, 106002 (2016)]

Maximally chaotic systems



S. Sachdev, Phys. Rev. Lett. **105**, 151602 (2010),
Phys. Rev. X **5**, 041025 (2015);
J. Maldacena and D. Stanford,
Phys. Rev. D **94**, 106002 (2016); …

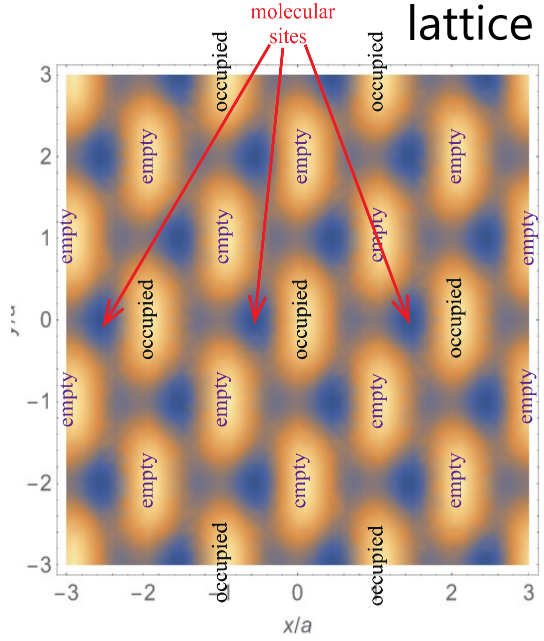
J. S. Cotler, G. Gur-Ari, M. Hanada, J. Polchinski, P. Saad, S. H. Shenker, D. Stanford, A. Streicher, and MT, JHEP **1705**(2017)118; T. Nosaka and T. Numasawa, 1912.12302; Y. Jia and J. J. M. Verbaarschot, JHEP **2007**(2020)193; …

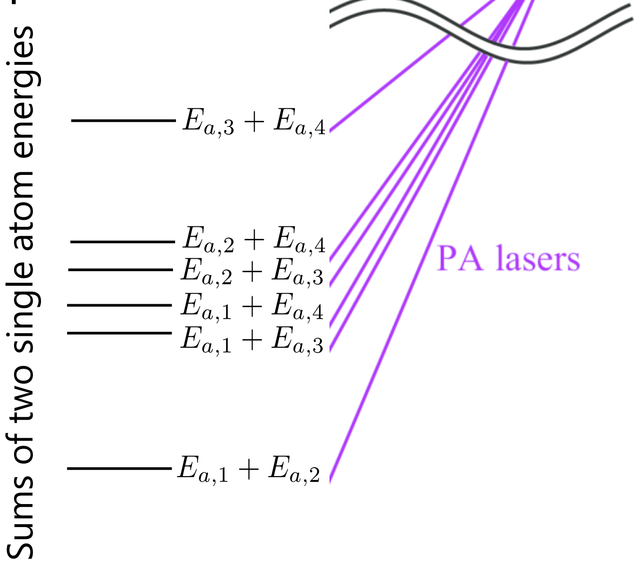
A. Almheiri and J. Polchinski, JHEP **1511**(2015)014;
P. Saad, S. H. Shenker, and D. Stanford, arXiv:1903.11115;
D. Stanford and E. Witten, arXiv:1907.03363; …

Proposals for experimental realization of SYK

[I. Danshita, M. Hanada, MT: PTEP **2017**, 083I01 (2017)]

Ultracold fermions in optical lattice + photoassociation lasers





s: molecular levels

$$\hat{H}_{\rm m}=\sum_{s=1}^{n_s}\left\{\nu_s\hat{m}_s^\dagger\hat{m}_s+\sum_{i,j}g_{s,ij}\left(\hat{m}_s^\dagger\hat{c}_i\hat{c}_j-\hat{m}_s\hat{c}_i^\dagger\hat{c}_j^\dagger\right)\right\}.$$

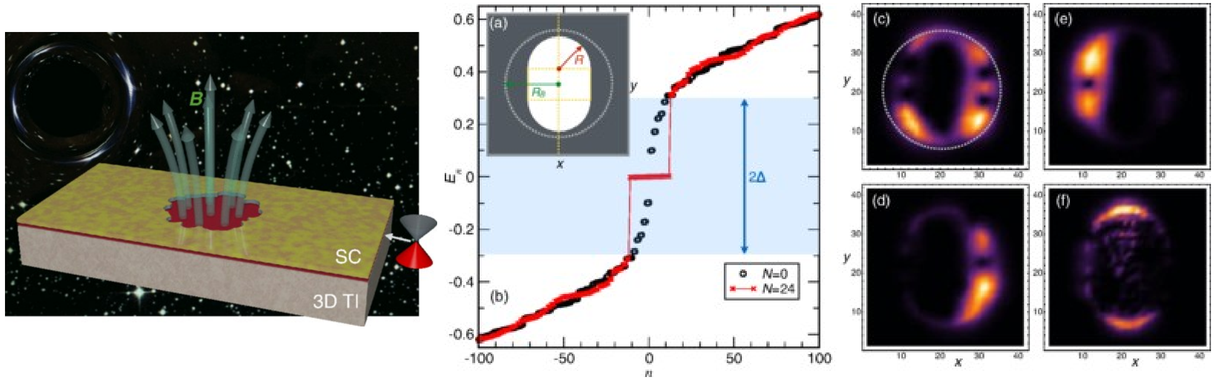
$|\nu_s|\gg|g_{s,ij}|$

$$\hat{H}_{\rm eff}=\sum_{s,i,j,k,l}\frac{g_{s,ij}g_{s,kl}}{\nu_s}\hat{c}_i^\dagger\hat{c}_j^\dagger\hat{c}_k\hat{c}_l.$$

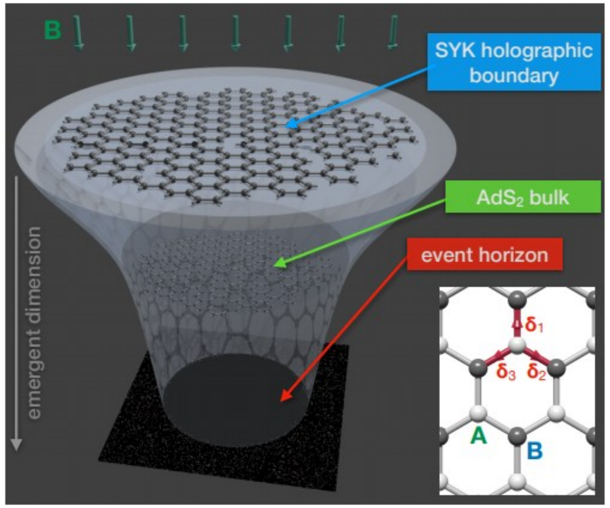
Approximately Gaussian if many levels are used

[D. I. Pikulin and M. Franz, PRX **7**, 031006 (2017)]

N quanta of magnetic flux through a nanoscale hole



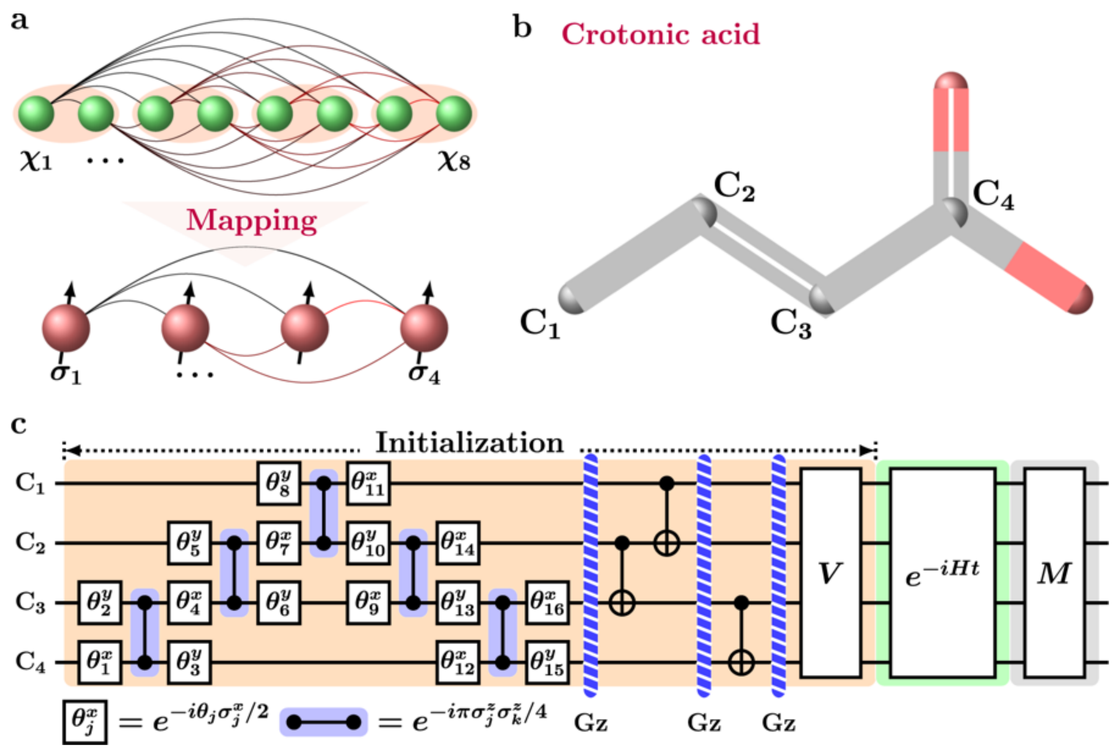
[A. Chen, R. Ilan, F. de Juan, D.I. Pikulin, M. Franz, PRL **121**, 036403 (2018)]



Graphene flake with an irregular boundary in magnetic field

NMR experiment for the SYK model

"Quantum simulation of the non-fermi-liquid state of Sachdev-Ye-Kitaev model" Zhihuang Luo, Yi-Zhuang You, Jun Li, Chao-Ming Jian, Dawei Lu, Cenke Xu, Bei Zeng and Raymond Laflamme, npj Quantum Information **5**, 53 (2019)

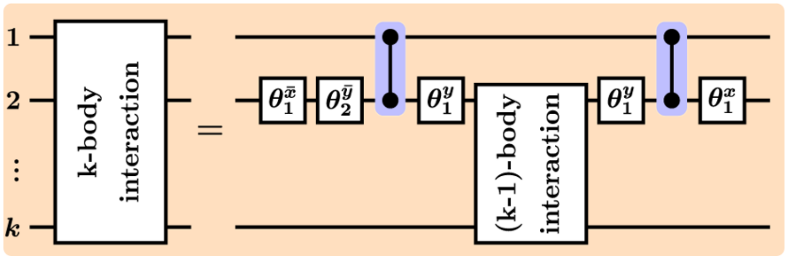


$$H = \frac{J_{ijkl}}{4!}\chi_i\chi_j\chi_k\chi_l + \frac{\mu}{4}C_{ij}C_{kl}\chi_i\chi_j\chi_k\chi_l$$

$$\chi_{2i-1} = \frac{1}{\sqrt{2}}\sigma_x^1\sigma_x^2\cdots\sigma_x^{i-1}\sigma_z^i,\ \chi_{2i} = \frac{1}{\sqrt{2}}\sigma_x^1\sigma_x^2\cdots\sigma_x^{i-1}\sigma_y^i.$$

$$H = \sum_{s=1}^{70} H_s = \sum_{s=1}^{70} a^s_{ijkl}\sigma^1_{\alpha_i}\sigma^2_{\alpha_j}\sigma^3_{\alpha_k}\sigma^4_{\alpha_l}$$

$$e^{-iH\tau} = \left(\prod_{s=1}^{70} e^{-iH_s\tau/n}\right)^n + \sum_{s<s'}\frac{[H_s, H_{s'}]\tau^2}{2n} + O(|a|^3\tau^3/n^2),$$



Numerically diagonalizing SYK

$$\hat{H} = \sum_{1 \le a < b < c < d \le 2N} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d$$

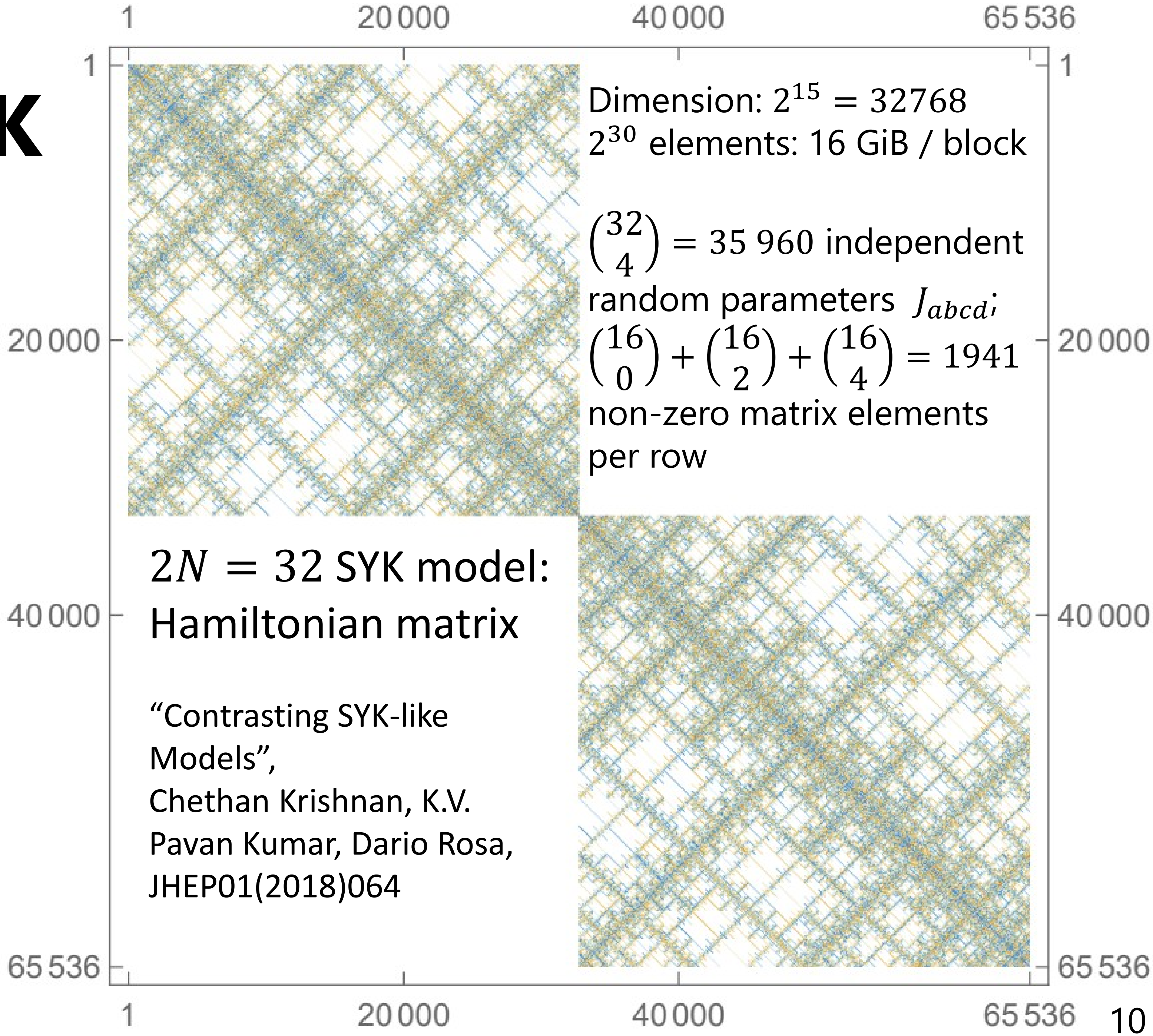
Introduce N complex fermions

$$\hat{c}_j = \frac{(\chi_{2j-1} + \mathrm{i}\chi_{2j})}{\sqrt{2}}, \; j = 1, 2, \dots, N$$

$\chi\chi\chi\chi$ preserves parity of complex fermion number

$$\begin{pmatrix} H_{\mathrm{E}} & O \\ O & H_{\mathrm{O}} \end{pmatrix}$$

➔ Numerically diagonalize H_{E} and H_{O}, 2^{N-1}-dimensional Hermitian matrices



Dimension: $2^{15} = 32768$
2^{30} elements: 16 GiB / block

$\binom{32}{4} = 35\,960$ independent random parameters J_{abcd};
$\binom{16}{0} + \binom{16}{2} + \binom{16}{4} = 1941$ non-zero matrix elements per row

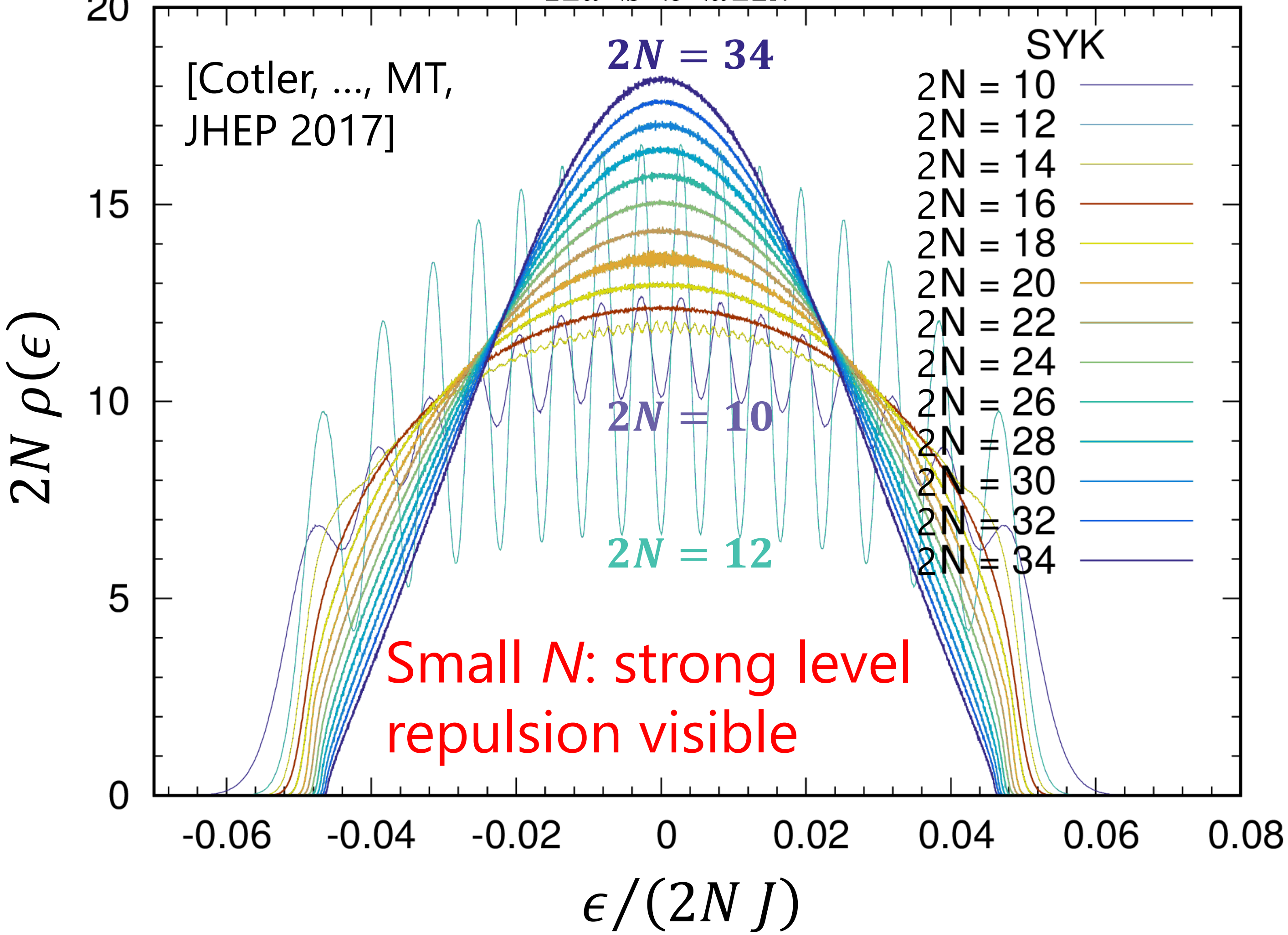
$2N = 32$ SYK model: Hamiltonian matrix

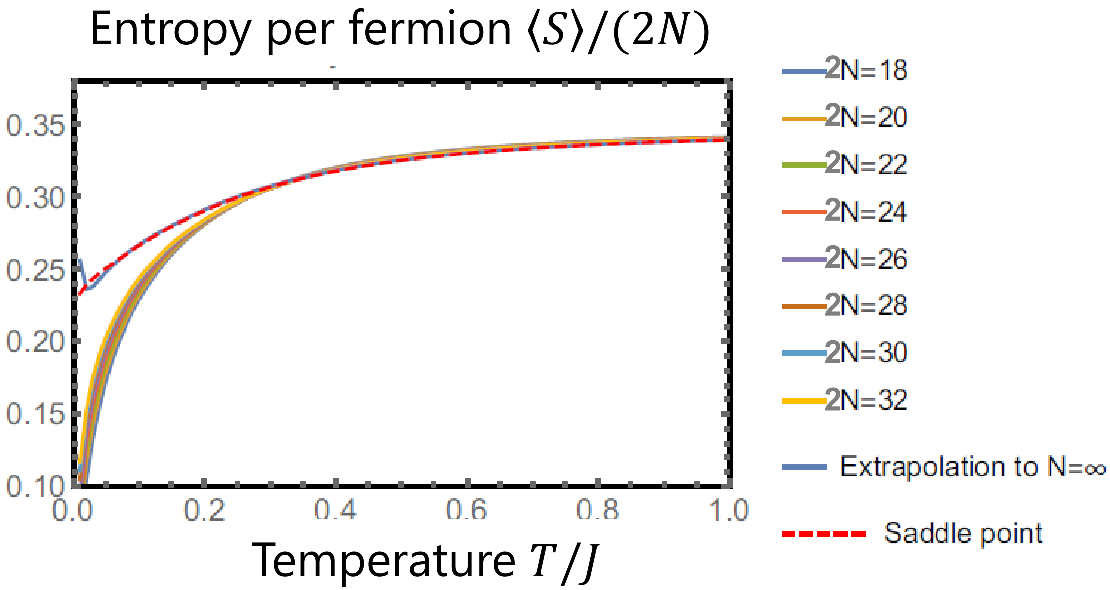
"Contrasting SYK-like Models", Chethan Krishnan, K.V. Pavan Kumar, Dario Rosa, JHEP01(2018)064

Eigenvalue spectrum and entropy

$$\hat{H} = \frac{\sqrt{3!}}{(2N)^{3/2}} \sum_{1\le a<b<c<d\le 2N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d$$

J_{abcd} : Gaussian and variance $\sigma^2 = J^2$





Entropy extrapolated to large N: finite in the low T limit

➔Quantify level correlation?

cf. BGS conjecture (random matrix-like level correlation is expected for chaotic systems)

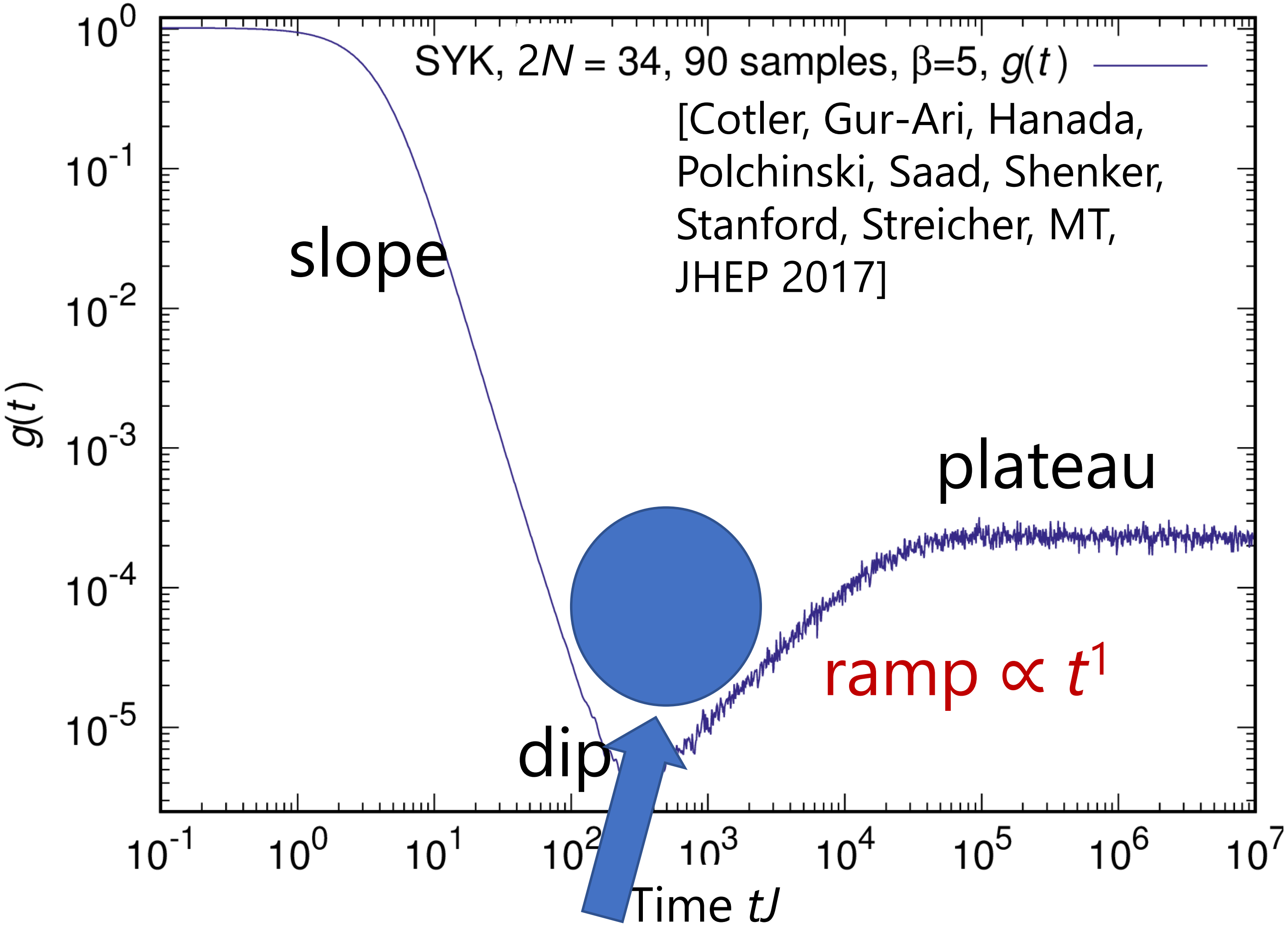
cf. Analytical spectral density for large N [A. M. García-García and J. J. M. Verbaarschot: PRD **94**, 126010 (2016), PRD **96**, 066012 (2017)]

Spectral form factor $g(\beta,t) = \frac{\langle |Z(\beta,t)|^2\rangle_{\{J\}}}{\langle Z(\beta)\rangle_{\{J\}}^{\ 2}}$

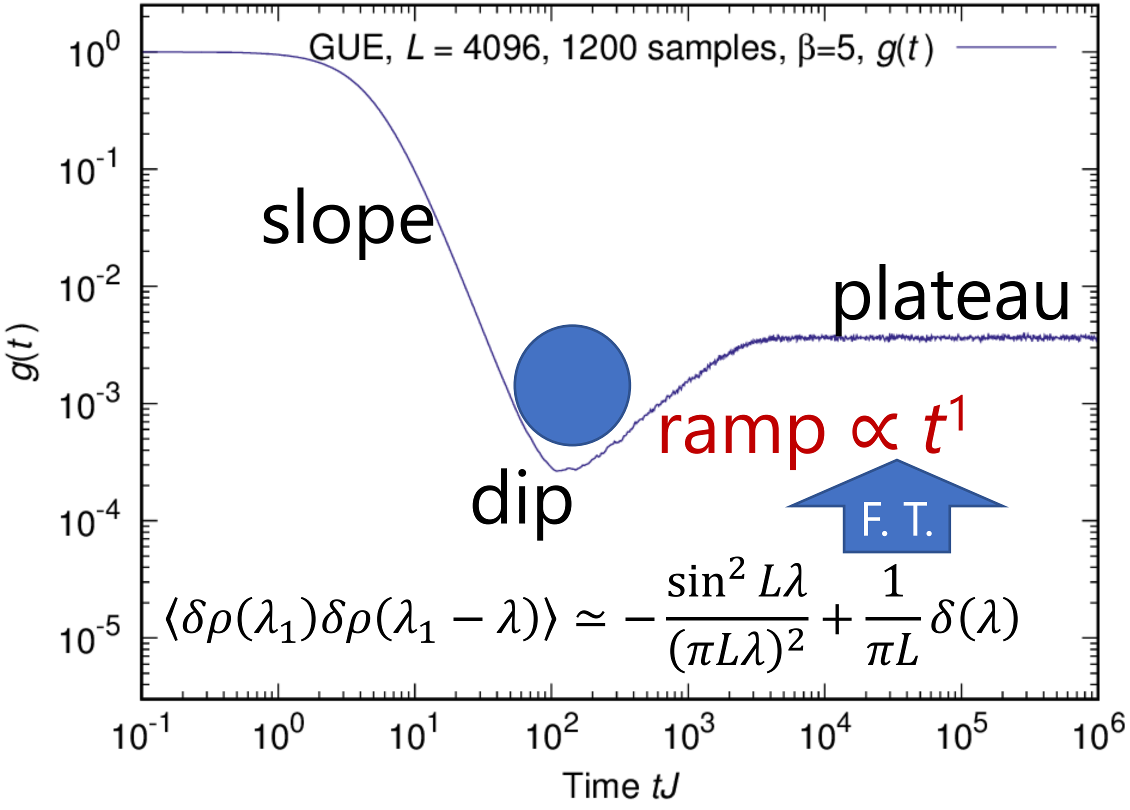
Partition function

$$Z(\beta,t) = Z(\beta+\mathrm{i}t) = \mathrm{Tr}\left(\mathrm{e}^{-\beta\hat{H}-\mathrm{i}\hat{H}t}\right)$$

Fourier transform of the spectrum



Gaussian unitary ensemble (dense random matrix)

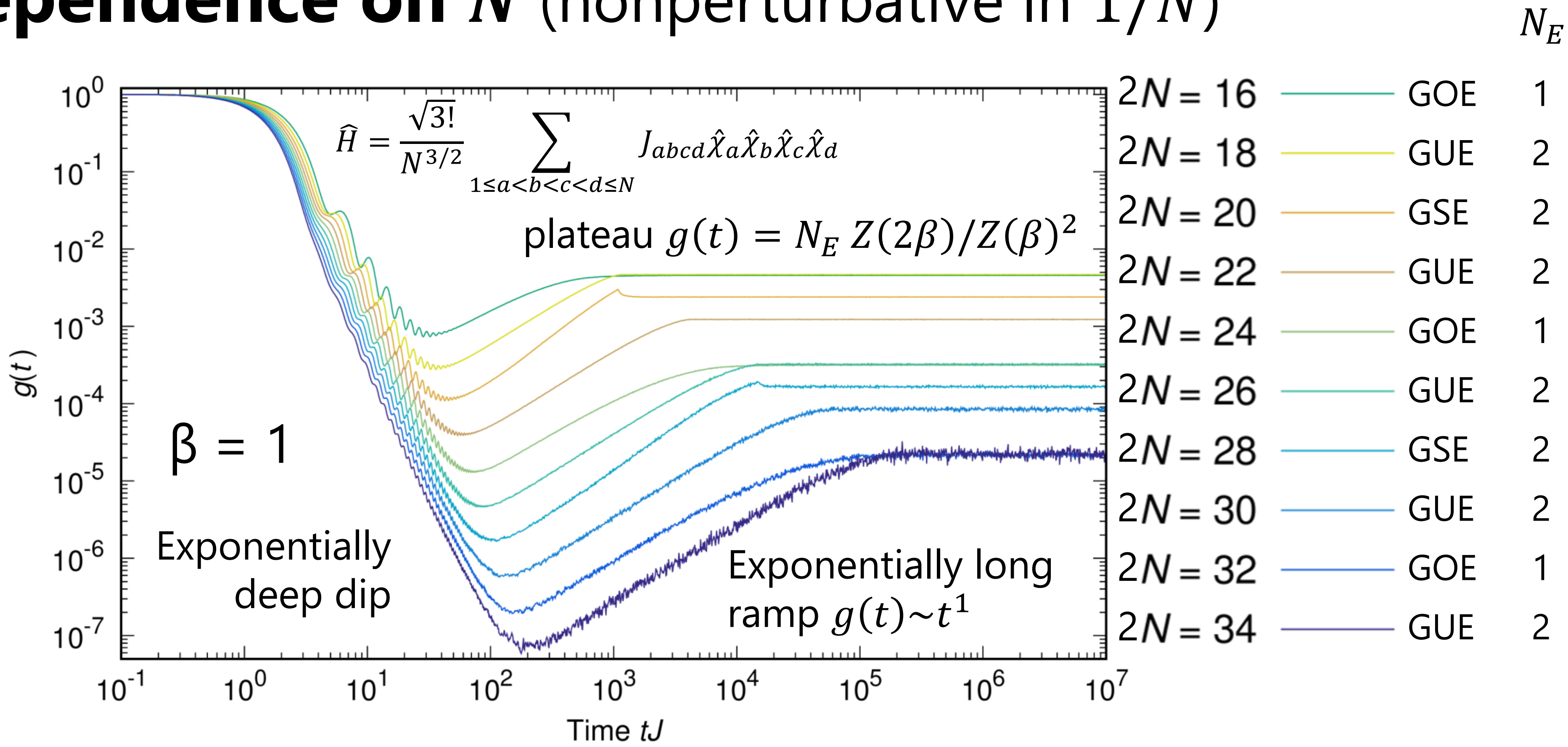


$$\langle \delta\rho(\lambda_1)\delta\rho(\lambda_1-\lambda)\rangle \simeq -\frac{\sin^2 L\lambda}{(\pi L\lambda)^2} + \frac{1}{\pi L}\delta(\lambda)$$

"correlation hole", as observed for dense random matrices

Cotler et al., JHEP **1705**(2017)118

$g(t)$: **Dependence on N** (nonperturbative in $1/N$)



Classification of SPT order in class BDI: reduced from Z to Z_8 by interaction
[L. Fidkowski and A. Kitaev: PRB **81**, 134509 (2010); PRB **83**, 075103 (2011)]

Many-body level statistics 🡨 corresponding (dense) random matrix ensemble

$N_\chi \pmod 8$	0	1	2	3	4	5	6	7
qdim	1	$\sqrt{2}$	2	$2\sqrt{2}$	2	$2\sqrt{2}$	2	$\sqrt{2}$
lev. stat.	GOE	GOE	GUE	GSE	GSE	GSE	GUE	GOE

[Y.-Z. You, A. W. W. Ludwig, and Cenke Xu, PRB **95**, 115150 (2017)]

Sparse (or pruned) SYK

$$\hat{H} = \sum_{a<b<c<d} x_{abcd} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d \,, x_{abcd} = \begin{cases} 1 & (\text{probability } p) \\ 0 & (\text{probability } 1-p) \end{cases}, P(J_{abcd}) = \frac{\exp\left(-\frac{J_{abcd}^2}{2J^2}\right)}{\sqrt{2\pi J^2}}$$

$K_{\mathrm{cpl}} = \binom{2N}{4} p$: Number of non-zero x_{abcd}

$K_{\mathrm{cpl}} \sim \mathcal{O}(1) N$ enough for
- Random matrix-like behavior
- Large entropy per fermion at low T !

$$p \sim \frac{4!}{(2N)^3} = \mathcal{O}(N^{-3})$$

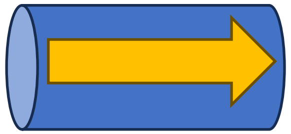
- Talk by Brian Swingle at Simons Center (18 September 2019)
- "Sparse Sachdev-Ye-Kitaev model, quantum chaos and gravity duals" A. M. García-García, Y. Jia, D. Rosa, J. J. M. Verbaarschot, Phys. Rev. D **103**, 106002 (2021)
- "A Sparse Model of Quantum Holography" S. Xu, L. Susskind, Y. Su, and B. Swingle, arXiv:2008.02303
- "Spectral Form Factor in Sparse SYK models" E. Cáceres, A. Misobuchi, and A. Raz, JHEP **2208**, 236 (2022)

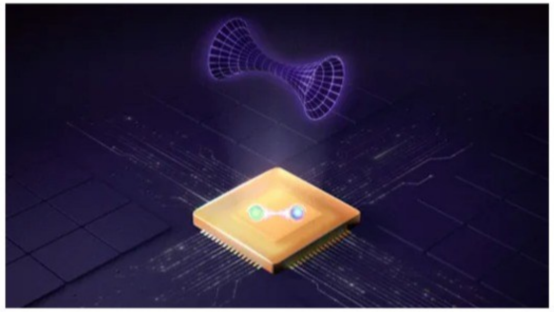
Article | Published: 30 November 2022

Traversable wormhole dynamics on a quantum processor

Daniel Jafferis, Alexander Zlokapa, Joseph D. Lykken, David K. Kolchmeyer, Samantha I. Davis, Nikolai Lauk, Hartmut Neven & Maria Spiropulu

Nature 612, 51–55 (2022) | Cite this article

$\mathrm{SYK_L}$  $\mathrm{SYK_R}$



Quanta Magazine (30 November 2022)

QUANTUM GRAVITY

Physicists Create a Wormhole Using a Quantum Computer

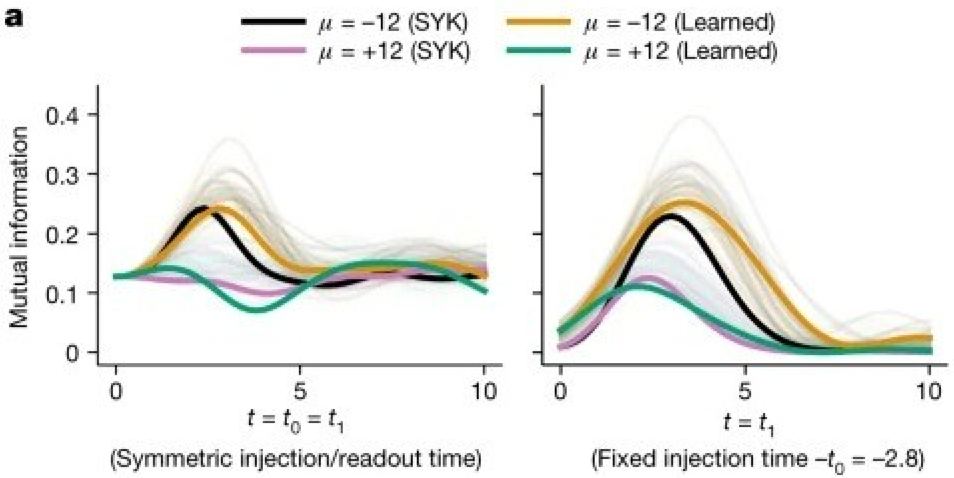
By NATALIE WOLCHOVER | NOVEMBER 30, 2022 |

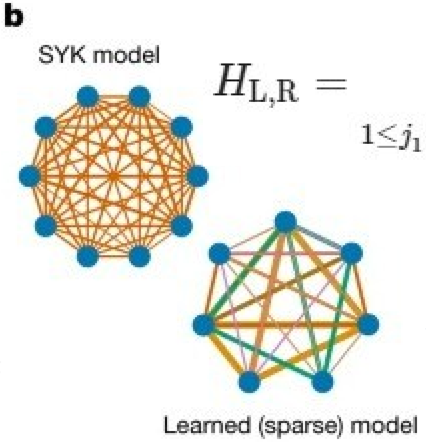
3 |

The unprecedented experiment explores the possibility that space-time somehow emerges from quantum information.

Theory (with dense SYK): J. Maldacena and X.-L. Qi, "Eternal traversable wormhole" arXiv:1804.00491

Fig. 2: Learning a traversable wormhole Hamiltonian from the SYK model.





$$H_{\mathrm{L,R}} = \sum_{1\le j_1<\cdots<j_q\le N} J_{j_1\ldots j_q}\psi^{j_1}_{\mathrm{L,R}}\cdots\psi^{j_q}_{\mathrm{L,R}},$$

Machine learning

$$H_{\mathrm{L,R}} = -0.36\psi^1\psi^2\psi^4\psi^5 + 0.19\psi^1\psi^3\psi^4\psi^7 - 0.71\psi^1\psi^3\psi^5\psi^6 + 0.22\psi^2\psi^3\psi^4\psi^6 + 0.49\psi^2\psi^3\psi^5\psi^7,$$

➔ Realized on the Google Sycamore processor (nine-qubit circuit of 164 two-qubit, 295 single-qubit gates)

➔ Much debate (e.g. comment by Kobrin, Schuster, and Yao (arXiv:2302.07897), reply 2303.15423, …)

Masaki Tezuka, Onur Oktay, Masanori Hanada, Enrico Rinaldi, and Franco Nori, Phys. Rev. B **107**, L081103 (2023)

Sparse (or pruned) SYK with interaction $= \pm 1$

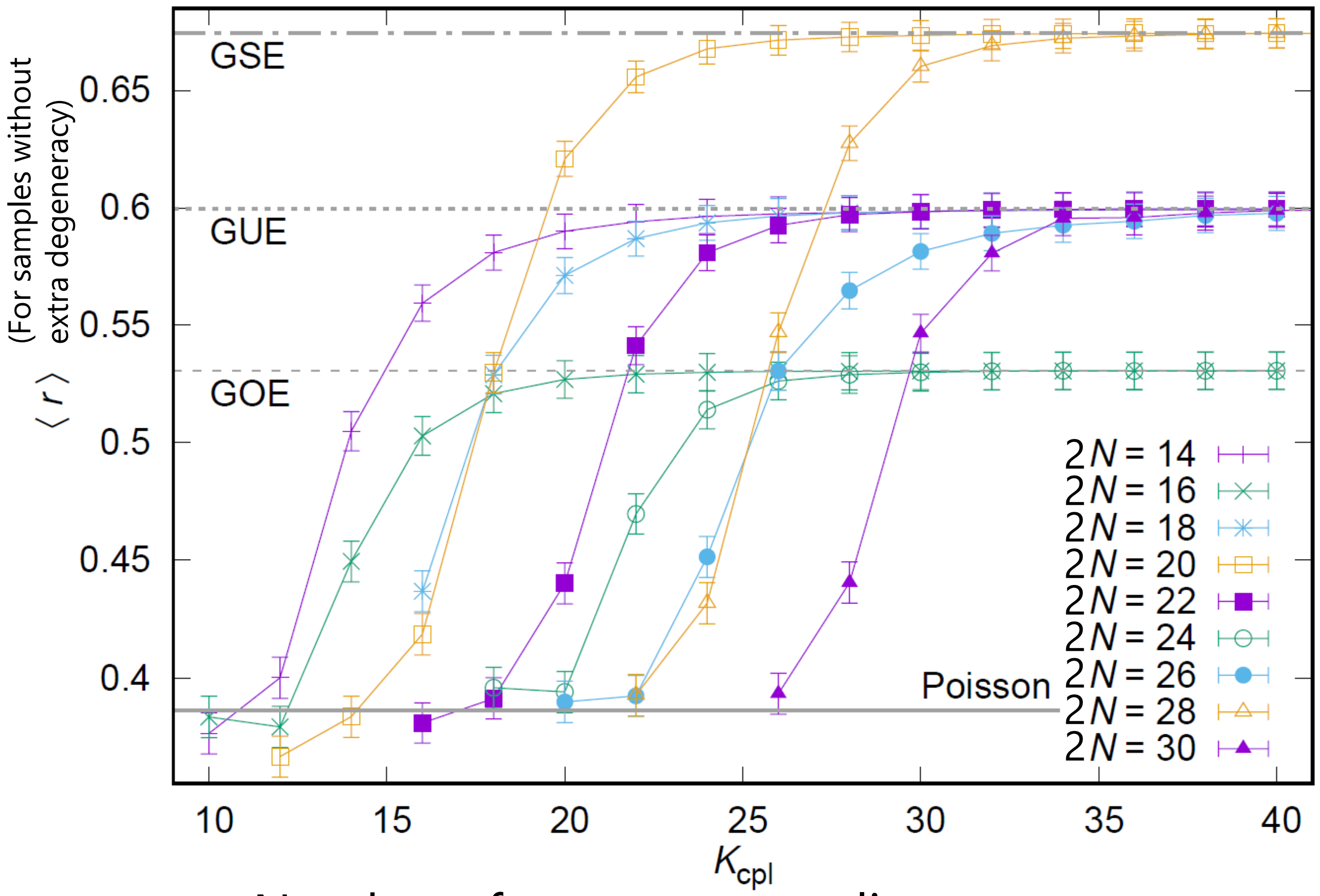
$$\hat{H} = C_{N,p} \sum_{1 \le a<b<c<d \le 2N} x_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d \,, x_{abcd} = \begin{cases} 1 & (\text{probability } p/2) \\ -1 & (\text{probability } p/2) \\ 0 & (\text{probability } 1-p) \end{cases}$$

Random-matrix statistics for $K_{\mathrm{cpl}} = \binom{2N}{4} p \gtrsim 2N$.

cf. Non-Gaussian disorder average [T. Krajewski, M. Laudonio, R. Pascalie, and A. Tanasa, PRD **99**, 126014 (2019)]; Kitaev's talk (2015)

x_{abcd} can be taken to be +1 at finite $p \ll 1$ (unary sparse SYK, see appendix of our PRB Letter), however at $p = 1$, the model is not chaotic [P. H. C. Lau, C.-T. Ma, J. Murugan, and MT, J. Phys. A. **54**, 095401 (2021)] and integrable [S. Ozaki and H. Katsura, PRR **7**, 013092 (2025)]

Neighboring gap ratio $\langle r \rangle$: approaches RMT value as K_{cpl} is increased



Number of non-zero coupling terms

Majorana SYK: $2N$ mod 8 periodicity of symmetry

- $[H, T] = 0$ for $2N \bmod 8 = 0, 4$
  - $2N \bmod 8 = 0$: $T^2 = +1$; GOE
  - $2N \bmod 8 = 4$: $T^2 = -1$; GSE
- No such antiunitary operator T for
  - $2N \bmod 8 = 2, 6$; GUE

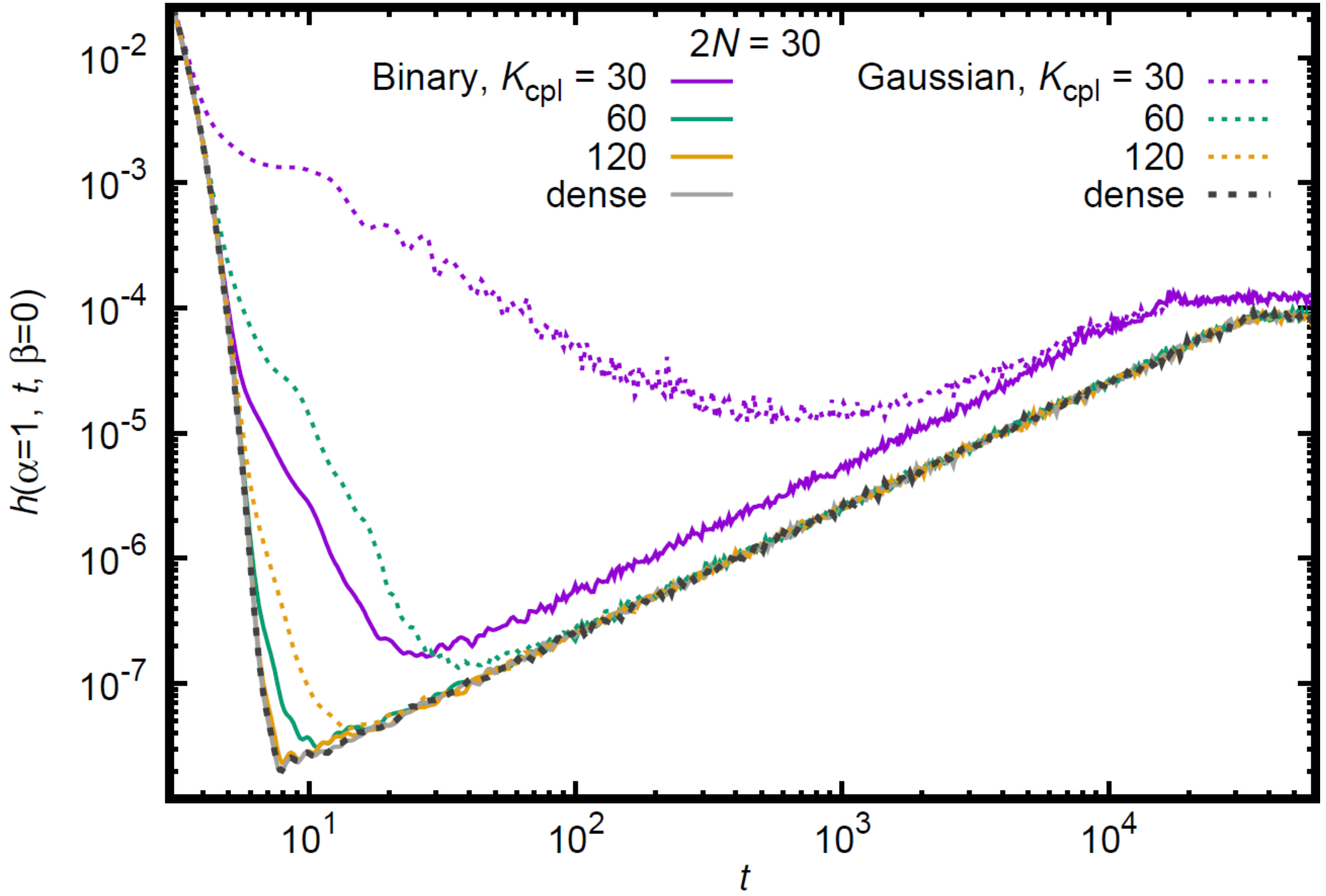
Neighboring gap ratio

$$r = \frac{\min(e_{i+1} - e_i\,,\ \ e_{i+2} - e_{i+1})}{\max(e_{i+1} - e_i\,,\ \ e_{i+2} - e_{i+1})}$$

	Poisson	GOE	GUE	GSE
$\langle r \rangle$	2log 2 – 1 = 0.38629…	0.5307(1)	0.599750 4209(1)	0.6744(1)

[Y. Y. Atas *et al.* PRL 2013]
[S. M. Nishigaki PTEP 2024]

Modified SFF (focus on band center)



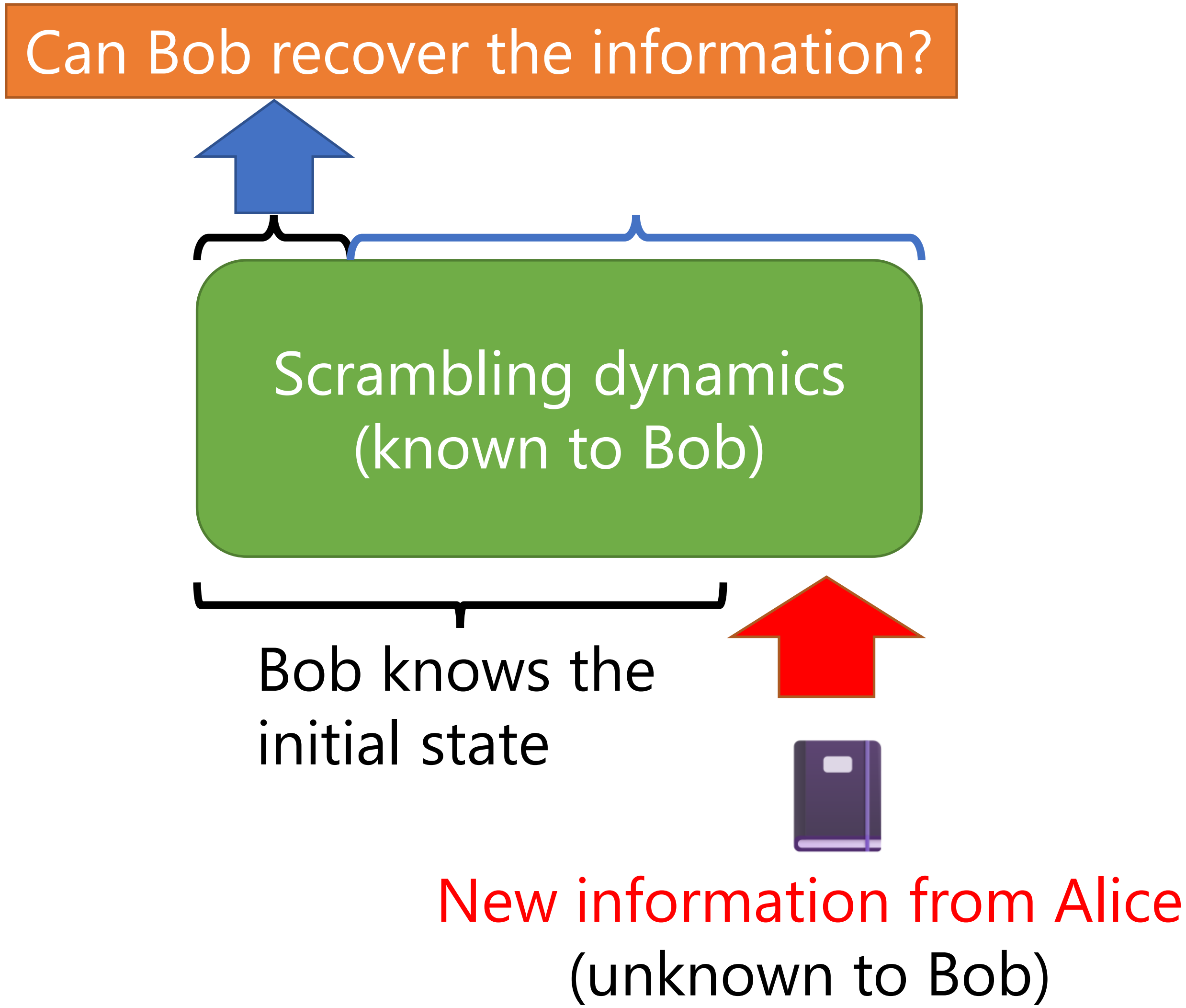
$$h(\alpha, t, \beta) = \frac{|Y(\alpha, t, \beta)|^2}{Y(\alpha, 0, \beta)^2},$$

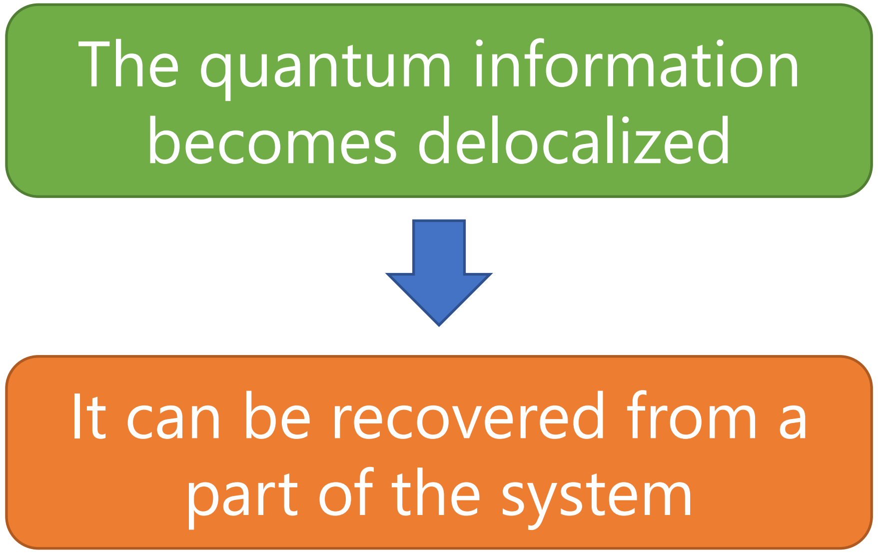
$$Y(\alpha, 0, \beta) = \sum_j e^{-\alpha\epsilon_j^2-(\beta+it)\epsilon_j}$$

- Spectral rigidity comparable to Gaussian-coupling sparse SYK with twice as large $K_{\rm cpl}$

Quantum error correction

(also known as information scrambling)



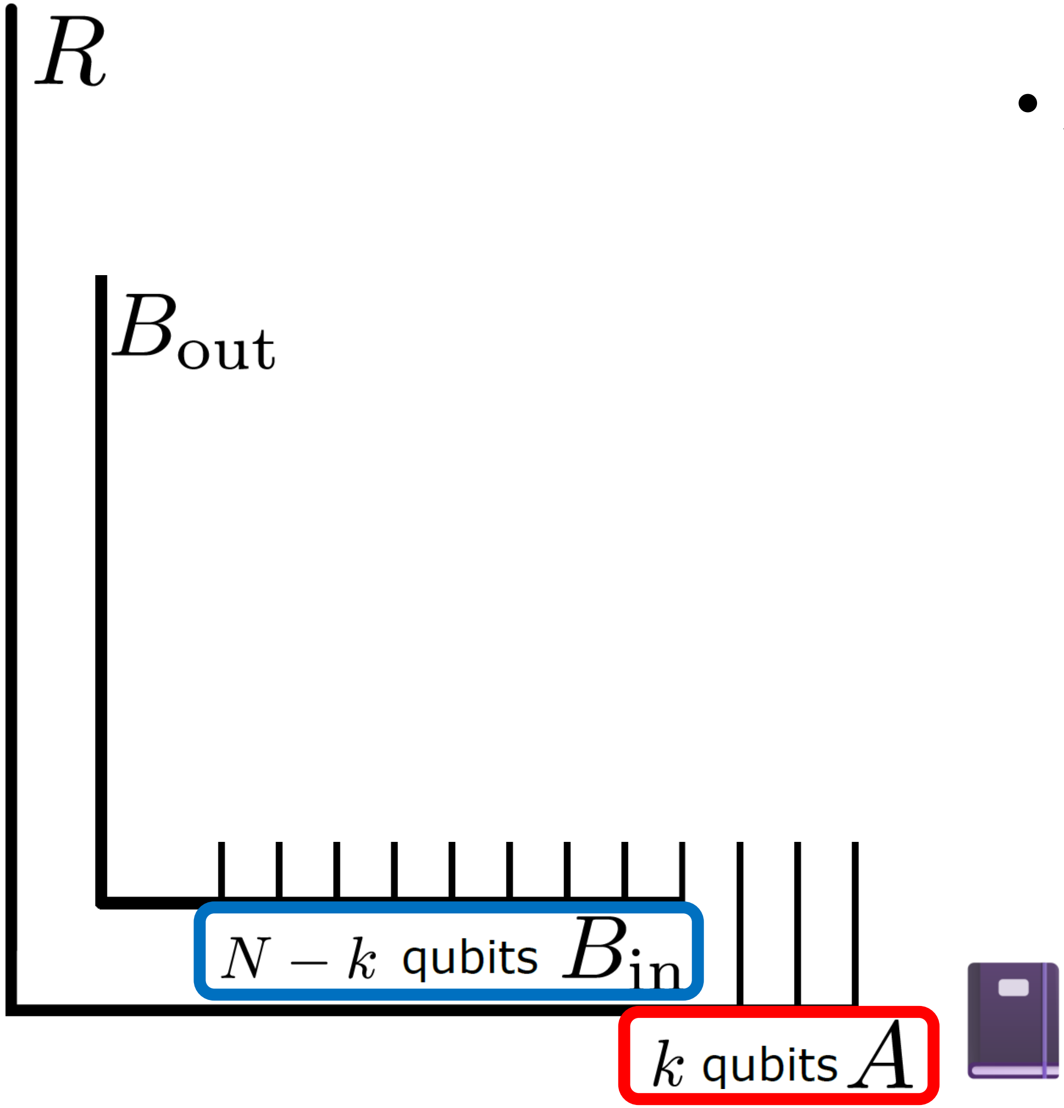


No-cloning theorem:
It is not possible to create two accurate copies of arbitrarily given quantum information!

After the recovery process, **the reminder of the system** should **lose correlation with the input**!

P. Hayden and J. Preskill, JHEP 2007

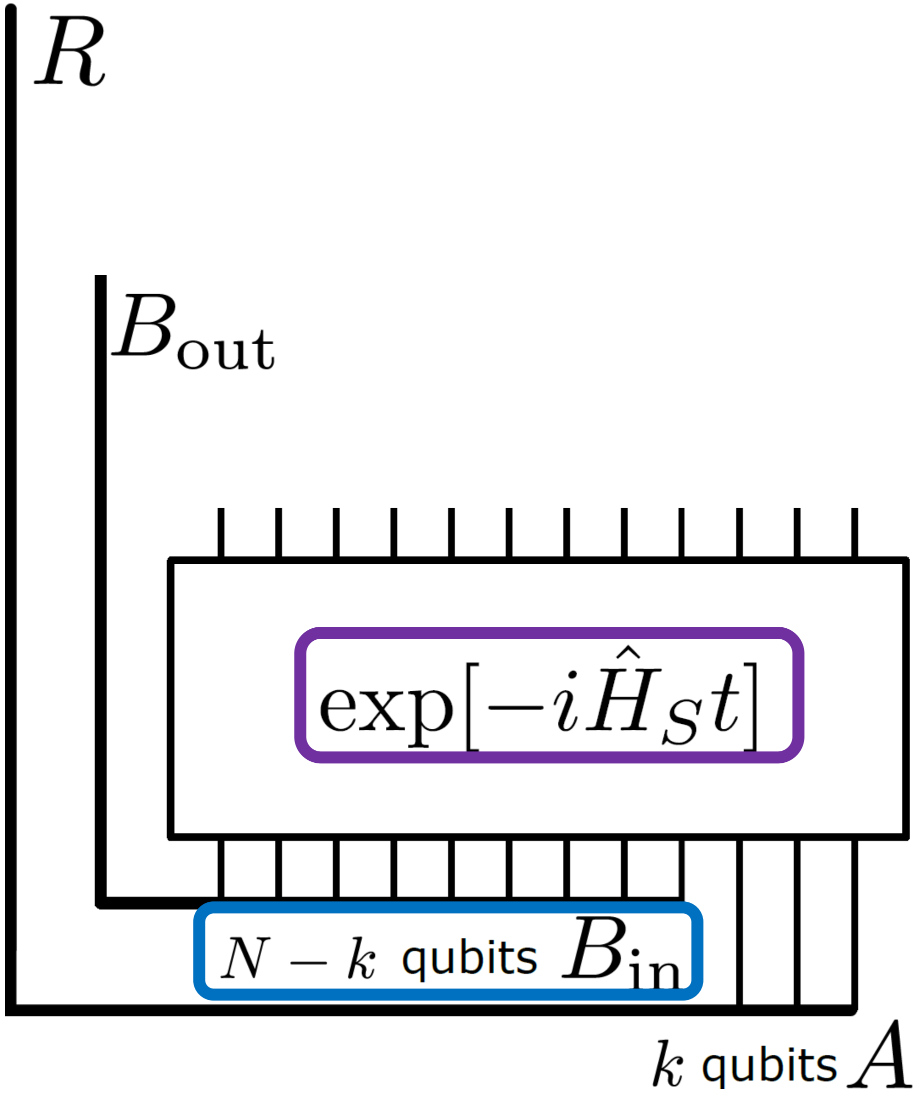
Quantum error correction: The Hayden-Preskill protocol



- Alice: throws k-qubit quantum information A into a box B_{in}

P. Hayden and J. Preskill, JHEP 2007

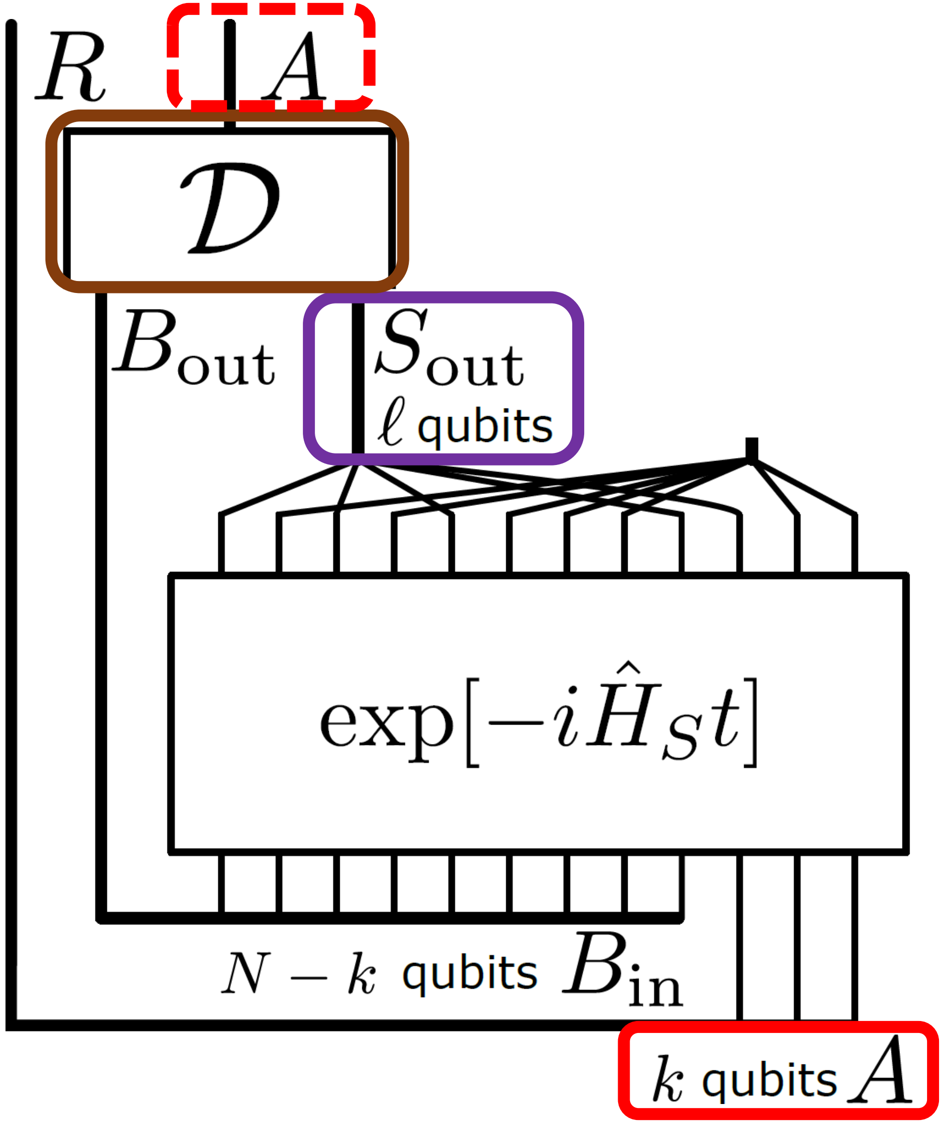
Quantum error correction: The Hayden-Preskill protocol



- Alice: throws k-qubit quantum information A into a box $B_{\rm in}$
- Bob: knows the original state of $B_{\rm in}$ and the Hamiltonian $\hat{H}_S$ of $S = A + B_{\rm in}$

P. Hayden and J. Preskill, JHEP 2007

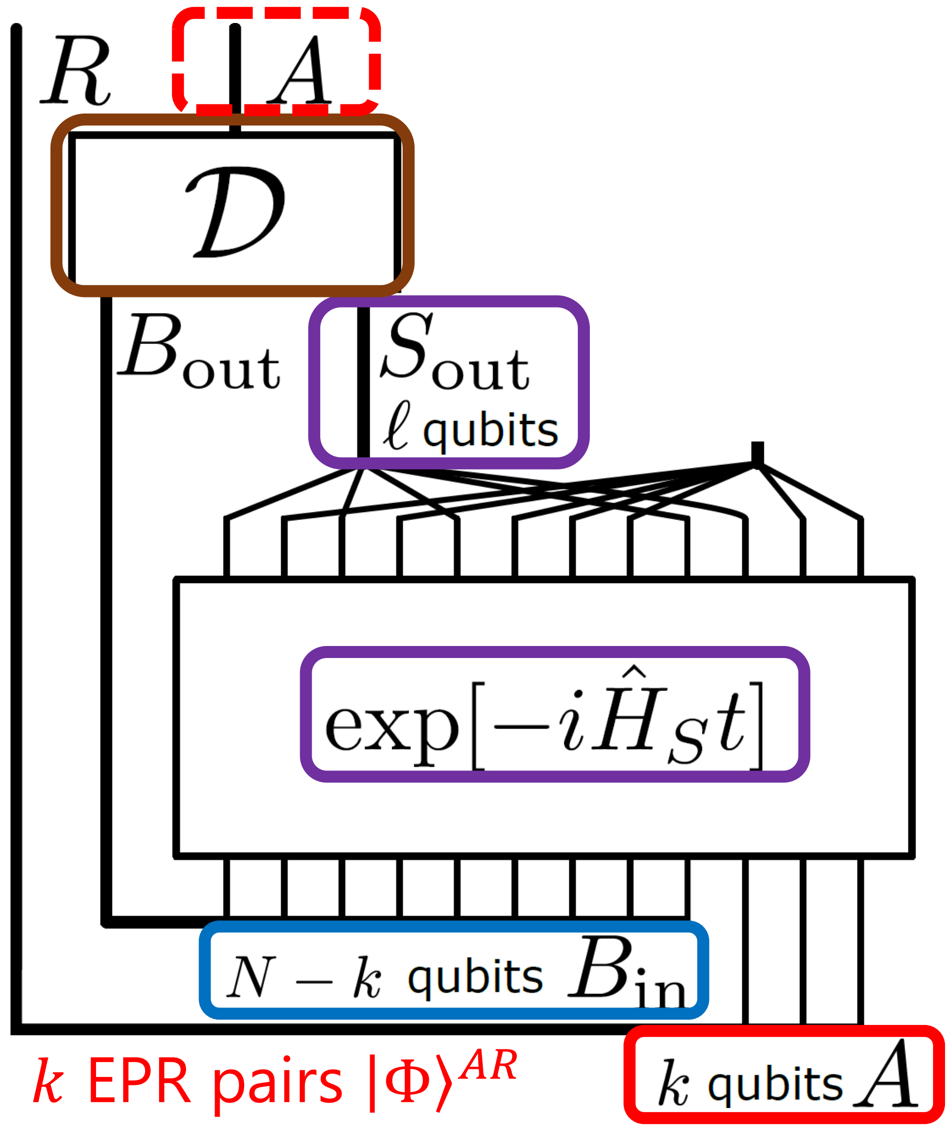
Quantum error correction: The Hayden-Preskill protocol



- Alice: throws k-qubit quantum information A into a box $B_{\rm in}$
- Bob: knows the original state of $B_{\rm in}$ and the Hamiltonian $\widehat{H}_S$ of $S = A + B_{\rm in}$
- Bob obtains ℓ qubits $S_{\rm out}$ after time t. Can Bob decode ($\mathcal{D}$) Alice's secret?

P. Hayden and J. Preskill, JHEP 2007

Quantum error correction: The Hayden-Preskill protocol

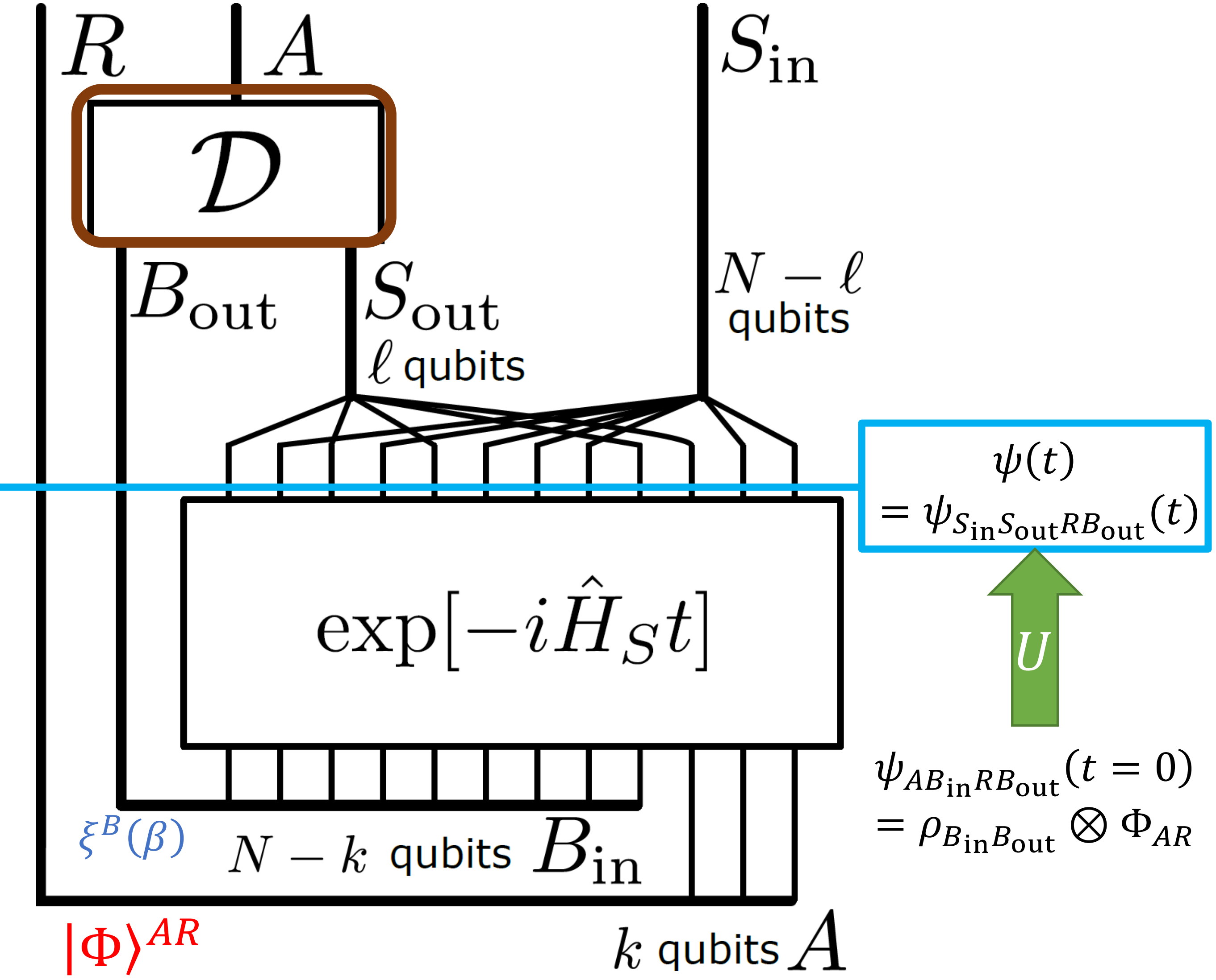


- Alice: throws k-qubit quantum information A into a box B_{in}
- Bob: knows the original state of B_{in} and the Hamiltonian $\hat{H}_S$ of $S = A + B_{\text{in}}$
- Bob obtains ℓ qubits S_{out} after time t. Can Bob decode ($\mathcal{D}$) Alice's secret?

Black holes: information recovery for $\ell \sim k$
[Hayden and Preskill, JHEP 2007]

Circular unitary (Haar) ensemble was assumed

Quantum error correction: The Hayden-Preskill protocol



Recovery error $\Delta_{\hat{H}}(t,\beta)$ among any $\mathcal{D}$ is hard to compute…

Decoupling approach

For $\mathcal{D}$ to succeed, no correlation is allowed between S_{in} and R

$$\rho_{S_{\mathrm{in}}R} = \mathrm{Tr}_{B_{\mathrm{out}},S_{\mathrm{out}}} |\psi(t)\rangle\langle\psi(t)|$$

Decoding error estimate

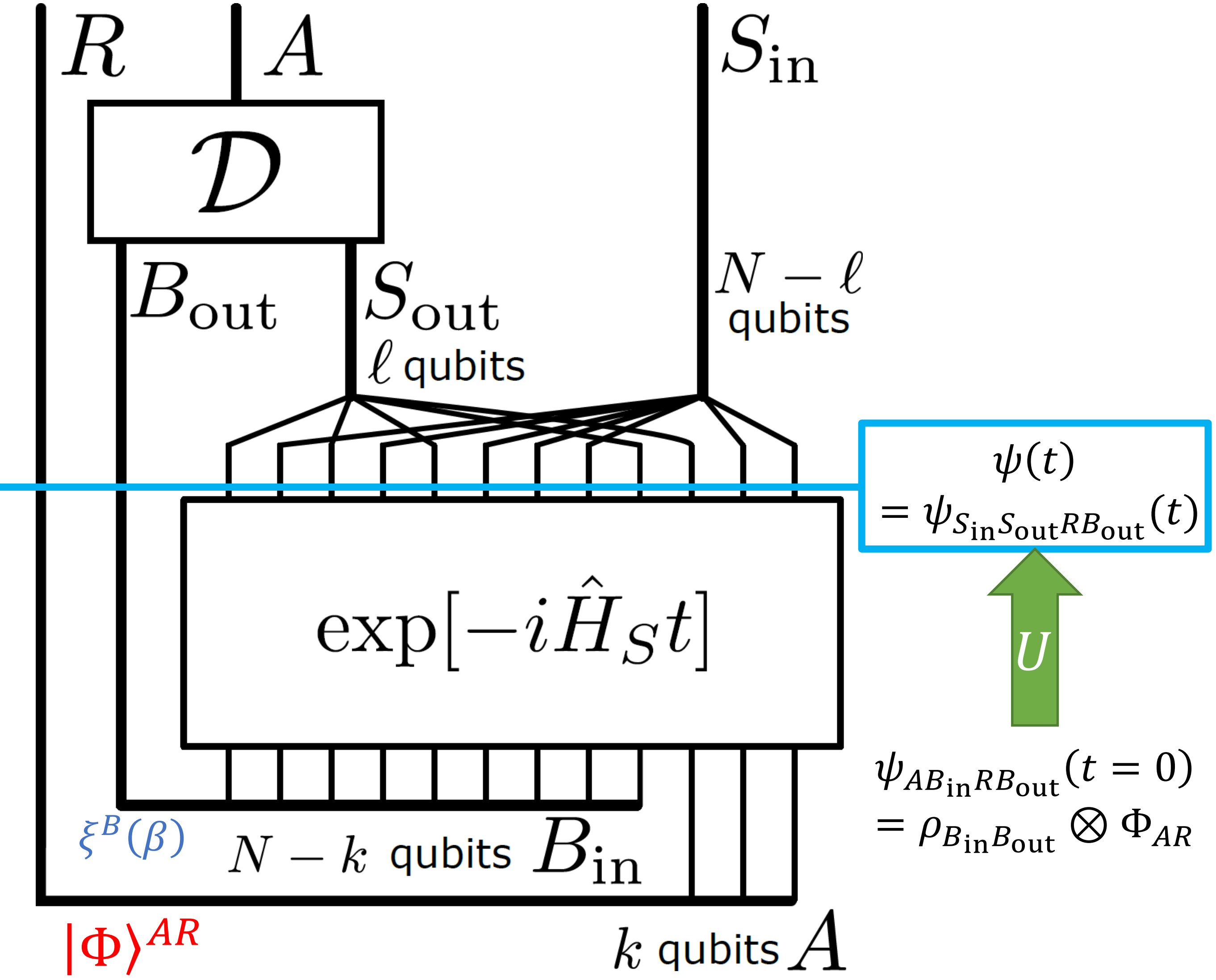
$$\bar{\Delta}_{\hat{H}}(t,\beta) \equiv \min\left\{1, \sqrt{\left\| \rho_{S_{\mathrm{in}}R} - \rho_{S_{\mathrm{in}}} \otimes \frac{I_R}{d_R} \right\|_1}\right\}$$

$(\geq \Delta_{\hat{H}}(t,\beta))$

$$\rho_{S_{\mathrm{in}}} = \mathrm{Tr}_R \rho_{S_{\mathrm{in}}R}$$

$$\|M\|_1 \equiv \mathrm{Tr}\sqrt{M^\dagger M}$$

Quantum error correction: The Hayden-Preskill protocol



<u>Haar random unitary case</u>:

$$\bar{\Delta}_{\rm Haar}(\beta) = \min\left\{1, 2^{\frac{1}{2}(\ell_{\rm Haar,th}(\beta)-\ell)}\right\}$$

$$\ell_{\rm Haar,th}(\beta) = \frac{N+k-H(\beta)}{2} \xrightarrow{\beta\to 0} k$$

$H(\beta)$: Renyi-2 entropy of $\xi^B(\beta)$

$\bar{\Delta}_{\rm Haar}$ exponentially decreases as function of ℓ after $\ell \approx k$ [HP recovery]

P. Hayden and J. Preskill, JHEP 2007

[Y. Nakata and MT, PRR 6, L022021 (2024)]

Our numerical study:

- **SYK-type Hamiltonians**
- One-dimensional spin chains

➔ Characterization of chaotic Hamiltonian dynamics

Error estimate for the SYK model

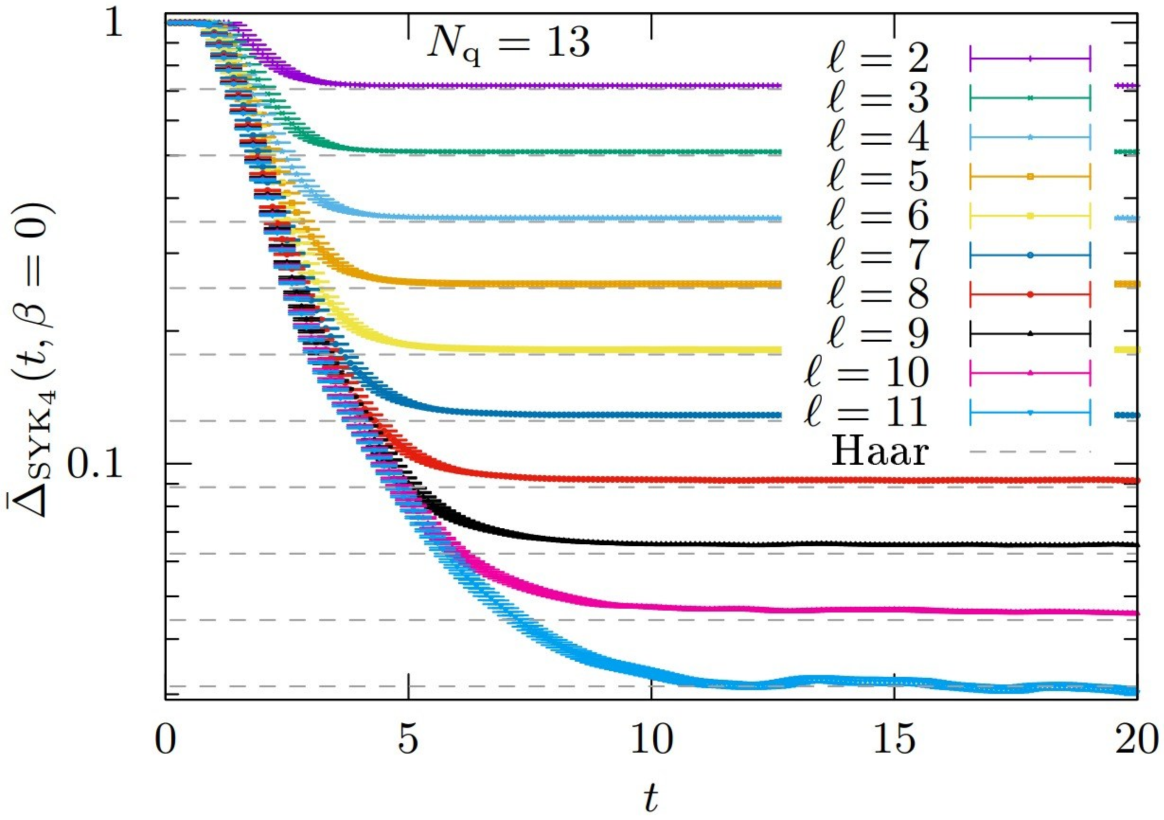
$$\hat{H} = \sum_{1 \le a < b < c < d \le 2N} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d$$

[Kitaev 2015][Sachdev & Ye 1993]

$\hat{\chi}_{a=1,2,\ldots,2N}$: $2N$ Majorana fermions ($\{\hat{\chi}_a, \hat{\chi}_b\} = 2\delta_{ab}$)

J_{abcd} : independent Gaussian random couplings ($\overline{J_{abcd}^{\,2}} = J^2$, $\overline{J_{abcd}} = 0$);

Normalization hereafter: SYK half-bandwidth $\sqrt{\frac{\langle \mathrm{Tr}\, \hat{H}^2 \rangle}{2^N}} = 1$



➔ **$\overline{\Delta}$ reaches the Haar value quickly ($t \sim \sqrt{N}$)**

[Yoshifumi Nakata and MT, PRR **6**, L022021 (2024)]

Models for $\hat{H}_S$ and quantum error correction (QEC)

1. SYK-like long-range couplings

Gaussian dense SYK_4

$$\hat{H} = \sum_{a<b<c<d} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d$$

sparsify →

Binary coupling sparse SYK

$$\hat{H} \propto \sum_{\substack{(a,b,c,d)\in P \\ \#P\sim N}} (\pm 1)\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d$$

[PRB 2023]

Error decays to ~ Haar value in $t \sim \sqrt{N}$

Add SYK_2 term ↓

SYK_{4+2}

$$\hat{H} = \cos\theta \sum_{a<b<c<d} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d + \sin\theta \sum_{a<b} K_{ab}\hat{\chi}_a\hat{\chi}_b$$

[PRL **120**, 241603; PRR **3**, 013023; PRL **127**, 030601]

$\delta \propto \tan\theta \ll 1$: SYK_4, $\delta = \mathcal{O}(1)$: chaotic spectrum but eigenstates restricted in Fock space, $\delta \gg 1$: many-body localization

Error increases before many-body localization

2. One-dimensional spin chains

Ising chain + uniform field

$$\hat{H}_{\mathrm{Ising}} = -J\sum_{\langle j,k\rangle} S_j^z S_k^z - g\sum_j S_j^x - h\sum_k S_k^z$$

$g = 0$ or $h = 0$: integrable, far from integrable lines: chaotic

Heisenberg chain + random field

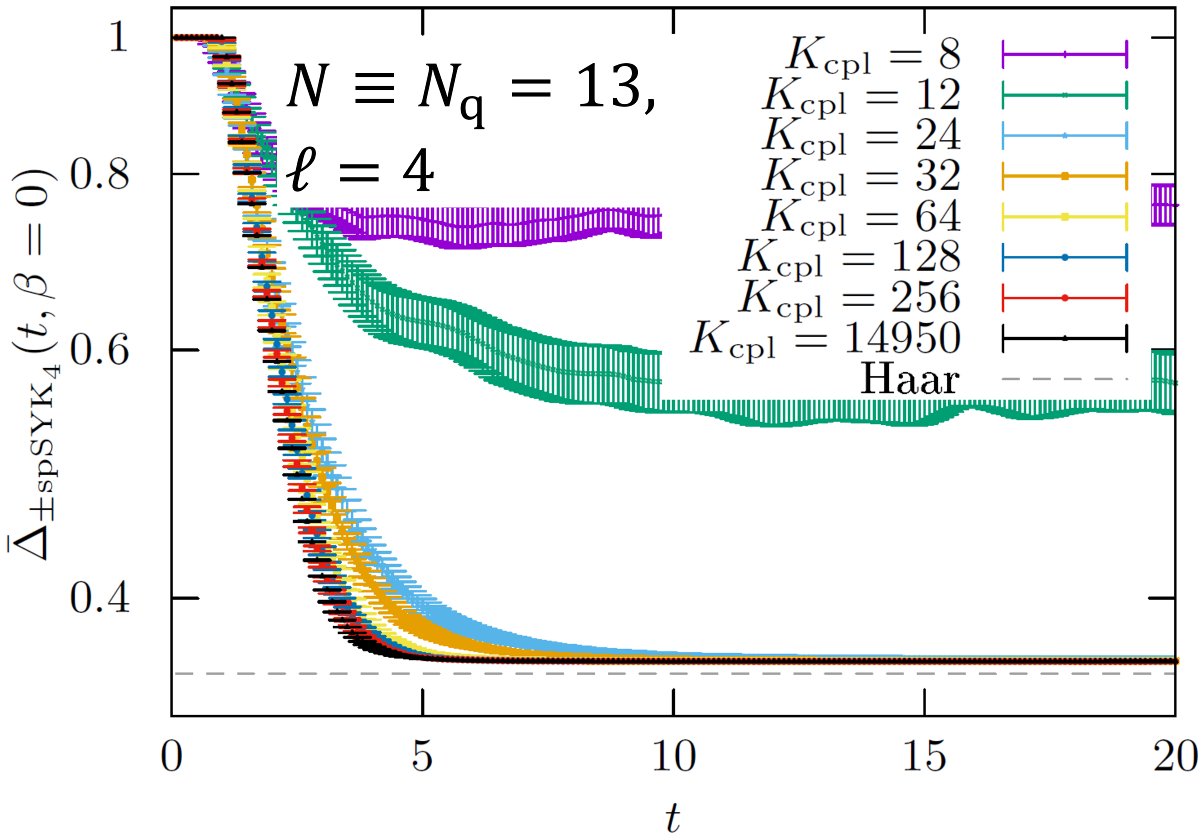
$$\hat{H}_{\mathrm{XXZ}} = \sum_{\langle j,k\rangle} S_j \cdot S_k + \sum_j h_j S_j^z,\ h_j \in [-W, W]$$

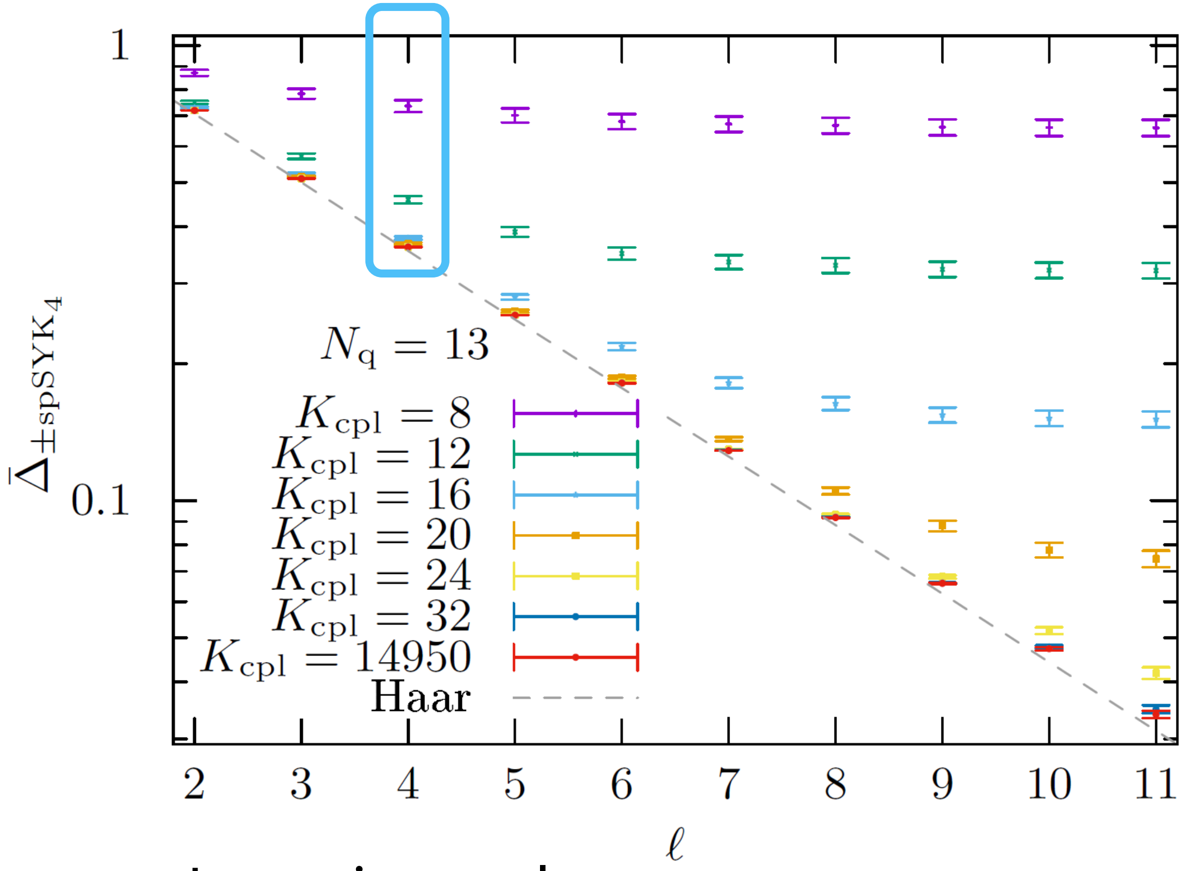
$W \ll 1$: integrable, $W \sim 1$: chaotic, $W \gg 4$: MBL(?)

Efficient QEC not observed even for chaotic cases

[Yoshifumi Nakata and MT, PRR **6**, L022021 (2024)]

$\overline{\Delta}_{\hat{H}}(t,\beta)$ for binary-coupling sparse SYK



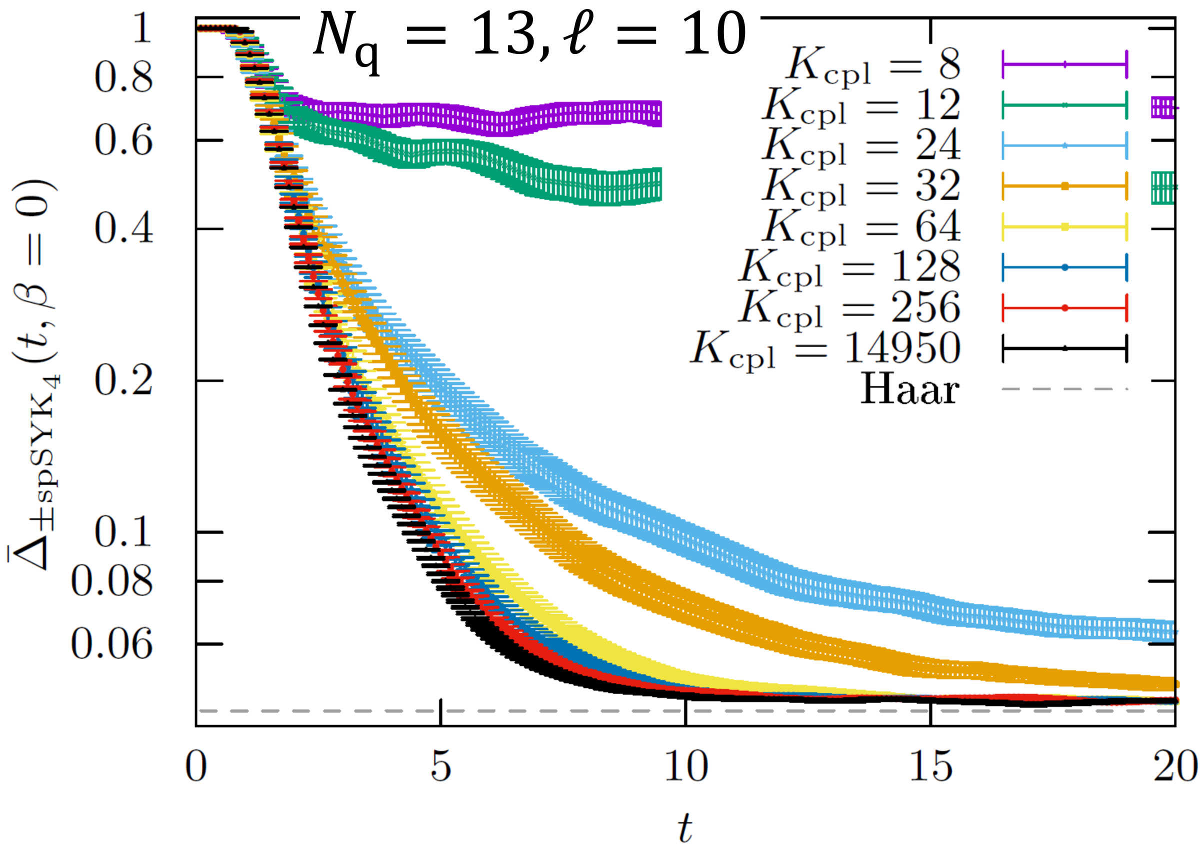


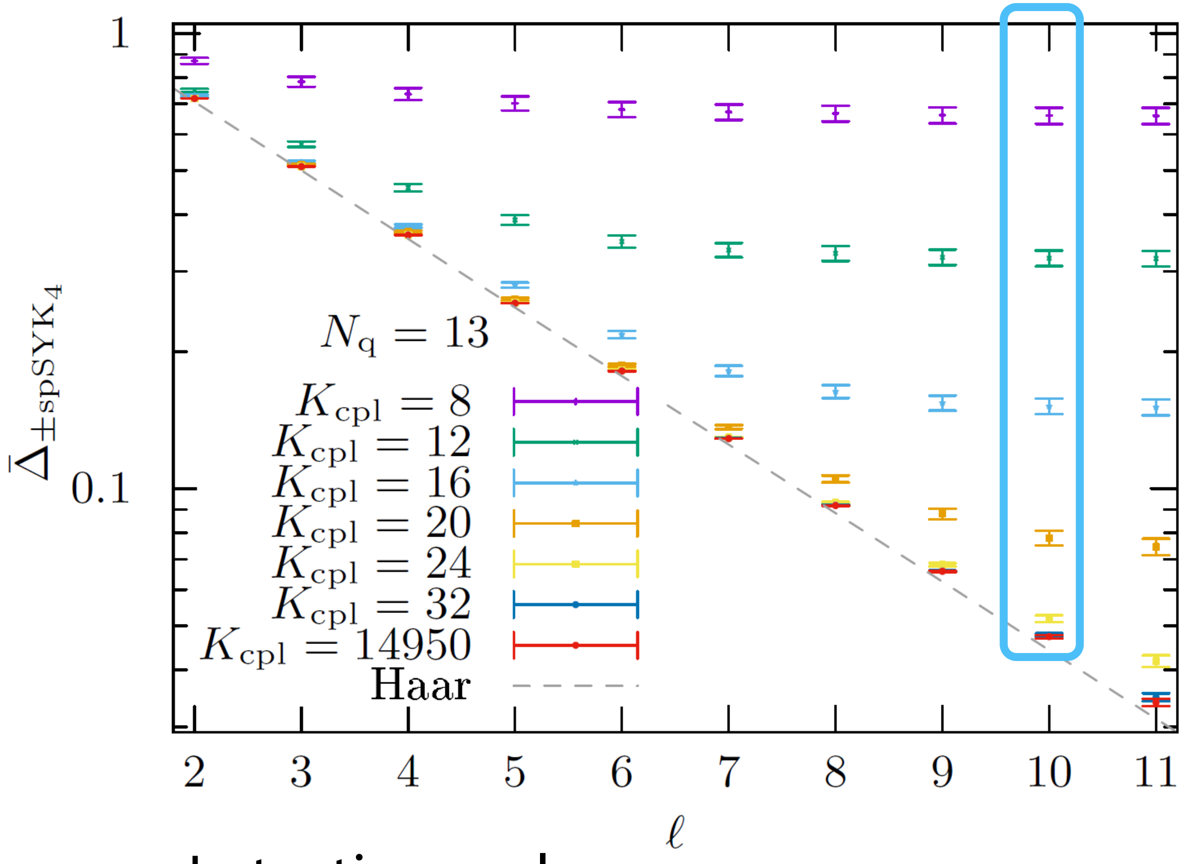
Time dependence:
approach (binary-coupling & Gaussian) dense model as K_{cpl} is increased

Late-time value:

very close to the Haar value $2^{\frac{1-\ell}{2}}$, indistinguishable for $K_{cpl} \gtrsim 3N$

$\overline{\Delta}_{\widehat{H}}(t,\beta)$ for binary-coupling sparse SYK



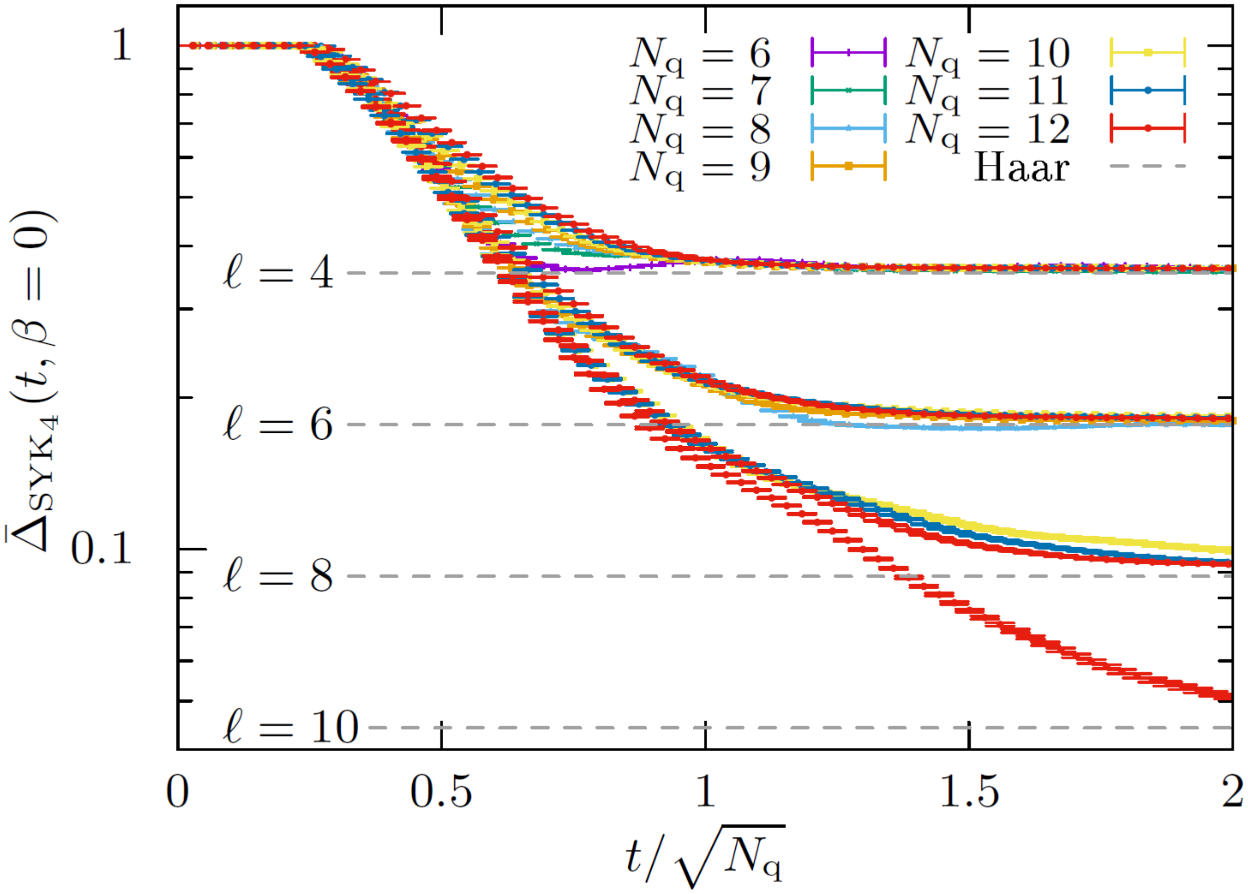


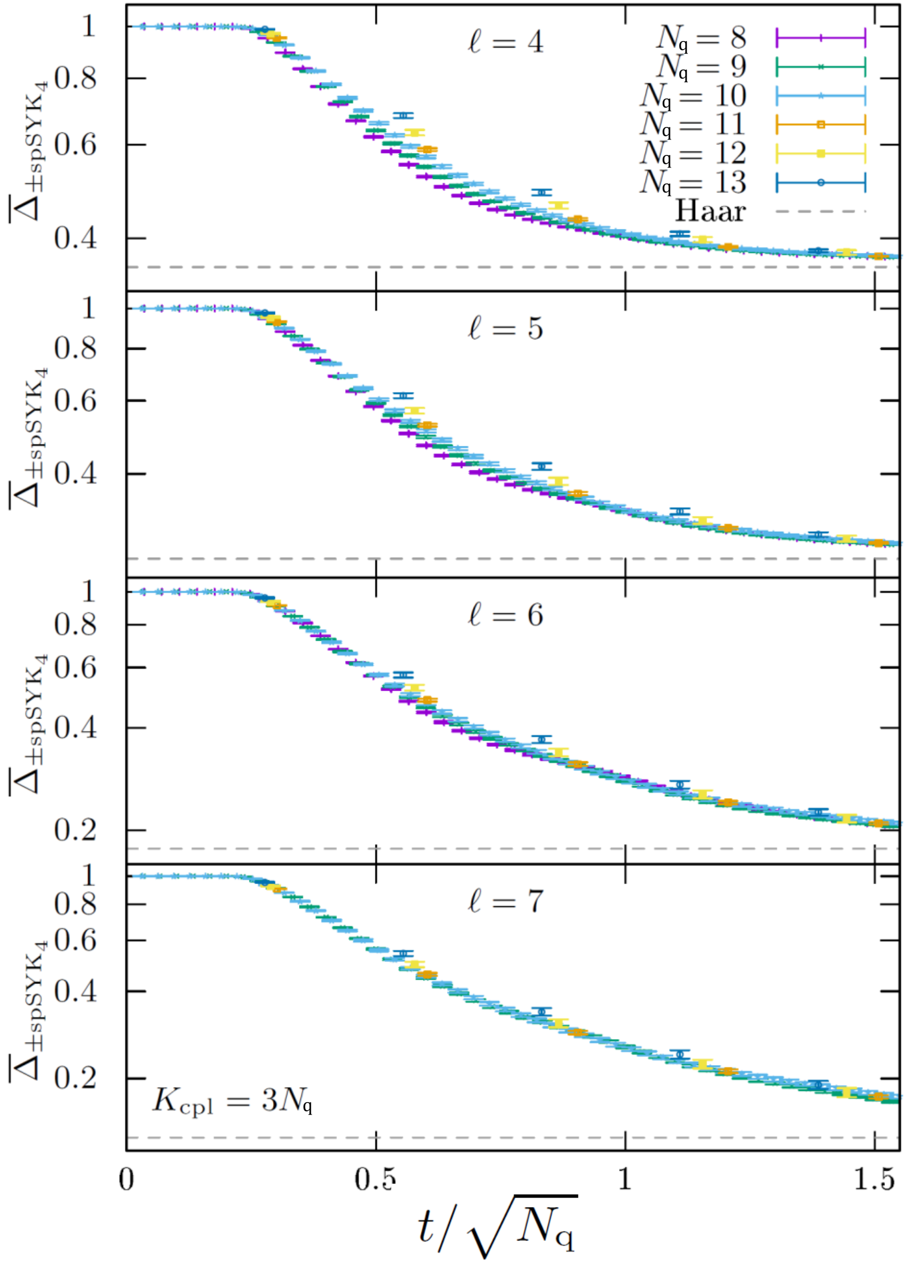
Time dependence:
approach (binary-coupling & Gaussian) dense model as K_{cpl} is increased

Late-time value:

very close to the Haar value $2^{\frac{1-\ell}{2}}$, indistinguishable for $K_{cpl} \gtrsim 3N$

Time scale for scrambling





Normalization: SYK half-bandwidth $\sqrt{\frac{\langle \mathrm{Tr}\, \widehat{H}^2 \rangle}{2^{N_q}}} = 1$, $\hbar = 1$

- The Haar value $\overline{\Delta} = 2^{\frac{k-\ell}{2}}$ is reached after $t \sim \mathcal{O}(\sqrt{N})$

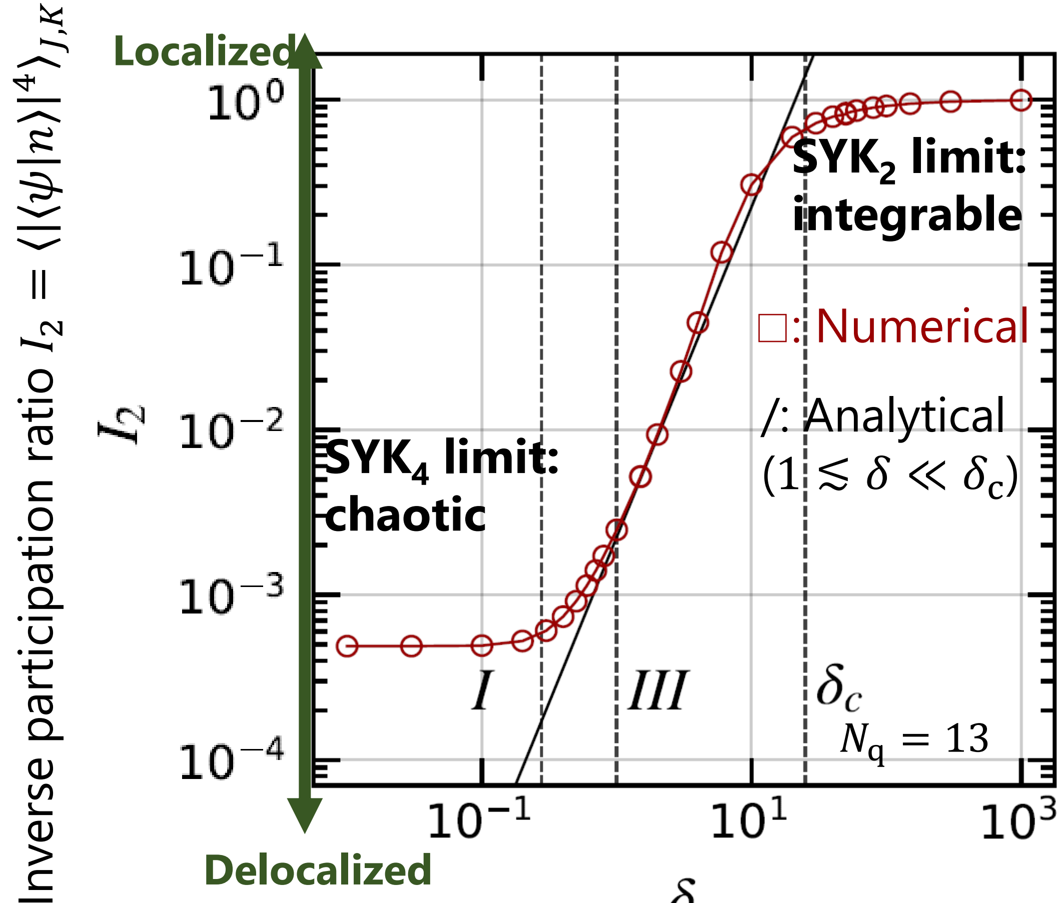
SYK_{4+2}

$$\hat{H} = \sum_{1\le a<b<c<d}^{N_{\mathrm{Maj}}=2N} J_{abcd}{}' \hat{\chi}'_a \hat{\chi}'_b \hat{\chi}'_c \hat{\chi}'_d + i \sum_{1\le a<b}^{N_{\mathrm{Maj}}} K_{ab} \hat{\chi}'_a \hat{\chi}'_b = - \sum_{1\le a<b<c<d}^{2N} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d + \sum_{1\le j\le N}^{N} v_j (2\hat{n}_j - 1)$$

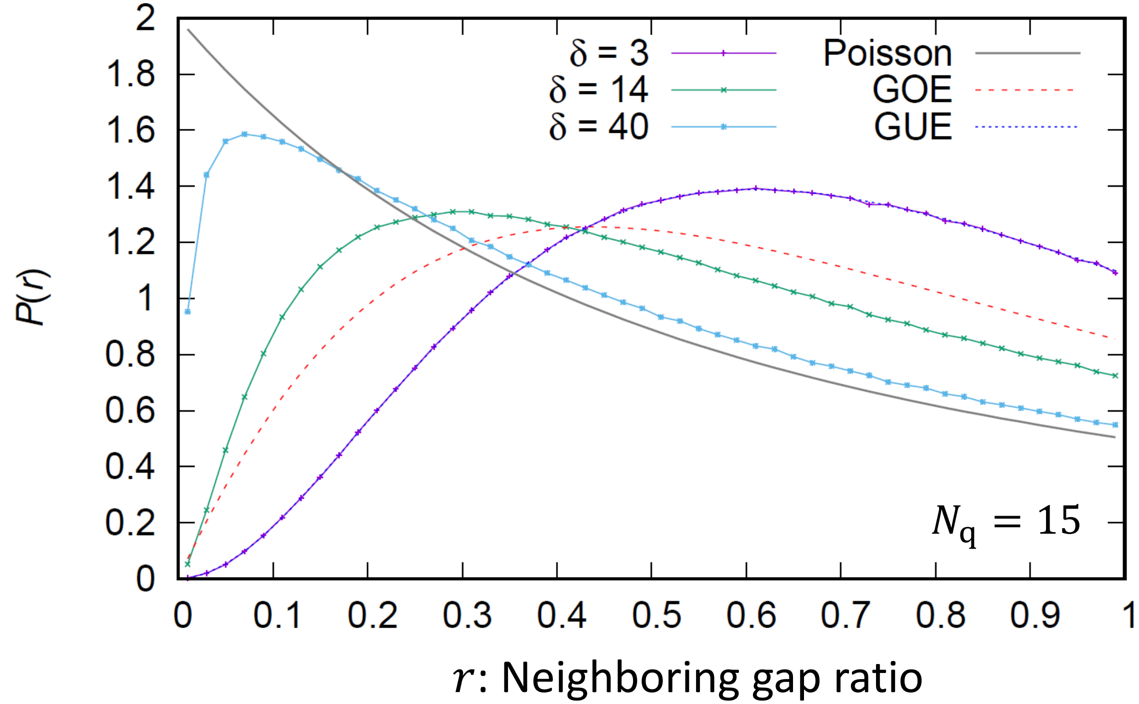
Normalization of J_{abcd}, v_j (mass of complex fermion $(\hat{\chi}_{2j-1} + i\hat{\chi}_{2j})/\sqrt{2}$):
SYK_4 bandwidth = 1, width of v_j distribution = δ

Chaos-integrable transition [A. M. García-García, A. Romero-Bermúdez, B. Loureiro, and MT, PRL **120**, 241603 (2018)]
Localization in many-body Fock-space [F. Monteiro, T. Micklitz, MT, and A. Altland, PRResearch **3**, 013023 (2021)]

Eigenstate localization in the Fock space



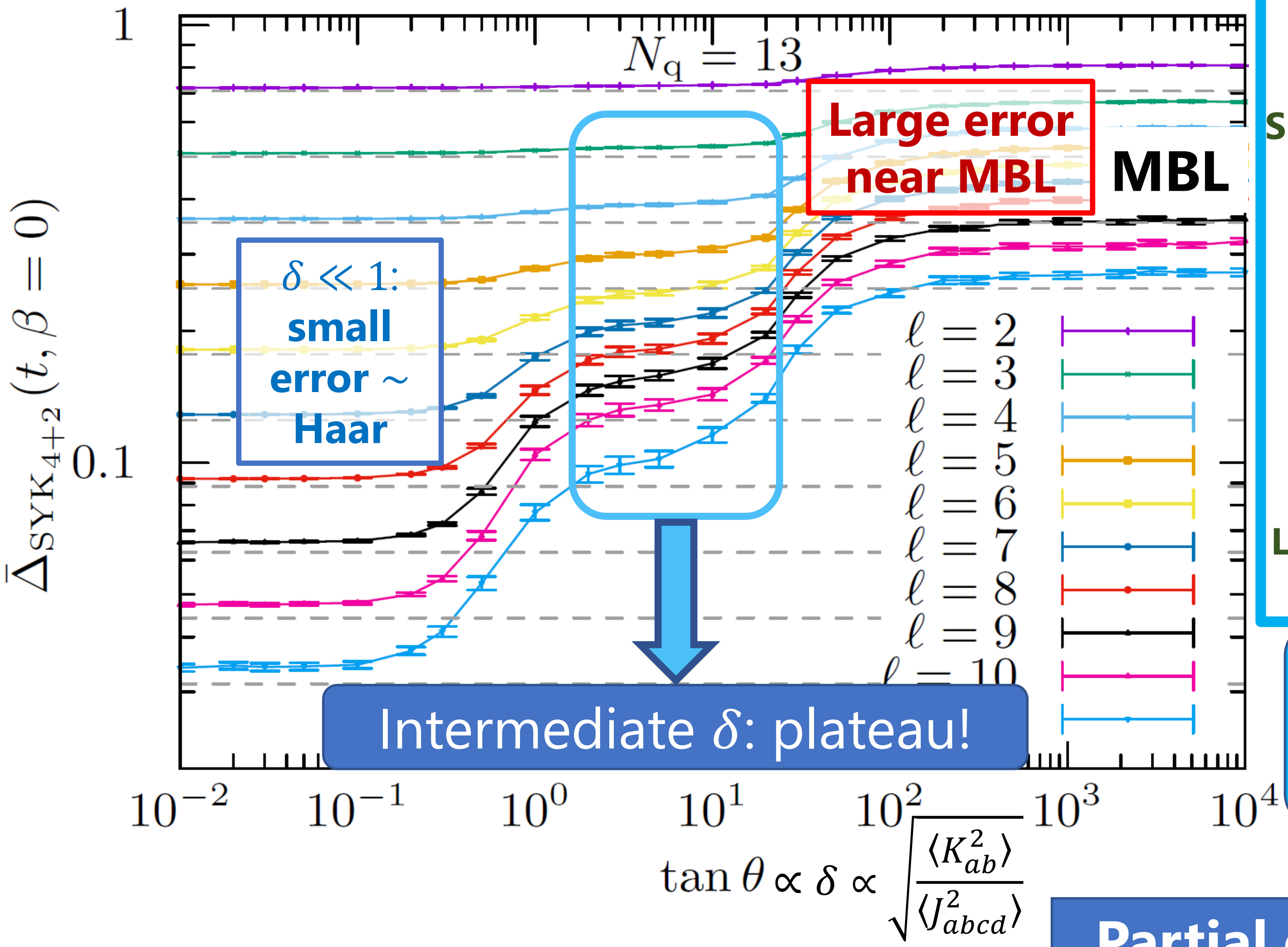
Distribution of r



Random-matrix like even for $\delta > 1$ (eigenstates are nearly localized in the Fock space)

Late-time error estimate for SYK_{4+2}

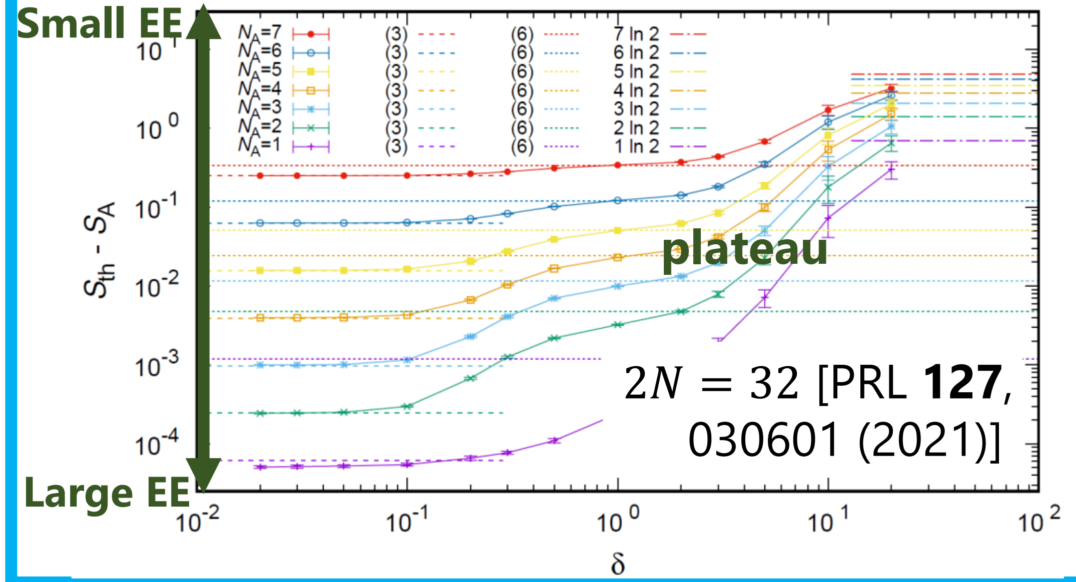
[Y. Nakata and MT, PRR **6**, L022021 (2024)]



Intermediate δ: plateau!

$$\hat{H}_{\mathrm{SYK}_{4+2}} = \cos\theta\, \hat{H}_{\mathrm{SYK}_4} + \sin\theta\, \hat{H}_{\mathrm{SYK}_2}$$

Understood by considering **eigenstate entanglement entropy (EE)**



Eigenstates in **restricted** part of Fock space, where they are thermally distributed

Partial decoupling, incomplete decrease of error [Nakata et al., 1903.05796, 2007.00895]

Spin chains in chaotic regime

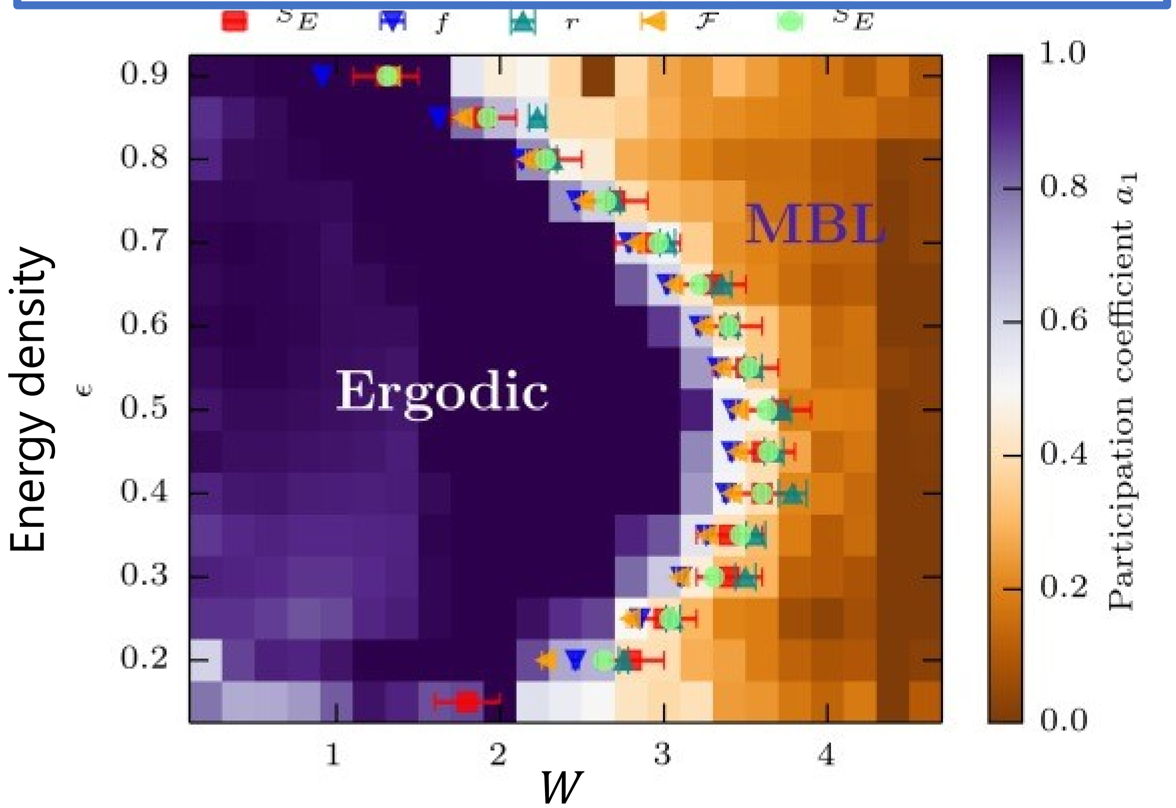
- Heisenberg + random field

$$\hat{H}_{XXZ} = \frac{J}{4}\sum_{j,\alpha=x,y,z}\hat{\sigma}_j^\alpha\hat{\sigma}_{j+1}^\alpha + \sum_j \frac{h_j^z}{2}\hat{\sigma}_j^z$$

$$J = 1, \left|h_j^z\right| \in [-W : W]$$

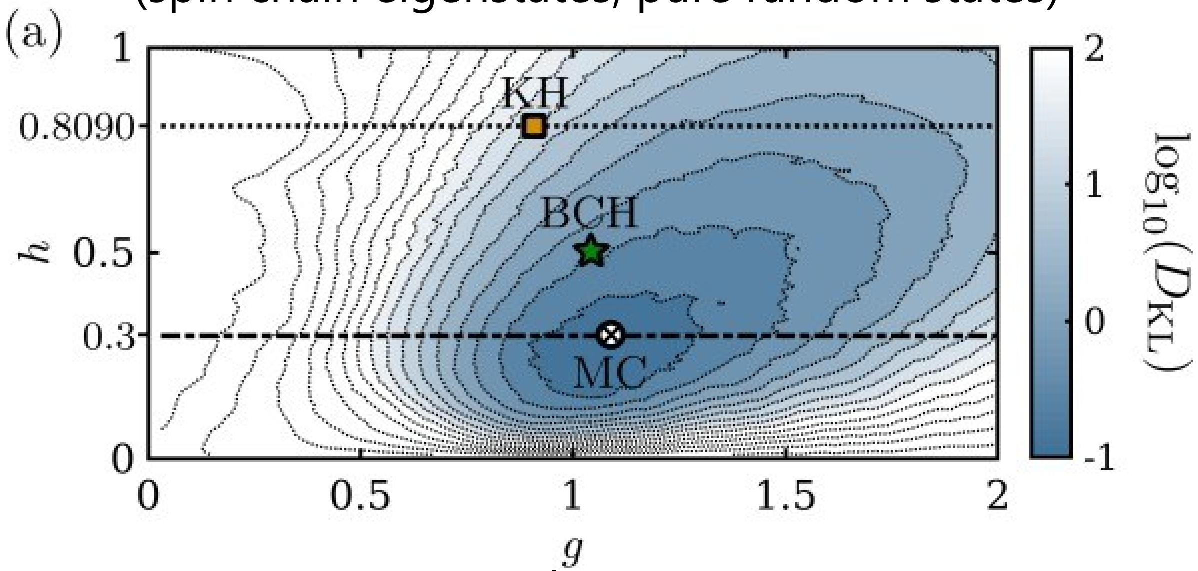
- Mixed-field Ising

$$\hat{H}_{\text{Ising}} = \sum_j (\hat{\sigma}_j^z\hat{\sigma}_{j+1}^z + g\hat{\sigma}_j^x + h\hat{\sigma}_j^z)$$



[Luitz, Laflorencie, Alet PRB 2015]

D_{KL}: KL divergence between probability distributions of entanglement entropy (spin chain eigenstates, pure random states)



KH: Kim-Huse, BCH: Banuls-Cirac-Hastings, MC: Most chaotic

[Rodriguez-Nieva, Jonay, Khemani PRX **14**, 031014 (2024)]

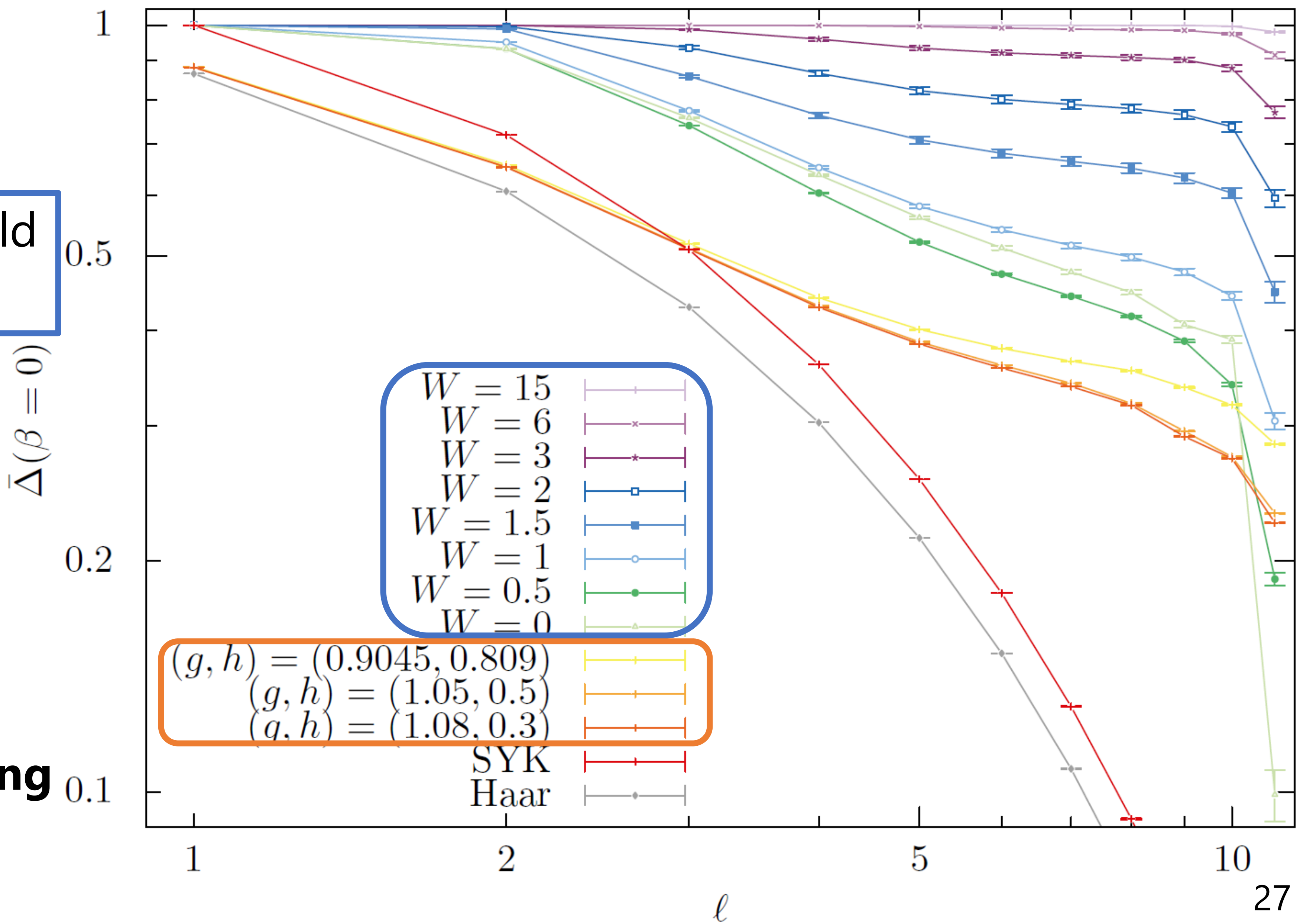
Spin chains

- Heisenberg + random field
  $J = 1, |h_j^z| \in [-W : W]$

- Mixed-field Ising
  $g\hat{x} + h\hat{z}$

Recovery errors are large
(No exponential decay)
Not information scrambling

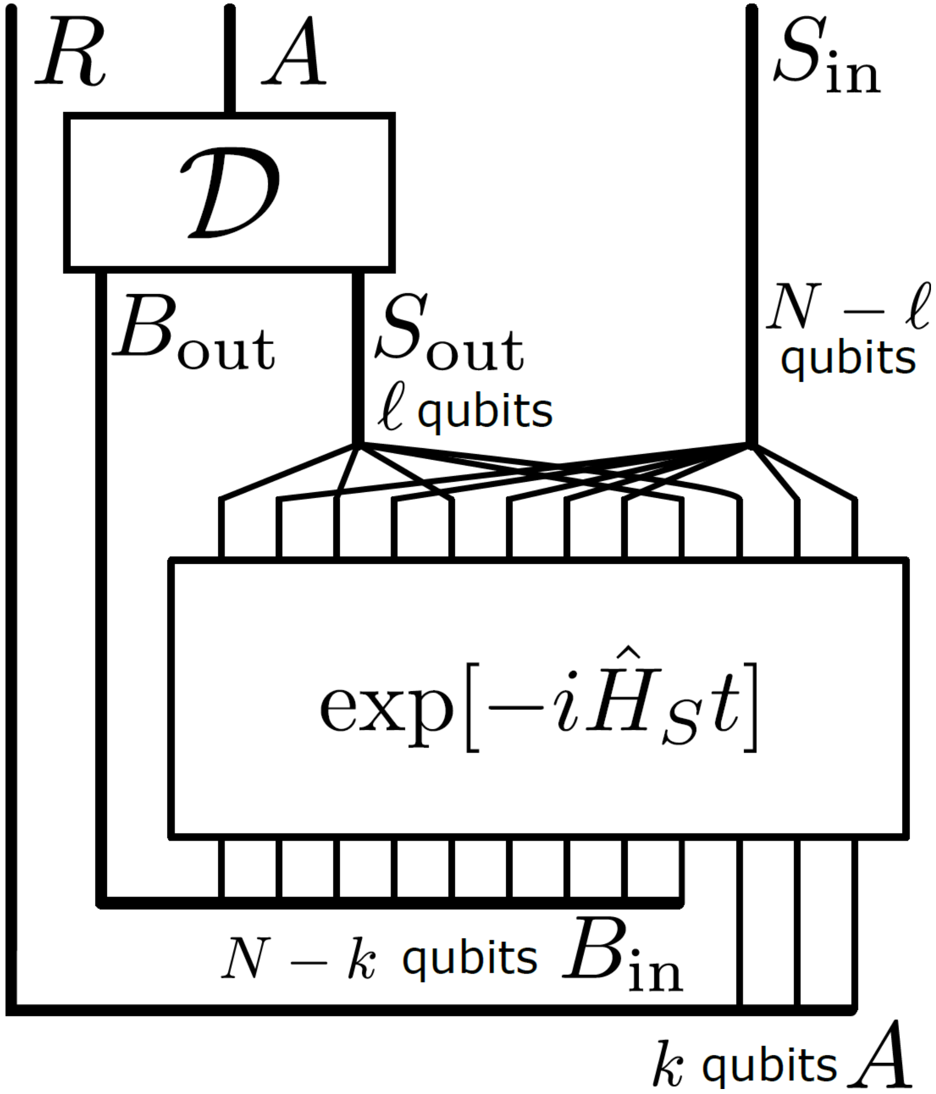
$N = 12$ spins, late-time



Summary so far

Masaki Tezuka, Onur Oktay, Masanori Hanada, Enrico Rinaldi, and Franco Nori, Phys. Rev. B **107**, L081103 (2023)

Yoshifumi Nakata and M. Tezuka, Phys. Rev. Research **6**, L022021 (2024)



- Proposed sparse SYK with coupling = ±1
- Studied quantum error correction by Hamiltonian dynamics
- SYK & sparse SYK: almost unchanged scrambling properties if spectrum is random matrix-like
- SYK4+2: suffers from wavefunction localization in Fock space; plateau for intermediate θ
- Spin chains: no Haar-like exponential decay of error as ℓ is increased, even in chaotic region

Random-coupling spin models

M. Hanada, A. Jevicki, X. Liu, E. Rinaldi, and MT, JHEP **05**(2024)280
cf. Swingle & Winer PRB **109**, 094206

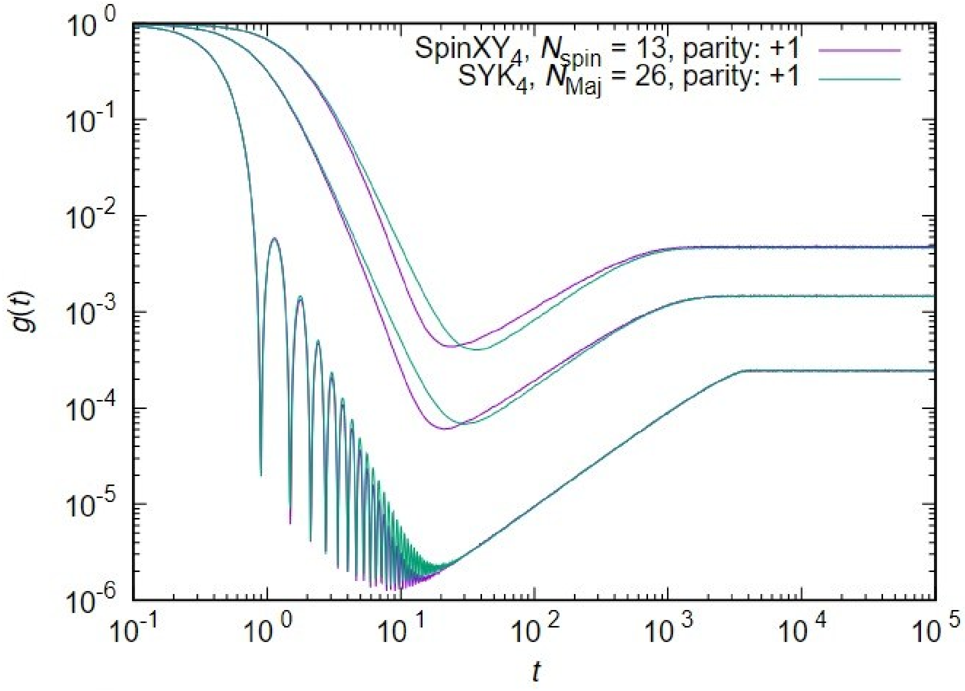
Consider N quantum spins ($S = 1/2$) with all-to-all interactions

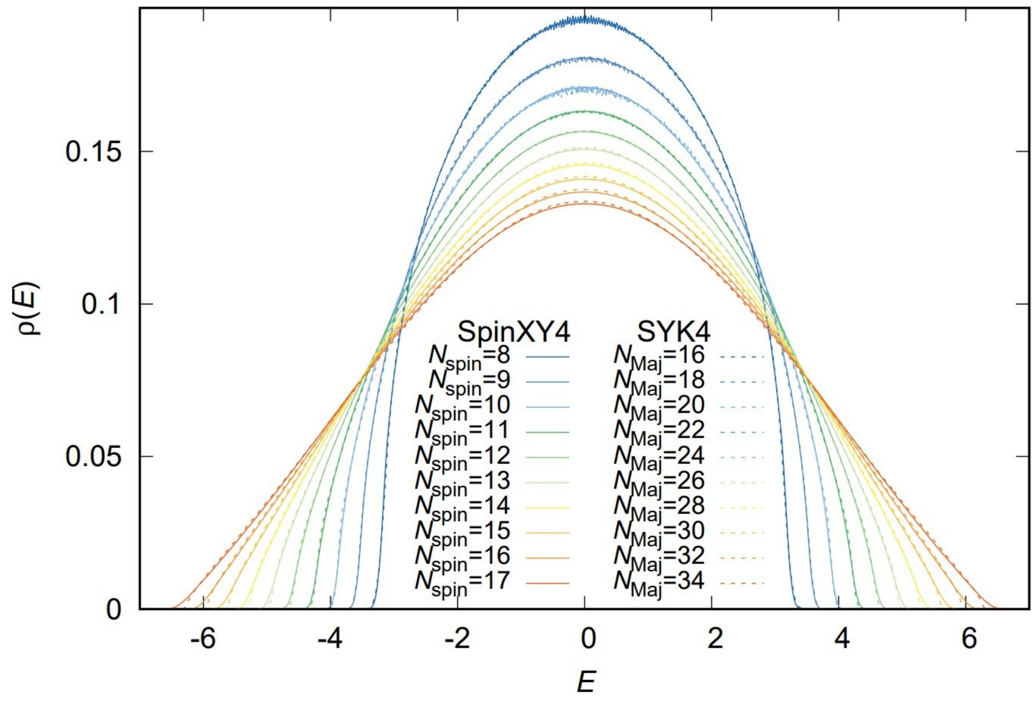
$$\hat{H} = \sum_{1 \le a < b < c < d \le 2N} i^{\eta_{abcd}} J_{abcd} \hat{O}_a \hat{O}_b \hat{O}_c \hat{O}_d$$

$\hat{O}_{2j-1} = \hat{\sigma}_j^x,\ \hat{O}_{2j} = \hat{\sigma}_j^y$

η_{abcd}: number of pairs of indices on the same spin

➔ Random-matrix behavior with density of states similar to the SYK_4 model





➔ Also, we may change the number of interacting spins, sparcify, forbid $\eta > 0$ terms, etc.

Spin operators vs Majorana fermions

- $2N$ spin operators
  - $O_1 = \sigma_{1,x} = \sigma_x \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \otimes 1$
  - $O_2 = \sigma_{1,y} = \sigma_y \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \otimes 1$
  - $O_3 = \sigma_{2,x} = 1 \otimes \sigma_x \otimes 1 \otimes \cdots \otimes 1 \otimes 1$
  - $O_4 = \sigma_{2,y} = 1 \otimes \sigma_y \otimes 1 \otimes \cdots \otimes 1 \otimes 1$

    $\vdots$
  - $O_{2N-1} = 1 \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \sigma_x$
  - $O_{2N} = 1 \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \sigma_y$
- $O_i^2 = 1$
- $O_{i,\alpha}\, O_{j,\beta}$ is hermitian and $[O_{i,\alpha}, O_{j,\beta}] = 0$ if $i \neq j$ $(\alpha, \beta = x, y)$
- $\mathrm{i}O_{i,x}O_{i,y} = -O_{i,z}$ is hermitian

- $2N$ Majorana fermions
  - $\chi_1 = \sigma_x \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \otimes 1$
  - $\chi_2 = \sigma_y \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \otimes 1$
  - $\chi_3 = \sigma_z \otimes \sigma_x \otimes 1 \otimes \cdots \otimes 1 \otimes 1$
  - $\chi_4 = \sigma_z \otimes \sigma_y \otimes 1 \otimes \cdots \otimes 1 \otimes 1$

    $\vdots$
  - $\chi_{2N-1} = \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \cdots \otimes \sigma_z \otimes \sigma_x$
  - $\chi_{2N} = \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \cdots \otimes \sigma_z \otimes \sigma_y$
- $\chi_i^2 = 1$
- $\mathrm{i}\chi_i\chi_j$ is hermitian if $i \neq j$
- Satisfy $\{\chi_i, \chi_j\} = \chi_i\chi_j + \chi_j\chi_i = 2\delta_{ij}$
  - Because $\sigma_i\sigma_j = \delta_{ij}1 + \mathrm{i}\sum_k \epsilon_{ijk}\sigma_k$

M. Hanada, A. Jevicki, X. Liu, E. Rinaldi, and M. Tezuka, JHEP**05**(2024)280.

SpinXY4 model vs SYK4

$\eta_{ijkl} \in \{0,1,2\}$: number of spins whose both x, y components are accessed by (i,j,k,l)

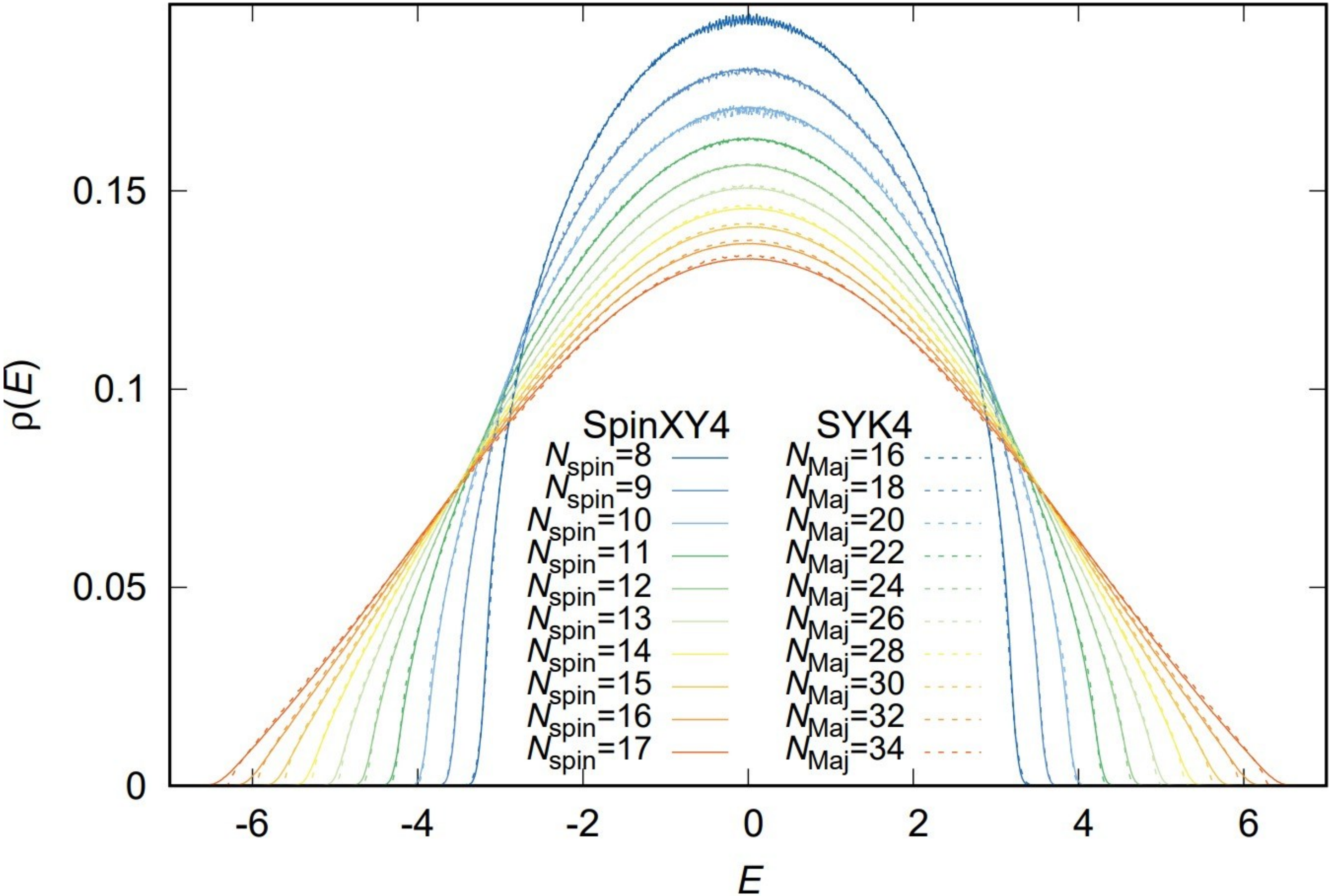
- $H_{\mathrm{SpinXY}_4} = C \sum_{ijkl} \mathrm{i}^{\eta_{ijkl}} J_{ijkl} O_i O_j O_k O_l$
- $2N$ spin operators
  - $O_1 = \sigma_{1,x} = \sigma_x \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \otimes 1$
  - $O_2 = \sigma_{1,y} = \sigma_y \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \otimes 1$
  - $O_3 = \sigma_{2,x} = 1 \otimes \sigma_x \otimes 1 \otimes \cdots \otimes 1 \otimes 1$
  - $O_4 = \sigma_{2,y} = 1 \otimes \sigma_y \otimes 1 \otimes \cdots \otimes 1 \otimes 1$

    $\vdots$
  - $O_{2N-1} = 1 \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \sigma_x$
  - $O_{2N} = 1 \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \sigma_y$
- $O_i^2 = 1$
- $O_{i,\alpha}\, O_{j,\beta}$ is Hermitian and $[O_{i,\alpha}, O_{j,\beta}] = 0$ if $i \neq j$ $(\alpha, \beta = x, y)$
- $\mathrm{i} O_{i,x} O_{i,y} = -O_{i,z}$ is hermitian

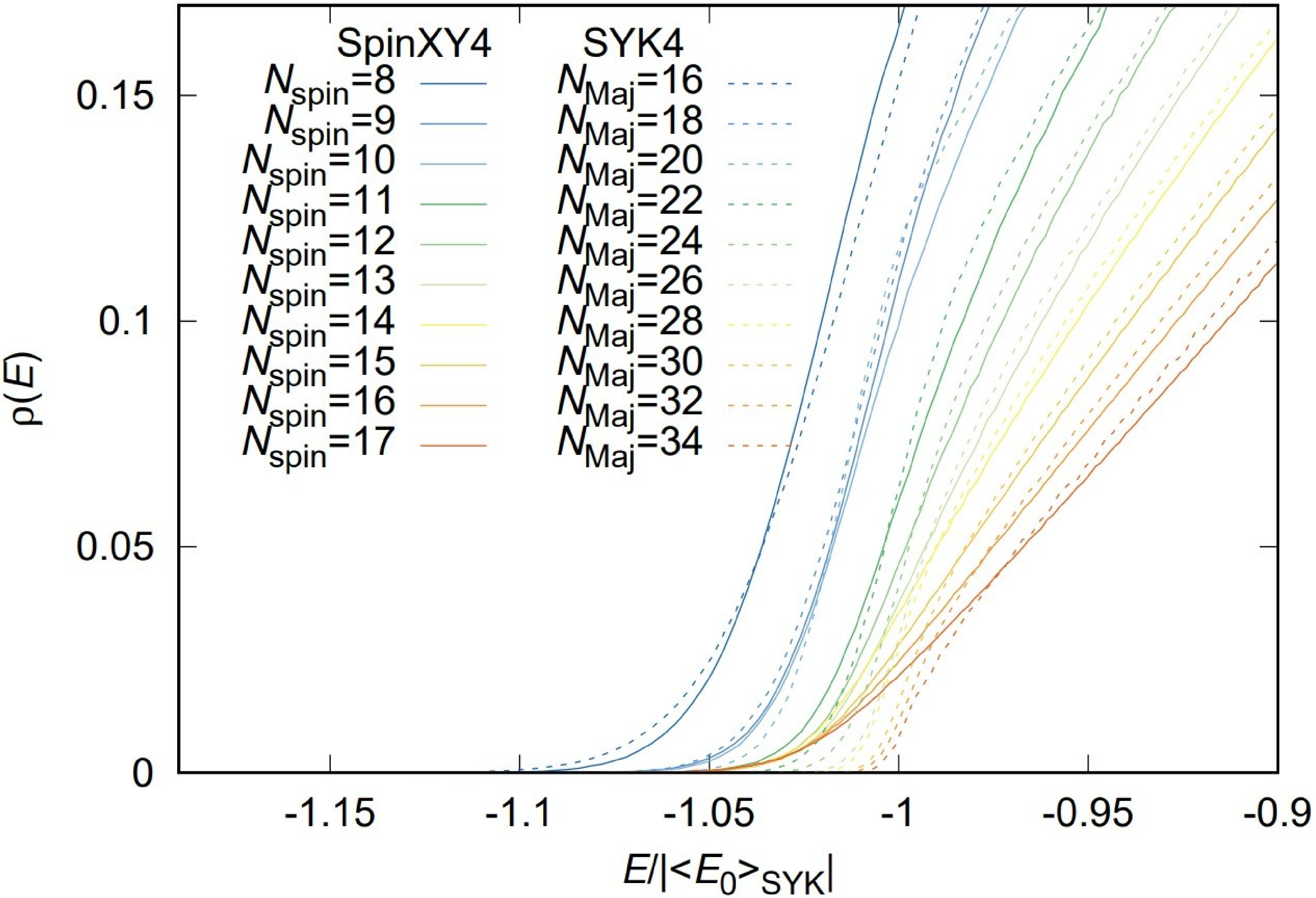
- $H_{\mathrm{SYK}_4} = C \sum_{ijkl} J_{ijkl} \chi_i \chi_j \chi_k \chi_l$
- $2N$ Majorana fermions
  - $\chi_1 = \sigma_x \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \otimes 1$
  - $\chi_2 = \sigma_y \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \otimes 1$
  - $\chi_3 = \sigma_z \otimes \sigma_x \otimes 1 \otimes \cdots \otimes 1 \otimes 1$
  - $\chi_4 = \sigma_z \otimes \sigma_y \otimes 1 \otimes \cdots \otimes 1 \otimes 1$

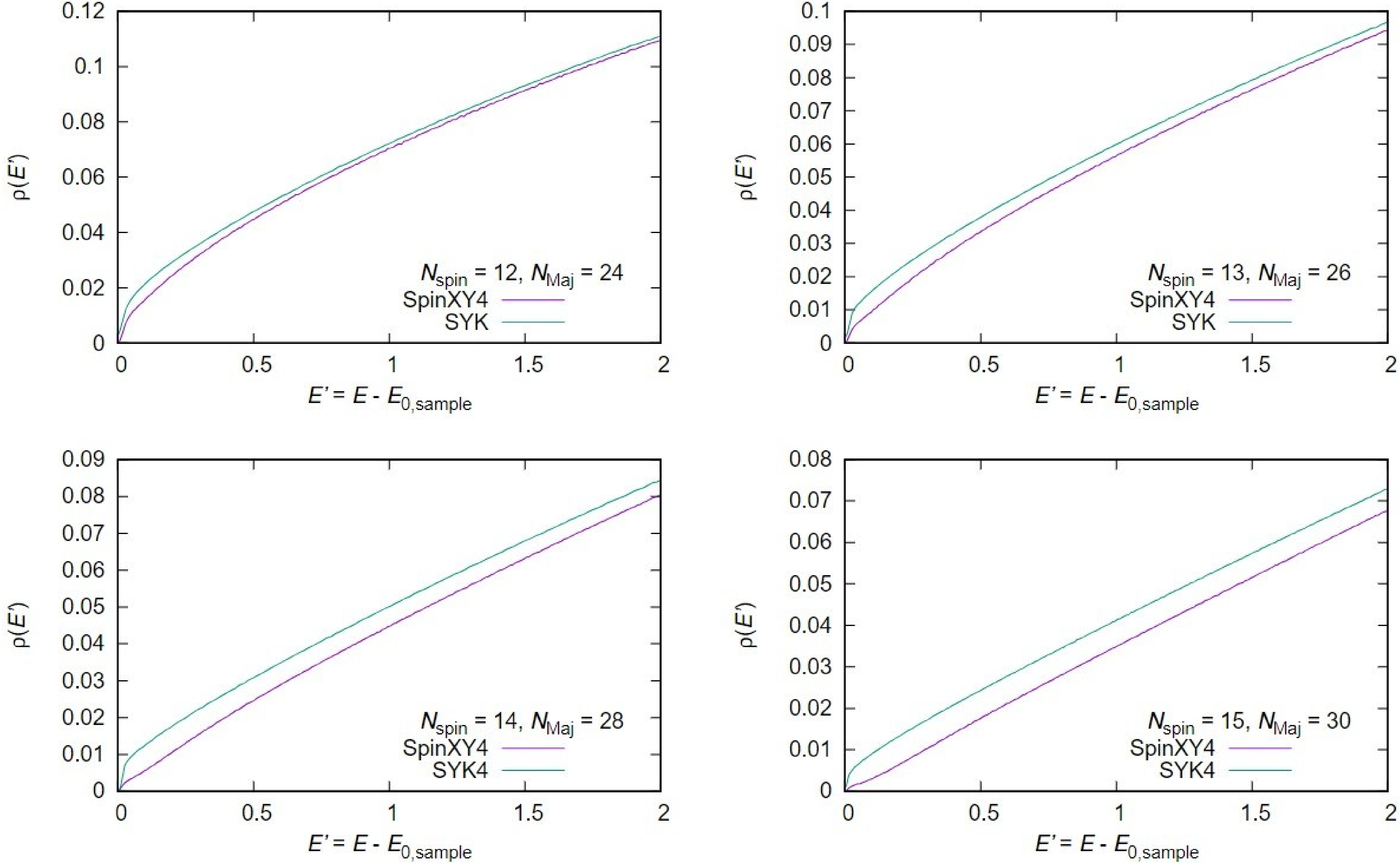
    $\vdots$
  - $\chi_{2N-1} = \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \cdots \otimes \sigma_z \otimes \sigma_x$
  - $\chi_{2N} = \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \cdots \otimes \sigma_z \otimes \sigma_y$
- $\chi_i^2 = 1$
- $\mathrm{i}\chi_i \chi_j$ is hermitian if $i \neq j$
- Satisfy $\{\chi_i, \chi_j\} = \chi_i \chi_j + \chi_j \chi_i = 2\delta_{ij}$
  - Because $\sigma_i \sigma_j = \delta_{ij} 1 + \mathrm{i} \sum_k \epsilon_{ijk} \sigma_k$

Density of states



Density of states: softer edge





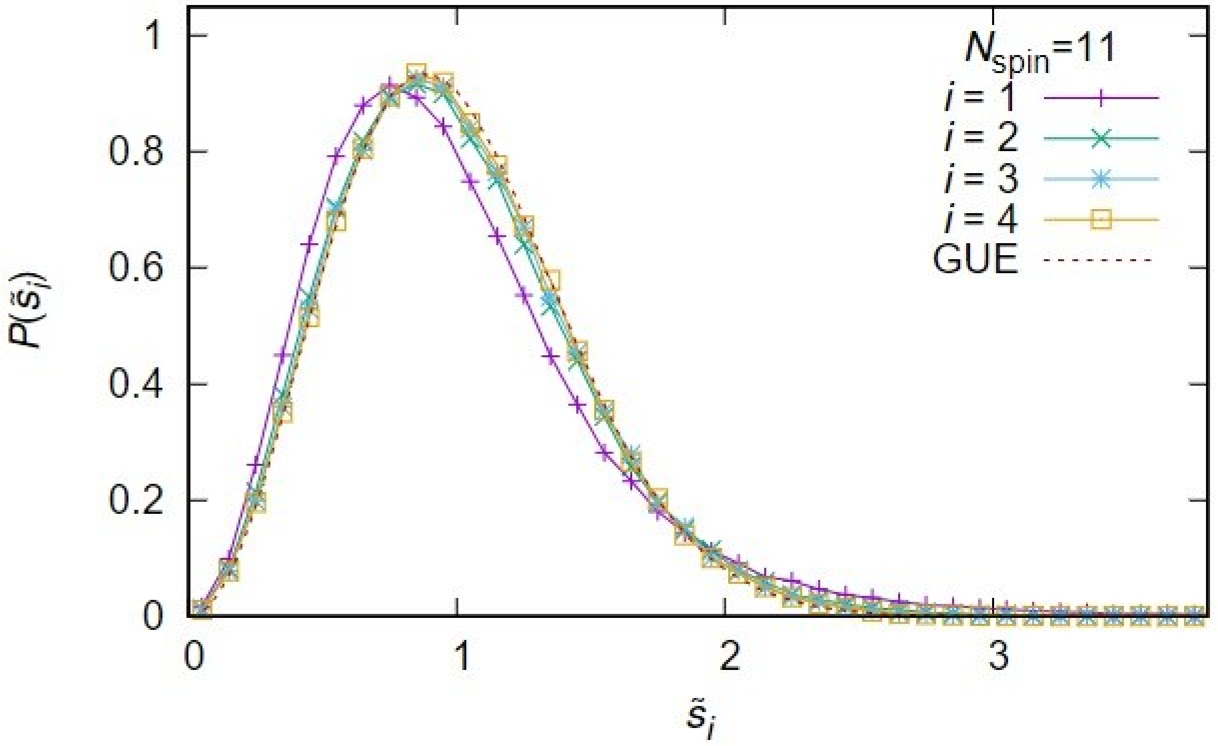
ρ(E')
E' = E - E0,sample
Nspin = 12, NMaj = 24
SpinXY4
SYK
Nspin = 13, NMaj = 26
SpinXY4
SYK
Nspin = 14, NMaj = 28
SpinXY4
SYK4
Nspin = 15, NMaj = 30
SpinXY4
SYK4

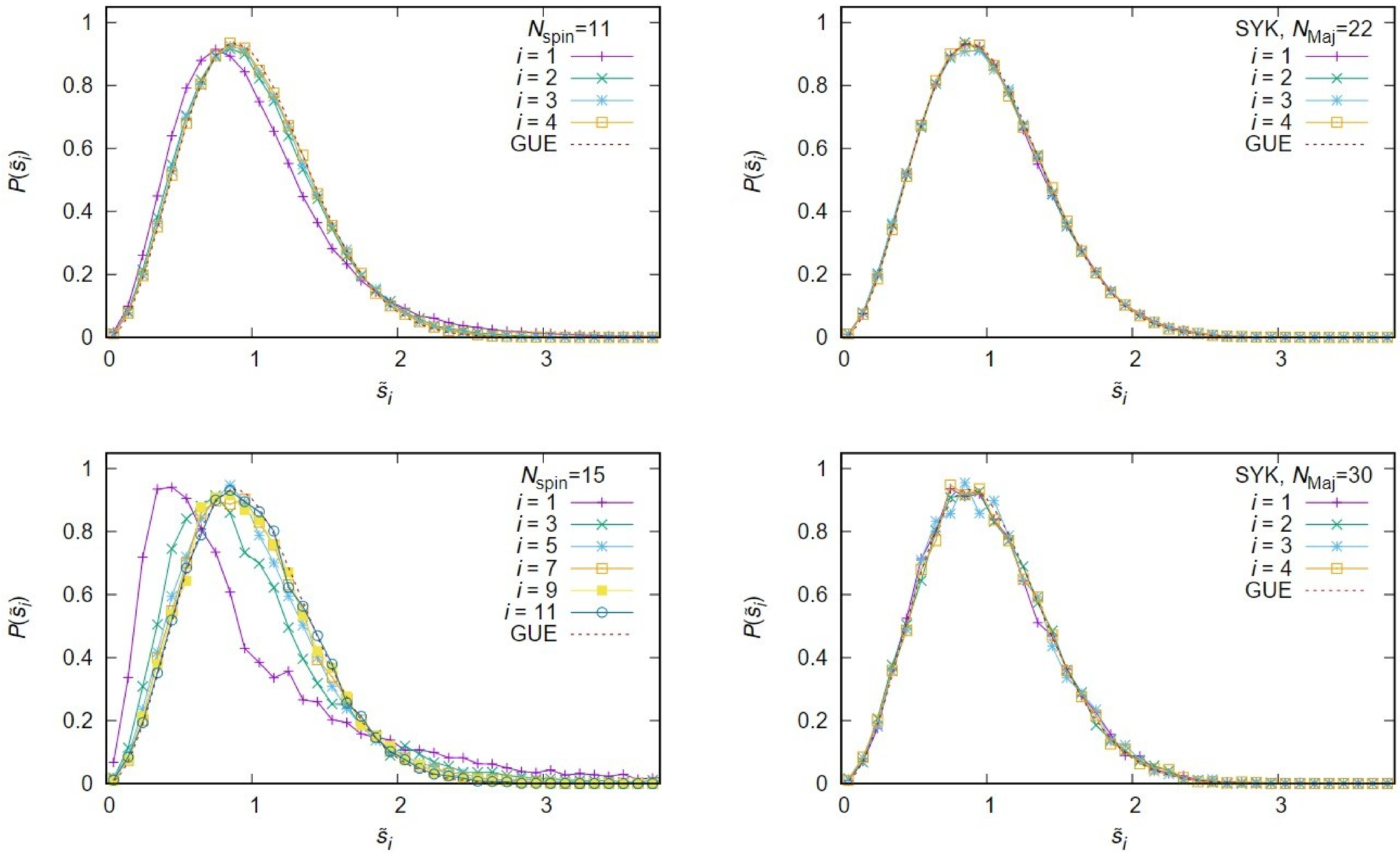
Level spacing and correlations

Eigenstate energies in one parity sector: $E_1 < E_2 < E_3 < \cdots < E_{2^{N_{\mathrm{spin}}-1}}$

Level spacings: $s_1 = E_2 - E_1, s_2 = E_3 - E_2, s_3 = E_4 - E_3, \ldots$

- Compare against random-matrix results (No particular symmetry: GUE)
- "Fixed-i" unfolding: $\tilde{s}_i = s_i / \langle s_i \rangle_{\{J\}}$
- Average of $\tilde{s}_i = 1$
- GUE: $P(s) \propto s^2$ for $s \ll 1$,
  $P(s) \sim e^{-s^2}$ for $s \gg 1$





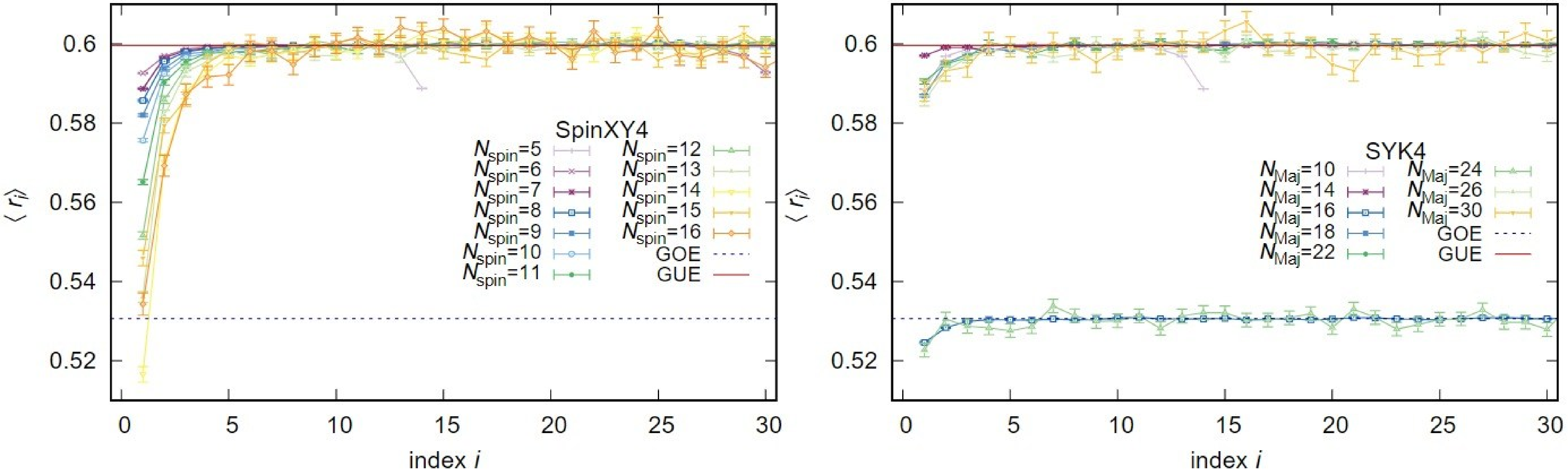
N_{spin}=11
i = 1
i = 2
i = 3
i = 4
GUE
$P(\tilde{s}_i)$
$\tilde{s}_i$
SYK, N_{Maj}=22
i = 1
i = 2
i = 3
i = 4
GUE
$P(\tilde{s}_i)$
$\tilde{s}_i$
N_{spin}=15
i = 1
i = 3
i = 5
i = 7
i = 9
i = 11
GUE
$P(\tilde{s}_i)$
$\tilde{s}_i$
SYK, N_{Maj}=30
i = 1
i = 2
i = 3
i = 4
GUE
$P(\tilde{s}_i)$
$\tilde{s}_i$

Neighboring gap ratio

$$r_i = \frac{\min(\tilde{s}_i, \tilde{s}_{i+1})}{\max(\tilde{s}_i, \tilde{s}_{i+1})}$$

$$\langle r \rangle = \begin{cases} 2\log 2 - 1 = 0.38629\ldots \ (\text{Poisson}) \\ 0.5307(1)\ (\text{GOE}) \\ 0.5997504209(1)\ (\text{GUE}) \\ 0.6744(1)\ (\text{GSE}) \end{cases}$$

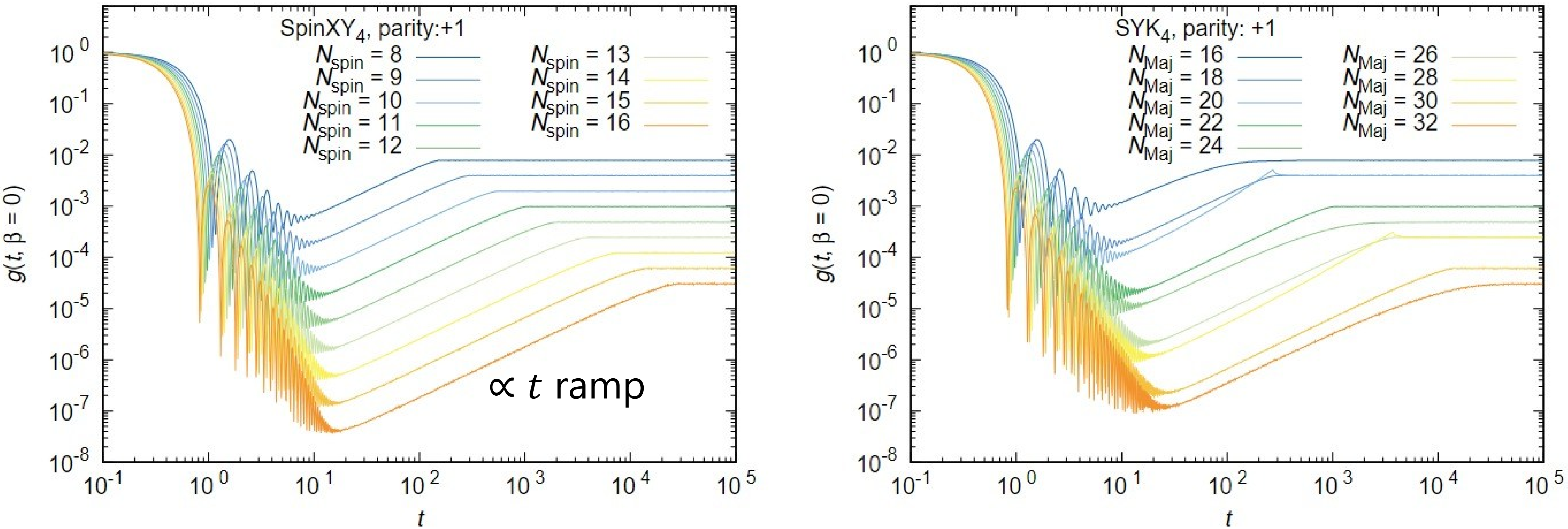
[Y. Y. Atas et al., PRL 2013]
[S. M. Nishigaki, PTEP 2024]



Spectral form factor

N_{Maj} mod 8	0	2	4	6
SpinXY4	GUE	GUE	GUE	GUE
SYK_4	GOE	GUE	GSE	GUE

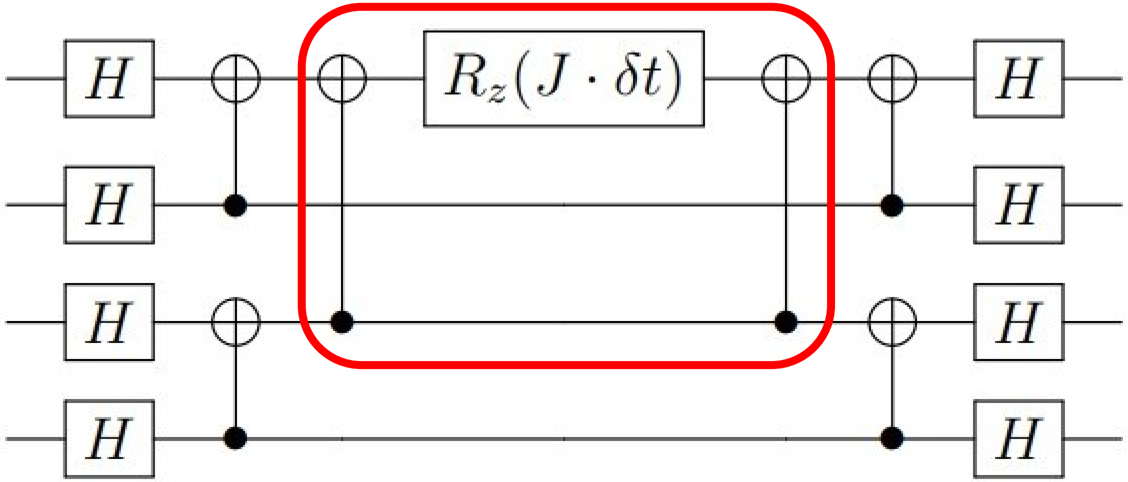
$$g(t,\beta) = \frac{\langle |Z(t,\beta)|^2 \rangle_J}{\langle |Z(0,\beta)|^2 \rangle_J}, Z(t,\beta) = \sum_j \exp\left(-(\beta + it)E_j\right).$$



M. Hanada, A. Jevicki, X. Liu, E. Rinaldi, and M. Tezuka, JHEP**05**(2024)280.

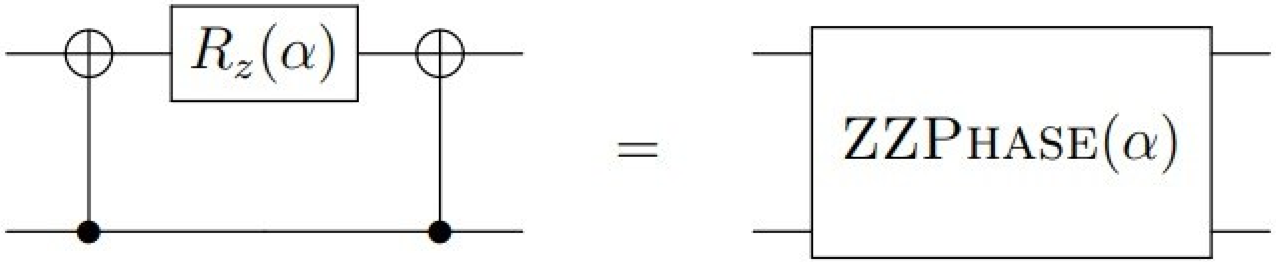
Toward quantum simulation

Code the Hamiltonian time evolution into a circuit using single-qubit and two-qubit quantum gates.
Example: $\widehat{U} = e^{-iJ\delta t\hat{\sigma}_{1,x}\hat{\sigma}_{2,x}\hat{\sigma}_{3,x}\hat{\sigma}_{4,x}}$ for $J\delta t \ll 1$



H: Hadamard gate
$R_z(\alpha) = e^{-\frac{1}{2}i\alpha\hat{\sigma}_z}$

Quantinuum H-series:
600-1000 2-qubit gates
[K. Yamamoto et al.
Phys. Rev. Res. **6**, 013221 (2024)]



Singular value statistics in non-Hermitian SYK

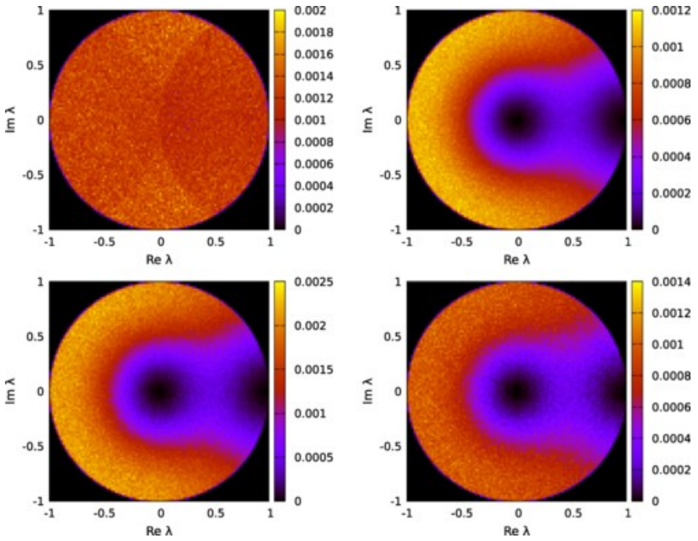
Non-Hermitian Hamiltonian: studied as an effective theory for open quantum systems

- Eigenvalues are complex-valued
- 38-fold symmetry classes (Hermitian: 10-fold Altland-Zirnbauer classes)

Bernard & LeClair 2002; Kawabata, Shiozaki, Ueda, & Sato PRX 2019

Complex eigenvalue statistics

- Distance and angle between nearest neighbors
- Two-dimensional distributions



[A. M. García-García, L. Sá, and J. J. M. Verbaarschot, PRX **12**, 021040 (2022)]

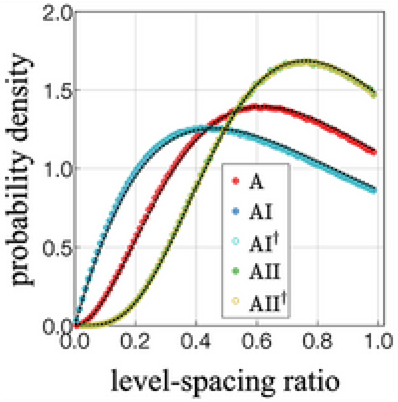
Complex spacing ratio for $2N = 20, q = 2, 3, 4, 6$

Singular value statistics

[Kawabata, Xiao, Ohtsuki, and Shindou, PRX Quantum **4**, 040312 (2023)]

- Singular values are non-negative
- One-dimensional distribution

Singular value decomposition (SVD) $H = U\Lambda V^\dagger$
U, V: unitary, $\Lambda \geq 0$: diagonal
Singular values of H:
|eigenvalues| of $\widetilde{H} \equiv \begin{pmatrix} 0 & H \\ H^\dagger & 0 \end{pmatrix}$



Sparse non-Hermitian SYK model

$$H_{\mathrm{nSYK}}^{\mathrm{sparse}} = \sum_{1 \le a<b<c<d \le N} x_{abcd}(J_{abcd} + \mathrm{i}\, M_{abcd})\psi_a\psi_b\psi_c\psi_d$$

$$\{\psi_a, \psi_b\} = \delta_{ab}$$

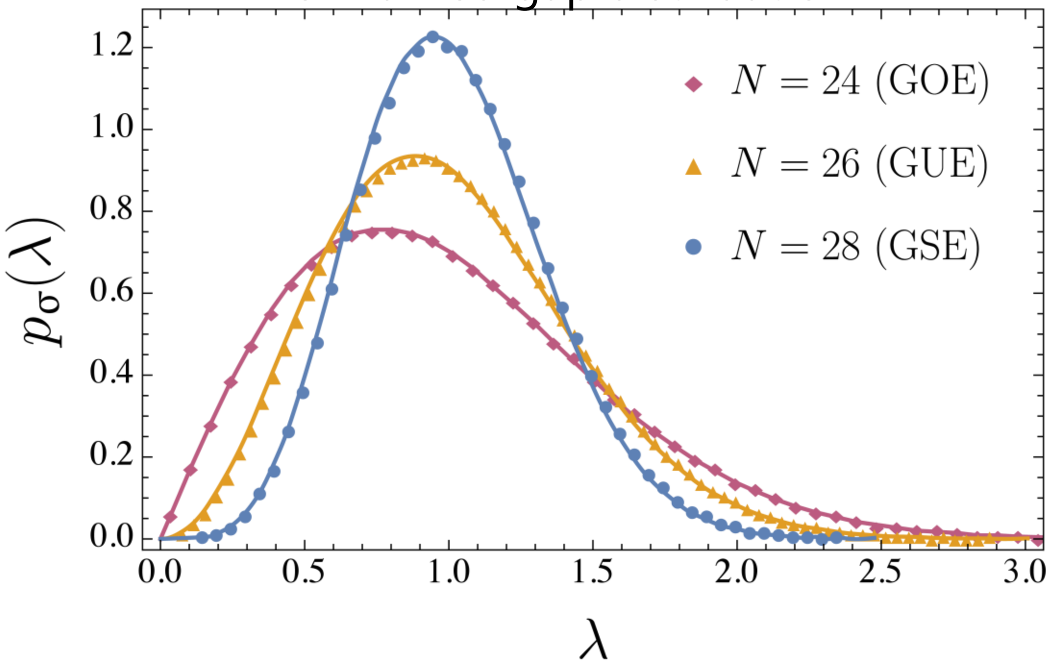
$$x_{abcd} = \begin{cases} 1 & (\text{probability } p) \\ 0 & (\text{probability } 1-p) \end{cases} \qquad \langle J_{abcd}^2 \rangle = \langle M_{abcd}^2 \rangle = \frac{6}{pN^3}$$

The dense case ($p = 1$) is random-matrix like

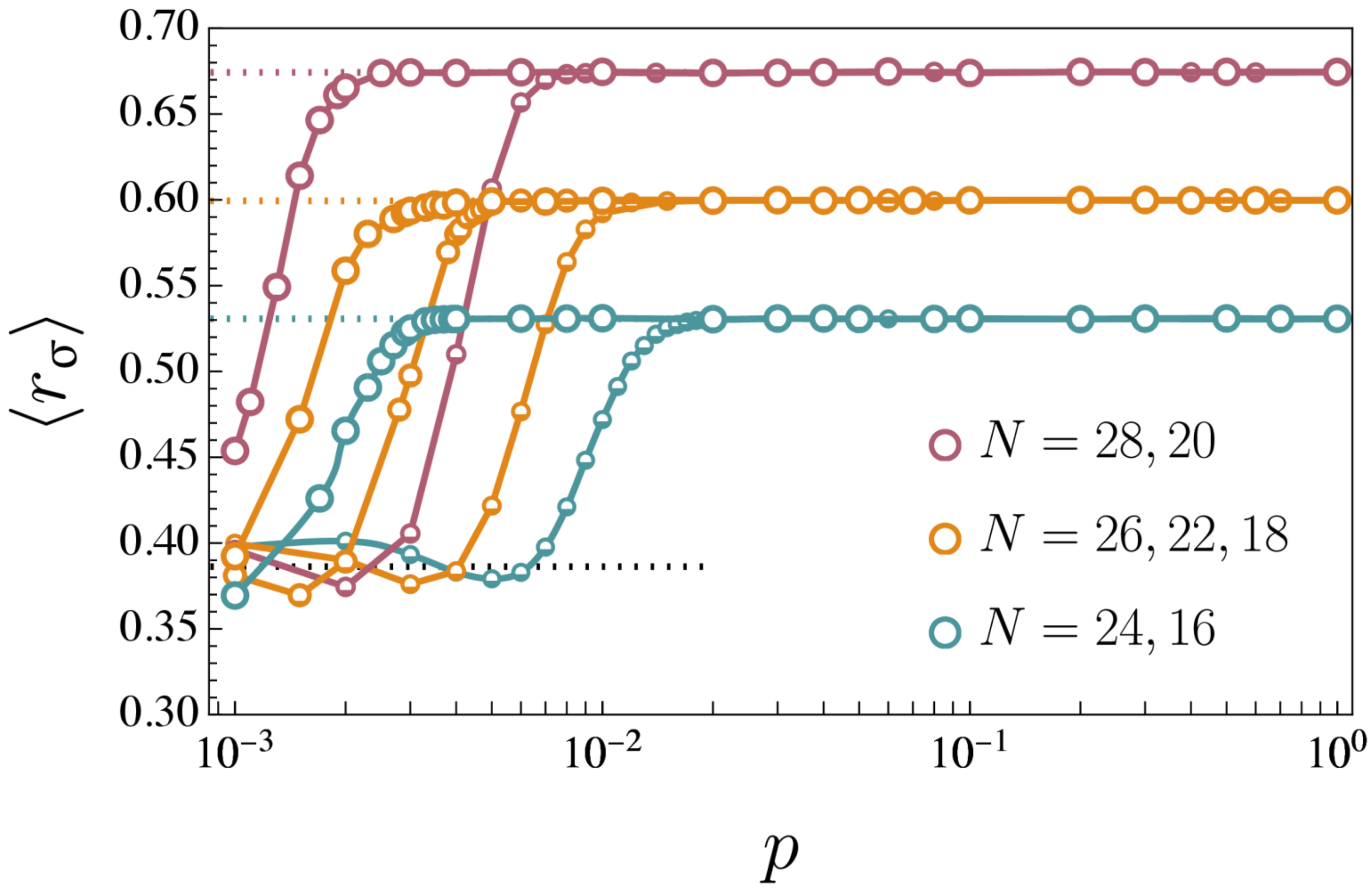
Averaged neighboring gap ratio

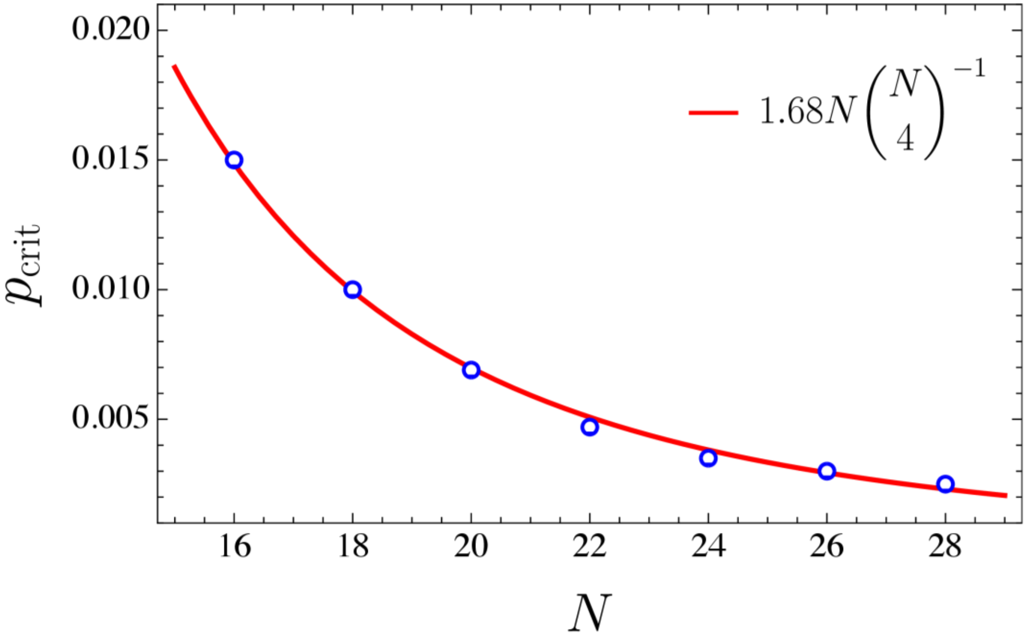
System	$N = 20$	$N = 22$	$N = 24$	$N = 26$	$N = 28$	$N = 30$
$\langle r \rangle_{\mathrm{RMT}}$	0.6744	0.5996	0.5307	0.5996	0.6744	0.5996
$\langle r_\sigma \rangle_{\mathrm{nSYK}}$	0.6744	0.5997	0.5307	0.5996	0.6745	0.5995

Normalized gap distribution



Gap ratio for singular value spectrum





Similar scaling to the Hermitian case

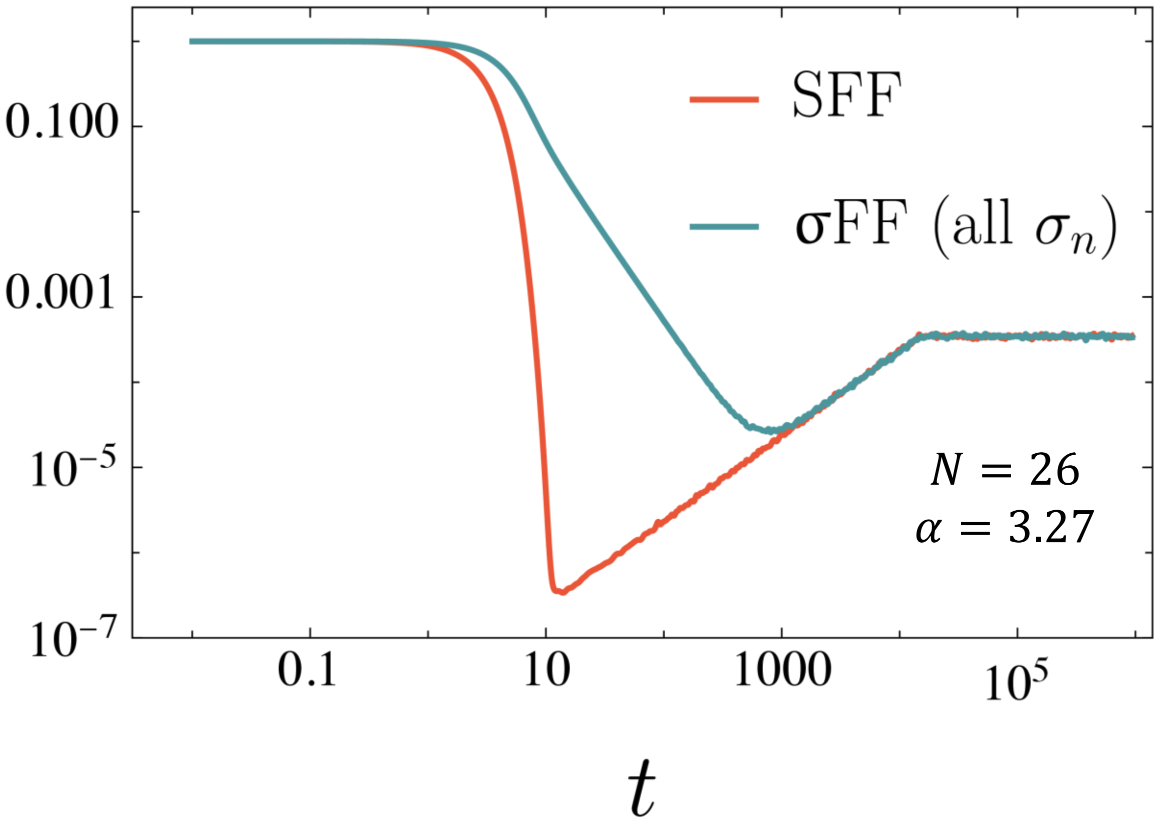
P. Nandy, T. Pathak, and M. Tezuka, Phys. Rev. B **111**, L060201 (2025)

Singular form factor σFF

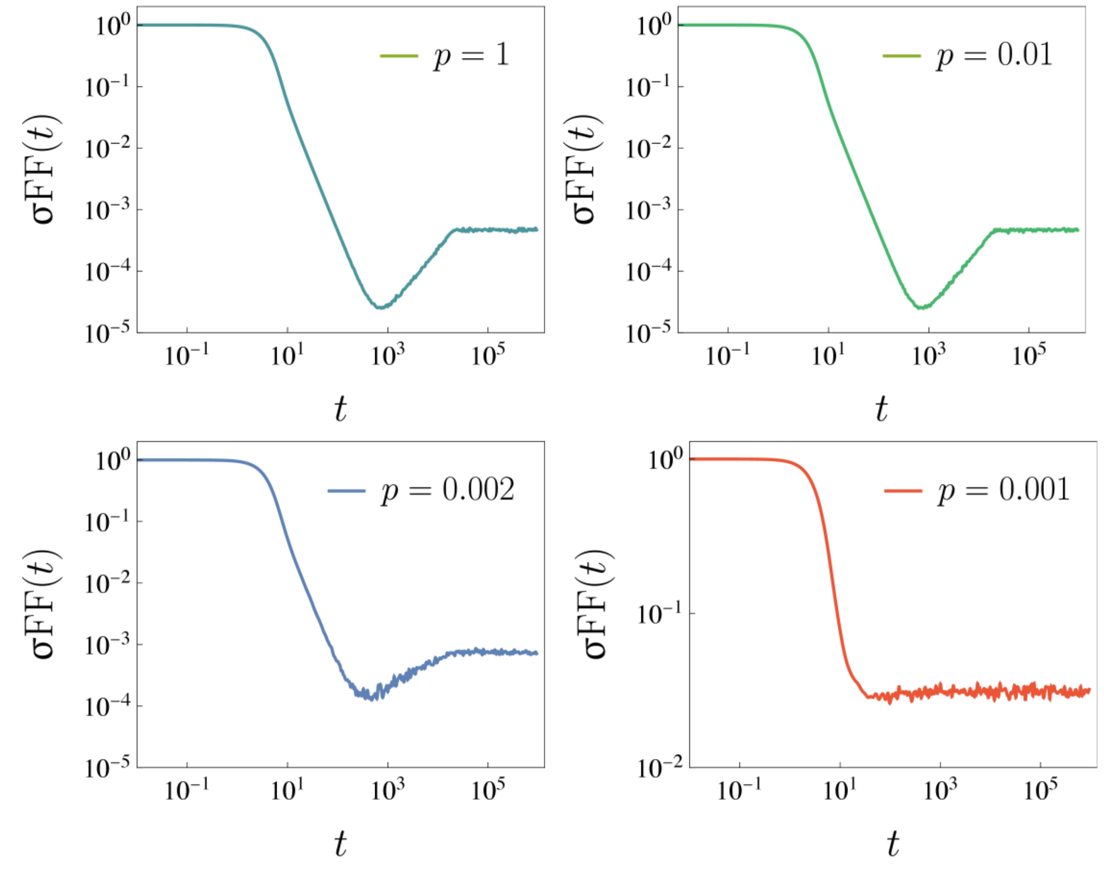
$$\sigma\mathrm{FF}(t) = \left\langle \frac{|Y_\sigma(\alpha,t)|^2}{|Y_\sigma(\alpha,0)|^2} \right\rangle, Y_\sigma(\alpha,t) = \sum_n e^{-\alpha\sigma_n^2 - \mathrm{i}\sigma_n t}$$

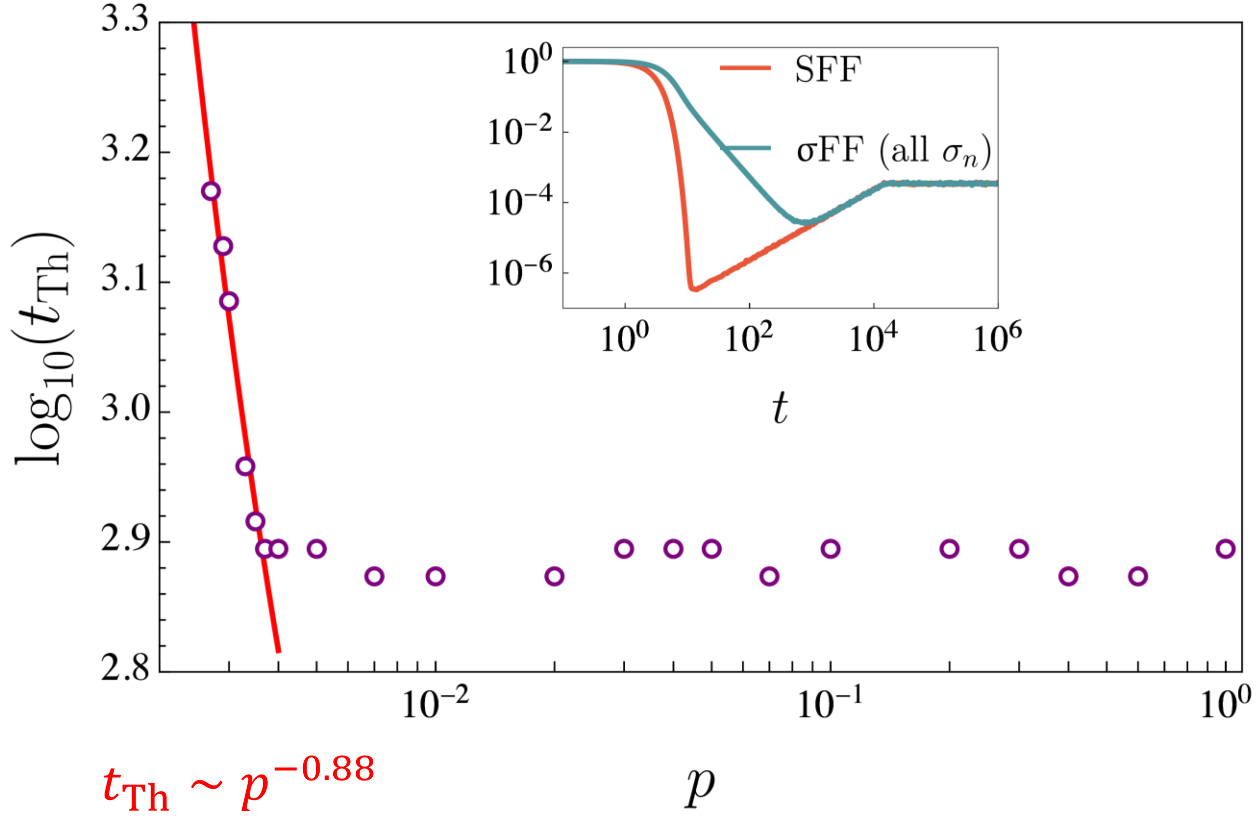
α: filtering parameter

Hermitian SYK



Singular form factor: ramp time vs p





$t_{\mathrm{Th}} \sim p^{-0.88}$

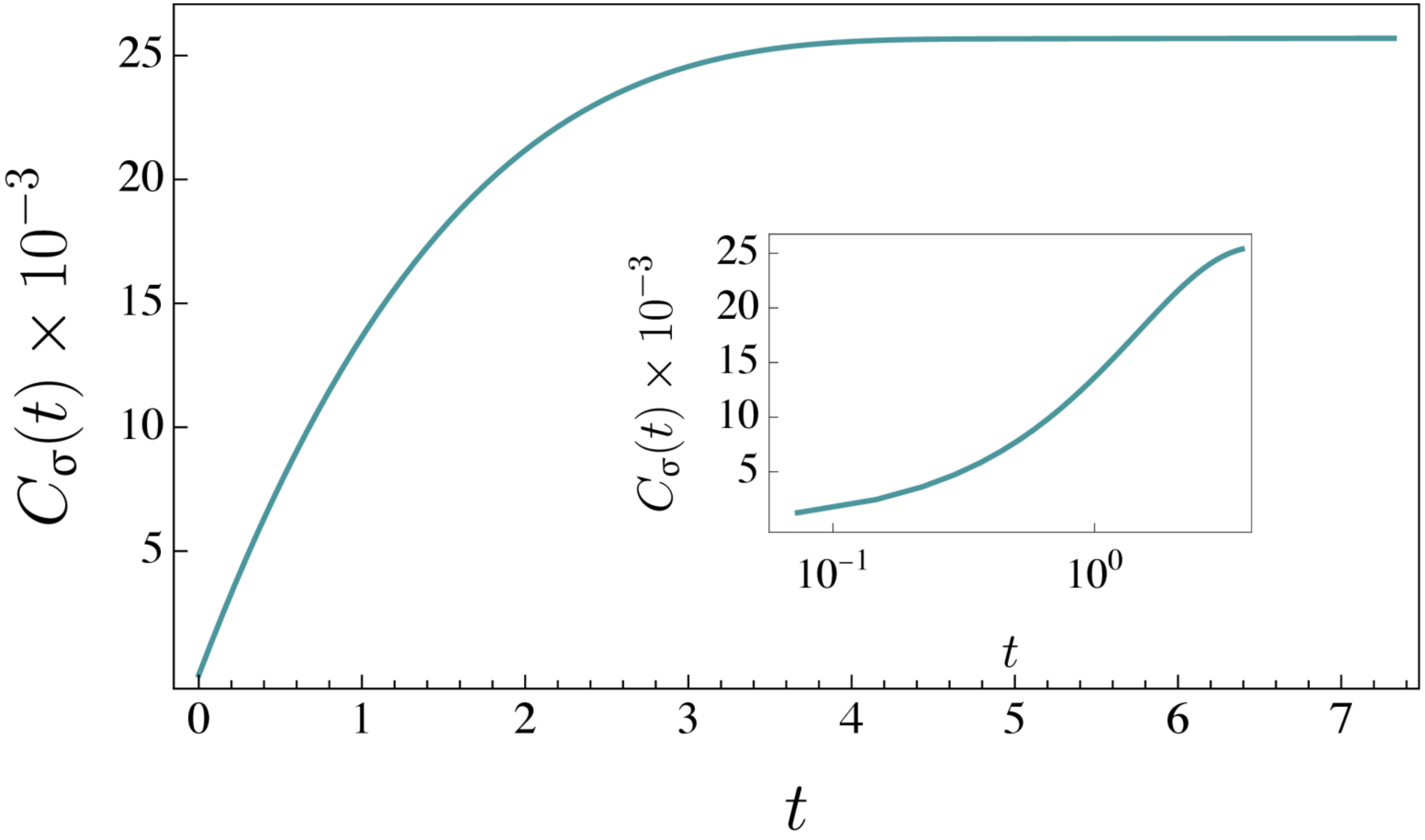
($t_{\mathrm{Th}} \sim p^{-2}$ for Hermitian.

[Orman, Gharibyan, and Preskill, arXiv: 2403.13884])

Singular complexity

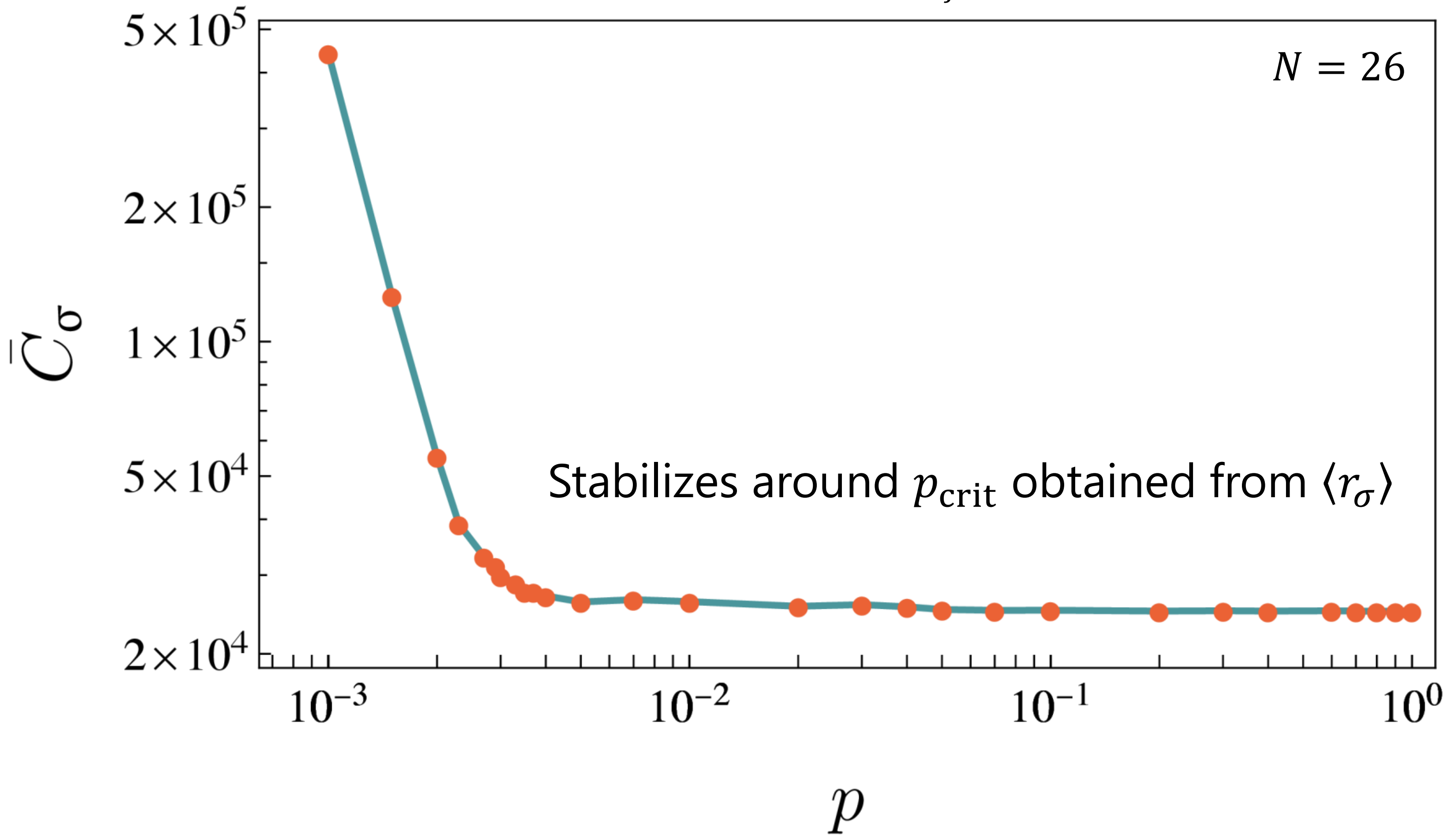
$$C_\sigma(t) = \frac{1}{L^2} \sum_{\epsilon_i \neq \epsilon_j} \frac{\sin^2 \frac{t(\sigma_i - \sigma_j)}{2}}{\left[\frac{\sigma_i - \sigma_j}{2}\right]^2}$$

Defined in an analogy to the **spectral complexity** proposed to be a dual quantity of the Einstein-Rosen bridge [L. V. Iliesiu, M. Mezei & G. Sárosi, JHEP07(2022)073]



Dependence of late-time complexity on p

$$\bar{C}_\sigma = \lim_{t_f\to\infty} \frac{1}{t_f}\int_0^{t_f} C_\sigma(t)dt \;= \frac{2}{L^2}\sum_{\sigma_i\neq\sigma_j} \frac{1}{\left(\sigma_i-\sigma_j\right)^2}$$



Summary

• Binary sparse SYK

$\mathcal{O}(N)$ terms sufficient for RMT-like spectral correlation
M. Tezuka, O. Oktay, E. Rinaldi, M. Hanada, and F. Nori, Phys. Rev. B **107**, L081103 (2023)

Sachdev-Ye-Kitaev (SYK) model

$$\hat{H} = \sum_{1 \le a<b<c<d \le 2N} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d$$

Gaussian random distribution

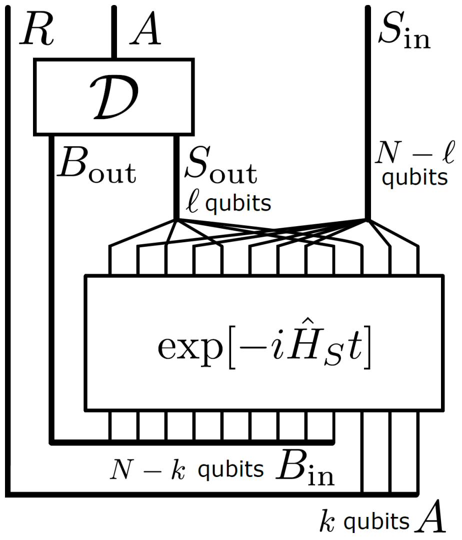
Majorana fermions

• Quantum error correction in SYK-like models

Decoding error estimate:

- **Exponentially small** as ℓ is increased after short time for **SYK** and **binary-coupling sparse SYK**, if spectrum is RMT-like (with $\mathcal{O}(N)$ terms)
- **Does not become small** for **SYK$_{4+2}$**, even after long time, where eigenstate localization proceeds **before** spectral correlation departs from RMT

Hayden-Preskill protocol



Y. Nakata and M. Tezuka, PRR **6**, L022021 (2024).

• Randomly-coupled Pauli spins

$$\hat{H} \propto \sum_{a<b<c<d} i^{\eta_{abcd}} J_{abcd} \hat{O}_a \hat{O}_b \hat{O}_c \hat{O}_d \,,$$

$\hat{O}_{2j-1} = \hat{\sigma}_{j,x}$, $\hat{O}_{2j} = \hat{\sigma}_{j,y}$

Energy spectrum: mostly RMT statistics
(Ground state: spin-glass??)
Easier to implement in quantum computer

M. Hanada, A. Jevicki, X. Liu, E. Rinaldi, and M. Tezuka, JHEP**05**(2024)280.

• Sparse non-Hermitian SYK

Singular form factor and complexity ~ dense model for $\mathcal{O}(N)$ terms P. Nandy, T. Pathak, and M. Tezuka, Phys. Rev. B **111**, L060201 (2025)

Backup

Chaotic dynamics in quantum systems?

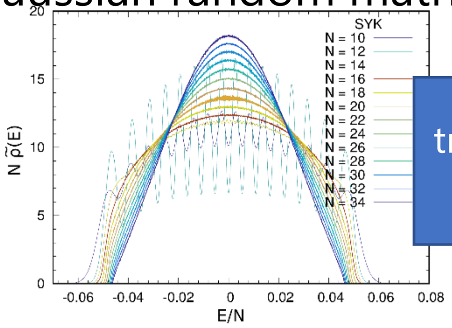
- (classical) Chaos: small change in initial condition leads to exponential difference at later time in deterministic dynamics
- **Quantum dynamics is linear in initial condition.**
  - $\frac{i}{\hbar}\frac{\partial}{\partial t}|\psi(t)\rangle = \widehat{\mathcal{H}}|\psi(t)\rangle$ should not change drastically if $\psi \mapsto \psi + \delta\psi$?
- Still, we can have **exponential decay** of (anti)commutators
  - $[A(t), B(t=0)] = A(t)B(0) - B(0)A(t)$
  - $|[A(t), B(t=0)]|^2 = \left(A(t)B(0) - B(0)A(t)\right)\left(A(t)B(0) - B(0)A(t)\right)^\dagger$
  - OTOC: $\left\langle\psi\middle||[A(t), B(t=0)]|^2\middle|\psi\right\rangle \simeq 1 - e^{2\lambda t}$ λ: Lyapunov exponent
- **Energy eigenvalues** of $\widehat{\mathcal{H}}$: have **random-matrix like correlation**
  - [Wigner][Berry and Tabor][Bohigas, Giannoni, Schmit] …
- Most quantum many-body systems are not integrable
  - Should be chaotic (after the symmetry sector is fixed)?

Chaos and scrambling in quantum many-body systems

<u>Energy spectrum in chaotic systems</u>
≈ random matrix level statistics

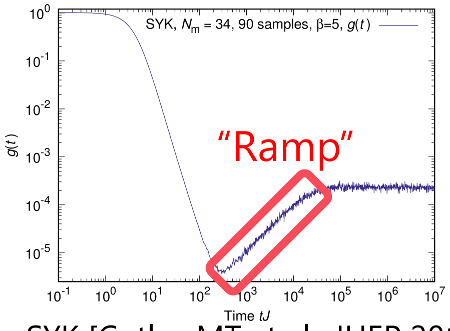
Wigner, …
BGS conjecture

Density of states:
not so universal
(cf. Wigner semi-circle law for Gaussian random matrices)



Fourier transformation of two-point function

Spectral form factor



SYK [Cotler, MT et al., JHEP 2017]

Neighboring gap ratio

$$r = \frac{\min(e_{i+1} - e_i\,,\ \ e_{i+2} - e_{i+1})}{\max(e_{i+1} - e_i\,,\ \ e_{i+2} - e_{i+1})}$$

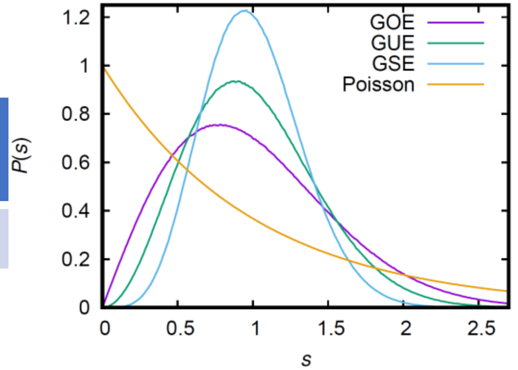
	Uncorrelated	GOE ($\mathbb{R}$)	GUE ($\mathbb{C}$)	GSE ($\mathbb{H}$)
$\langle r \rangle$	0.38629	0.5307	0.59975	0.6744

[Atas et al., PRL 2013]
[Nishigaki, PTEP 2024]

Normalized gap distribution

$s_j = \frac{e_{j+1} - e_j}{\Delta(\bar{e})}$: universal

G*E: Gaussian ensembles



$e^{-it\hat{H}}$
$\approx 1 - it\hat{H} - \cdots$

<u>Scrambling dynamics:</u>
Delocalization of quantum information
cf. OTOC, Lyapunov spectrum

$e^{-it\hat{H}}, |\epsilon_j| t \gg 2\pi$

$(\hbar = 1)$

Circular unitary ensemble
(level repulsion on unit circle)
not realized by t-independent
Hamiltonian time evolution

[D. A. Roberts and B. Yoshida, 1610.04903]

Early time — Late time

ΔE large — ΔE small

Majorana (real) fermions

- Particle = antiparticle
- Creation operator = annihilation operator $\chi_a^\dagger = \chi_a$
- Anticommutation relation $\{\chi_a, \chi_b\} \equiv \chi_a \chi_b + \chi_b \chi_a = 2\delta_{ab}$ in this talk
  - $\{\chi_a, \chi_b\} = \delta_{ab}$ is also used in the literature
- Two Majorana fermions correspond to one complex (Dirac) fermion
  - $\chi_+ = \hat{c} + \hat{c}^\dagger, \chi_- = i\left(\hat{c}^\dagger - \hat{c}\right) \Leftrightarrow \hat{c} = \frac{\chi_+ + i\chi_-}{2}, \hat{c}^\dagger = \frac{\chi_+ - i\chi_-}{2}$
  - $\{\chi_+, \chi_+\} = 2\chi_+^2 = 2\left\{\hat{c}, \hat{c}^\dagger\right\} = 1, \{\chi_-, \chi_-\} = 2\chi_-^2 = 2\left\{\hat{c}, \hat{c}^\dagger\right\} = 1, \{\chi_+, \chi_-\} = 0$
- Does not conserve the number of complex fermions
  - $\chi_a \chi_b$ conserves the parity (even or odd) of the number
- Topological superconductor, quantum spin liquid, …

Feynman diagrams

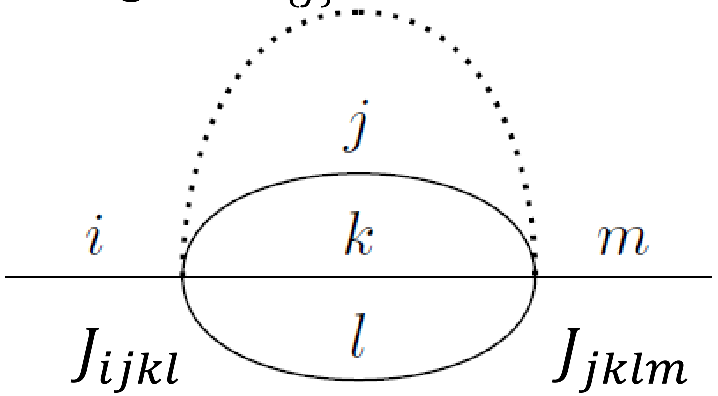
[J. Polchinski and V. Rosenhaus, JHEP **1604** (2016) 001]
[J. Maldacena and D. Stanford, PRD **94**, 106002 (2016)]

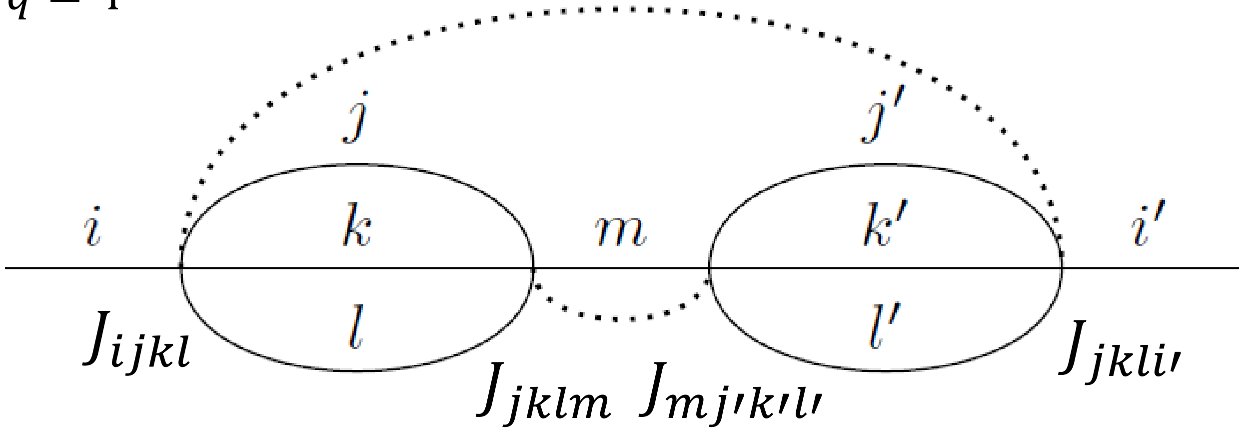
$$\hat{H}_{\mathrm{SYK4}} = \frac{\sqrt{3!}}{(2N)^{3/2}} \sum_{1\le a<b<c<d\le 2N} \underbrace{J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d}_{q\,=\,4}$$

$$\{\hat{\chi}_a, \hat{\chi}_b\} = 2\delta_{ab}$$

$$\langle J_{abcd}\rangle^2 = J^2 = 1$$

Sample average $\langle \cdots \rangle_{\{J\}}$





$$\sum_{jkl} \langle J_{ijkl} J_{jklm} \rangle_{\{J\}} \propto \frac{(2N)^3}{3!}\delta_{im}$$

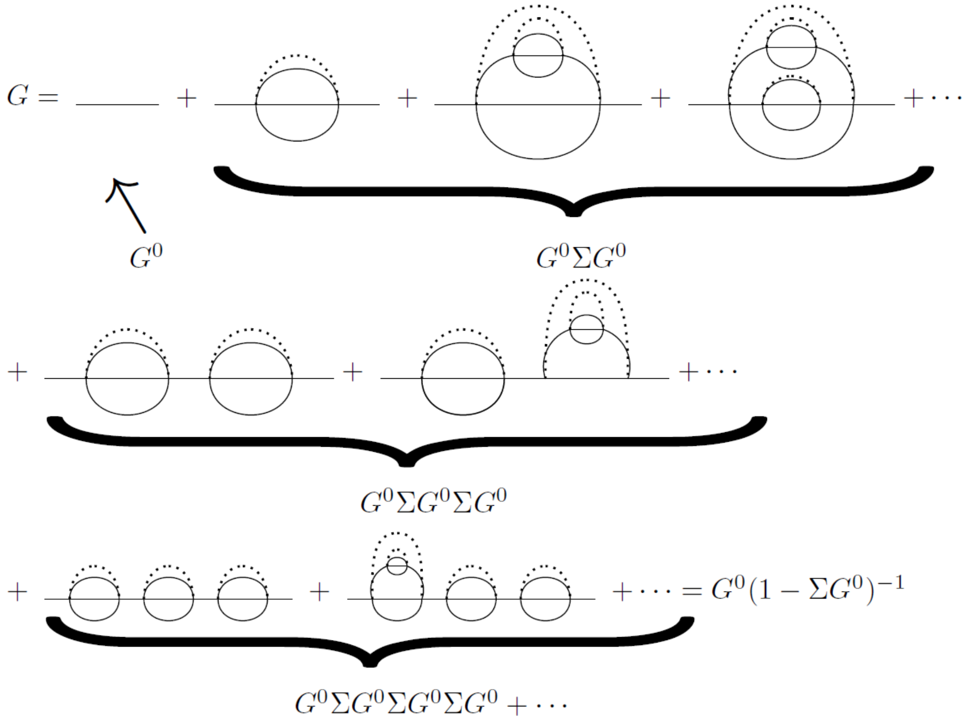
$$\sum_{m\neq i}\sum_{jklj'k'l'} \langle J_{ijkl} J_{jklm} J_{mj'k'l'} J_{j'k'l'i'} \rangle_{\{J\}} \propto N^4 \delta_{ii'}$$

➡ $O(N^0)$ contribution

➡ $O(N^{-2})$ contribution

Large-*N*: "Melon diagrams" dominate

Dominant diagrams in the $N \gg 1$ limit



[Sachdev and Ye 1993],
[Parcollet and Georges 1999], …

$$G(1 - \Sigma G_0) = G_0$$

$$G^{-1} = {G_0}^{-1} - \Sigma$$

$$G(i\omega)^{-1} = i\omega - \Sigma(i\omega)$$

↑ Figure from [I. Danshita, M. Tezuka, and M. Hanada: Butsuri **73**(8), 569 (2018)]

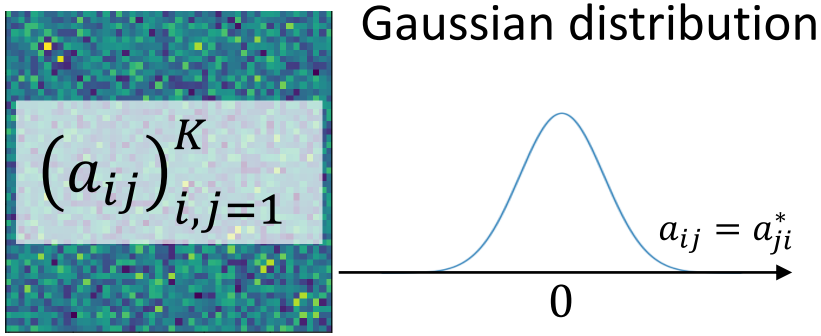
Low-energy behavior: as expected for a theory dual to 1+1d gravity

S. Sachdev, Phys. Rev. X **5**, 041025 (2015);
J. Maldacena and D. Stanford, Phys. Rev. D **94**, 106002 (2016);
Antal Jevicki, Kenta Suzuki, and Junggi Yoon, JHEP07(2016)007;

Jackiw-Teitelboim (JT) gravity: 1+1d dilaton gravity
near the horizon of a near-extremal black hole

Gaussian random matrices

Real ($\beta = 1$): Gaussian Orthogonal Ensemble (GOE)
Complex ($\beta = 2$): G. Unitary E. (GUE)
Quaternion ($\beta = 4$): G. Symplectic E. (GSE)



$$\text{Density} \propto e^{-\frac{\beta K}{4}\mathrm{Tr}H^2} = \exp\left(-\frac{\beta K}{4}\sum_{i,j}^{K}|a_{ij}|^2\right)$$

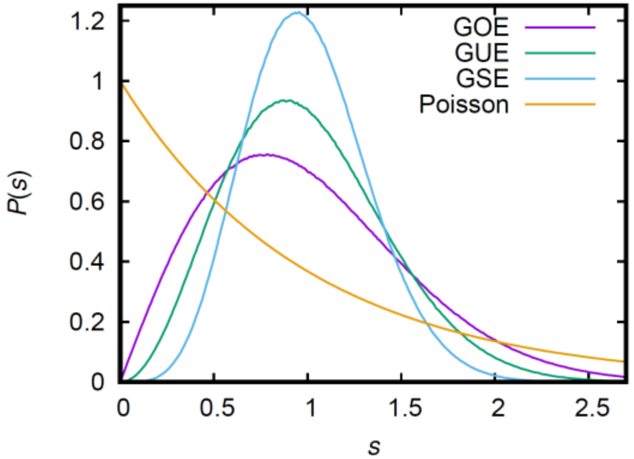
Joint distribution function for energy eigenvalues $\{e_j\}$

Level repulsion

$$p(e_1, e_2, \dots, e_K) \propto \prod_{1\le i<j\le K} |e_i - e_j|^{\beta} \prod_{i=1}^{K} e^{-\beta K e_i^2/4}$$

Distribution of normalized level separation $s_j = \frac{e_{j+1} - e_j}{\Delta(\bar{e})}$

GOE/GUE/GSE: $P(s) \propto s^{\beta}$ at small s, has e^{-s^2} tail



Uncorrelated: $P(s) = e^{-s}$ (Poisson distribution)

Neighboring gap ratio

$$r = \frac{\min(e_{i+1} - e_i\,,\ \ e_{i+2} - e_{i+1})}{\max(e_{i+1} - e_i\,,\ \ e_{i+2} - e_{i+1})}$$

	Uncorrelated	GOE	GUE	GSE
$\langle r \rangle$	2log 2 – 1 = 0.38629...	0.5307(1)	0.5997504209(1)	0.6744(1)

[Y. Y. Atas *et al.* PRL 2013]
[S. M. Nishigaki PTEP 2024]

➔ SYK model: level correlation ($P(s), P(r), \langle r \rangle$, etc.) indistinguishable from corresponding Gaussian ensemble

Majorana SYK4 with
$N \equiv 0 \pmod 8$: GOE
$N \equiv 2, 6 \pmod 8$: GUE
$N \equiv 4 \pmod 8$: GSE

[Fidkowski and Kitaev PRB 2010, 2011]
[You, Ludwig, and Xu PRB 2017]

Extra degeneracy for small $K_{\mathrm{cpl}} \lesssim N$

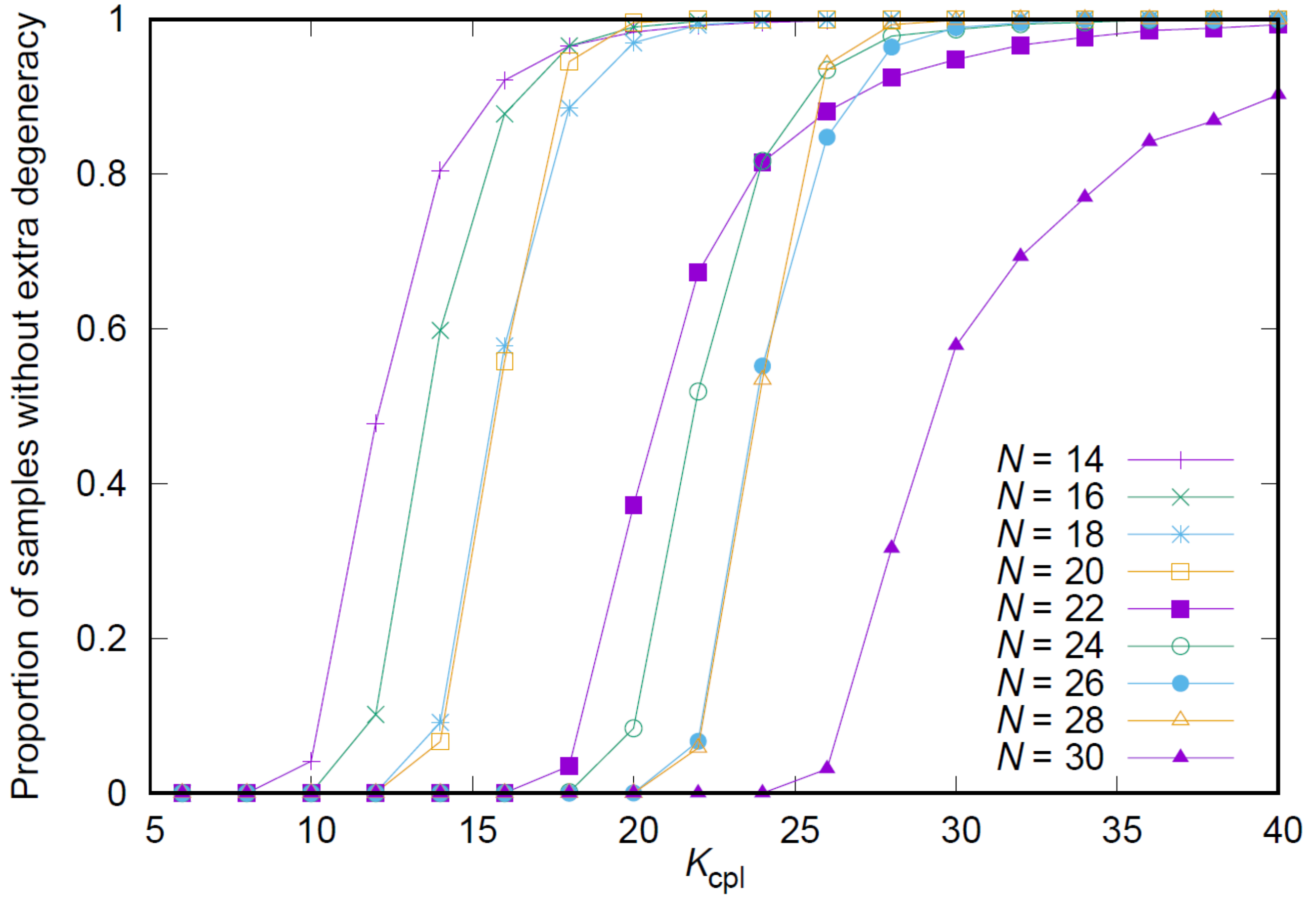
$$\hat{H} = C_{N,p} \sum_{1 \le a<b<c<d \le N} x_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d \qquad x_{abcd} = \begin{cases} 1 & (\text{probability } p/2) \\ -1 & (\text{probability } p/2) \\ 0 & (\text{probability } 1-p) \end{cases}$$

- If only few x_{abcd} are nonzero, some products of $\hat{\chi}_j$ can (anti)commute with the Hamiltonian [A. M. García-García *et al.*, PRD 2021]
- Simple example: if both $\hat{\chi}_{2k}$ and $\hat{\chi}_{2k+1}$ do not appear in $\hat{H}$
  - The state of the qubit k does not change the energy
  - Twofold extra degeneracy

In the following, we take $C_{N,p} = 1/\sqrt{K_{\mathrm{cpl}}}$ so that the variance of $\{\epsilon_j\}$ is 1 (rather than $\mathcal{O}(N)$):

$$\mathrm{Tr}\hat{H}^2 = C_{N,p}^2 \sum_{\substack{abcd \\ a'b'c'd'}} x_{abcd} x_{a'b'c'd'} \mathrm{Tr} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d \hat{\chi}_{a'} \hat{\chi}_{b'} \hat{\chi}_{c'} \hat{\chi}_{d'} = C_{N,p}^2 K_{\mathrm{cpl}} 2^{\frac{N}{2}} = 2^{\frac{N}{2}}.$$

$K_{\mathrm{cpl}} \gtrsim N$: extra degeneracy disappears



2^{24} eigenvalues ($2^{17} - 2^9$ samples)

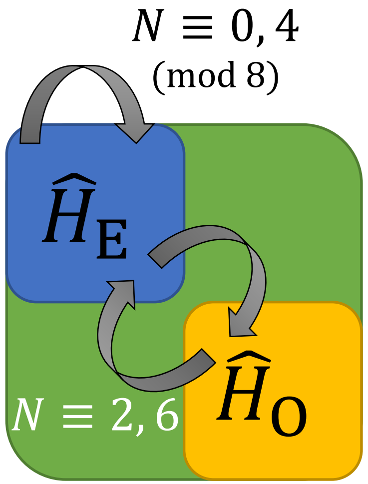
N mod 8 classification of Majorana SYK$_{q=4}$

$$\hat{H} = \frac{\sqrt{3!}}{N^{3/2}} \sum_{1\le a<b<c<d\le N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d$$

SPT phase classification for class BDI, 1D:
$\mathbb{Z} \rightarrow \mathbb{Z}_8$ due to interaction
[L. Fidkowski and A. Kitaev, PRB 2010, PRB 2011]

Introduce $N/2$ complex fermions $\hat{c}_j = \frac{(\hat{\chi}_{2j-1}+\mathrm{i}\hat{\chi}_{2j})}{\sqrt{2}}$

$\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d$ respects the complex fermion parity
Even ($\hat{H}_\mathrm{E}$) and odd ($\hat{H}_\mathrm{O}$) sectors: $L = 2^{N/2-1}$ dimensions



N mod 8	0	2	4	6
η	-1	+1	+1	-1
$\hat{X}^2$	**+1**	+1	**-1**	-1
$\hat{X}$ maps H_E to	H_E	H_O	H_E	H_O
Class	**AI**	**A+A**	**AII**	**A+A**
Gaussian ensemble	**GOE ($\mathbb{R}$)**	**GUE ($\mathbb{C}$)**	**GSE ($\mathbb{H}$)**	**GUE ($\mathbb{C}$)**

$$\hat{X} = \hat{K}\prod_{j=1}^{N/2}(\hat{c}_j^\dagger + \hat{c}_j) \qquad \hat{X}\hat{c}_j\hat{X} = \eta\hat{c}_j^\dagger \qquad [\hat{X}, \hat{H}] = 0$$

[Y.-Z. You, A. W. W. Ludwig, and C. Xu, PRB **95**, 115150 (2017)]; [F. Sun and J. Ye, PRL **124**, 244101 (2020)] for generic q and SUSY cases; …

Also see [A. M. Garcia-Garcia, L. Sa, J. J. M. Verbaarschot, PRX **12**, 021040 (2022)] for classification of non-hermitian SYK: 19 out of 38 [Kawabata-Shiozaki-Ueda-Sato] classes identified

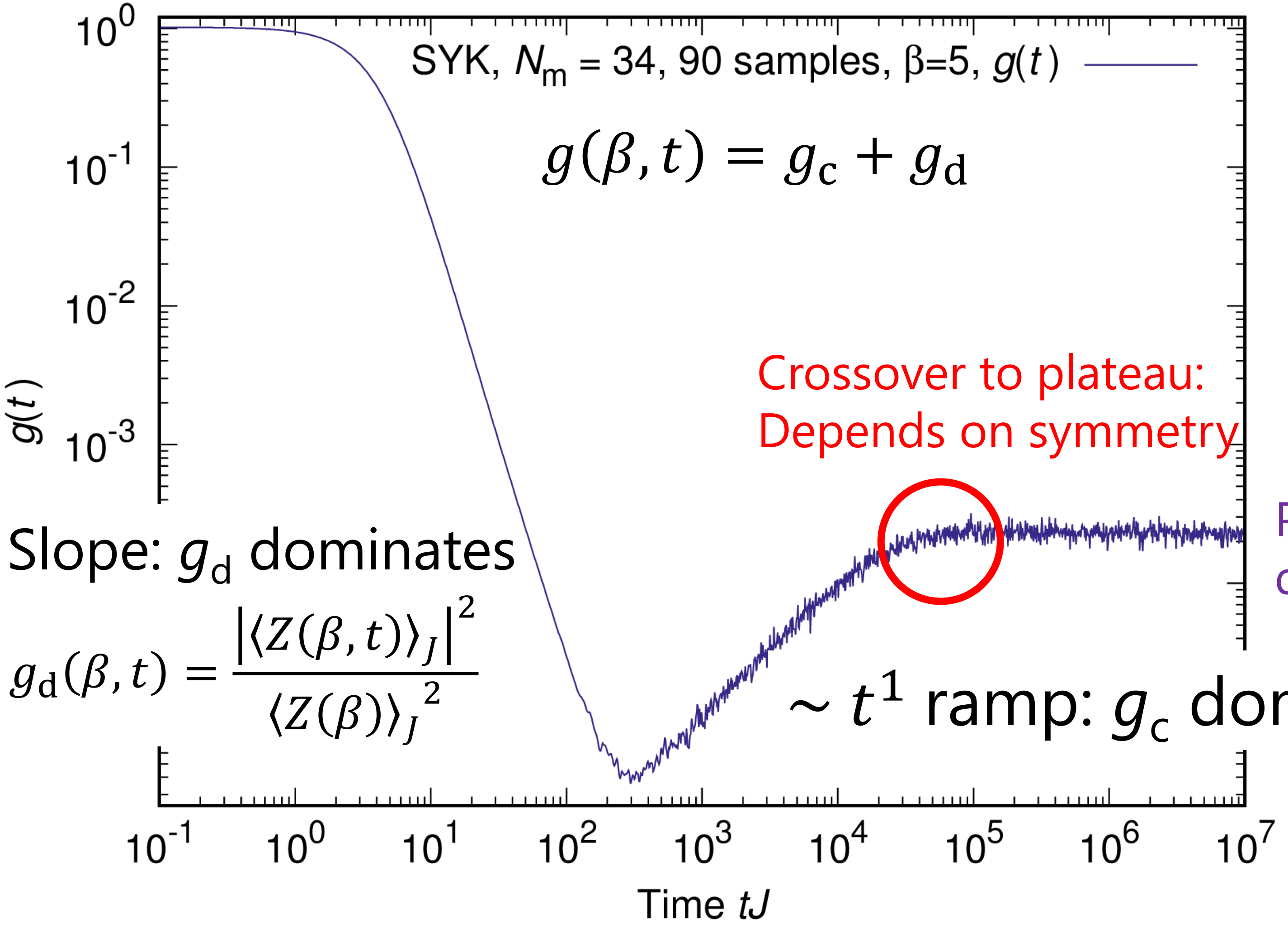
SYK: sparse matrix, but energy spectral statistics strongly resemble that of the corresponding (dense) Gaussian ensemble

[Cotler, …, MT, JHEP 2017]

Slope-dip-ramp-plateau structure

Cotler, Gur-Ari, Hanada, Polchinski, Saad, Shenker, Stanford, Streicher, and MT, JHEP
1705(2017)118

$$Z(\beta,t) = \mathrm{Tr}\left(e^{-\beta\hat{H}-i\hat{H}t}\right)$$



$$g(\beta,t) = g_c + g_d$$

Crossover to plateau:
Depends on symmetry

Slope: g_d dominates

$$g_d(\beta,t) = \frac{\left|\langle Z(\beta,t)\rangle_J\right|^2}{\langle Z(\beta)\rangle_J^{\;2}}$$

$\sim t^1$ ramp: g_c dominates

$$g_c(\beta,t) = \frac{\langle |Z(\beta,t)|^2\rangle_J - \left|\langle Z(\beta,t)\rangle_J\right|^2}{\langle Z(\beta)\rangle_J^{\;2}}$$

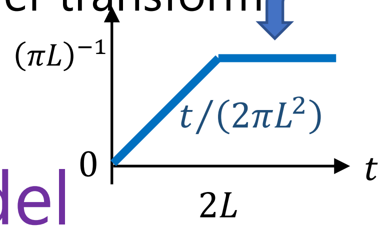
$$\sim \iint d\lambda_1 d\lambda_2 \langle \delta\rho(\lambda_1)\delta\rho(\lambda_2)\rangle e^{it(\lambda_1-\lambda_2)}$$

$$\rho(\lambda) = \sum_j \delta(\epsilon_j - \lambda)$$

Plateau height:
determined by degeneracy

$$R(\lambda) = \langle \delta\rho(\lambda_1)\delta\rho(\lambda_1-\lambda)\rangle = -\frac{\sin^2 L\lambda}{(\pi L\lambda)^2} + \frac{1}{\pi L}\delta(\lambda)$$

Fourier transform



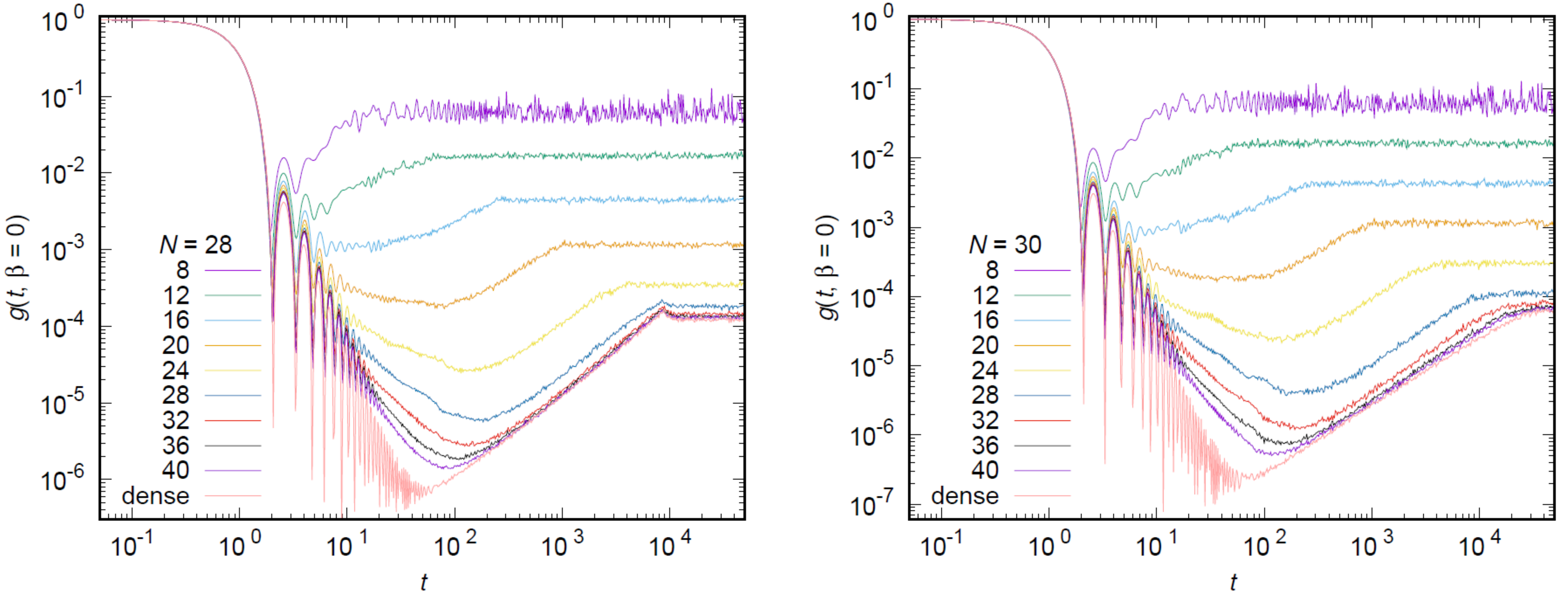
Random matrix theory
(GUE)

Exponentially long $\sim t^1$ ramp
➔ Rigid spectrum of the Sachdev-Ye-Kitaev model

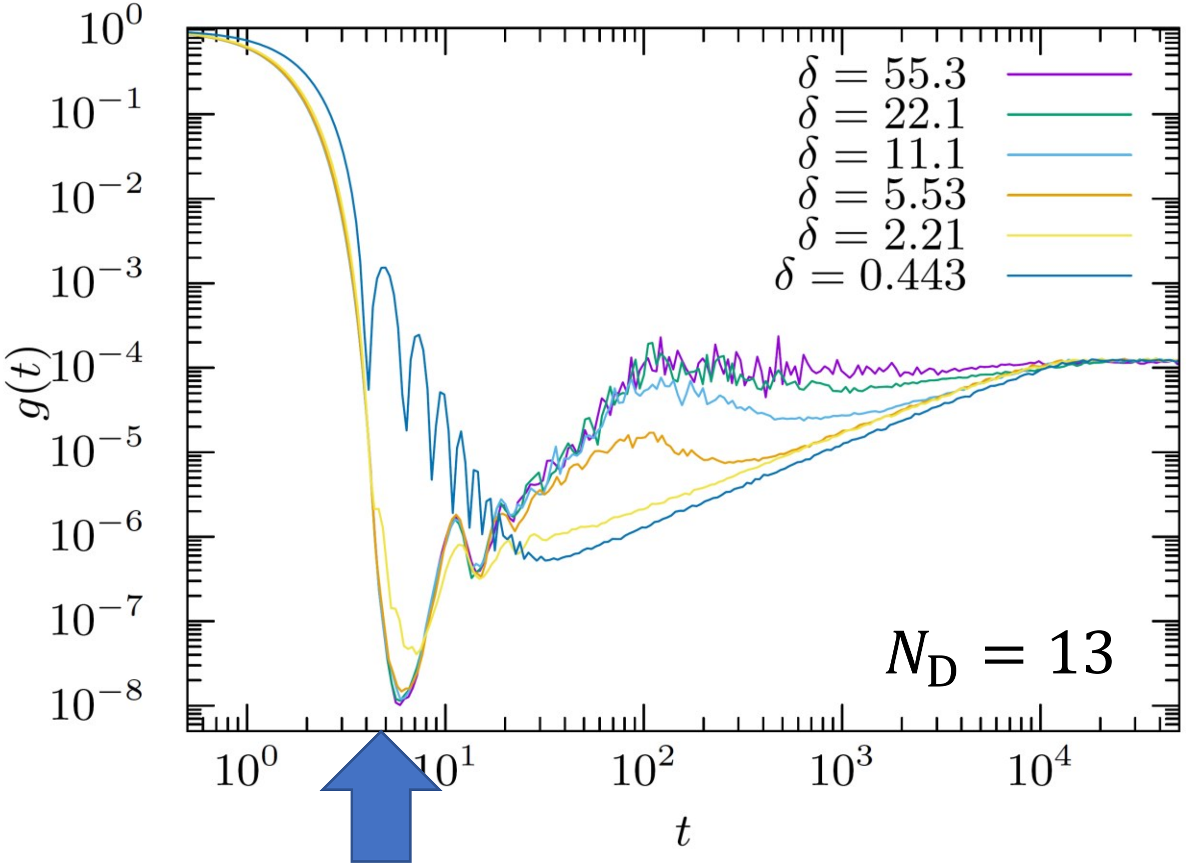
M. Tezuka, O. Oktay, E. Rinaldi, M. Hanada, and F. Nori, arXiv:2208.12098 (PRB Letter 2023)

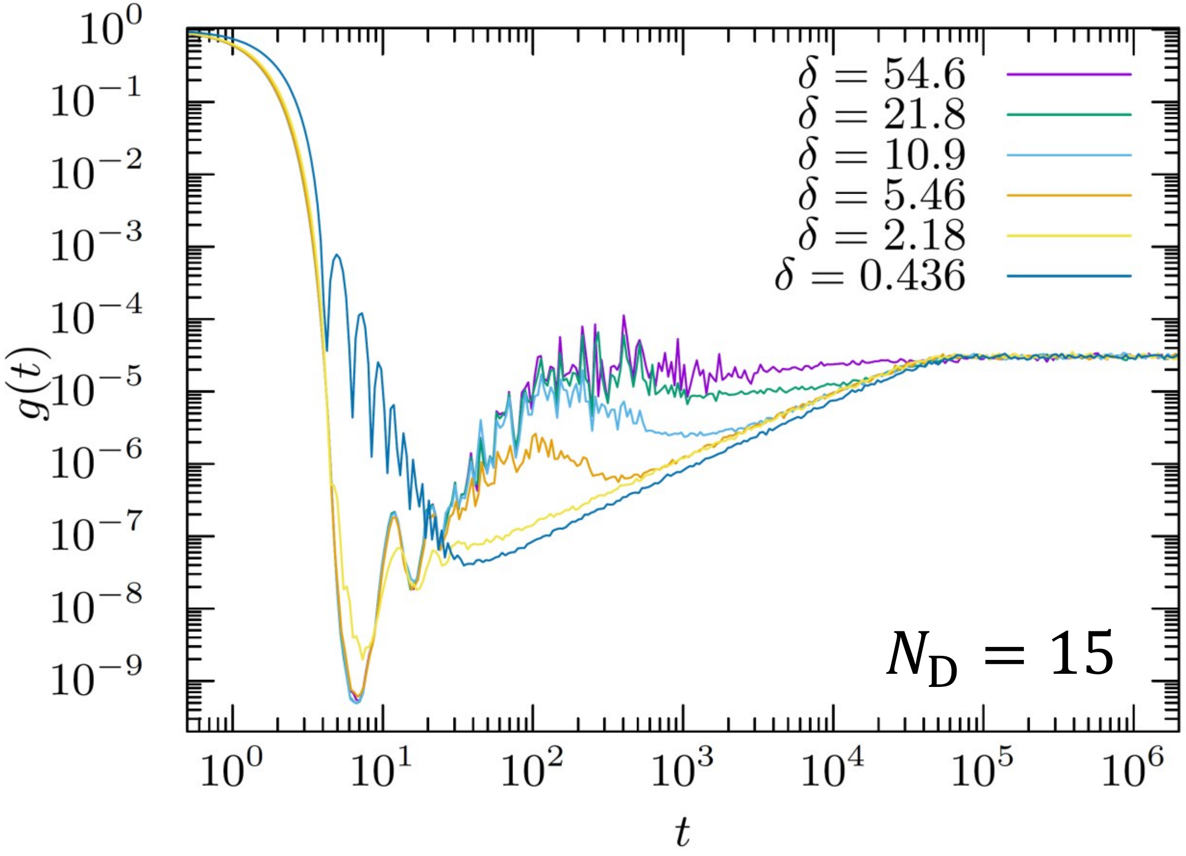
Spectral form factor

Clear ramp for $K_{\rm cpl} \gtrsim N$, coincides with the dense SYK as $N \to$ large



SYK$_{4+2}$: spectral form factor

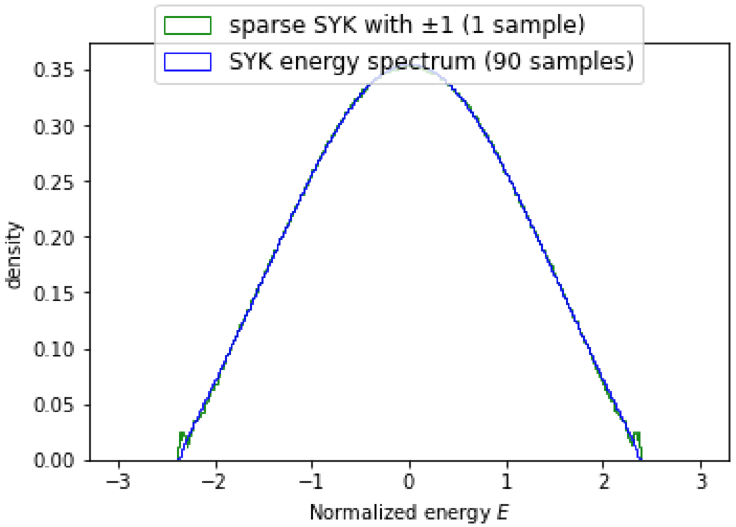


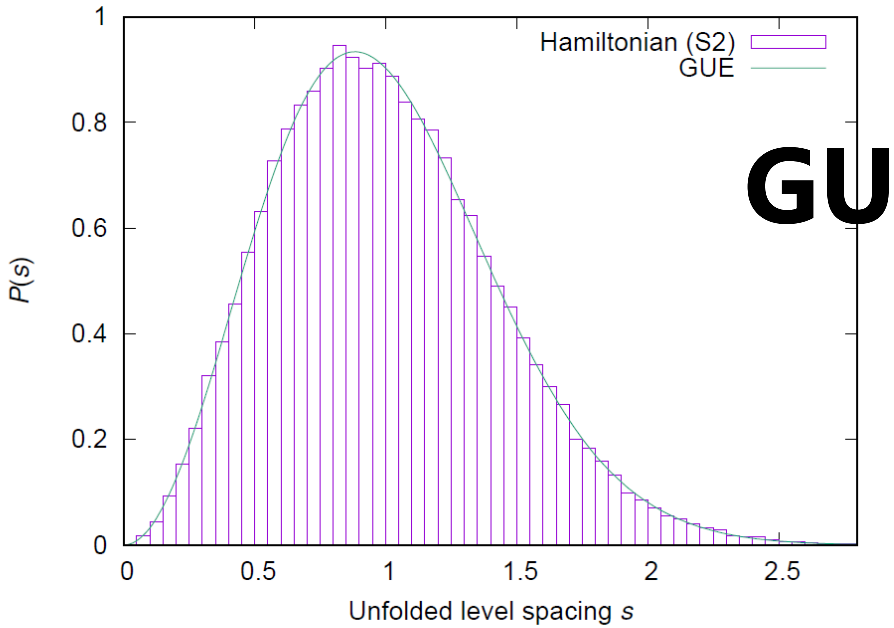


This dip (not directly followed by ramp) appears for SYK2 (+ uniform SYK4).
see 1812.04770 and 2003.05401 for detailed discussion

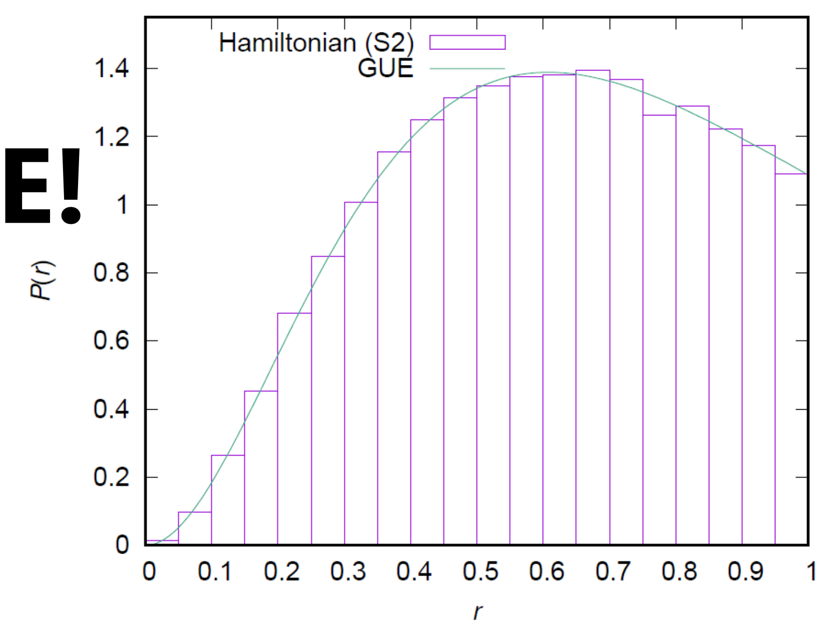
$\hat{H} = (\cos\theta)\hat{H}_{\mathrm{SYK}_4} + (\sin\theta)\hat{H}_{\mathrm{SYK}_2}, \delta = \tan\theta$
1.57×10^7 eigenvalues (1920 samples for $N_{\mathrm{D}} = 13$)

$2N = 34, K_{\mathrm{cpl}} = 36$, one sample





GUE!



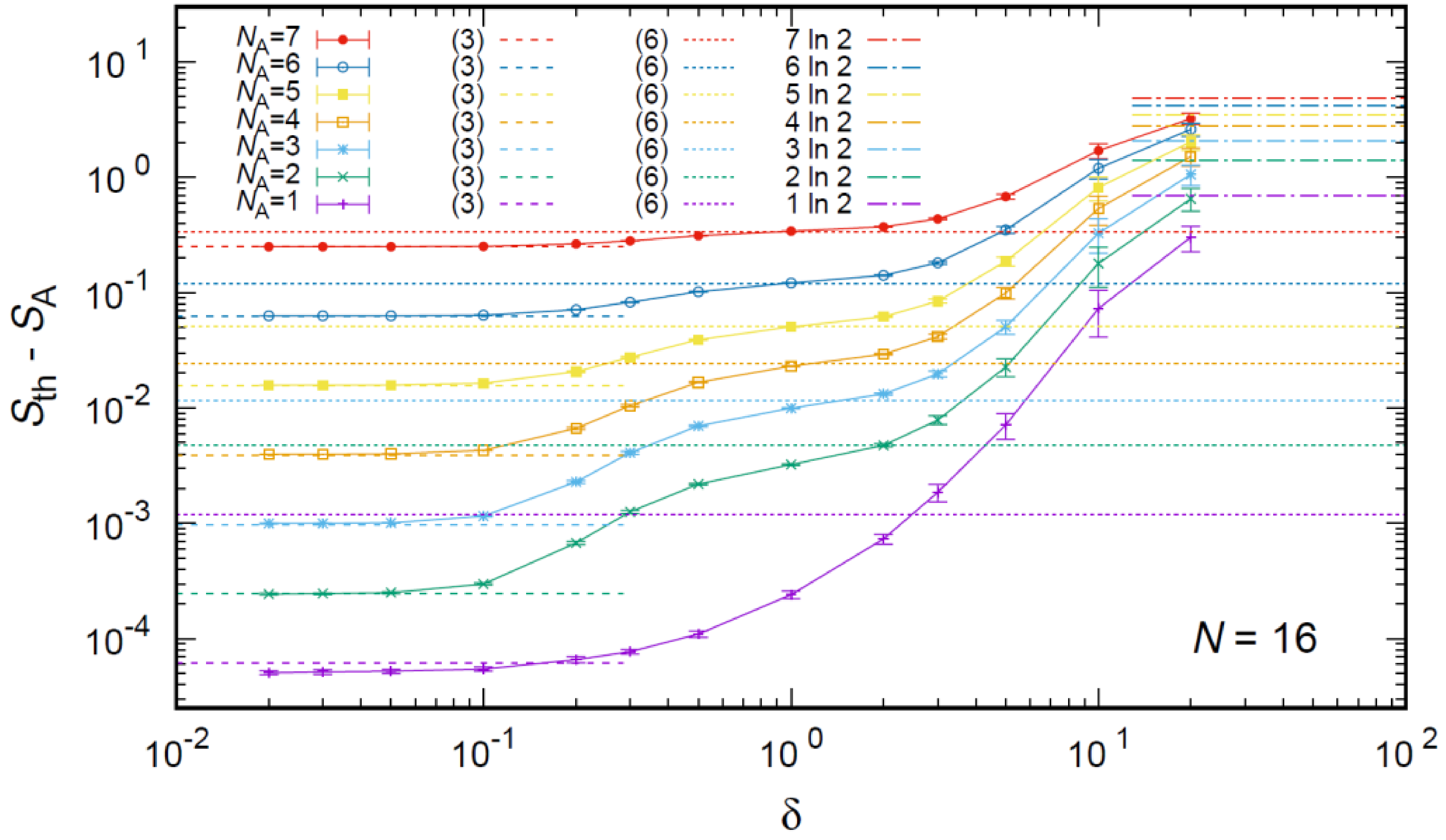
$$\begin{aligned}\mathcal{H} = {} & \chi_0\chi_5\chi_{19}\chi_{27} + \chi_0\chi_6\chi_{21}\chi_{23} - \chi_0\chi_9\chi_{14}\chi_{24} - \chi_0\chi_{14}\chi_{18}\chi_{30} - \chi_0\chi_{14}\chi_{20}\chi_{25} - \chi_1\chi_2\chi_{16}\chi_{22} \\ & + \chi_1\chi_{18}\chi_{22}\chi_{23} + \chi_2\chi_4\chi_5\chi_{15} + \chi_2\chi_{13}\chi_{16}\chi_{21} + \chi_2\chi_{14}\chi_{19}\chi_{24} + \chi_2\chi_{20}\chi_{27}\chi_{33} + \chi_2\chi_{22}\chi_{31}\chi_{32} \\ & + \chi_3\chi_4\chi_5\chi_{29} - \chi_3\chi_8\chi_{14}\chi_{28} - \chi_3\chi_8\chi_{29}\chi_{31} + \chi_3\chi_{21}\chi_{26}\chi_{29} - \chi_3\chi_{22}\chi_{25}\chi_{33} + \chi_4\chi_7\chi_{13}\chi_{30} \\ & - \chi_4\chi_9\chi_{14}\chi_{17} - \chi_5\chi_6\chi_{17}\chi_{29} + \chi_5\chi_{12}\chi_{29}\chi_{31} - \chi_5\chi_{13}\chi_{19}\chi_{24} - \chi_5\chi_{14}\chi_{22}\chi_{31} - \chi_5\chi_{17}\chi_{31}\chi_{33} \\ & + \chi_5\chi_{20}\chi_{30}\chi_{31} - \chi_6\chi_{23}\chi_{27}\chi_{29} + \chi_7\chi_{12}\chi_{13}\chi_{18} + \chi_8\chi_{10}\chi_{24}\chi_{28} - \chi_9\chi_{12}\chi_{20}\chi_{33} + \chi_{10}\chi_{11}\chi_{28}\chi_{32} \\ & + \chi_{10}\chi_{21}\chi_{27}\chi_{29} - \chi_{12}\chi_{20}\chi_{22}\chi_{24} + \chi_{14}\chi_{17}\chi_{26}\chi_{27} - \chi_{15}\chi_{24}\chi_{26}\chi_{27} - \chi_{16}\chi_{18}\chi_{23}\chi_{27} - \chi_{18}\chi_{24}\chi_{30}\chi_{32}\end{aligned}$$

2^{16} dimensions/parity; dense SYK: 46376 terms ➔ randomly chose $K_{\mathrm{cpl}} = 36$, half +1, half -1

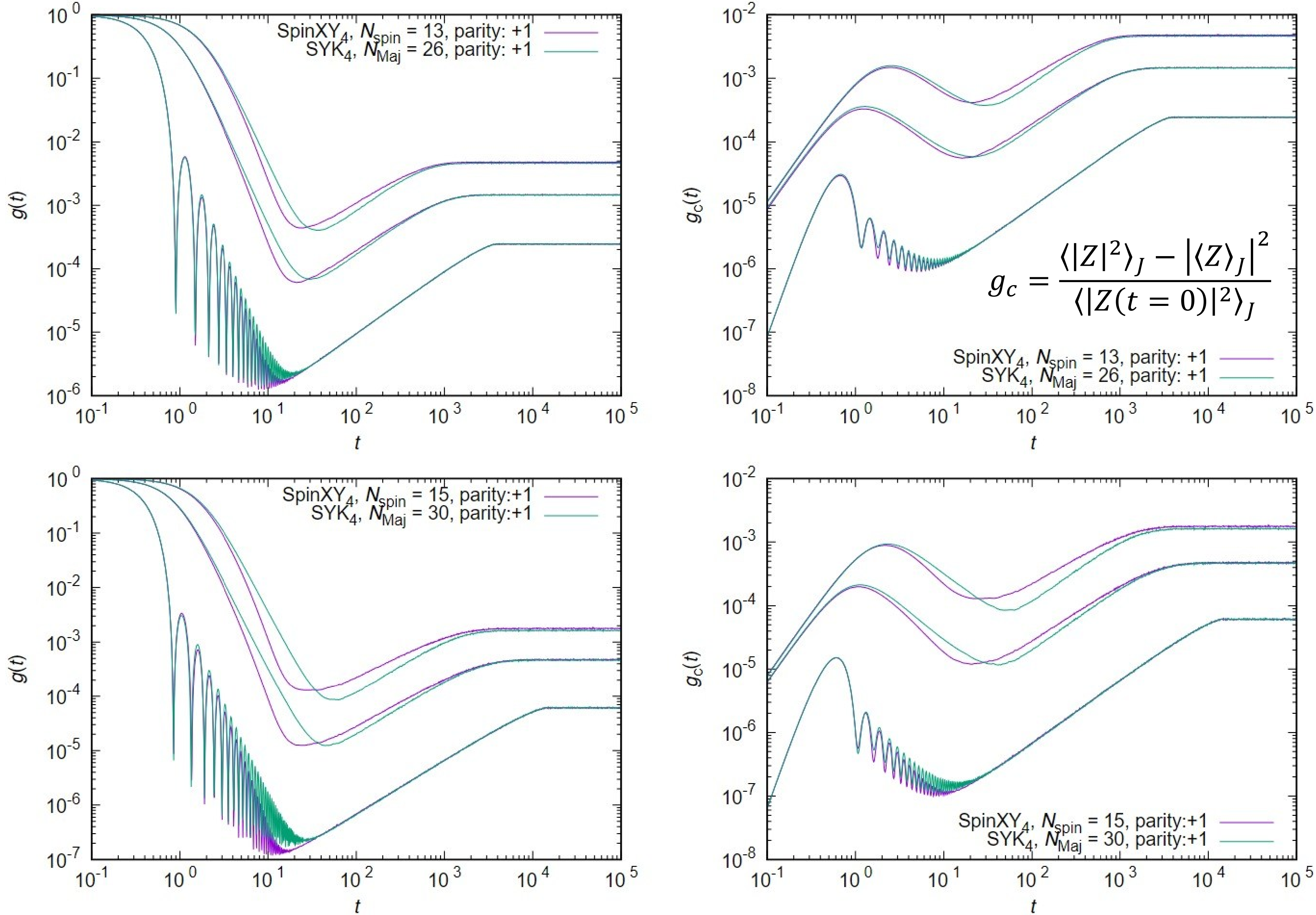
Unary sparse SYK

- $\hat{H} = C_{N,p} \sum_{1\leq a<b<c<d\leq N} x_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d\,, x_{abcd} = \begin{cases} 1 & (\text{probability } p) \\ 0 & (\text{probability } 1-p) \end{cases}$
- Reordering Majorana fermions: flips about half of the signs of x_{abcd}
- Similar statistics as binary sparse SYK expected unless p is very large

➔ Numerically checked (see supplemental materials of our paper)



N_A=7
N_A=6
N_A=5
N_A=4
N_A=3
N_A=2
N_A=1
(3)
(6)
7 ln 2
6 ln 2
5 ln 2
4 ln 2
3 ln 2
2 ln 2
1 ln 2
N = 16
$S_{th} - S_A$
δ



$$g_c = \frac{\langle |Z|^2 \rangle_J - \left| \langle Z \rangle_J \right|^2}{\langle |Z(t=0)|^2 \rangle_J}$$

$\beta = 2$

$\beta = 1$

$\beta = 0$

$\beta = 2$

$\beta = 1$

$\beta = 0$

Edwards-Anderson parameter

Standard tool to see if a given system has a spin-glass phase or not

$$q_{\mathrm{zEA}}(j) = \frac{1}{N_{\mathrm{spin}}} \sum_i \left| \langle \psi_j | \hat{\sigma}_{i,z} | \psi_j \rangle \right|^2$$

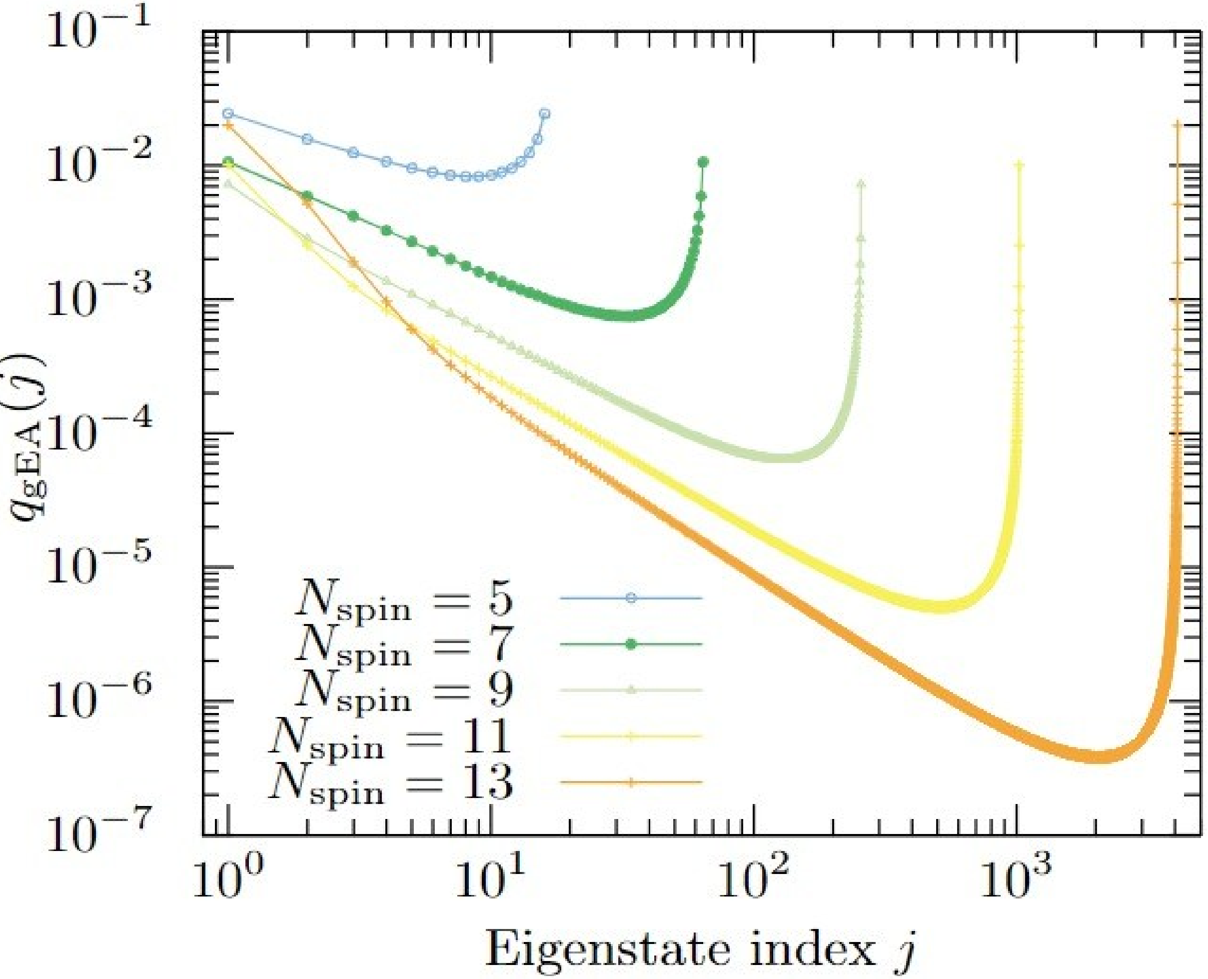
Squared norms of matrix elements averaged over spin
increase for spin glass as N_{spin} is increased

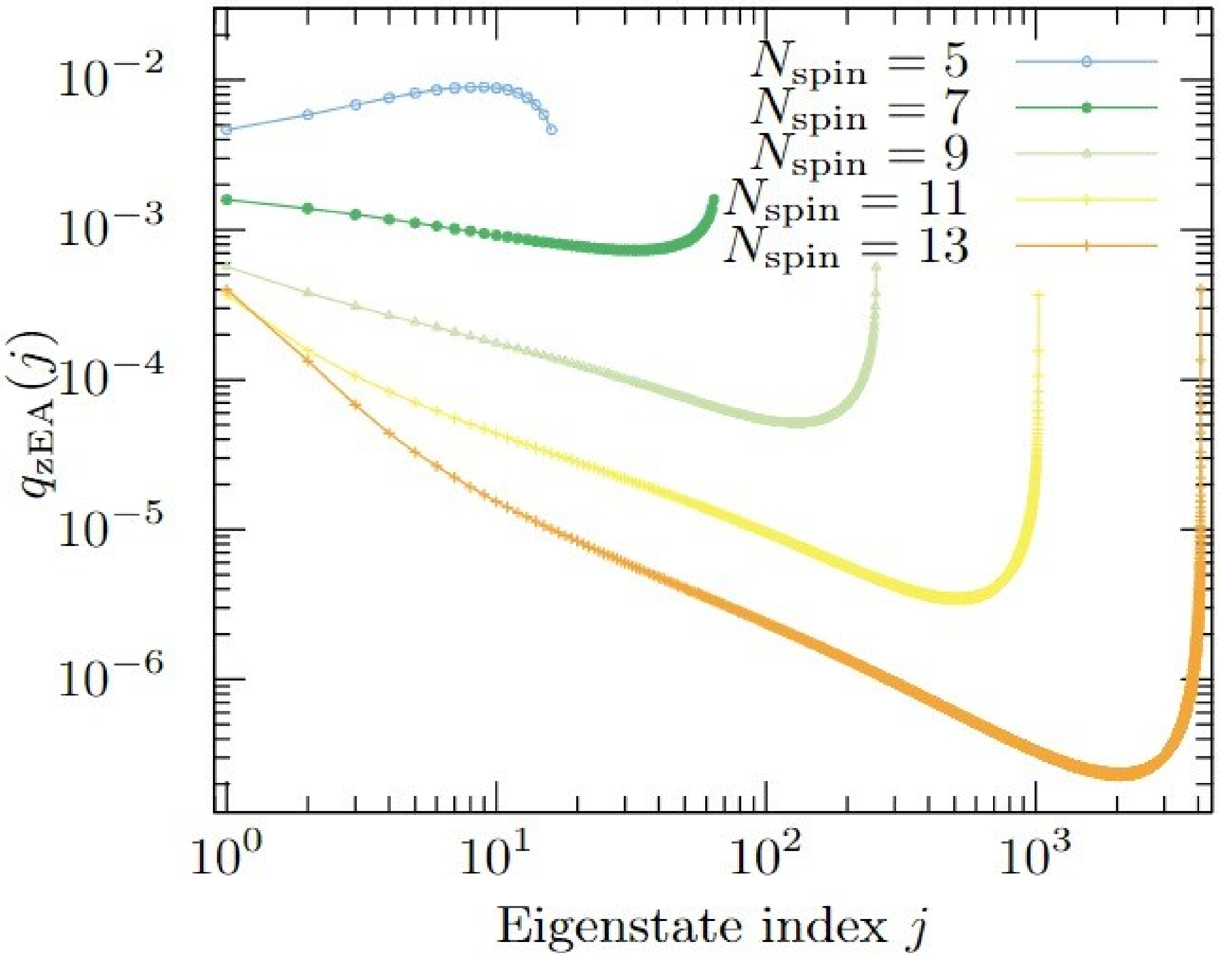
Another choice (generalization):

$$q_{\mathrm{gEA}}(j) = \frac{1}{N_{\mathrm{spin}}} \sum_i \sum_{\alpha=x,y} \left| \left\langle \psi_j^{(\mathrm{O})} \middle| \hat{\sigma}_{i,\alpha} \middle| \psi_j^{(\mathrm{E})} \right\rangle \right|^2$$

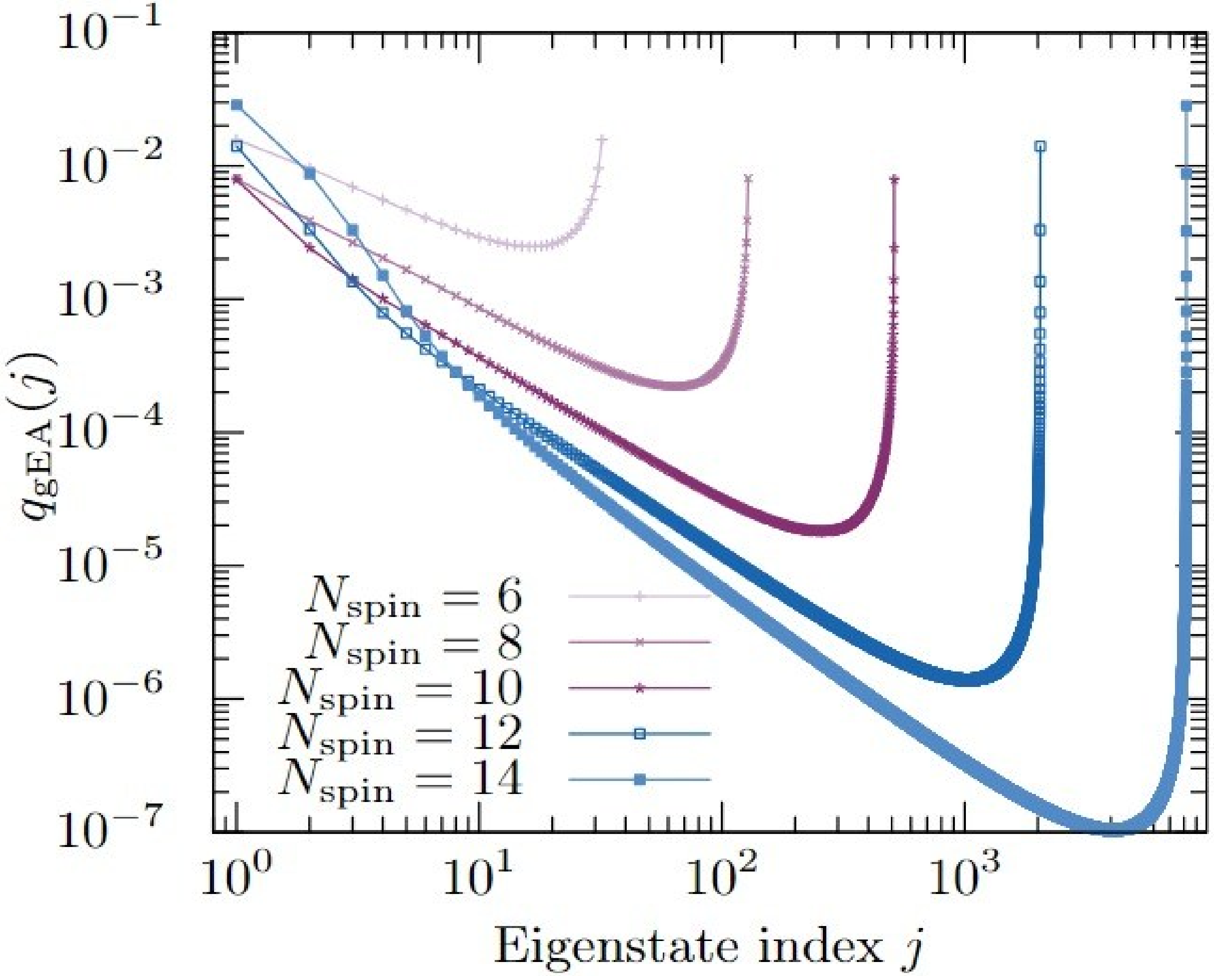
Matrix elements between j-th eigenvectors (sorted by energy)

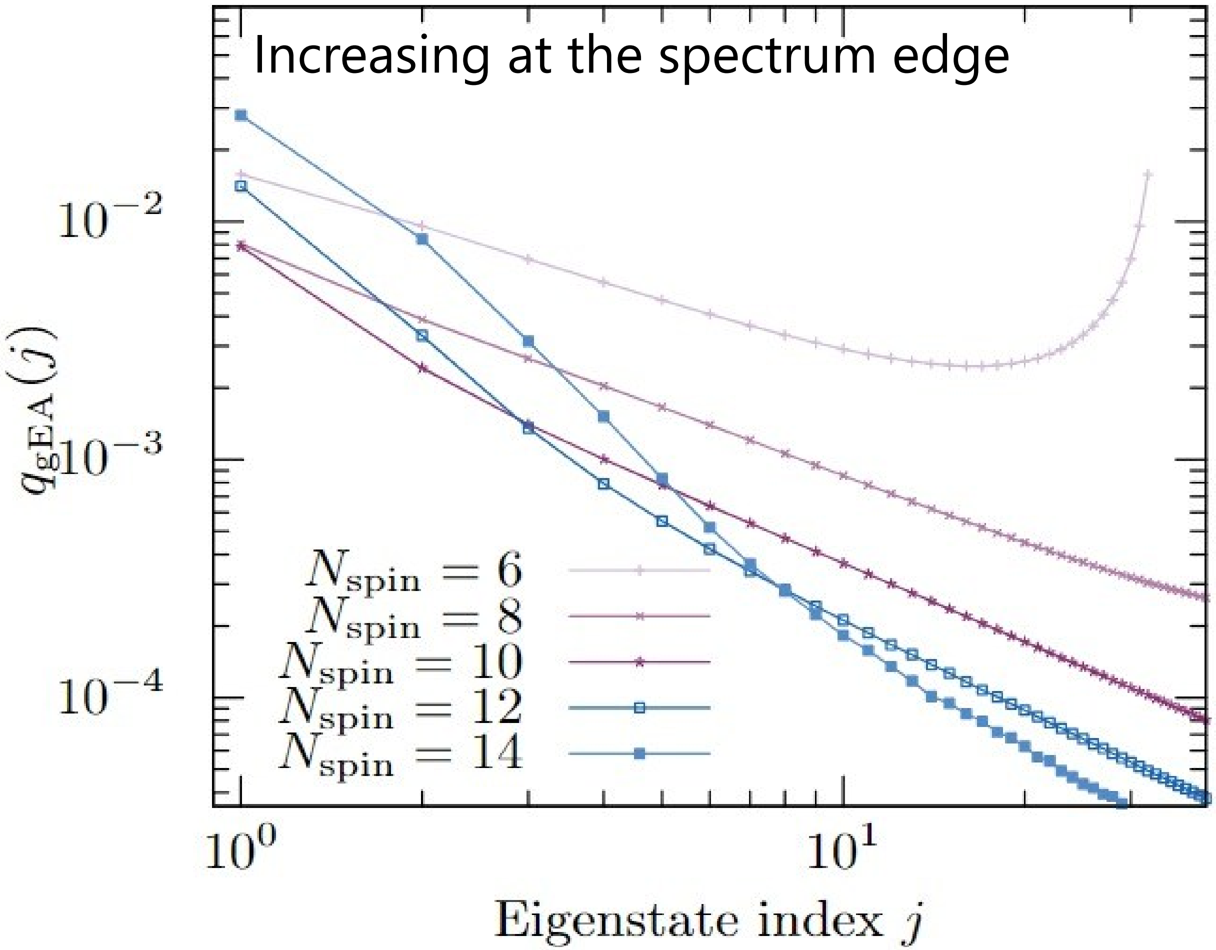
gEA and zEA for odd N_{spin}



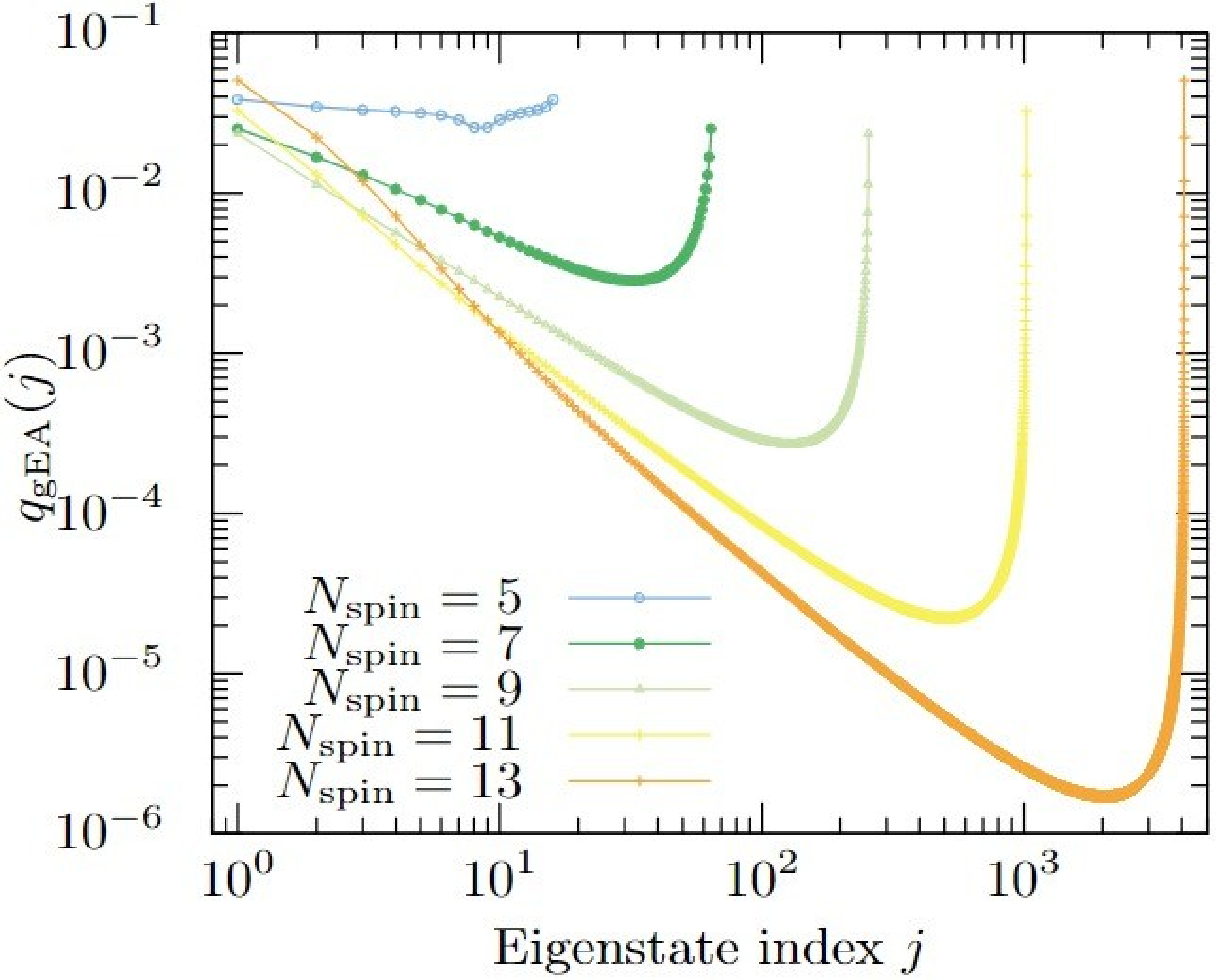


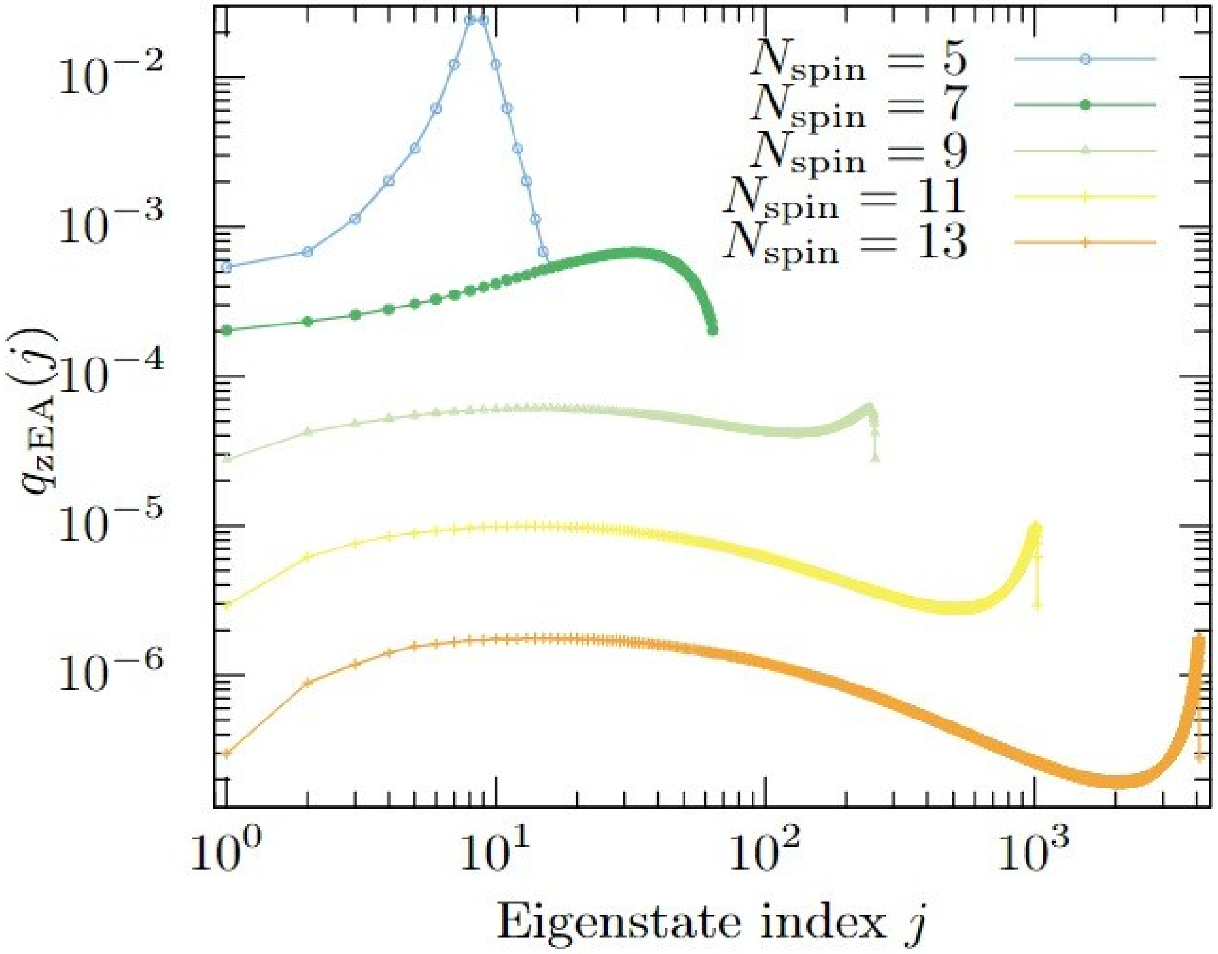
gEA for even N_{spin}



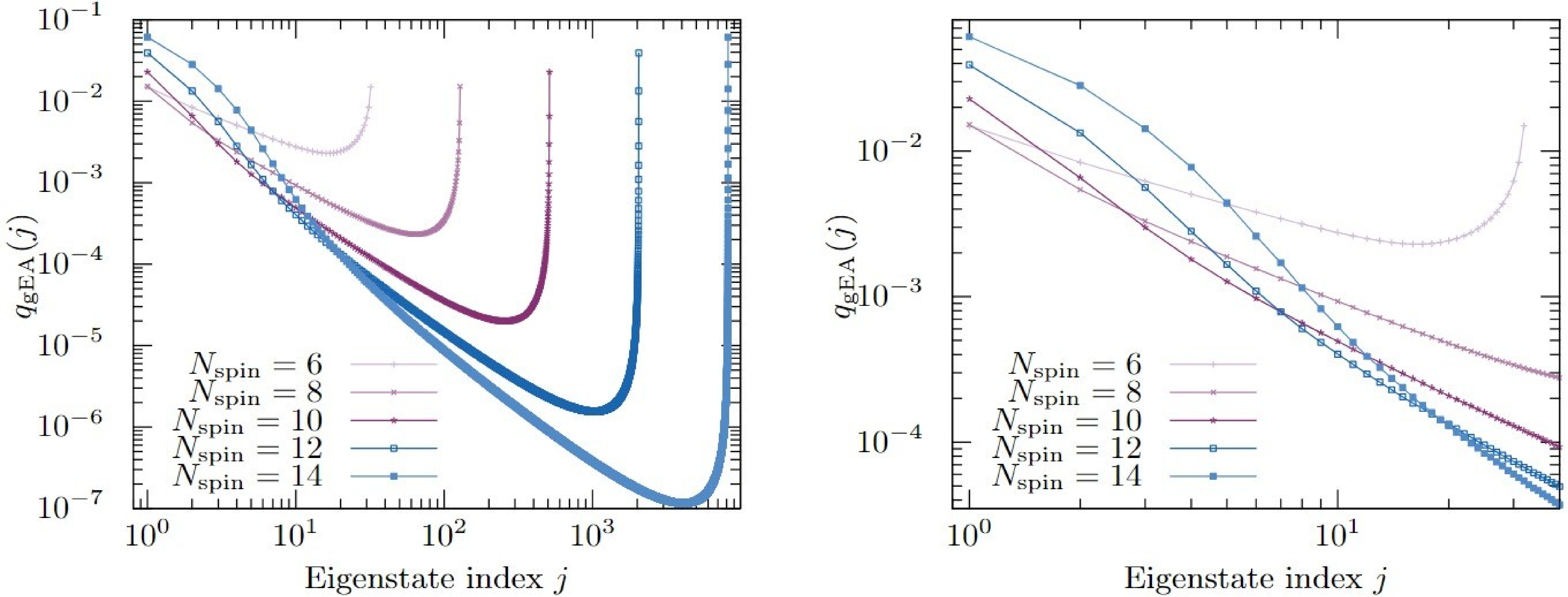


gEA and zEA for odd N_{spin}: $J_{abcd} = 0$ for $\eta_{abcd} > 0$





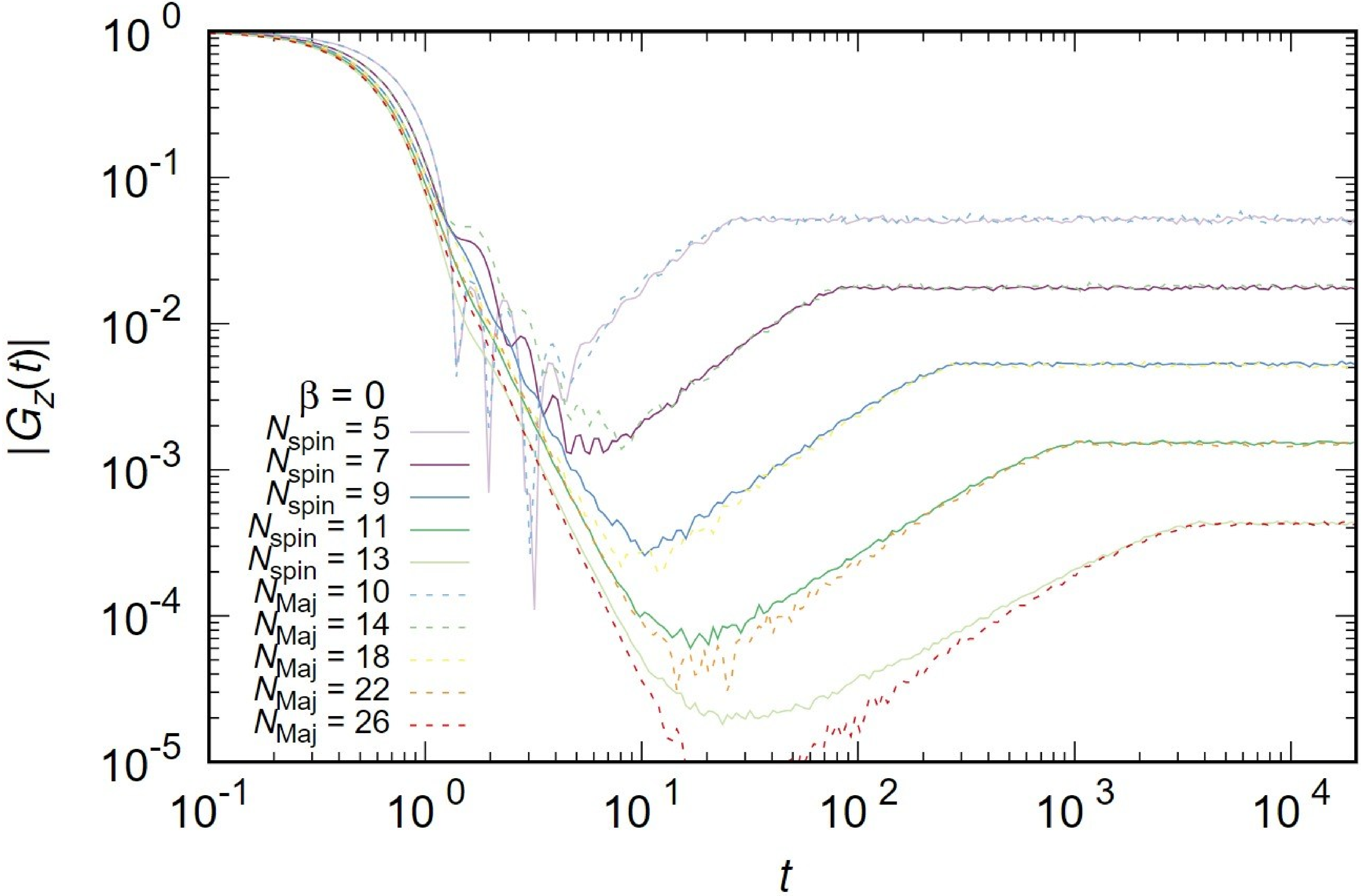
gEA for even N_{spin}: $J_{abcd} = 0$ for $\eta_{abcd} > 0$



Two-point correlated function

$$G_z(t) = \frac{1}{N_{\text{spin}}} \sum_{j=1}^{N_{\text{spin}}} \left\langle \hat{\sigma}_{j,z}(t) \hat{\sigma}_{j,z}(0) \right\rangle_{\beta,J}$$

$$= \frac{1}{N_{\text{spin}}} \frac{1}{\langle Z(\beta) \rangle_J} \left\langle \sum_{E,E'} e^{-\beta E + i(E-E')t} \sum_{j=1}^{N_{\text{spin}}} \left| \left\langle E \middle| \hat{\sigma}_{j,z} \middle| E' \right\rangle \right|^2 \right\rangle_J$$


$|G_z(t)|$
t
$\beta = 0$
$N_{spin} = 5$
$N_{spin} = 7$
$N_{spin} = 9$
$N_{spin} = 11$
$N_{spin} = 13$
$N_{Maj} = 10$
$N_{Maj} = 14$
$N_{Maj} = 18$
$N_{Maj} = 22$
$N_{Maj} = 26$