Quantum Gravity and Information in Expanding Universe

Higher-derivative Fermionic Theories

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Quantum Gravity and Information in Expanding Universe

dS/CFT correspondence and Information

Toward "dS/CFT"

- ✴ Analytic continuation from AdS/CFT correspondence
  - ✓ Higher spin dS/CFT correspondence

    [Anninos, Hartman, Strominger, 2011], [Gim Seng Ng, Strominger, 2012], …
    [Das, Das, Jevicki, Ye, 2012], …

  - ✓ dS_3/CFT_2 based based on W_N minimal model

    [Hikida, Nishioka, Takayanagi, Taki, 2021, 2022], [Chen, Chen, Hikida, 2022], …
    [Chen, Hikida, Taki, Uetoko, 2024]

- ✴ Worldline Observer

  [Anninos, Hartnoll and Hofman, 2011], …, [Witten, 2023], [Loganayagam, Shetye, 2023], …
  [Anninos, Galante, Maneerat, 2024], …

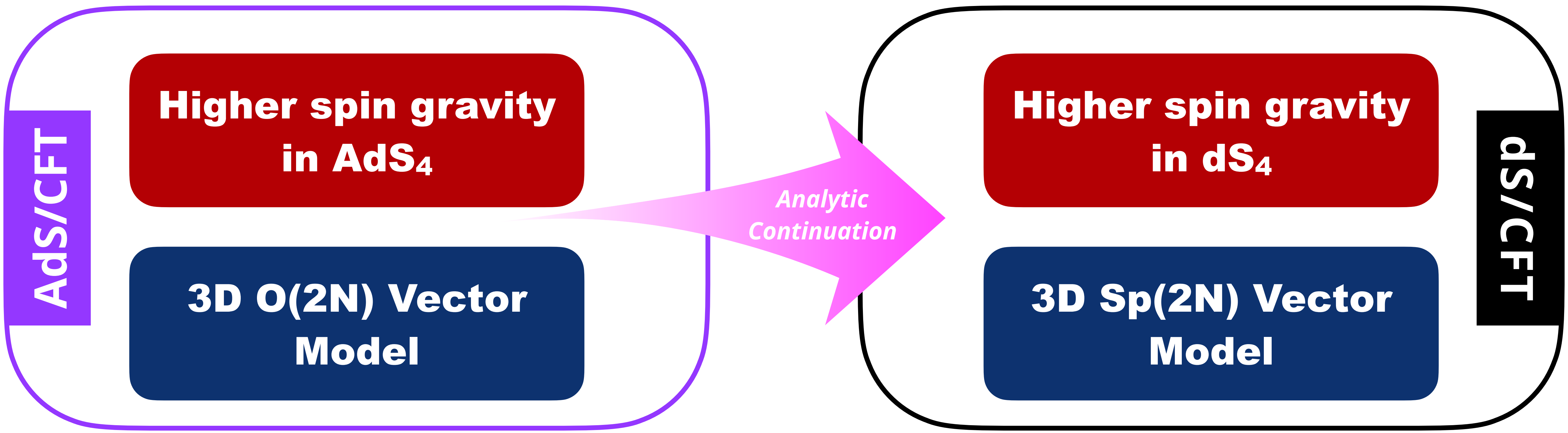
- ✴ Cosmological correlator

  [McFadden, Skenderis, 2011], [Mata, Raju, Trivedi, 2012], [Arkani-Hamed, Maldacena, 2015], …
  [Arkani-Hamed, Baumann, Lee, Pimentel, 2018], [Jain, Kundu, Kundu, Mehta, Sake, 2022], …

Old Story about dS/CFT

✳ Higher spin dS/CFT correspondence

[Anninos, Hartman, Strominger, 2011], [Gim Seng Ng, Strominger, 2012], [Das, Das, Jevicki, Ye, 2012], …



✳ Analytic continuation: $N \longrightarrow -N$

✳ The matching of correlation functions follows from AdS/CFT.

✳ Geometric interpretation is obscure.

✓ e.g. Where is the boundary?

Sp(2N) Model

$$S = \frac{1}{2}\int d^d x \; \partial^\mu \psi_i \partial_\mu \psi_i$$

- ✴ called as "symplectic fermion" or "anti-commuting scalar"
- ✴ In the old time, there have been many works as an exotic field theory.
  (e.g. negative central charge, Logarithmic CFT)

  [Kausch,1995], [Gaberdiel, Kausch, 1999], [Kausch,2000], [LeClair, Neubert, 2007], [Robinson, Kapit, LeClair, 2009]…

Curiosities at $c = -2$

Horst G. Kausch[1]

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, U.K.

20 October 1995

- ✴ This is a higher-derivative theory
  - ✓ Usual fermion (anti-commuting field): single-derivative
  - ✓ Symplectic fermion: two-derivative

Higher-derivative Theory

- ✳ Ostrogradsky Instability
  - ✓ The Lagrangian with higher-(time-)derivatives has larger Hilbert space than we often expect.

$$\text{e.g.} \quad L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2 - \frac{\epsilon}{2}\ddot{x}^2$$

  - ✓ "Usually", Hamiltonian of the higher-derivative theory is unbounded from below.

$$\text{e.g.} \quad H = a_1^\dagger a_1 - a_2^\dagger a_2$$

- ✳ This instability is also involved with
  - ✓ Non-normalizable vacuum
  - ✓ Negative norm state etc
- ✳ One might be able to avoid one of them by some "trick". But one cannot evade all of them.

Is it TRUE in general?

Is it TRUE in general?

What about symplectic Fermion?

Is it TRUE in general?

What about symplectic Fermion?

Review: Higher-derivative fermion

OUTLINE

I. Fermionic Higher-derivative Toy Model

II. “Higher-derivative” Theory from Field Redefinition

III. 2D Symplectic Fermion

IV. Implication to α-vacua

Fermionic Higher-derivative Toy Model

$T\bar{T}$-deformed Fermionic Theories Revisited

Kyungsun Lee, Piljin. Yi and JY
arXiv: 2104.09529

Kyungsun Lee
KIAS

Piljin Yi
KIAS

Quantum Mechanical Toy Model

- Consider additional term to the (0+1)-dim free fermion

$$L = \frac{i}{2}\bar{\psi}\dot{\psi} - \frac{i}{2}\dot{\bar{\psi}}\psi + m\bar{\psi}\psi - \lambda\dot{\bar{\psi}}\dot{\psi}$$

  - ✓ the extra term is the same as kinetic term of boson

- From Lagrangian, the conjugate momentum can be obtained:

$$\pi = \frac{\overleftarrow{\delta} L}{\overleftarrow{\delta}\dot{\psi}} = \frac{i}{2}\bar{\psi} - \lambda\dot{\bar{\psi}} \qquad \bar{\pi} = \frac{\overrightarrow{\delta} L}{\overrightarrow{\delta}\dot{\bar{\psi}}} = -\frac{i}{2}\psi - \lambda\dot{\psi}$$

  - ✓ $\dot{\psi}$ can be expressed in terms of canonical variables!

$$\dot{\psi} = -\frac{1}{\lambda}\left(\bar{\pi} + \frac{i}{2}\psi\right) \qquad \dot{\bar{\psi}} = -\frac{1}{\lambda}\left(\pi - \frac{i}{2}\bar{\psi}\right)$$

  - ✓ No constraint ➡ No elimination of D.o.F. : Fermion “Doubling”

$$\{\, \psi \,,\, \bar{\psi} \,,\, \pi \,,\, \bar{\pi} \,\}$$

Quantization

* Quantization: Canonical anti-commutation relation

$$\pi + \frac{i}{2}\bar{\psi} \;=\; i(\cosh\theta\bar{b} + \sinh\theta\bar{c}) \qquad \bar{\pi} - \frac{i}{2}\psi \;=\, - i(\cosh\theta b + \sinh\theta c)$$

$$\pi - \frac{i}{2}\bar{\psi} \;=\, - i(\sinh\theta\bar{b} + \cosh\theta\bar{c}) \qquad \bar{\pi} + \frac{i}{2}\psi \;=\; i(\sinh\theta b + \cosh\theta c)$$

$$\{\psi, \pi\} = i \qquad \{\bar{\psi}, \bar{\pi}\} = - i \qquad \Longrightarrow \qquad \{b, \bar{b}\} = 1 \qquad \{c, \bar{c}\} = - 1$$

✓ θ : (real) parameter for Bogoliubov transformation generated by $G = i(\bar{b}c + b\bar{c})$

✓ Cannot avoid minus sign in RHS of $\{c, \bar{c}\} = - 1$

* Hamiltonian becomes

$$H = -\left(me^{2\theta} + \frac{1}{\lambda}\sinh^2\theta\right)\bar{b}b - \left(me^{2\theta} + \frac{1}{\lambda}\cosh^2\theta\right)\bar{c}c - \left(me^{2\theta} + \frac{1}{\lambda}\cosh\theta\sinh\theta\right)(\bar{b}c + \bar{c}b)$$

Q: Does this Hamiltonian is Hermitian?

$$H = -\left(me^{2\theta} + \frac{1}{\lambda}\sinh^2\theta\right)\bar{b}b - \left(me^{2\theta} + \frac{1}{\lambda}\cosh^2\theta\right)\bar{c}c$$

$$+\left(me^{2\theta} + \frac{1}{\lambda}\cosh\theta\sinh\theta\right)\left(\bar{b}c + \bar{c}b\right)$$

If you have two students,

Hamiltonian

Lagrangian

$$H = -\left(me^{2\theta} + \frac{1}{\lambda}\sinh^2\theta\right)\bar{b}b - \left(me^{2\theta} + \frac{1}{\lambda}\cosh^2\theta\right)\bar{c}c$$
$$+\left(me^{2\theta} + \frac{1}{\lambda}\cosh\theta\sinh\theta\right)(\bar{b}c + \bar{c}b)$$

$$H_{ab} \equiv \langle a | H | b \rangle$$ in Fock space

$$L = \frac{i}{2}\bar{\psi}\dot{\psi} - \frac{i}{2}\dot{\bar{\psi}}\psi + m\bar{\psi}\psi - \lambda\dot{\bar{\psi}}\dot{\psi}$$

$$Z = \int d\psi d\bar{\psi} e^{-\int_0^\beta d\tau\, L}$$

If you have two students,

Hamiltonian

$$E_1 = 0$$

$$E_2 = \frac{1 - 2m\lambda - \sqrt{1 + 4m^2\lambda^2}}{2\lambda}$$

$$E_3 = \frac{1}{\lambda}$$

$$E_4 = \frac{1 - 2m\lambda + \sqrt{1 + 4m^2\lambda^2}}{2\lambda}$$

: real for all m, λ

Lagrangian

$$E_1 = 0$$

$$E_2 = \frac{1 - \sqrt{1 + 4m\lambda}}{2\lambda}$$

$$E_3 = \frac{1}{\lambda}$$

$$E_4 = \frac{1 + \sqrt{1 + 4m\lambda}}{2\lambda}$$

: can be complex for some m, λ

Q: Does this Hamiltonian is Hermitian?

$$H = -\left(me^{2\theta} + \frac{1}{\lambda}\sinh^2\theta\right)\bar{b}b - \left(me^{2\theta} + \frac{1}{\lambda}\cosh^2\theta\right)\bar{c}c$$

$$+\left(me^{2\theta} + \frac{1}{\lambda}\cosh\theta\sinh\theta\right)(\bar{b}c + \bar{c}b)$$

If you answer the Hamiltonian is Hermitian,

then you would think the result of  is correct

Negative Norm State

- From anti-commutation relation

$$\{b, \bar{b}\} = 1 \qquad \{c, \bar{c}\} = -1$$

the norm of the excited state $c^{\dagger}|0\rangle$ is

$$\langle 0 | c\bar{c} | 0 \rangle = -\langle 0 | 0 \rangle - \langle 0 | \bar{c} c | 0 \rangle = -\langle 0 | 0 \rangle$$

✓ Either the vacuum or $\bar{c}|0\rangle$ has negative norm!

- This is similar to Ostrogradsky instability for higher derivative theory.

Is this model pathological?

Too Early to conclude because

Resolution of Negative Norm

- Define J operator: unitary and Hermitian $J^\dagger = J^{-1} = J$

$$J = e^{i\pi \bar{c}c}$$

$$JcJ = -c \qquad JbJ = b$$

$$J\bar{c}J = -\bar{c} \qquad J\bar{b}J = \bar{b}$$

c.f.) $(-1)^F$ in SUSY

[LeClair, Neubert, 2007], [Robinson, Kapit, LeClair, 2009]

- Define J-inner product

$$\langle \mathcal{O} \rangle_J \equiv \langle J \mathcal{O} \rangle$$

- J-inner product is positive-definite

$$\Big| \bar{c} | 0 \rangle \Big|_J^2 = \langle 0 | cJ\bar{c} | 0 \rangle = -\langle 0 | c\bar{c} | 0 \rangle = \langle 0 | 0 \rangle$$

Ad hoc?

Relation to Path Integral

- ✴ You might think that the J-norm is an artificial ad hoc modification to save the theory.
  - ✓ We showed that the J-norm (not the "ordinary" norm) follows from the path integral formalism.
- ✴ The connection between operator formalism and path integral formalism can be found by inserting the completeness relation into the transition amplitude. e.g. $\langle \bar{\eta}_f | e^{-iTH} | \eta_i \rangle$
  - ✓ Completeness relation: $1 = |0\rangle\langle 0| - \bar{c}\,|0\rangle\langle 0|\,c + \cdots$

    $$= |0\rangle\langle 0| + \bar{c}\,|0\rangle\langle 0|\,c\,J + \cdots$$
  - ✓ for example, $\mathrm{tr}(J\,e^{-\beta H}) = \int_{\psi(0)=-\psi(\beta),\bar{\psi}(0)=-\bar{\psi}(\beta)} D\psi\, D\bar{\psi}\;\; e^{-S}$ at finite temperature

Ad hoc?

No

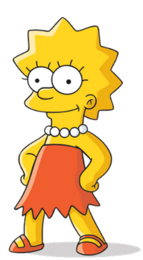
Energy Spectrum

- ✳ If you define a model by Lagrangian, the correct operator formalism of the model should use J-norm. It is not a choice.

- ✳ There are two ways to get the energy spectrum:

  - ✓ With Fock state $|a\rangle$, diagonalize the matrix $M_{ab} \equiv \langle a|H|b\rangle_J$
    cf) biorthogonal basis [1308.2609] [quant-ph/0306040], $\cdots$

  - ✓ Find energy eigenstates directly without bra state $H|E\rangle = E|E\rangle$

- ✳ The result of  is correct.

Lagrangian

$$E_1 = 0$$

$$E_2 = \frac{1 - \sqrt{1 + 4m\lambda}}{2\lambda}$$

$$E_3 = \frac{1}{\lambda}$$

$$E_4 = \frac{1 + \sqrt{1 + 4m\lambda}}{2\lambda}$$

Wait a second....

Could the energy be complex?

$$E_1 = 0$$

$$E_2 = \frac{1 - \sqrt{1 + 4m\lambda}}{2\lambda}$$

$$E_3 = \frac{1}{\lambda}$$

$$E_4 = \frac{1 + \sqrt{1 + 4m\lambda}}{2\lambda}$$

when $4m\lambda < -1$?

J-Hermitian Adjoint

- With new J-inner product, we should define new J-Hermitian adjoint.

$$\mathcal{O}^{\dagger_J} \equiv J\mathcal{O}^{\dagger}J \qquad \text{so that} \qquad \langle \Phi \,|\, \mathcal{O}\Psi \rangle_J = \langle\, \mathcal{O}^{\dagger_J}\,\Phi \,|\, \Psi \rangle_J$$

- Then, the Hermiticity of an operator should be defined with J-Hermitian adjoint.

An operator $\mathcal{O}$ is J-Hermitian iff $\mathcal{O}^{\dagger_J} = \mathcal{O}$

Bi-orthogonal State

✴ Instead of inserting the operator J in the inner product, it is more convenient to define new bra state by using J-Hermitian adjoint. (double-bracket notation)

$$|\Phi\rangle\rangle = \mathscr{O}|0\rangle\rangle = |\Phi\rangle \longrightarrow \langle\langle\Phi| \equiv \langle\langle 0|\mathscr{O}^{\dagger_J}$$

✓ The overlaps of double-bracket bra and ket states gives the J-inner product.

$$\langle\langle\Phi|\mathscr{O}|\Psi\rangle\rangle = \langle\Phi|\mathscr{O}|\Psi\rangle_J$$

✓ Use double-bracket, and forget the insertion of J

Q: Does this Hamiltonian is Hermitian?

$$H = -\left(me^{2\theta} + \frac{1}{\lambda}\sinh^2\theta\right)\bar{b}b - \left(me^{2\theta} + \frac{1}{\lambda}\cosh^2\theta\right)\bar{c}c$$

$$+\left(me^{2\theta} + \frac{1}{\lambda}\cosh\theta\sinh\theta\right)\left(\bar{b}c + \bar{c}b\right)$$

Q: Does this Hamiltonian is ~~Hermitian~~?

J-Hermitian?

$$H = -\left(me^{2\theta} + \frac{1}{\lambda}\sinh^2\theta\right)\bar{b}b - \left(me^{2\theta} + \frac{1}{\lambda}\cosh^2\theta\right)\bar{c}c$$

$$+\left(me^{2\theta} + \frac{1}{\lambda}\cosh\theta\sinh\theta\right)\boxed{(\bar{b}c + \bar{c}b)}$$

No

Wait a second....

Could the energy be complex?

$$E_1 = 0$$

$$E_2 = \frac{1 - \sqrt{1 + 4m\lambda}}{2\lambda}$$

$$E_3 = \frac{1}{\lambda}$$

$$E_4 = \frac{1 + \sqrt{1 + 4m\lambda}}{2\lambda}$$

when $4m\lambda < -1$?

Since Hamiltonian is not J-Hermitian,

it is not surprising to have complex energy.

When do we have real spectrum?

- For a special value of θ, the Hamiltonian becomes J-Hermitian.

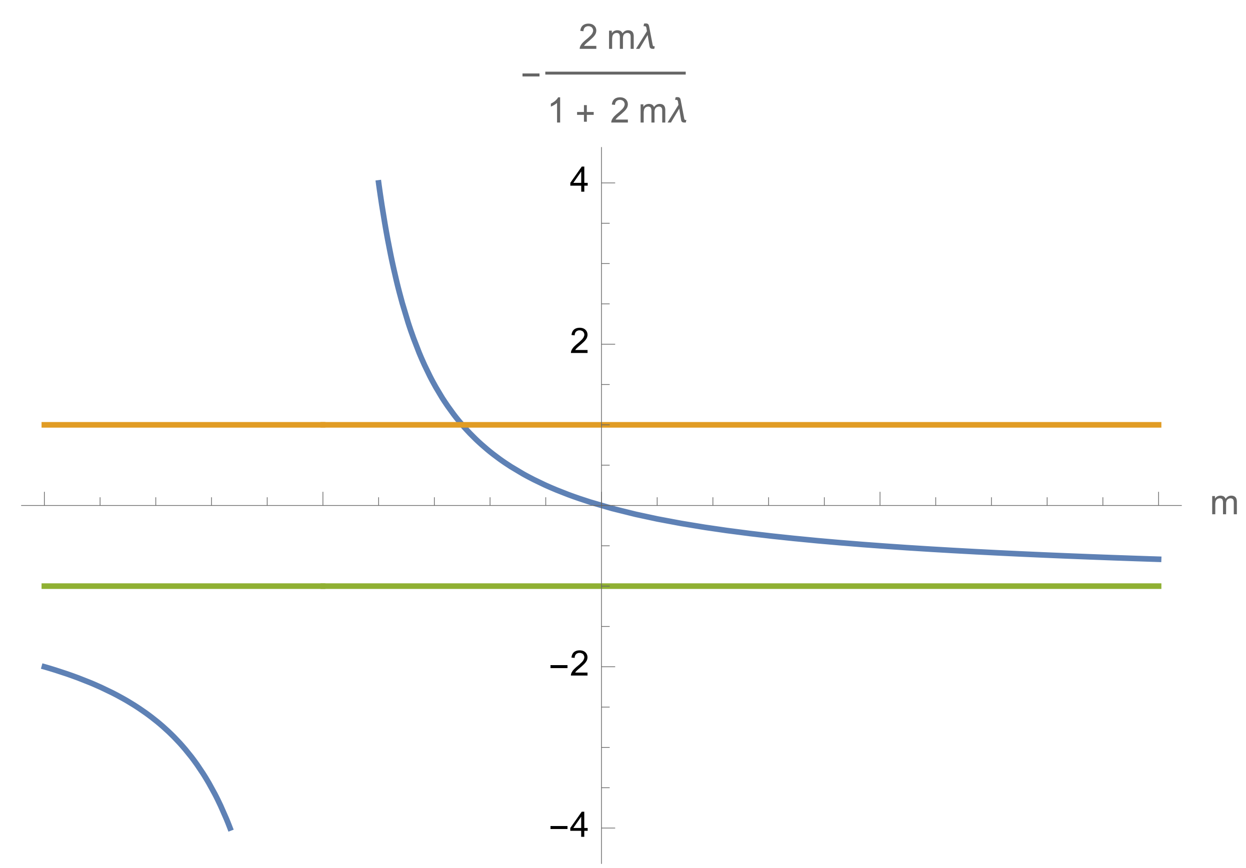
$$H = -\left(me^{2\theta} + \frac{1}{\lambda}\sinh^2\theta\right)\bar{b}b - \left(me^{2\theta} + \frac{1}{\lambda}\cosh^2\theta\right)\bar{c}c - \left(me^{2\theta} + \frac{1}{\lambda}\cosh\theta\sinh\theta\right)(\bar{b}c + \bar{c}b)$$

$$\tanh 2\theta = -\frac{2m\lambda}{2m\lambda + 1}$$

$$\Rightarrow \quad H = \frac{1 - \sqrt{1 + 4m\lambda}}{2\lambda}\bar{b}b - \frac{1 + \sqrt{1 + 4m\lambda}}{2\lambda}\bar{c}c$$

- This special value exists only when $4m\lambda > -1$
  : identical to the condition for real energy spectrum.

$$E_{2/4} = \frac{1 \mp \sqrt{1 + 4m\lambda}}{2\lambda}$$



As λ goes to 0....

$$L = \frac{i}{2}\bar{\psi}\dot{\psi} - \frac{i}{2}\dot{\bar{\psi}}\psi + m\bar{\psi}\psi - \lambda\dot{\bar{\psi}}\dot{\psi}$$

as $\lambda \to 0$

$$L = \frac{i}{2}\bar{\psi}\dot{\psi} - \frac{i}{2}\dot{\bar{\psi}}\psi + m\bar{\psi}\psi$$

: ordinary free fermion

- The small λ expansion of the Hamiltonian

$$H = \frac{1-\sqrt{1+4m\lambda}}{2\lambda}\bar{b}b - \frac{1+\sqrt{1+4m\lambda}}{2\lambda}\bar{c}c$$

$$H = \left(-m + m^2\lambda + \cdots\right)b^\dagger b - \left(\frac{1}{\lambda} + m - m^2\lambda + \cdots\right)c^\dagger c$$

diverges as $\lambda \to 0$

Decoupled!!

We are exploring more higher-derivative theories.

(See Mehta's poster)

Xavier Bekaert
University of Tours

Abhishek Mehta
APCTP ➞ Kyung Hee University

“Higher-derivative” Theory from Field Redefinition

One section in the paper, "$T\bar{T}$ deformation of $\mathcal{N} = (1,1)$ off-shell supersymmetry and partially broken supersymmetry"

Kyungsun Lee and JY
arXiv: 2306.08030

Kyungsun Lee
KIAS

Equivalence Theorem in Path Integral

- In the path integral, the physics (or, specifically, physical observables like S-matrix, correlation functions) should not depends on the field redefinition.

Field Redefinition: $\phi \longrightarrow \phi[\Phi]$

$$\int \mathscr{D}\phi\, e^{iS[\phi]} \longrightarrow \int \mathscr{D}\Phi \left[\frac{\delta\phi}{\delta\Phi}\right] e^{iS\left[\phi[\Phi]\right]}$$

“Higher-derivative” Theory from Free Theory

- Let us consider a free scalar field

$$L = -\frac{1}{2}\partial_\mu \phi \partial^\mu \phi$$

- Under a field redefinition $\phi = \partial^2 \Phi$, the Lagrangian has higher-derivatives

$$L = -\frac{1}{2}(\partial_\mu \partial^2 \Phi)(\partial^\mu \partial^2 \Phi)$$

- According to the equivalence theorem in QFT, it should describe the free scalar field theory.

Does this free theory have Ostrogradsky instability?

$$L = -\frac{1}{2}(\partial_\mu \partial^2 \Phi)(\partial^\mu \partial^2 \Phi)$$

Does this free theory have Ostrogradsky instability?

$$L = -\frac{1}{2}(\partial_\mu \partial^2 \Phi)(\partial^\mu \partial^2 \Phi)$$

No.

Probably most of you know the answer because it is an old problem.

Going back to Free Fermion

- (0+1)-dim Free Complex Fermion

$$L = -i\dot{\bar{\psi}}\psi + m\bar{\psi}\psi$$

- Let's take field redefinition

$$\psi = \eta + i\lambda\dot{\eta} \qquad\qquad \bar{\psi} = \bar{\eta}$$

  - ✓ Then we have

$$L = \left(\frac{i}{2} + i\lambda m\right)\bar{\eta}\dot{\eta} - \frac{i}{2}\dot{\bar{\eta}}\eta + \lambda\dot{\bar{\eta}}\dot{\eta} + m\bar{\eta}\eta$$

What is wrong with this field redefinition?

Comparison of Phase Space

- Comparison of Phase space

$$L = -i\dot{\bar{\psi}}\psi + m\bar{\psi}\psi$$

Constraints

$$\pi = 0 \qquad \bar{\pi} + i\bar{\psi} = 0$$

Phase Space

$$\psi, \bar{\psi}, \pi, \bar{\pi}$$

Comparison of Phase Space

- Comparison of Phase space

$$L = - i\dot{\bar{\psi}}\psi + m\bar{\psi}\psi$$

Constraints

$$\pi = 0 \qquad \bar{\pi} + i\bar{\psi} = 0$$

Phase Space

$$\psi, \bar{\psi}, \cancel{\pi}, \cancel{\bar{\pi}}$$

Comparison of Phase Space

- Comparison of Phase space

$$L = -i\dot{\bar{\psi}}\psi + m\bar{\psi}\psi$$

Constraints

$$\pi = 0 \qquad \bar{\pi} + i\bar{\psi} = 0$$

Phase Space

$$\psi, \bar{\psi}, \pi, \bar{\pi}$$

$$L = \left(\frac{i}{2} + i\lambda m\right)\bar{\eta}\dot{\eta} - \frac{i}{2}\dot{\bar{\eta}}\eta + \lambda\dot{\bar{\eta}}\dot{\eta} + m\bar{\eta}\eta$$

Comparison of Phase Space

- Comparison of Phase space

$$L = -i\dot{\bar{\psi}}\psi + m\bar{\psi}\psi$$

Constraints

$$\pi = 0 \qquad \bar{\pi} + i\bar{\psi} = 0$$

Phase Space

$$\psi\,,\,\bar{\psi}\,,\,\cancel{\pi}\,,\,\cancel{\bar{\pi}}$$

$$L = \left(\frac{i}{2} + i\lambda m\right)\bar{\eta}\dot{\eta} - \frac{i}{2}\dot{\bar{\eta}}\eta + \lambda\dot{\bar{\eta}}\dot{\eta} + m\bar{\eta}\eta$$

No constraint

Phase Space

$$\eta\,,\,\bar{\eta}\,,\,\pi\,,\,\bar{\pi}$$

Resolution of this problem

$$\int \mathscr{D}\phi \; e^{iS[\phi]} \longrightarrow \int \mathscr{D}\Phi \left[\frac{\delta\phi}{\delta\Phi}\right] e^{iS\left[\phi[\Phi]\right]}$$

Jacobian

as path integral measure

Jacobian in Path Integral

- Jacobian from the change of variable in the integration

$$y = y[x] \qquad\qquad \int dy\, f[y] = \int dx\, \frac{dy}{dx}\, f[y[x]]$$

- Jacobian should be taken into account in the field redefinition of path integral

$$\Phi = \Phi[\varphi]$$

$$\int D\Phi\, e^{-S[\Phi]} = \int D\varphi \left|\frac{\partial\Phi}{\partial\varphi}\right| e^{-S[\Phi[\varphi]]}$$

$$\psi = \psi[\eta]$$

$$\int D\psi\, e^{-S[\psi]} = \int \frac{D\eta}{\left|\frac{\partial\psi}{\partial\eta}\right|}\, e^{-S[\psi[\eta]]}$$

BRST Symmetry

[Slavnov, 1990]

✳ Exponentiate the Jacobian [Alfaro, Damgaard, 1990][Bastianelli, 1990]

$$\Phi = \Phi[\varphi] \qquad \int D\Phi\, e^{-S[\Phi]} = \int D\varphi \left|\frac{\partial \Phi}{\partial \varphi}\right| e^{-S[\Phi[\varphi]]}$$

$$L_{tot} = L[\Phi[\varphi]] + \bar{b}\frac{\partial \Phi}{\partial \varphi} b$$: (fermi) ghost

$$\psi = \psi[\eta] \qquad \int D\psi\, e^{-S[\psi]} = \int \frac{D\eta}{\left|\frac{\partial \psi}{\partial \eta}\right|} e^{-S[\psi[\eta]]}$$

$$L_{tot} = L[\psi[\eta]] + \bar{\gamma}\frac{\partial \psi}{\partial \eta}\gamma$$: boson

✳ BRST symmetry from the field redefinition "gauge transformation"

$$\begin{aligned} \delta\varphi &= \epsilon\, b \\ \delta b &= 0 \\ \delta\bar{b} &= -\epsilon\frac{\delta S}{\delta \Phi} \end{aligned}$$

$$\begin{aligned} \delta\eta &= \epsilon\, \gamma \\ \delta\gamma &= 0 \\ \delta\bar{\gamma} &= \epsilon\frac{\delta S}{\delta \psi} \end{aligned}$$

Back to Free Fermion

* Full Lagrangian including exponentiated Jacobian

$$L = i\lambda m\bar{\eta}\dot{\eta} - i\dot{\bar{\eta}}\eta + \lambda\dot{\bar{\eta}}\dot{\eta} + m\bar{\eta}\eta + \bar{\gamma}(1 + i\lambda\partial)\gamma$$

* BRST Symmetry

$$\delta\eta = \epsilon\,\gamma \qquad \delta\bar{\gamma} = \epsilon\big(-i\dot{\bar{\eta}} + m\bar{\eta}\big) \qquad \delta\bar{\eta} = \delta\gamma = 0$$

  - ✓ BRST charge: $Q = -\big(i\lambda m\bar{\eta} + \lambda\dot{\bar{\eta}}\big)\gamma$

* The same Lagrangian can be obtained by another field redefinition from free fermion: $\psi = \eta$ and $\bar{\psi} = \bar{\eta} - i\lambda\dot{\bar{\eta}}$ (up to total derivative)
  - ✓ The corresponding BRST symmetry:

$$\delta\bar{\eta} = \bar{\epsilon}\bar{\gamma} \qquad \delta\gamma = -\bar{\epsilon}\big(-i\dot{\eta} - m\eta\big) \qquad \delta\eta = \delta\bar{\gamma} = 0$$

  - ✓ BRST charge: $\bar{Q} = \bar{\gamma}\big(i\lambda m\eta - \lambda\dot{\eta}\big)$

Similar Canonical Quantization

- Conjugate momentum

$$\pi = \frac{\delta L}{\delta \dot{\eta}} = i\lambda m\bar{\eta} + \lambda\dot{\bar{\eta}} \qquad \bar{\pi} = \frac{\delta L}{\delta \dot{\bar{\eta}}} = -i\eta + \lambda\dot{\eta}$$

$$\Pi = \frac{\delta L}{\delta \dot{\gamma}} = i\lambda\bar{\gamma} \qquad \bar{\Pi} = \frac{\delta L}{\delta \dot{\bar{\gamma}}} = 0$$

: 2nd class constraints

- Canonical (anti-)commutation relation

$$\{\eta, \pi\} = i \qquad \{\bar{\eta}, \bar{\pi}\} = -i \qquad [\gamma, \bar{\gamma}] = \frac{1}{\lambda}$$

from Dirac bracket

✓ Transformation to Oscillators (with Bogoliubov parameter θ)

$$\{b, \bar{b}\} = 1 \qquad \{c, \bar{c}\} = -1 \qquad [a, \bar{a}] = 1$$

Hamiltonian

- Hamiltonian and BRST charge (for a specific θ)

  (for all value of m and λ, one can always find $\theta \in \mathbb{R}$.)

$$H = -m\bar{b}b + \frac{1}{\lambda}\bar{c}c - \frac{1}{\lambda}\bar{a}a$$

$$Q = i\left(\frac{1-m\lambda}{\lambda}\right)^{\frac{1}{2}}\bar{c}a \qquad \overline{Q} = -i\left(\frac{1-m\lambda}{\lambda}\right)^{\frac{1}{2}}\bar{a}c$$

✓ Hamiltonian can be expressed as

$$H = -m\bar{b}b + \frac{1}{1-m\lambda}\{\overline{Q}, Q\}$$

Q: Is Q and $\overline{Q}$ Hermitian conjugate to each other?

Hermitian conjugate?

$$Q = i\left(\frac{1-m\lambda}{\lambda}\right)^{\frac{1}{2}}\bar{c}a \qquad \Longleftrightarrow \qquad \overline{Q} = -i\left(\frac{1-m\lambda}{\lambda}\right)^{\frac{1}{2}}\bar{a}c$$

Q2: Hamiltonian is bounded from below or above?

$$H = -m\bar{b}b + \frac{1}{\lambda}\bar{c}c - \frac{1}{\lambda}\bar{a}a$$

VS

$$H = -m\bar{b}b + \frac{1}{1-m\lambda}\{\overline{Q}, Q\}$$

Recall:

Need J-Hermitian conjugation

$$\{c, \bar{c}\} = -1$$

$$\langle \Phi | \mathcal{O} \Psi \rangle_J = \langle \mathcal{O}^{\dagger_J} \Phi | \Psi \rangle_J$$

$$\mathcal{O}^{\dagger_J} \equiv J \mathcal{O}^{\dagger} J$$

A1: $\mathcal{Q}$ is **NOT** J-Hermitian conjugate to $\overline{\mathcal{Q}}$, **but...**

$$\mathcal{Q} = i\left(\frac{1-m\lambda}{\lambda}\right)^{\frac{1}{2}}\bar{c}a \qquad \overline{\mathcal{Q}} = -i\left(\frac{1-m\lambda}{\lambda}\right)^{\frac{1}{2}}\bar{a}c$$

$$\mathcal{Q}^{\dagger_J} = -\overline{\mathcal{Q}}$$

A2: Hamiltonian is bounded from above

$$H = -m\bar{b}b + \frac{1}{\lambda}\bar{c}c - \frac{1}{\lambda}\bar{a}a$$

$$H = -m\bar{b}b + \frac{1}{1-m\lambda}\{\overline{Q}, Q\}$$

$$\langle\Psi|\{\overline{Q}, Q\}|\Psi\rangle_J = -\left\|Q|\Psi\rangle\right\|_J^2$$

Spectrum

- Fock space

$$|n_b, n_c; n_a\rangle \; \frac{1}{\sqrt{n_a!}} \bar{b}^{n_b} \bar{c}^{n_c} \bar{a}^{n_a} |0,0;0\rangle$$

$$(\; n_b, n_c = 0,1 \;\; , \quad n_a = 0,1,2,\cdots \;)$$

- Physical states is annihilated by Q and $\overline{Q}$

$$Q \;=\; i\left(\frac{1-m\lambda}{\lambda}\right)^{\frac{1}{2}} \bar{c} a$$

$$\overline{Q} \;= -\, i\left(\frac{1-m\lambda}{\lambda}\right)^{\frac{1}{2}} \bar{a} c$$

$$H|0,0;0\rangle = 0|0,0;0\rangle$$

$$H|1,0;0\rangle \;= -\, m|1,0;0\rangle$$

Reproduce
the spectrum of free fermion

- The same conclusion can be obtained by BRST cohomology with Q.

2D Symplectic Fermion

Central Charge

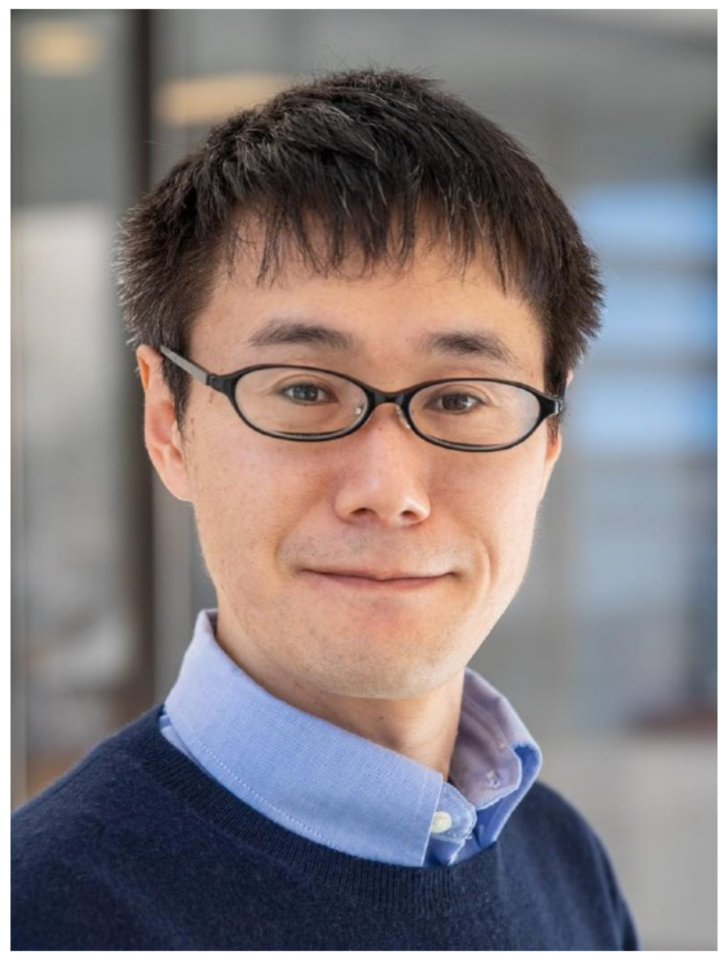
* In CFT_2, the central charge is a good indicator of the non-unitarity.
* We often say that

  CFT_2 with negative central charge is non-unitary.

Does negative central charge always imply non-unitarity?

Unitarity of Symplectic Fermion in α-vacua with Negative Central Charge

Shinsei Ryu and JY
arXiv: 2208.12169



Shinsei Ryu
Princeton University

Model

- Two-dimensional Euclidean symplectic fermion (or in other words, anti-commuting scalar)

$$S = \int dz d\bar{z} \left(2\partial\bar{\psi}\bar{\partial}\psi + 2\bar{\partial}\bar{\psi}\partial\psi\right)$$

  - ✓ We consider "NS section": $\psi(\tau,0) = -\psi(\tau,\ell)$

- Quantization:

half-integer

From mode expansion $\psi(z,\bar{z}) = \frac{i}{\sqrt{4\pi}} \sum_{n>0} \frac{1}{n}\left(b_n z^{-n} - c_{-n} z^{n} + \bar{b}_n \bar{z}^{-n} - \bar{c}_{-n}\bar{z}^{n}\right)$

anti-commutation : $\{b_n, b_m\} = |n|\delta_{n+m,0}$ $\{c_n, c_m\} = -|n|\delta_{n+m,0}$

cf) free scalar field in CFT_2

The Same Story

- The anti-commutation relation $\{c_n, c_m\} = -|n|\delta_{n+m,0}$ leads to negative norm state
- The negative norm state can be cured by $J = exp[i\pi \sum_{n>0} \frac{1}{n} c_{-n} c_n]$

  [LeClair, Neubert, 2007], [Robinson, Kapit, LeClair, 2009]

- We have to use J-norm, J-Hermitian adjoint etc., and the J-norm follows from the path integral.
- Positive norm, real energy eigenvalues: Unitarity!

Virasoro Symmetry

- Central charge $c = -2$ from OPE of EMT.

$$T(z)T(w) \sim \frac{(-1)}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}$$

- Virasoro generators

$$L_n = \frac{1}{2}\sum_{m>0}(b_{n-m} + c_{n-m})(b_m - c_m) + \frac{1}{2}\sum_{m>n}(b_{n-m} - c_{n-m})(b_m + c_m) \qquad \text{for } n \neq 0$$

$$L_0 = \sum_{m>0}(b_{-m}b_m - c_{-m}c_m) - \frac{1}{8}$$

vacuum energy density comes from $\langle T(z)\rangle_J = -\frac{1}{8z^2}$

- One can explicitly check the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \qquad \text{with } c = -2$$

Well-known Proposition:

CFT_2 with negative central charge should have negative norm states.

Negative central charge

$$c = -2$$

VS

Positive-definite J-norm

Counterexample

- Well-known proposition in CFT$_2$

$$\langle h|\, L_n L_{-n}\,|h\rangle = \langle h|\,[L_n, L_{-n}]\,|h\rangle = \left(2nh + \frac{c}{12}n(n^2-1)\right)\langle h|h\rangle$$

- Loophole of the proposition with J-norm

  - ✓ J-Hermitian adjoint of L_n is not L_{-n} $(n \neq 0)$ i.e. $L_n^{\dagger_j} \neq L_{-n}$

$$L_n = \frac{1}{2}\sum_{m>0}(b_{n-m}+c_{n-m})(b_m-c_m) + \frac{1}{2}\sum_{m>n}(b_{n-m}-c_{n-m})(b_m+c_m)$$

  - ✓ One has to take J-norm of the state $L_{-n}|h\rangle$

$$\Big|L_{-n}|h\rangle\Big|_J = \langle h|J\, L_{-n}^{\dagger_J} L_{-n}|h\rangle = \langle h|L_n\, J\, L_{-n}|h\rangle$$

  - ✓ We cannot use Virasoro algebra to prove the proposition.

The symplectic Fermion is a counterexample of the proposition:

Unitary CFT_2 with negative central charge

Negative Entanglement Entropy in Unitarity Theory?

$$S_{EE}(\ell) = -\rho_{red}\log\rho_{red} = \frac{c}{3}\log\left(\frac{\ell}{\epsilon}\right)$$

Positive Entanglement Entropy

✴ Effective central charge, instead of central charge in Virasoro algebra, appears in EE. [1405.2804], [1502.03275], [1611.08506]

✓ Effective central charge is positive for symplectic fermion

$$S_{EE}(\ell) = \frac{c_{eff}}{3}\log\left(\frac{\ell}{\epsilon}\right) \qquad\qquad c_{eff} = c - 24\Delta_{min} = 1$$

✓ But, positive effective central charge does not always mean unitarity. e.g. Lee-Yang model $c = -\frac{22}{5}, c_{eff} = \frac{2}{5}$

Implication to α-vacua

Generator of Bogoliubov Transformation

✴ Generator of Bogoliubov Transformation:

$$\mathcal{G}_\alpha = i\sum_{n>0}\frac{\alpha_n}{n}\left(b_{-n}c_{-n} + b_n c_n\right)$$

✓ parameterized by arbitrary $\alpha_n \in \mathbb{R} \quad (n > 0)$

✓ Hermitian, but NOT J-Hermitian

$$\mathcal{G}_\alpha^\dagger = \mathcal{G}_\alpha \qquad\qquad \mathcal{G}_\alpha^{\dagger_\mathcal{J}} = -\mathcal{G}_\alpha = \mathcal{G}_{-\alpha}$$

Bogoliubov Transformation

* Bogoliubov transformation of oscillators

$$\tilde{b}_n^{(\alpha)} \equiv e^{-i\mathcal{G}_\alpha} b_n e^{i\mathcal{G}_\alpha} = \cosh\alpha_n b_n - \sinh\alpha_n c_{-n}$$

$$\tilde{c}_{-n}^{(\alpha)} \equiv e^{-i\mathcal{G}_\alpha} c_{-n} e^{i\mathcal{G}_\alpha} = -\sinh\alpha_n b_n + \cosh\alpha_n c_{-n}$$

$$\tilde{b}_{-n}^{(\alpha)} \equiv e^{-i\mathcal{G}_\alpha} b_{-n} e^{i\mathcal{G}_\alpha} = \cosh\alpha_n b_{-n} - \sinh\alpha_n c_n$$

$$\tilde{c}_n^{(\alpha)} \equiv e^{-i\mathcal{G}_\alpha} c_n e^{i\mathcal{G}_\alpha} = -\sinh\alpha_n b_{-n} + \cosh\alpha_n c_n$$

✓ canonical transformation

✓ not J-unitary transformation, but similarity transformation

Different Mode Expansion

✴ Under the Bogoliubov transformation, the mode expansion of $\psi(z,\bar{z})$ becomes

$$\psi(z,\bar{z}) = \frac{i}{\sqrt{4\pi}}\sum_{n>0}\frac{1}{n}\left(b_n z^{-n} - c_{-n}z^{n} + \bar{b}_n\bar{z}^{-n} - \bar{c}_{-n}\bar{z}^{n}\right)$$

$$= \frac{i}{\sqrt{4\pi}}\sum_{n>0}\frac{1}{n}\Big[\left(\cosh\alpha_n\tilde{b}_n^{(\alpha)} + \sinh\alpha_n\tilde{c}_{-n}^{(\alpha)}\right)z^{-n} - \left(\sinh\alpha_n\tilde{b}_n^{(\alpha)} + \cosh\alpha_n\tilde{c}_{-n}^{(\alpha)}\right)z^{n}$$

$$+\left(\cosh\bar{\alpha}_n\tilde{\bar{b}}_n^{(\bar{\alpha})} + \sinh\bar{\alpha}_n\tilde{\bar{c}}_{-n}^{(\bar{\alpha})}\right)\bar{z}^{-n} - \left(\sinh\bar{\alpha}_n\tilde{\bar{b}}_n^{(\bar{\alpha})} + \cosh\bar{\alpha}_n\tilde{\bar{c}}_{-n}^{(\bar{\alpha})}\right)\bar{z}^{n}\Big]$$

✴ Let's say we expand $\psi(z,\bar{z})$ in this form from the beginning and we omit α and tilde,

$$\psi(z,\bar{z}) = \frac{i}{\sqrt{4\pi}}\sum_{n>0}\frac{1}{n}\Big[\left(\cosh\alpha_n b_n + \sinh\alpha_n c_{-n}\right)z^{-n} - \left(\sinh\alpha_n b_n + \cosh\alpha_n c_{-n}\right)z^{n}$$

$$+\left(\cosh\bar{\alpha}_n\bar{b}_n + \sinh\bar{\alpha}_n\bar{c}_{-n}\right)\bar{z}^{-n} - \left(\sinh\bar{\alpha}_n\bar{b}_n + \cosh\bar{\alpha}_n\bar{c}_{-n}\right)\bar{z}^{n}\Big]$$

Chooe different vacuum

J-Hermiticity of Hamiltonian

✳ In this mode expansion, the Hamiltonian (or, L_0) becomes

$$L_0 = \sum_{n>0} \left[\cosh 2\alpha_n (b_{-n}b_n - c_{-n}c_n) + \sinh 2\alpha_n (b_{-n}c_{-n} + c_n b_n) - 2n \sinh^2 \alpha_n\right]$$

✓ Hermitian, but NOT J-Hermitian

✓ Eigenvalue is not necessarily real in general.

✳ Direct diagonalization in the subspace $|0\rangle\,,\ |1\rangle \equiv \frac{1}{\sqrt{n}} b_{-n}|0\rangle\,,\ |2\rangle \equiv \frac{1}{\sqrt{n}} c_{-n}|0\rangle\,,\ |3\rangle \equiv \frac{1}{n} b_{-n}c_{-n}|0\rangle$

$$M_{ji} \equiv \langle j |\, J\, L_0\, | k \rangle \qquad M = \begin{pmatrix} -2n\sinh^2\alpha_n & 0 & 0 & -n\sinh 2\alpha_n \\ 0 & n & 0 & 0 \\ 0 & 0 & n & 0 \\ n\sinh 2\alpha_n & 0 & 0 & 2n\cosh^2\alpha_n \end{pmatrix} \sim \mathrm{diag}(0, n, n, 2n)$$

✓ reproduce the original spectrum

Lesson:

If there exists Bogoliubov transformation under which Hamiltonian is J-Hermitian, one can recover unitarity (real energy).

We are studying the various vacua of higher-derivative theories in the on-going works. [Bekaert, Mehta, JY]

α-vacua

- Bogoliubov transformation of the vacuum $|0\rangle$

$$|\alpha\rangle = \frac{e^{-i\mathcal{G}_\alpha}}{\sqrt{\mathcal{N}}}|0\rangle = \prod_{n>0}\left(\frac{\cosh\alpha_n + \sinh\alpha_n \frac{1}{n} b_{-n} c_{-n}}{\sqrt{\cosh 2\alpha_n}}\right)|0\rangle$$

✓ annihilated by $\tilde{b}_n\,,\ \ \tilde{c}_n \quad (n>0)$

✓ Similar to TFD state: maximally entangled state of Fock space $\mathcal{H}_b$ and $\mathcal{H}_c$ created by the oscillators b and c, respectively

e.g. reduced density matrix (by tracing out $\mathcal{H}_c$)

$$\rho_b = \bigotimes_{n>0}\left(\frac{\cosh^2\alpha_n}{\cosh 2\alpha_n}|0\rangle\langle 0| + \frac{\sinh^2\alpha_n}{\cosh 2\alpha_n} b_{-n}|0\rangle\langle 0|b_n\right)$$

: becomes thermal state for $\alpha_n = e^{-\frac{\beta|n|}{2}}$

α-vacua

- ✳ α-vacua

$$|\alpha\rangle = \frac{e^{-i\mathcal{G}_\alpha}}{\sqrt{\mathcal{N}}}|0\rangle = \prod_{n>0}\left(\frac{\cosh\alpha_n + \sinh\alpha_n \frac{1}{n} b_{-n}c_{-n}}{\sqrt{\cosh 2\alpha_n}}\right)|0\rangle$$

✓ $e^{-i\mathcal{G}_\alpha}$ is not J-unitary because $\mathcal{G}_\alpha$ is not J-Hermitian

$\mathcal{N} = \prod_{n>0}\cosh 2\alpha_n$: non-trivial normalization

cf) For usual TFD vacuum, $e^{-i\mathcal{G}_\alpha}$ is unitary and $\mathcal{N} = 1$

✓ Not invariant under J action i.e. $J|\alpha\rangle = |-\alpha\rangle \neq |\alpha\rangle$

✓ Namely, the vacuum $|0\rangle$ is the unique state that is invariant under J action

Correlation Function w.r.t. α-vacua

* Two point function with respect to α-vacuum with J-inner product

$$\langle \partial\bar{\psi}(z)\partial\psi(w)\rangle_{J,\alpha} = \frac{1}{4\pi z w}\sum_{n>0} n\left[\left(\frac{w}{z}\right)^n \frac{\cosh^2\alpha_n}{\cosh 2\alpha_n} - \left(\frac{z}{w}\right)^n \frac{\sinh^2\alpha_n}{\cosh 2\alpha_n}\right]$$

  - ✓ J-inner product is crucial because α-vacuum is not invariant.

    cf) J-inner product does not play a crucial role in correlation functions w.r.t. $|0\rangle$

  - ✓ A function of z/w: can diverges for $z = w$

* For the same value of $\alpha_n = \alpha$, we have

$$\langle \partial\bar{\psi}(z)\partial\psi(w)\rangle_{J,\alpha} = \frac{1}{8\pi}\frac{\sqrt{\frac{w}{z}} + \sqrt{\frac{z}{w}}}{(z-w)^2}$$

  - ✓ Independent of α

Naive Correlation Function w.r.t. α-vacua

- Two point function w.r.t. α-vacuum with naive inner product

$$\langle \partial\bar{\psi}(z)\partial\psi(w)\rangle_\alpha = \frac{1}{4\pi z w}\sum_{n>0} n\left[\left(\frac{w}{z}\right)^n \cosh^2\alpha_n + \left(\frac{z}{w}\right)^n \sinh^2\alpha_n + \left((zw)^n + (zw)^{-n}\right)\sinh\alpha_n \cosh\alpha_n\right]$$

  - ✓ power series of z/w and zw: can diverge at $z = w$ and $zw = 1$

- For the same value of $\alpha_n = \alpha$, we have

$$\langle \partial\bar{\psi}(z)\partial\psi(w)\rangle_{\alpha_n=\alpha} = \frac{1}{8\pi}\frac{\sqrt{\frac{z}{w}}+\sqrt{\frac{w}{z}}}{(z-w)^2}\cosh(2\alpha) + \frac{1}{8\pi}\frac{1}{w^2}\frac{\sqrt{zw}+\sqrt{\frac{1}{zw}}}{\left(z-\frac{1}{w}\right)^2}\sinh(2\alpha)$$

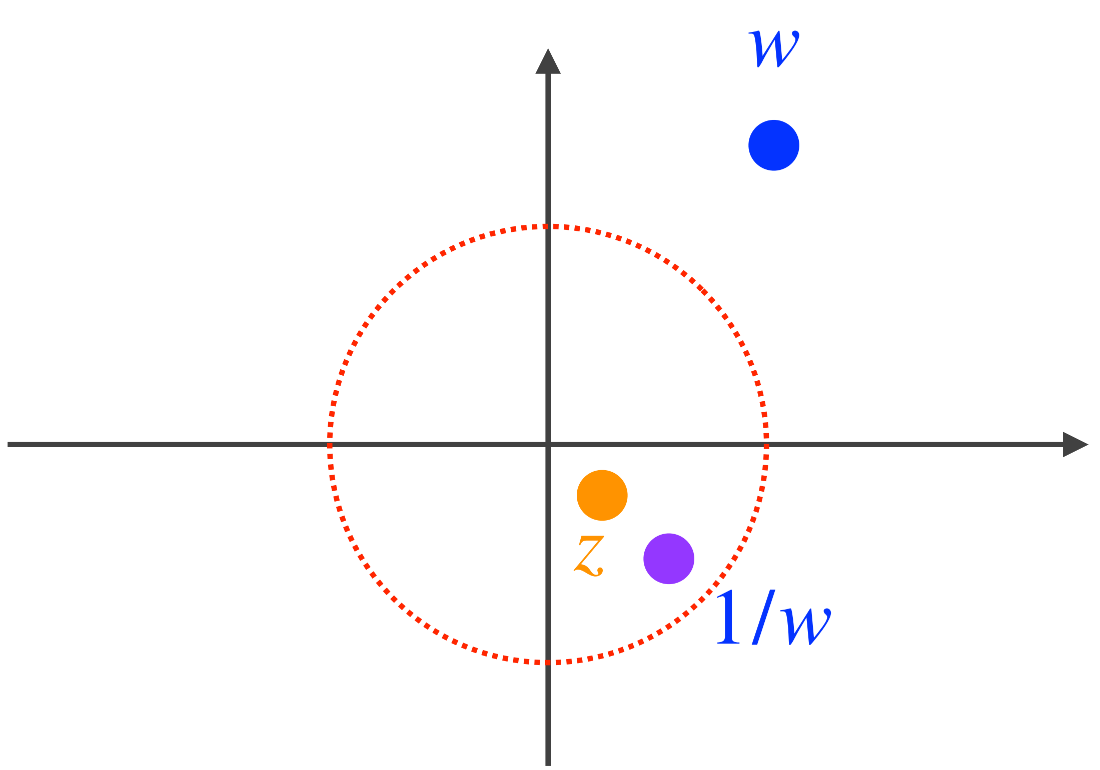
  - ✓ Depends on α
  - ✓ diverge at $z = w$ and $zw = 1$

Naive Correlation Function w.r.t. α-vacua

- For $\alpha_n = \alpha$, the naive two point function can be written as

$$\langle \partial\bar{\psi}(z)\partial\psi(w)\rangle_{\alpha_n=\alpha} = G_0(z,w)\cosh^2\alpha + \frac{1}{z^2w^2}G_0(1/z,1/w)\sinh^2\alpha + \frac{1}{2}\sinh 2\alpha\left[\frac{1}{w^2}G_0(z,1/w) + \frac{1}{z^2}G_0(1/z,w)\right]$$

  - ✓ where $G_0(z,w) = \langle 0 | \partial\bar{\psi}(z)\partial\psi(w) | 0\rangle$
  - ✓ Linear combination of two point functions $G_0(z,w)$ and $G_0(z,1/w)$



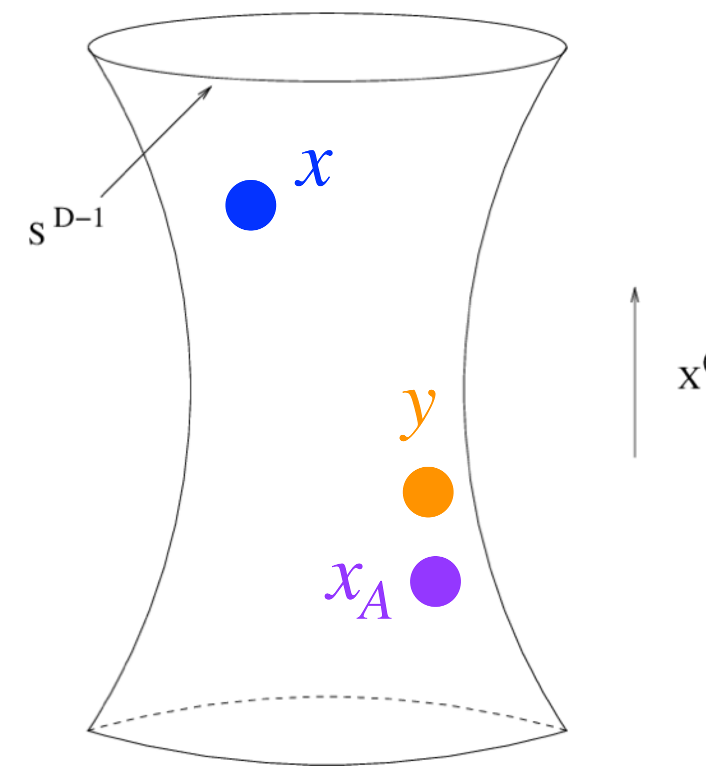
α-vacua in de Sitter Space

- Antipodal map in de Sitter space

X^a: embedding coordinate

x^μ: global dS coordinate

$$X^a(x) \xrightarrow{\text{Antipodal map}} X^a(x_A) = -X^a(x)$$



- Two-point function of massless conformally coupled scalar field in α-vacua of de Sitter space

$$G_\alpha(x,y) = G_E(x,y)\cosh^2\alpha + G_E(x_A,y_A)\sinh^2\alpha + \frac{1}{2}\sinh 2\alpha\left[G_E(x,y_A) + G_E(x_A,y)\right]$$

Two point function w.r.t. Bunch-Davies vacuum

Comparison

- Naive two point function in α-vacua of symplectic fermion

$$\langle \partial\bar{\psi}(z)\partial\psi(w)\rangle_{\alpha_n=\alpha} = G_0(z,w)\cosh^2\alpha + \frac{1}{z^2w^2}G_0(1/z,1/w)\sinh^2\alpha + \frac{1}{2}\sinh 2\alpha\left[\frac{1}{w^2}G_0(z,1/w) + \frac{1}{z^2}G_0(1/z,w)\right]$$

- Two point function in α-vacua of de Sitter space

$$G_\alpha(x,y) = G_E(x,y)\cosh^2\alpha + G_E(x_A,y_A)\sinh^2\alpha + \frac{1}{2}\sinh 2\alpha\left[G_E(x,y_A) + G_E(x_A,y)\right]$$

Summary

- In higher derivative (quadratic) fermi theory, the J-inner product and J-Hermitian conjugation is consistent with path integral formulation.
- Path integral measure can rescue the Ostrogradsky instability.
- CFT_2 with negative central charge can have non-negative norm.
- Expect that the symplectic fermion can shed light on the α-vacua problem in de Sitter space.
- Need to study interacting higher-derivative fermi theories carefully, which might be useful in understanding of dS/CFT correspondence.

Thank You