Discrete bulk spectrum in JT theory?

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Issues

Canonical quantization of gravity theories:

- Lorentzian approach vs Euclidean path integral.
- Diffeo (gauge) symmetries.
- Discrete bulk spectrum is essential for the finiteness of entropy.
  [In collaboration with CJ Kim & SH Yi]

JT Gravity dual to [Saad-Shenker-Stanford] Matrix Model

- The AdS2 dilaton gravity is consisting of

$$I = I_{top} + \frac{1}{16\pi G}\int_M d^2x\sqrt{g}\,\phi\,(R+2) + \frac{1}{8\pi G}\int_{\partial M} du\sqrt{-\gamma_{uu}}\;\phi(K-1)$$

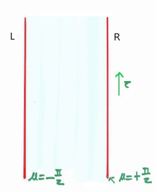
where

$$I_{top} \quad = \quad -\frac{S_0}{2\pi}\left[\frac{1}{2}\int_{M_E}\sqrt{g}R + \int_{\partial M_E}\sqrt{\gamma}K\right] \quad (\to e^{(2-2g-n)S_0})$$

- The equations of motion read

$$R+2=0 \; (\to AdS_2 \text{ with arbitrary genus } g\,)$$
$$\nabla_a\nabla_b\phi - g_{ab}\nabla^2\phi + g_{ab}\phi = 0$$

- The leading-order ($g = 0$) geometry is given by the gobal AdS_2



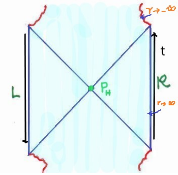
$$ds^2 = \frac{\ell^2}{\cos^2 \mu}\left(-d\tau^2 + d\mu^2\right)$$

where the coordinate μ is ranged from $-\pi/2$ to $+\pi/2$. (Two sided!)

- The vacuum (BH) solution for the dilaton field is given by

$$\phi = \bar{\phi}\, L \, \frac{\cos\tau}{\cos\mu} = \bar{\phi}\, r$$

- Make the coordinate transformation

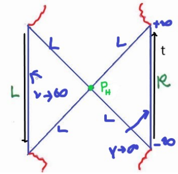


$$\frac{r}{L} = \frac{\cos\tau}{\cos\mu}\,, \qquad \tanh\frac{Lt}{\ell^2} = \frac{\sin\tau}{\sin\mu}$$

Two-sided black holes

- One is led to the corresponding AdS Rindler metric and the dilaton field:

$$ds^2 = -\frac{r^2 - L^2}{\ell^2}dt^2 + \frac{\ell^2}{r^2 - L^2}dr^2$$
$$\phi = \bar{\phi}\, r$$



The time $t\,(= u)$ runs over $(-\infty, \infty)$.

- The singularity is determined by $\Phi^2 = \phi_0 + \phi = 0$ where Φ^2 plays the role of radius squared of the transverse space from the viewpoint of the dimensional reduction from the higher dimensions as in

$$ds_4^2 = ds^2_{AdS_2} + r_0^2\, \Phi^2 (d\theta^2 + sin^2\theta d\varphi^2)$$

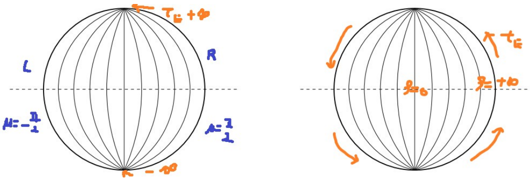
- The geometry is intrinsically two-sided!

Eclidean disk

- The Euclidean disk metric becomes

$$ds^2 = \frac{\ell^2}{\cos^2\mu}\left(d\tau_E^2 + d\mu^2\right)$$

where $-\infty < \tau_E < \infty$. This Euclidean disk may have two-sided interpretation.



- From the Rindler, the Euclidean disk metric becomes

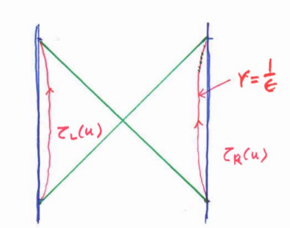
$$ds^2 = d\rho^2 + \sinh^2\rho\, dt_E^2$$

where $r = \cosh\rho$ and $t_E \sim t_E + 2\pi$.

Schwarzian description of the bulk – Bd particle picture

- Any bulk fluctuation can be reflected through the boundary fluctuation: This is nothing but the mode of reparameterization. These fluctuations are described by the Schwarzian dynamics.
- 1. First we take the dilaton as the size of transverse space and introduce a cut-off trajectory in term of the dilaton

$$\phi|_{\partial M} = \bar{\phi}\,\frac{1}{\epsilon}\ (= \bar{\phi}\, r)$$



- 2. The induced metric at the cut off trajectory will be fixed by

$$ds^2|_{\partial M} = -\frac{1}{\epsilon^2}du^2 \sim -\left(\frac{\tau'(u)\,du}{\cos\mu_{\partial M}(u)}\right)^2 + \cdots$$

- u is the physical (gauge-fixed) boundary time running from $(-\infty, +\infty)$. The remaining degrees are given by $\tau_r(u)$ and $\tau_l(u)$.

- Using the rules, evaluate the on shell gravity action. This leads to

$$S = -\mathcal{C}\int du\left[\left\{\tan\frac{\tau_l(u)}{2},u\right\}+\left\{\tan\frac{\tau_r(u)}{2},u\right\}\right]$$

$$\{f(u),u\} = -\frac{1}{2}\frac{f''^2}{f'^2}+\left(\frac{f''}{f'}\right)'$$

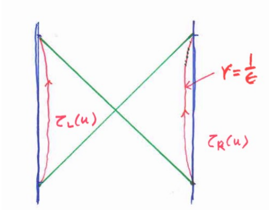
where $\mathcal{C}$ is the coupling with $\mathcal{C}=\bar{\phi}\,(=1/2)$.

- For our black hole spacetime, the Schwarzian equation of motion is solved by

$$\sin\tau(u)=\tanh\frac{2\pi}{\beta}u\,,\qquad \tau'(u)=\frac{2\pi}{\beta}\frac{1}{\cosh\frac{2\pi u}{\beta}}$$

where we set $\tau_l=\tau_r=\tau(u)$.

Therefore τ coordinate along the cut-off is ranged from $-\pi/2$ to $+\pi/2$.



Quantization [Penington-Witten 23]

- Let us try to quantize the L-R system which involves higher derivative terms

$$S = \int du\, L_r + \int du\, L_l\,, \qquad L_{r/l} = \frac{\mathcal{C}}{2}\left[\left(\frac{\tau''_{r/l}}{\tau'_{r/l}}\right)^2 - \tau'^2_{r/l}\right]$$

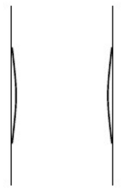
Introducing the Lagrange multiplier term, one may rewrite the above Lagrangian as

$$L_{r/l} = \frac{\mathcal{C}}{2}\chi'^2_{r/l} - \frac{1}{2\mathcal{C}}e^{2\chi_{r/l}} + p_{\tau_{r/l}}\left(\tau'_{r/l} - \frac{1}{\mathcal{C}}e^{\chi_{r/l}}\right)$$

$\Rightarrow$ Then by introducing the $p_{\chi_{r/l}}$ conjugated to the variable $\chi_{r/l}$, one finds left right Hamiltonians

$$L_{r/l} = p_{\tau_{r/l}}\tau'_{r/l} + p_{\chi_{r/l}}\chi'_{r/l} - H_{r/l}$$

$$H_{r/l} = \frac{1}{2\mathcal{C}}\left[p^2_{\chi_{r/l}} + 2p_{\tau_{r/l}}\, e^{\chi_{r/l}} + e^{2\chi_{r/l}}\right]$$



- Note $e^{\chi_{r/l}} = \mathcal{C}\tau'_{r/l} = \phi_{\partial M}\cos\mu_{\partial M}(u) \sim O(\frac{1}{\epsilon}\cdot\epsilon)$.

SL2 gauge symmetries

- The linear dependence in p_τ tells us that the Hamiltonian is unbounded from below. However (large diffeo) gauge symmetries will make the system well defined.
- SL2 ($\sim$ SO(2,1)) gauge symmetries: For this we use the embedding space coordinates of ads2: $-Y_1^2 + Y_2^2 + Y_3^3 = 1$ with

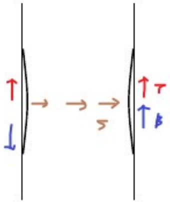
$$Y_1 = \tan\mu, \qquad Y_2 = \frac{\cos\tau}{\cos\mu}, \qquad Y_3 = \frac{\sin\tau}{\cos\mu}$$

$\Rightarrow$ SL2 rotation is an isometric transformation with Killing vectors [Lin-Maldacena-Zhao 19]

$\xi_1 = -\partial_\tau$ (τ-*translation*)

$\xi_2 = -\cos\tau\sin\mu\,\partial_\tau - \sin\tau\cos\mu\,\partial_\mu$ (*boost*)

$\xi_3 = -\sin\tau\sin\mu\,\partial_\tau + \cos\tau\cos\mu\,\partial_\mu$ (*scaling*)



- The SL2 gauge symmetries are generated by

$$\tilde{J}_i = J_i^{bulk} \ (bulk) + J_i^r \ (bd\text{-}R) + J_i^l \ (bd\text{-}L)$$

- Classically, one has the corresponding constraints, $\tilde{J}_i = 0$, and imposing these leads to consistent solutions of the left-right boundary dynamics.
- Quantum mechanically, imposing the constraints properly on the wave function

$$\tilde{J}_i \Psi = 0$$

leads to the full quantization of our gravity theory.

Gauge fixing and constraints [Penington-Witten 23]

- Our starting point is the unconstrained Hilbert space

$$\mathcal{H}_0 = L_2(\tau_r, \tau_I, \chi_r, \chi_I)$$

where the L_2 function is specified as a complex function of the four variables.

- We may use the quantization scheme based on the coinvariant equivalent classes defined by

$$\Psi \cong g\Psi$$

- We then introduce inner product by the integral

$$\langle \tilde{\Psi} | \Psi \rangle = \int dg\, (\tilde{\Psi}, g\Psi)$$

- This also ensures the constraints

$$\int dg (\tilde{\Psi}, \tilde{J}_i\, g\Psi) \cong 0$$

- One may set $\tau_r = \tau_l = 0$ and $\chi_r - \chi_l = 2q_{rel} = 0$ with $q = \chi_r + \chi_l$. Thus we choose a gauge-fixed wavefunction as

$$\Psi = \delta(\tau_r)\delta(\tau_l)\delta(q_{rel})\psi(q)$$

- The inner product is reduced to

$$\langle\tilde{\Psi}|\Psi\rangle = \int dq\, \tilde{\psi}^*(q)\psi(q)$$

- Our physical Hilbert space is reduced to $L_2(q)$. On this Hilbert space the left right Hamiltonian becomes

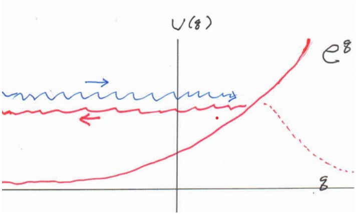
$$2\mathcal{C}H = 2\mathcal{C}H_{r/l} = p^2 + e^q$$

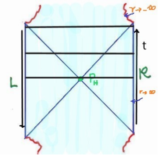
- $p = -i\partial_q$. $[H_r, H_l] = 0$. No factorization! [Harlow-Jafferis 18]

Liouville quantum-mechanics

- This is a Liouville quantum-mechanical system that involves an exponential potential $V(q) = e^q$.
- Note that the renormalized geodesic length between two boundary points $\tau_l\,(=0)$ and $\tau_r\,(=0)$ is given by

$$\ell_{ren} \equiv \ell_{bare}\,(\geq 0) - \cancelto{\infty_\epsilon}{\ln 2\phi|_r} - \cancelto{\infty_\epsilon}{\ln 2\phi|_l} = \ln\left(\frac{\cos^2\frac{\tau_r - \tau_l}{2}}{\mathcal{C}^2 \tau_l' \tau_r'}\right) = -q$$





Eigenvalue problem

- The left side is unbounded and the spectrum becomes of course continuous. Hence the density of state is ill defined.
- The corresponding eigenvalue problem

$$H\,\psi_s(q) = \frac{s^2}{2\mathcal{C}}\,\psi_s(q) \quad (s \in [0, \infty))$$

can be solved by

$$\psi_s(q) = N_s\,K_{2is}(2e^{q/2})\,, \qquad N_s = \frac{2}{\pi}(s \sinh 2\pi s)^{\frac{1}{2}}$$

- This satisfies the scattering normalization

$$\int_{-\infty}^{\infty} dq\,\psi_s^*(q)\psi_{s'}(q) = \delta(s - s')$$

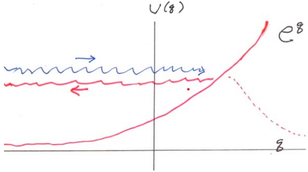
- In the scattering regime of $q \to -\infty$, the wavefunction behaves as

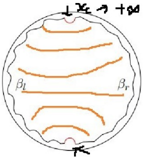
$$\psi_s \to \frac{\Gamma(-2is)}{\sqrt{2\pi}\,|\Gamma(-2is)|}\left(e^{isq} + R(s)e^{-isq}\right)$$

where the reflection amplitude may be identified as $R(s) = \frac{\Gamma(2is)}{\Gamma(-2is)}$.

- In the forbidden region of $q \to \infty$ (a small separation limit), the wavefunction decays doubly-exponentially as

$$\psi_s(q) \to N_s \sqrt{\frac{\pi}{4e^{q/2}}}\, e^{-2e^{q/2}}$$



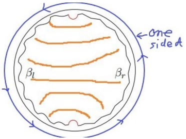


Disk partition function $Z_{0,1}$

- The relevant density of states is basically one-sided quantity whereas our physical Hilbert space is inherently two-sided.
- In this respect, currently there is no well-defined procedure computing the disk partition function based on the two-sided description. One can follow the proposal by [Lin-Maldacena-Rozenberg-Shan, 22]

$$Z(\beta) \propto \lim_{q_c \to \infty} \langle q_c | \, e^{-\frac{\beta}{2} H_{tot}} (= 2H) | q_c \rangle$$

- In this two-sided picture, one starts the evolution from an initial geodesic connecting two slightly separated boundary points somewhere on the bottom side. Namely $\Psi_I = \delta(q - q_c)$.



- Basically the evolution is based on the propagator with an appropriate initial state, which defines the path integral computation in the two-sided picture.

- With this regularization, one finds

$$Z_{disk}(\beta) \propto \lim_{q_c \to \infty} W(q_c) \frac{1}{2\pi^2} \int_0^\infty ds\, s \sinh 2\pi s\, e^{-\beta s^2}$$

where

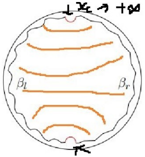
$$W(q) = 4\pi\, e^{-4e^{q/2} - q/2}$$

- The disk partition function may be identified as

$$Z_{disk}(\beta) = \mathrm{tr}\, e^{-\beta H} = \frac{e^{S_0}}{2\pi^2} \int_0^\infty ds\, s \sinh 2\pi s\, e^{-\beta s^2}$$

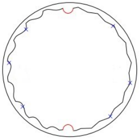
$$= \int_0^\infty ds \rho_{JT}(s)\, e^{-\beta E(s)} = \frac{e^{S_0}}{4\sqrt{\pi}\beta^{\frac{3}{2}}} e^{\frac{\pi^2}{\beta}}$$ [Stanford Witten 17]

where $\rho_{JT}(E) = \frac{e^{S_0}}{4\pi^2} \sinh 2\pi\sqrt{E}$ by an inverse Laplace transform.

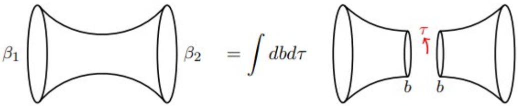


Disk correlation function ($g = 0$, $n = 1$) + double-trumpet ($g = 0, n = 2$)

- Euclidean disk correlation function can be computed using the two-sided picture. The result agrees with those of the Euclidean path integral [Penington,Witten].



- For the double trumpet geometries, purely two-sided description is not known. The result from Euclidean gravitational path integral is known. Two trumpets with size b and integrations. $\to Z_{2,0}$



SSS duality–a review

- The matrix model partition function $\mathcal{Z}$ (for an $N \times N$ Hermitian matrix H) is given by

$$\mathcal{Z} = \int dH \; e^{-N \mathrm{tr}\, U(H)}$$

- In the matrix model, it is convenient to introduce the so-called resolvent

$$R(E) = \mathrm{tr}\, \frac{1}{E - H}$$

which is related to the density of states as

$$R(E + i\epsilon) - R(E - i\epsilon) = -2\pi i \rho(E)$$

- Here, the density of states is defined by

$$\rho(E) \equiv \mathrm{tr}\, \delta(E - H) = \sum_{j=1}^{N} \delta(E - \lambda_j), \qquad Z = \mathrm{tr}\, e^{-\beta H}$$

where λ_i are the eigenvalues of the matrix H.

Double-scaling limit and genus expansions

- Taking into account the Vandermonde factor in the Hermitian matrix model, the large N saddle point equation is given by

$$U'(E) = \frac{2}{N} \fint d\lambda \, \frac{\rho(\lambda)}{(E-\lambda)}$$

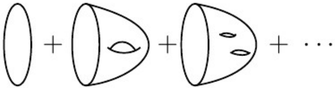
- As a specific JT density of states, SSS suggested the following expression:

$$\rho(E,a) = \frac{e^{S_0}}{4\pi^2} \sinh 2\pi \sqrt{E\left(1 - E/2a\right)}\,, \qquad N = \int_0^{2a} dE\, \rho(E,a)$$

  which determines $U(E,a)$ from the saddle point equation.
- The double-scaling limit is defined by $N \to \infty$ and $a \to \infty$, while keeping e^{S_0} finite (e^{-S_0} is the level spacing!). This leads to

$$\lim_{a\to\infty} \rho(E,a) = \rho_{JT}(E)$$

Topological genus expansion



- The connected part of n resolvents (the n connected boundaries in the geometric side) is encoded in the topological expansion

$$\langle R(E_1)\cdots R(E_n)\rangle_{conn} \simeq \sum_{g=0}^{\infty} e^{S_0(2-2g-n)} R_{g,n}(E_1,\cdots,E_n)$$

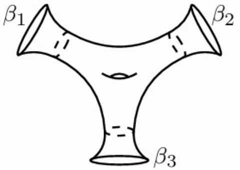
- This may be rewritten in terms of the correlators of the partition functions $\langle Z(\beta_1),\cdots,Z(\beta_n)\rangle_{conn}$.
- The genus zero resolvant

$$R_{0,1}(E) = y = -\pi i \hat{\rho}_{JT}(s=\sqrt{E})$$

gives us the so-called spectral curve of the matrix model. This plays the role of initial data for the topological recursion relation of the resolvent correlators. $\rightarrow$ Full perturvation theory!

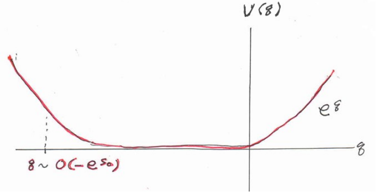
JT-SSS correspondence

$$Z_{1,3}(\beta_1, \beta_2, \beta_3) =$$



- These correlators are known to satisfy specific topological recursion relations and then related to Weil-Petersson volume of the moduli space of a genus g surface with n geodesic boundaries of length $b_1, \cdots, b_n$.
- According to the SSS duality, all such correlators can be determined completely by two initial inputs: disk partition function $Z_{disk}(\beta)$ and trumpet partition function $Z_{trumpet}(\beta, b)$.
- Both of these quantities are computed from Schwarzian boundary wiggles in Euclidean pure JT gravity. $\rightarrow$ A precise agreement!

Left confining potential

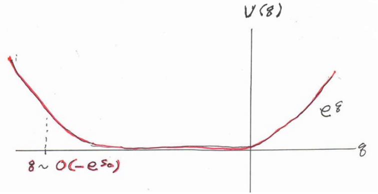


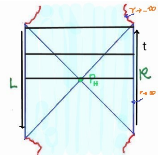
- The needs for the left confining potential may be argued in the following ways.
- First of all, the spectrum is continuous, which is in contradiction with **the finite density of states with the finite level spacing.**
- Note the complexity operator may be identified with $\ell_{ren} = -q$ where ℓ_{ren} is the geodesic length. With the Liouville Hamiltonian, we have

$$\frac{d^2}{dt^2}\langle q\rangle_{tfd} = -2\langle e^q\rangle_{tfd}$$

where we use the TFD state as an initial states.

Confining Potential in Lorentzian picture





- As $\ell_{ren} = -q$ becomes large, the force in the right side becomes negligible and

$$\ell_{ren} = -\langle q \rangle_{tfd} \sim C_1\, t$$

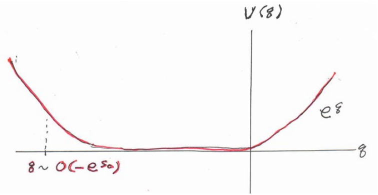
  with C_1 to be an O(1) positive coefficient. Even including the perturbative and nonperturbative contributions the above behaviors continue until $t \ll e^{S_0}$.

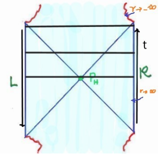
- It was further shown that [Iliesiu, Mezei and Sarosi 21]

$$\ell_{ren} = -\langle q \rangle_{tfd} \to e^{S_0}\, C_2$$

  as $t \gg e^{S_0}$ where C_2 is another O(1) positive coefficient, which has a nonperturbative nature and universal for any QM with a discrete spectrum.

Left confining potential W



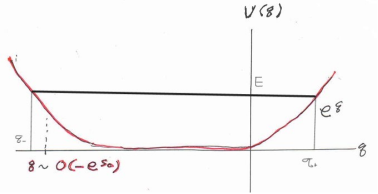


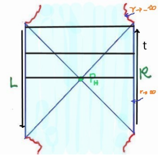
- With the confining potential the spectrum naturally becomes discrete. We assume

$$V(q) = e^q + W(q)$$

  and determine the form of W explicitly.
- The left confining potential $W(q)$ becomes $O(1)$ only when $\ell_{ren} = -q$ becomes of $O(e^{S_0})$.
- As e^{S_0} goes to infinity, the effect of the confining potential disappears completely leading to the continuous spectrum.

Confining potential W





- Let us obtain the shape of the left-confining potential W which reproduce the desired JT density of states

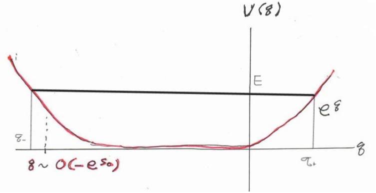
$$\rho_{JT}(E) = e^{S_0}\hat{\rho}(E), \qquad \hat{\rho}(E) = \frac{1}{4\pi^2}\sinh 2\pi\sqrt{E}$$

- The density of states in the semiclassical limit with the left right confining potential is given by

$$\frac{1}{\pi}\frac{d}{dE}\int_{q_-}^{q_+} dq\sqrt{E - V(q)} = e^{S_0}\hat{\rho}(E)$$

where the left and right turning points $q_{\mp}$ are defined by the relation $E = V(q_{\pm})$.

Left Confining Potential



- Now let us consider the potential of the form

$$V(q) = e^q + W(X(q))$$

with

$$X(q) = e^{-S_0}\left[\log(1 + e^{-q-a}) + v(q)\right]$$

without loss of any generality. $a, v(q) \sim O(1)$.

- One then finds $q_+ = O(1)$ and $q_- = -O(e^{S_0})$.
- Therefore we get

$$\frac{1}{\pi}\frac{d}{dE}\int_0^{X_0} dX\sqrt{E - W(X)} = \hat{\rho}(E)$$

where $W(X_0) = E$. Abel's first integral!

JT solution to string equation [Okuyama and Sakai 19] [Johnson and Rosso 20] [Johnson 22]

- This is solved by

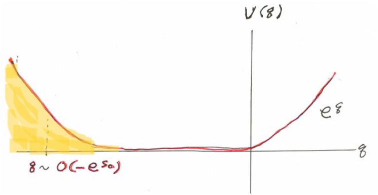
$$2\pi X = \sqrt{W(X)} I_1(2\pi\sqrt{W(X)})$$

  which is the JT solution to the "string equation".
- For small X, one finds

$$W(X) = 2X + O(X^2)$$

- For the large X,

$$W(X) = \left[\frac{1}{2\pi} \ln\left((2\pi)^{3/2} X\right)\right]^2 (1 + O(\ln \ln X / \ln X))$$



Formulation of the string equation [Johnson 22]

- With the technique of orthogonal polynomials, one can get a new formulation of MM perturbation theory with a fictitious quantum system

$$\langle Z(\beta)\rangle = \int_{-\infty}^{\infty} dE \langle \rho(E)\rangle e^{-\beta E} = \int_{-\infty}^{\mu} dy \, \langle y|e^{-\beta\mathcal{H}}|y\rangle$$

where $\mathcal{H} = -\hbar^2\partial_y^2 + u(y)$ with $\hbar = e^{-S_0}$.

- Now in the semiclassical limit with $\hbar \to 0$, one finds

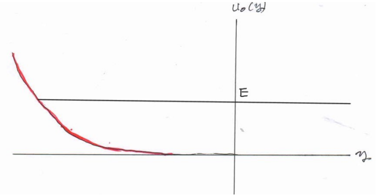
$$\rho(E) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\mu} dy \frac{1}{\sqrt{E - u_0(y)}}$$

- With $\rho(E) = \rho_{JT}(E)$ and $\mu = 0$, one finds

$$y + \frac{\sqrt{u_0}}{2\pi} I_1(2\pi\sqrt{u_0}) = 0$$

which agrees with our equation with the replacement $-y \to X$ and $u_0(y) \to W(X)$.

The QM is unphysical!

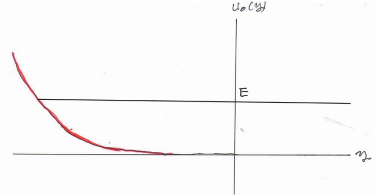


- This quantum mechanics problem is rather unconventional and unphysical.
- Note one needs the relation

$$\int_{-\infty}^{\mu} dy|y\rangle\langle y| = \int_{-\infty}^{\infty} dE|E\rangle\langle E|$$

the eigenvalue problem for $\mathcal{H}\psi(y,E) = E\psi(y,E)$ in the conventional framework of quantum mechanics must be supplemented by a self-adjoint boundary condition at $y = \mu$, which is, in fact, NOT the case here.

This QM is unphysical!



- If a boundary condition or some wall potential were imposed, the spectrum would become discrete, as the configuration space would then be confined to the region $y \in (-\infty, \mu)$.
- However, it is clear that E is continuous because there is no right confining potential.

Krylov-Complexity Dynamics [Balasubramanian,caputa, Magan,Wu 22]

- Krylov spread complexity: $C - C_0 = \langle l_{ren} \rangle = -\langle q \rangle$
- We can take the thermofield double state

$$|\psi(t)\rangle_{tfd} = \frac{1}{\sqrt{Z}} \sum_n e^{-(\frac{\beta}{2}+it)E_n} |n, n\rangle$$

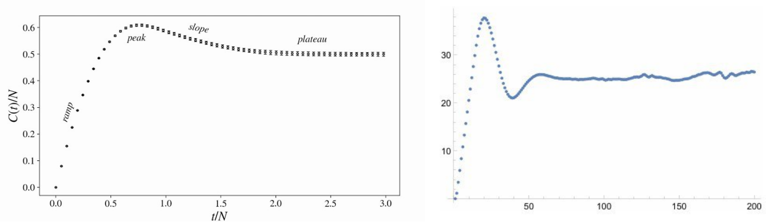
  as an initial state.
- We have

$$\frac{d}{dt}\langle q \rangle_{tfd} = -i\langle [q, H] \rangle_{tfd} = 2\langle p \rangle_{tfd}$$
$$\frac{d}{dt}\langle p \rangle_{tfd} = -i\langle [p, H] \rangle_{tfd} = -\langle V' \rangle_{tfd}$$

- One has a cancellation in the late time ($t \gg e^{S_0}$)

$$2\langle p \rangle_{tfd} = -i\langle [q, H] \rangle_{tfd}$$
$$= \frac{i}{Z} \sum_{m,n} e^{-\frac{\beta}{2}(E_m+E_n)} e^{it(E_m-E_n)} (E_m - E_n) \langle m, m|q|n, n\rangle \simeq 0$$

Krylov spread complexity Balasubramanian, Kar, Li, Parrikar,2022



- Left panel: For $N \times N$ Gaussian ensemble, we depict the time evolution of the complexity. One can see the patterns of **ramp-peak-slope-plateau.**
- We show that $C = \ell_{ren} = -\langle q \rangle$ follows the basically the same pattern with the confining potential adding the random potential part. (See the right panel!)

Higher genus contributions + higher dimensions?

- The theory will be defined with the ensemble average

$$\langle \lambda_n \rangle = \int_{(v,a)} da\, \mathcal{D}v\; \mathcal{P}(v,a)\, \lambda_n(v,a)\ (n = 1, 2, \cdots)$$

  with the weight $\mathcal{P}(v,a)$. These weight should be determined by the original gravity theory or the matrix theory.
- For the higher genus contributions, further work is necessary.
- This should be generalized to higher dimensions since the concept of complexity ($\sim$ wormhole volume) is universal!

Thanks a lot for your attention!