Higher-Order Krylov State Complexity in Random Matrix Quenches

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The complexity crew in GIST

Talk based on

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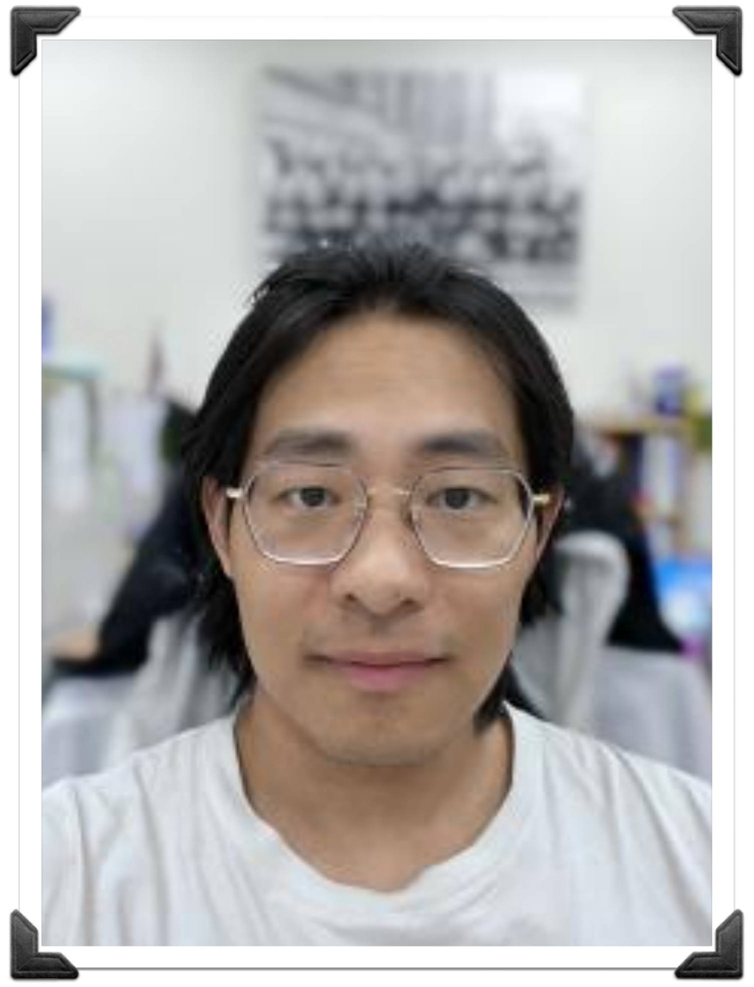

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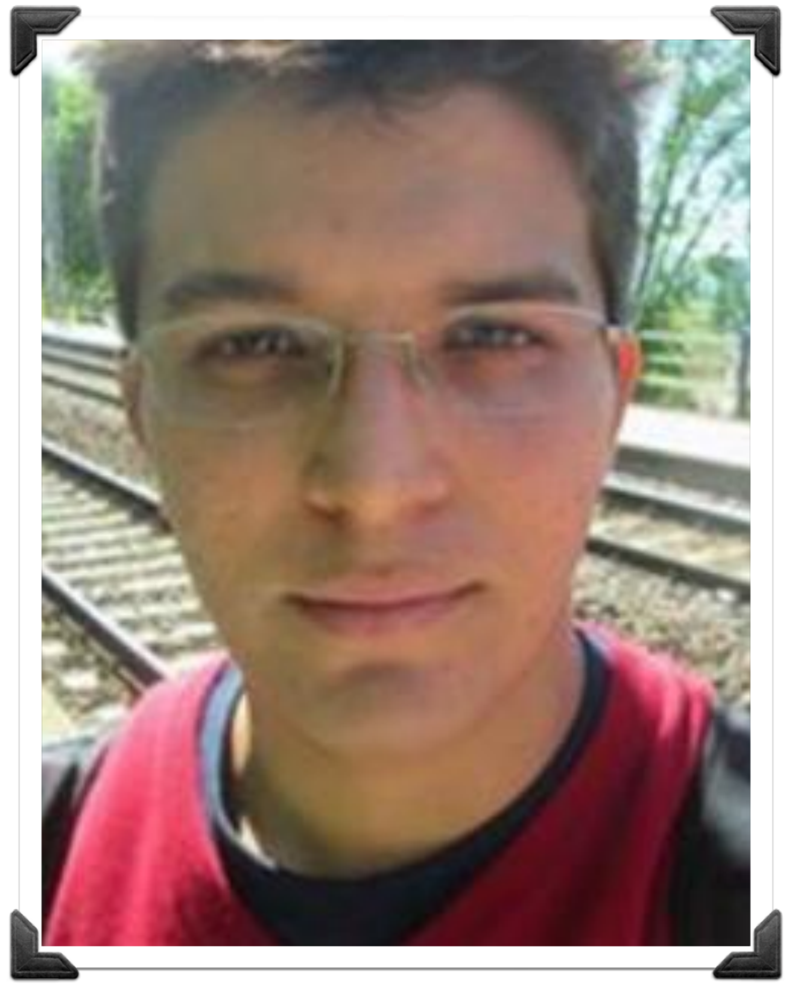

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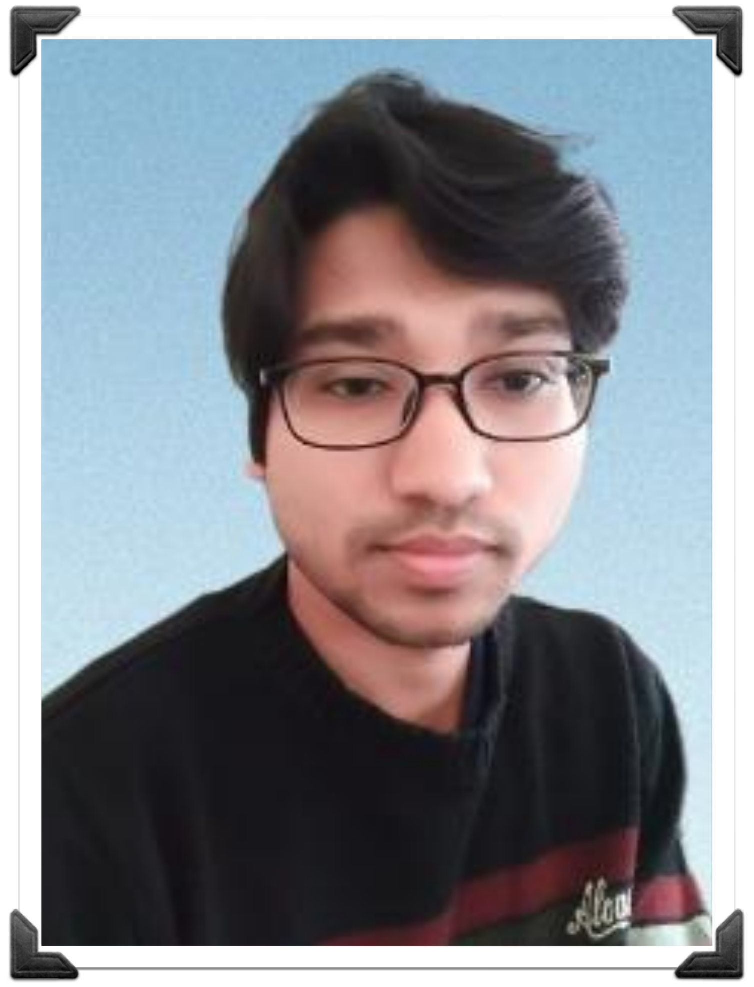


Kuntal Pal

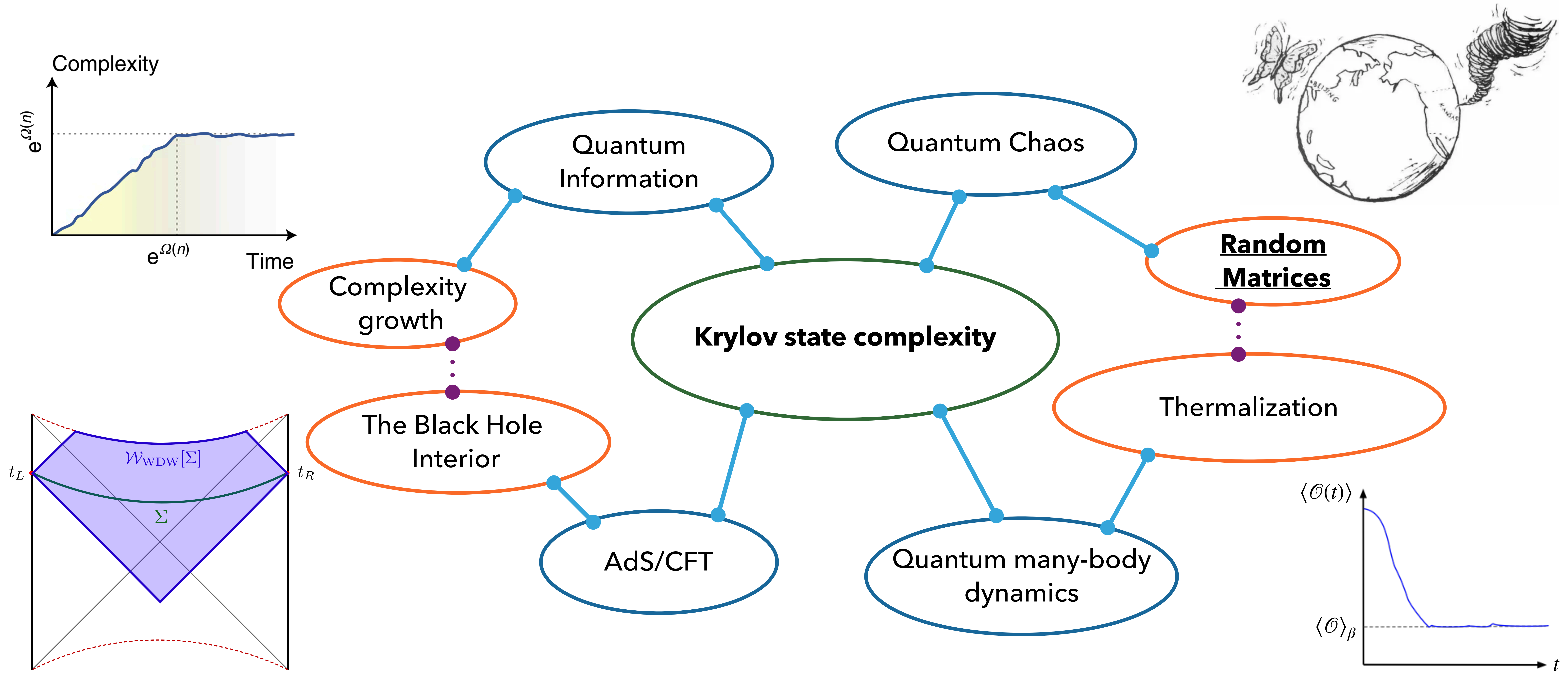
Background and Motivation

Recently, there has been a renewed interest in studying state and operator dynamics in **Krylov space.** This has been a fruitful pursuit, leading to novel measures of state and operator complexity and new avenues to study quantum chaos in many-body systems and holography.

- Relation to out-of-time-order correlators (OTOCs) and a new conjectured universal chaos bound (universal operator growth hypothesis: [Parker, Cao, Avdoshkin, Scaffidi, Altman (2019)]).
- Connections with holographic complexity in the context of DSSYK/JT gravity ([Rabinovici, Sánchez-Garrido, Shir, Sonner (2023)], [Balasubramanian, Magan Nandi, Wu (2024)]) and momentum-complexity growth rate correspondence ([Caputa, Chen, McDonald, Simón, Strittmatter (2024)]).
- New tools to study long-time quantum chaos and encoding of RMT behavior (e.g. spectral rigidity) ([Balasubramanian, Magan, Wu (2022, 2023)], [Erdmenger, Jian, Xian (2023)], [Alishahiha, Banerjee, Javad Vasli (2024),…]).
- New connections between quantum chaos and quantum computation ([Craps, Evnin, Pascuzzi (2023)]).
- New approaches to study operator growth in open quantum systems ([Bhattacharya, Nandy, Nath, Sahu (2022, 2023), Bhattacharjee, Nandy, Pathak (2023), Nandy, Pathak, Tezuka (2024),…])

Background and Motivation

- **Krylov state complexity**, also known as spread complexity [Balasubramanian, Caputa, Magan, Wu (2022)], has played a central role in the previous developments.



This Talk

In this talk, I will discuss Krylov state complexity and its higher-order generalizations in the context of quantum quenches involving **random matrices**.

(1) Krylov state complexity and signatures of quantum chaos

(2) Higher-order generalizations and statistics of the spreading operator

(3) Random matrix quenches

(4) Results

The Lanczos Algorithm

- Basic idea: study the time evolution of states in dynamical quantum-mechanical systems. For a time-independent Hamiltonian H:

$$i\partial_t|\psi(t)\rangle = H|\psi(t)\rangle \quad (|\psi_0\rangle \equiv |\psi(t=0)\rangle) \implies |\psi(t)\rangle = e^{-iHt}|\psi_0\rangle \equiv \sum_{n\geq 0}\frac{(-it)^n}{n!}H^n|\psi_0\rangle = \sum_{n\geq 0}\frac{(-it)^n}{n!}|\psi_n\rangle$$

- The states $|\psi_n\rangle := H^n|\psi_0\rangle$ form a basis of the **Krylov subspace** $\mathscr{K}$ associated with $|\psi_0\rangle$, a subspace of the full Hilbert space $\mathscr{H}$.

- Using the **Lanczos algorithm**, it is possible to construct an orthonormal basis (**Krylov basis** $\mathbf{K}$) in Krylov subspace $\mathscr{K}$ which brings the Hamiltonian H to a Hessenberg (or tridiagonal) form ([Viswanath & Müller (1994)]).

Initial state and dynamics	→	Unnormalized Basis in Krylov space $\mathscr{K}$	→	Lanczos algorithm	→	**Krylov basis**
$\{\|\psi_0\rangle, H\}$		$\|\psi_n\rangle := H^n\|\psi_0\rangle$		$\begin{cases} \|A_{n+1}\rangle = (H-a_n)\|K_n\rangle - b_n\|K_{n-1}\rangle \\ \|K_n\rangle = b_n^{-1}\|A_n\rangle \quad (n>0)\end{cases}$ $b_n := \langle A_n\|A_n\rangle^{1/2} \quad a_n := \langle K_n\|H\|K_n\rangle$		$\mathbf{K} := \{\|K_n\rangle\}$ $\langle K_m\|K_n\rangle = \delta_{mn}$ (with $\|K_0\rangle = \|\psi_0\rangle$ and $b_0 = 0$))

The Lanczos Algorithm

- The Lanczos algorithm also yields the **Lanczos coefficients** $\{a_n, b_n\}$

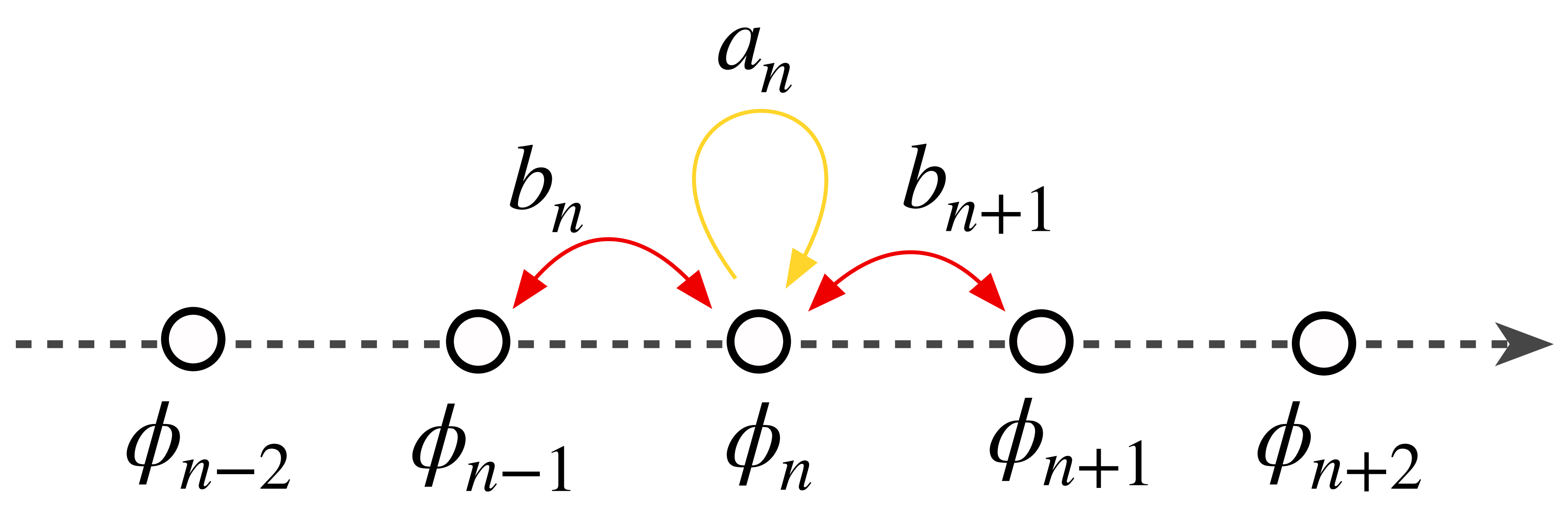
$$\langle\psi_m|H|\psi_n\rangle \sim \begin{pmatrix} *_{11} & *_{12} & *_{13} & *_{14} & \cdots \\ *_{21} & *_{22} & *_{23} & *_{24} & \cdots \\ *_{31} & *_{32} & *_{33} & *_{34} & \cdots \\ *_{41} & *_{42} & *_{43} & *_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \xrightarrow{\text{Lanczos algorithm}} \langle K_m|H|K_n\rangle \sim \begin{pmatrix} a_0 & b_1 & 0 & 0 & \cdots \\ b_1 & a_1 & b_2 & 0 & \cdots \\ 0 & b_2 & a_2 & b_3 & \cdots \\ 0 & 0 & b_3 & a_3 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

(in general, not an orthonormal basis) — (orthonormal basis)

- In the Krylov basis $\mathbf{K}$, the coefficients $\{\phi_n(t)\}$ of the time-evolved state $|\psi(t)\rangle$ have the interpretation of **probability amplitudes**:

$$|\psi(t)\rangle = \sum_{n\geq 0} \phi_n(t)\,|K_n\rangle \qquad \begin{cases} \phi_n(t) := \langle K_n|\psi(t)\rangle = \langle K_n|e^{-iHt}|\psi_0\rangle \quad \in \mathbb{C} \\ \sum_{n\geq 0} |\phi_n(t)|^2 \equiv \sum_{n\geq 0} p_{\mathbf{K}}(n,t) = 1 \quad \forall t \end{cases}$$

and the Schrödinger equation describes the hopping of a particle on a 1-dimensional lattice (the **Krylov chain**):



$$i\partial_t\phi_n(t) = a_n\phi_n(t) + b_{n+1}\phi_{n+1}(t) + b_n\phi_{n-1}(t)$$

Key: $\phi_0(t) := \langle\psi_0|\psi(t)\rangle$ (Survival Amplitude)

Krylov State Complexity

Initially, the state is localized in Krylov space $|\psi_0\rangle = |K_0\rangle$. During time evolution, it "spreads" in Krylov space, acquiring contributions from more Krylov basis states $|K_n\rangle$.

A way of measuring the spread of $|\psi_0\rangle$ in the Krylov space $\mathscr{K}$ is by computing the **average position** of the time-evolved state $|\psi(t)\rangle$ in the Krylov basis $\mathbf{K}$:

$$C_\psi(t) := \sum_{n\geq 0} n\,|\langle K_n|\psi(t)\rangle|^2 = \sum_{n\geq 0} n\,p_{\mathbf{K}}(n,t) = \sum_{n\geq 0} n\,|\phi_n(t)|^2$$

This is the Krylov state complexity of $|\psi_0\rangle$. One can also view it as the expectation value of the **spreading operator** $\hat{n}_\psi : \mathscr{K} \to \mathscr{K}$, given by $\hat{n}_\psi = \sum_{n\geq 0} n\,|K_n\rangle\langle K_n|$ in the time-evolved state $|\psi(t)\rangle$

$$\langle \hat{n}_\psi\rangle_t := \langle\psi(t)|\hat{n}_\psi|\psi(t)\rangle = \langle\psi_0|\hat{n}_\psi(t)|\psi_0\rangle = \sum_{n\geq 0} n\,|\phi_n(t)|^2 \equiv C_\psi(t) \qquad \left(\hat{n}_\psi(t) = e^{itH}\hat{n}_\psi e^{-itH}\right)$$

Cost Function Interpretation of Krylov State Complexity

In [Balasubramanian, Caputa, Magan, Wu (2022)] it was argued that the Krylov basis $\mathbf{K}$, generated by applying the Lanczos algorithm to $\{\,|\psi_n\rangle = H^n|\psi_0\rangle\}_{n\geq 0}$, minimizes the complexity **cost functional**

$$C_{\mathbf{B}}(t) = \sum_{n\geq 0} n\,|\langle B_n|\psi(t)\rangle|^2 = \sum_{n\geq 0} n\,p_{\mathbf{B}}(n,t)$$

over all possible choices of complete, orthonormal, and ordered bases $\mathbf{B} = \{\,|B_n\rangle\}_{n\geq 0}$, namely

$$C_\psi(t) = \min_{\mathbf{B}}\{C_{\mathbf{B}}(t)\}$$

This was shown to hold over a finite interval of time around $t = 0$ in continuous time evolution, using arguments related to the Taylor series coefficients of $C_{\mathbf{B}}(t)$, as well as for all times in discrete time evolution implemented by sequences of unitaries.

Krylov state complexity of the TFD state

Krylov state complexity can be defined for *any* state $|\psi_0\rangle$. However, one interesting state to consider as the initial state is the TFD state. This allows for comparison with other spectral quantities, such as the spectral form factor (SFF).

$$|\psi_0\rangle = |\text{TFD}(\beta)\rangle := \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\frac{\beta E_n}{2}} |n\rangle \otimes |n\rangle \qquad \left(Z(\beta) = \text{tr}\left(e^{-\beta H}\right) = \sum_n e^{-\beta E_n} \right)$$

In this case, the first probability amplitude $\phi_0(t)$ (given by the return amplitude of the TFD state) is directly related to the SFF:

$$\phi_0(t) := \langle \text{TFD}(\beta) | \text{TFD}(\beta + 2it) \rangle = \frac{Z(\beta + it)}{Z(\beta)} \sim \sqrt{\text{SFF}(\beta, t)} \qquad \text{SFF}(\beta, t) := \frac{|Z(\beta + it)|^2}{|Z(\beta)|^2} = \frac{1}{|Z(\beta)|^2} \sum_{n,m} e^{-\beta(E_n + E_m)} e^{it(E_n - E_m)}$$

The Krylov state complexity of the TFD state depends only on the spectrum of the theory $\{E_n, |n\rangle\}$ and β.

- At early times: $C_\psi(t) \approx b_1^2 t^2$

  (For any $|\psi_0\rangle$)

- Saturation value: $\lim_{t \to \infty} C_\psi(t) = \frac{d-1}{2}$

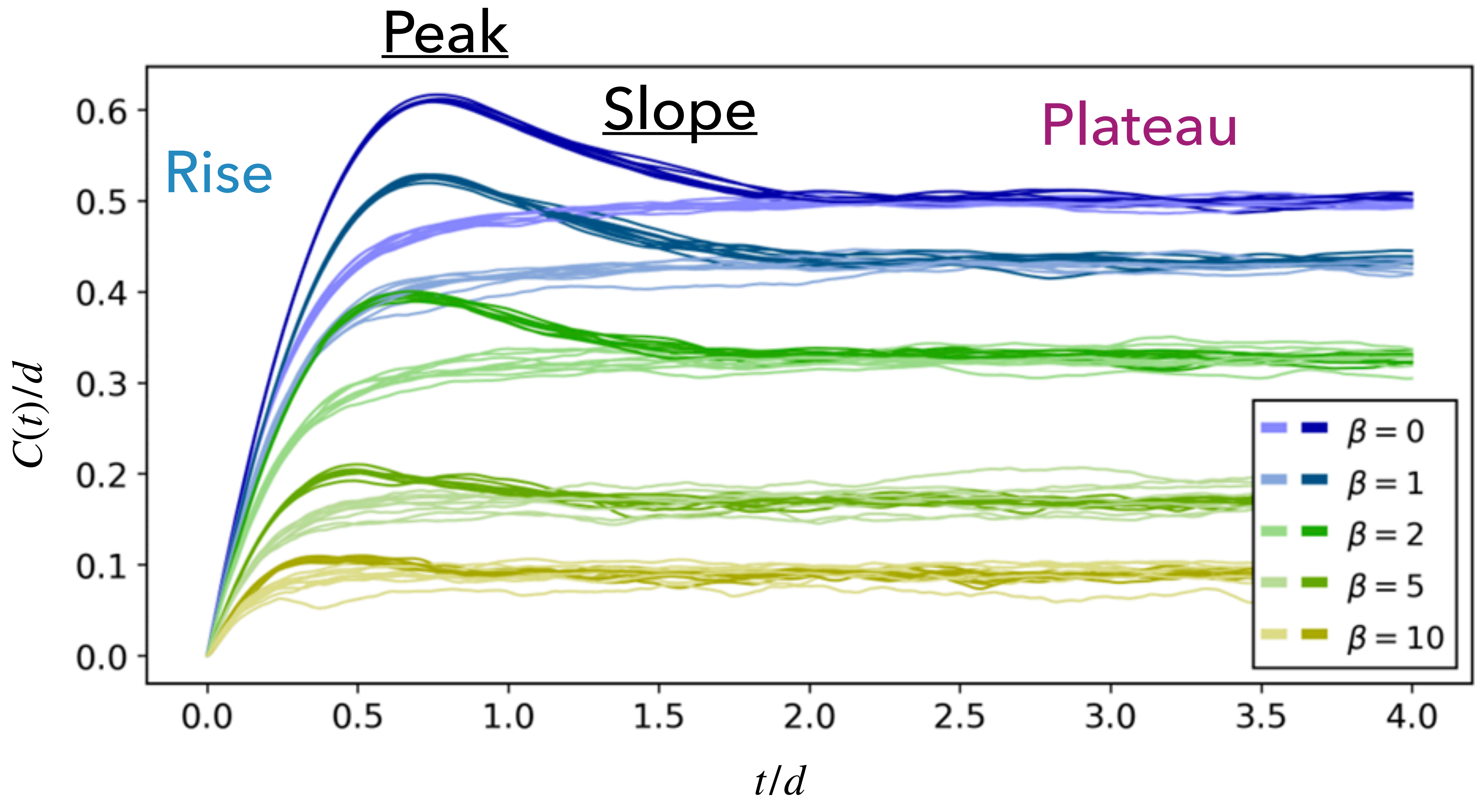
  (For $|\psi_0\rangle = |\text{TFD}(\beta = 0)\rangle$.)

  (d = Hilbert space dim.)

Signatures of Quantum Chaos

In [Balasubramanian, et al. (2022)] it was shown that the Krylov state complexity of the TFD state in Gaussian random matrix ensembles (GOE, GUE, GSE) has a prototypical shape akin to that of the **spectral form factor** (SFF):

Krylov state complexity of the time-evolved TFD state for realizations of GUE matrices of size $d \sim O(10^3)$ at finite β.



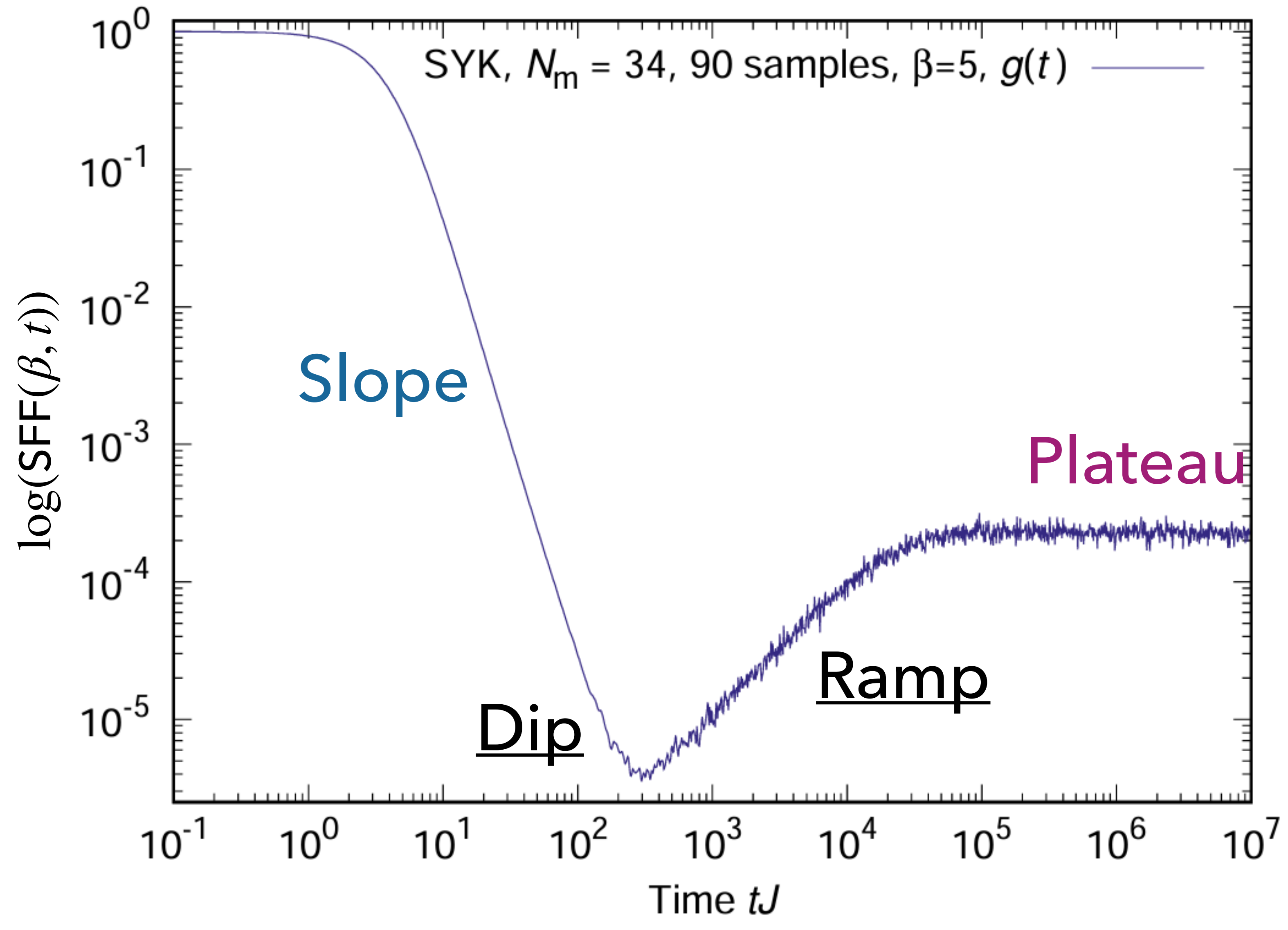
([Balasubramanian, Caputa, Magan, Wu (2022)])

In RMTs

$t_{dip} \sim O(\sqrt{d})$

$t_{peak} \lesssim O(d)$

The ramp in the SFF arises from spectral rigidity.

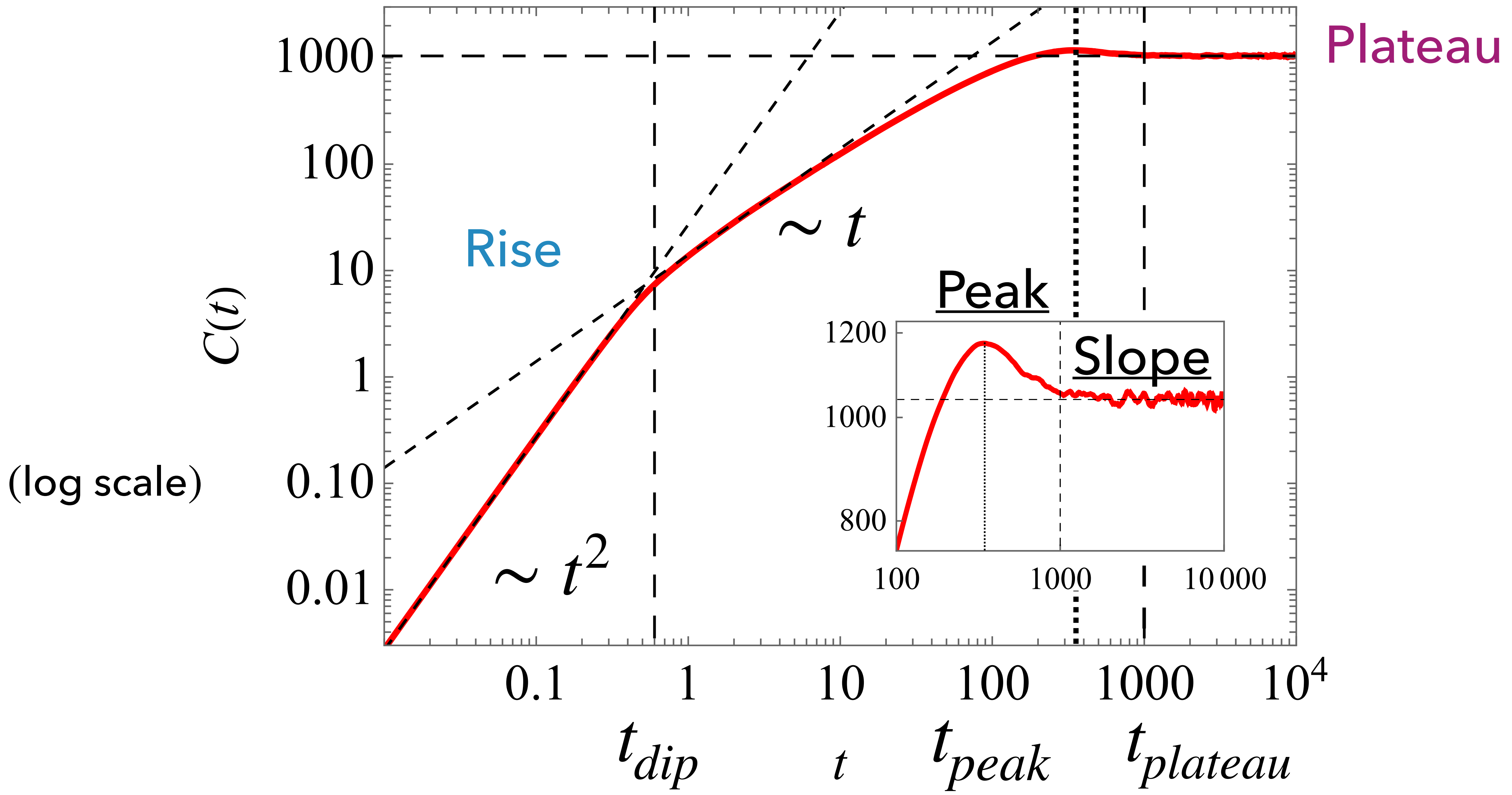


([Cotler, Gur-Ari, Hanada, Polchinski, Saad, Schenker, Stanford Streicher, Tezuka (2018)])

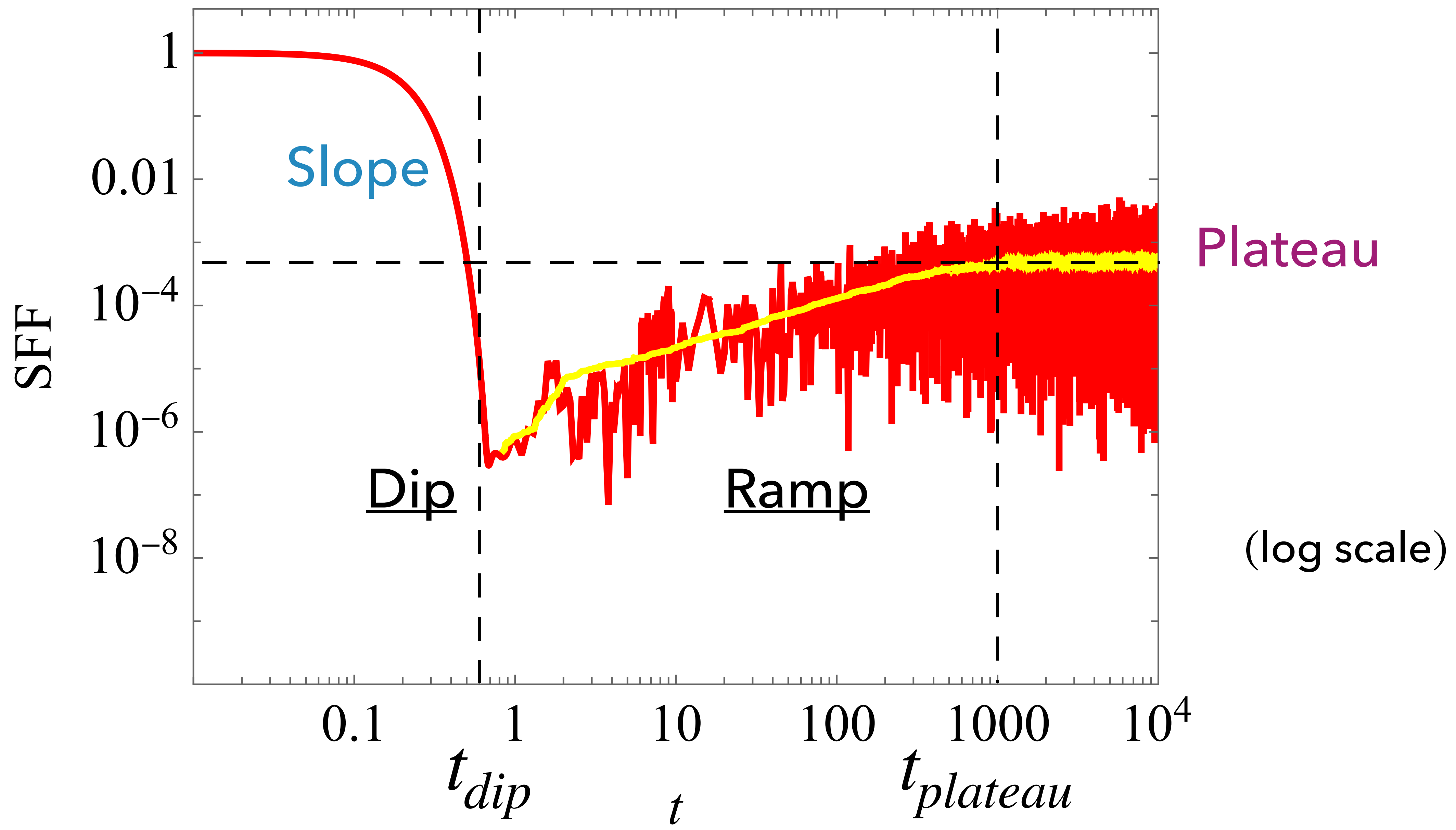
This observation has been reproduced in different settings, with the **peak** and subsequent **slope** before the plateau being the indication of energy-level repulsion and spectral rigidity.

For example, in [Camargo, Huh, Jahnke, Jeong, Kim & Nishida (2024)] it was shown that for the $\beta = 0$ TFD state in the even-parity sector of the mixed-field Ising model in the **chaotic regime** for $N = 12$ spins:

Krylov state complexity of TFD state



SFF



where: $t_{dip} \sim \mathcal{O}(1)$, $t_{plateau} \sim O(d)$, and $t_{dip} < t_{peak} < t_{slope} \lesssim t_{plateau}$ with $d \sim O(10^3)$ (c.f. RMTs where $t_{dip} \sim \sqrt{d}$).

Some Questions

Initial-state dependence of Krylov state complexity

1. Is the rise-peak-slope-plateau structure present only for the Krylov complexity of the TFD state?

Random matrix quenches

Information captured by the probability amplitudes

2. What information do the higher moments of the probability amplitudes $\{\phi_n(t)\}$ have access to?

Generalized Krylov state complexity

Mathematical origin of the peak

3. When it appears, can we provide a (semi-)analytic understanding of the peak (i.e. time-scale and value)?

Continuum approximation in Gaussian RMTs

Generalized Spreading Operator

Generalizations of the Krylov state complexity of the form

$$C_\psi^{(m)}(t) = \sum_{n\geq 0} n^m |\langle K_n | \psi(t)\rangle|^2 = \sum_{n\geq 0} n^m |\phi_n(t)|^2$$

with $m = 1,2,3,\ldots$, can be seen as as arising from the expectation value of the **generalized spreading operator** $\hat{n}_\psi^{(m)} : \mathscr{K} \to \mathscr{K}$, given by $\hat{n}_\psi^{(m)} = \sum_{n\geq 0} n^m |K_n\rangle\langle K_n|$ in the time-evolved state $|\psi(t)\rangle$

$$\langle \hat{n}_\psi^{(m)}\rangle_t := \langle \psi(t)|\hat{n}_\psi^{(m)}|\psi(t)\rangle = \langle \psi_0|\hat{n}_\psi^{(m)}(t)|\psi_0\rangle = C_\psi^{(m)}(t) \qquad \left(\hat{n}_\psi^{(m)}(t) = e^{itH}\hat{n}_\psi^{(m)}e^{-itH}\right)$$

These were introduced in the context of the statistics of operator measurements in quantum mechanics [Fu, Pal, Pal & Kim (2024)]. Consider the generating functional $G(\eta, t)$ where η is an auxiliary parameter:

$$G(\eta, t) := \sum_{n\geq 0} e^{\eta n} |\phi_n(t)|^2 \quad \Longrightarrow \quad \left.\frac{\mathrm{d}^m G(\eta, t)}{\mathrm{d}\eta^m}\right|_{\eta=0} = \sum_{n\geq 0} n^m |\phi_n(t)|^2 \equiv C_\psi^{(m)}(t) = \langle \hat{n}_\psi^{(m)}\rangle_t$$

Statistics of the Spreading Operator

One can analytically continue the generating functional to complex values of $\eta = -iu$ [Fu, Pal, Pal & Kim (2024)]

$$G(-iu,t) = \sum_{n\geq 0} e^{-iun}|\phi_n(t)|^2 = \sum_{m\geq 1} \frac{(-iu)^m}{m!}\langle \hat{n}_\psi^{(m)}\rangle_t \equiv \chi_{\hat{n}}(u,t) \quad ,$$

where $\chi_{\hat{n}}(u,t)$ is the characteristic function of the probability distribution $\{\phi_n(t)\}$ defined as

$$\chi_{\hat{n}}(u,t) = \langle\psi_0|e^{-iu\hat{n}_\psi(t)}|\psi_0\rangle = \sum_{m\geq 0}\frac{(-iu)^m}{m!}\langle\psi_0|(\hat{n}_\psi(t))^m|\psi_0\rangle = \sum_{n,m\geq 0}\frac{(-iu)^m}{m!}n^m|\phi_n(t)|^2$$

where $\hat{n}_\psi^{(m)}(t) = e^{itH}\hat{n}_\psi^{(m)}e^{-itH}$. The characteristic function $\chi_{\hat{n}}(u,t)$ is the "Fourier transform" of the probability distribution $P_{\hat{n}}$ of the spreading operator $\hat{n}_\psi$

$$P_{\hat{n}}(j,t) = \sum_{n\geq 0}|\phi_n(t)|^2\delta(j-n) \quad\Longrightarrow\quad \chi_{\hat{n}}(u,t) = \int \mathrm{d}j\, e^{-iuj}P_{\hat{n}}(j,t)$$

Higher-order Krylov state complexity

By similar arguments to the original work [Balasubramanian, et al. (2022)], the higher-order Krylov state complexities

$$C_\psi^{(m)}(t) = \sum_{n\geq 0} n^m |\phi_n(t)|^2$$

are also measures of **quantum complexity**, in the sense that the Krylov basis $\mathbf{K}$ minimizes cost functionals of the form:

$$C_{\mathbf{B}}^{(m)}(t) = \sum_{n\geq 0} n^m |\langle B_n | \psi(t)\rangle|^2 = \sum_{n\geq 0} n^m p_{\mathbf{B}}(n,t)$$

over all possible choices of complete, ordered, orthonormal bases **B**.

However, any arbitrary linear combination of these quantities may not be the minimum in the Krylov basis. For example, the variance $\sigma_{\hat{n}}^2(t) := \langle \hat{n}_\psi^{(2)}\rangle - \langle \hat{n}_\psi\rangle^2$ is **not** a measure of the quantum complexity of $|\psi(t)\rangle$ (in the above sense).

All of these have a universal early-time quadratic behaviour $C_\psi^{(m)}(t) \approx b_1^2 t^2 \sum_{n\geq 0} n^m \delta_{n1} + \mathcal{O}(t^3)$

Random Matrix Quenches

Quantum quenches provide a framework for investigating the non-equilibrium dynamics of closed, interacting quantum systems following a change in one or more of the system's parameters.

Consider a sudden quench protocol involving two random $d \times d$ matrices from a one parameter class of random matrices ($H_r(h)$) of the form ([Brandino, De Luca, Konik & Mussardo (2012)])

$$H_r(h) = \begin{pmatrix} A & hB \\ hB^\dagger & C \end{pmatrix}$$

$(H_r(h) \sim H_d + hV)$

(h breaks $\mathbb{Z}_2$ symm. of H_d)

e.g. Ising with transverse magnetic field

- Here, the matrices A , C are $(d/2) \times (d/2)$ symmetric matrices sampled from a normalized random matrix ensemble with measure

$$\mu(M) = \exp\left(-\frac{\tilde{\beta} d}{4} \text{tr}\left(M^2\right)\right) \qquad \begin{cases} \tilde{\beta} = 1 \text{ (GOE)} \\ \tilde{\beta} = 2 \text{ (GUE)} \end{cases}$$

- In the GOE case, the B_{ij} are real numbers drawn from a normal distribution with zero mean and variance $1/d$.
- In the GUE case, the B_{ij} are complex numbers $x_{ij} + iy_{ij}$, where both x_{ij} and y_{ij} are independently drawn from a normal distribution with zero mean and variance $1/(2d)$.

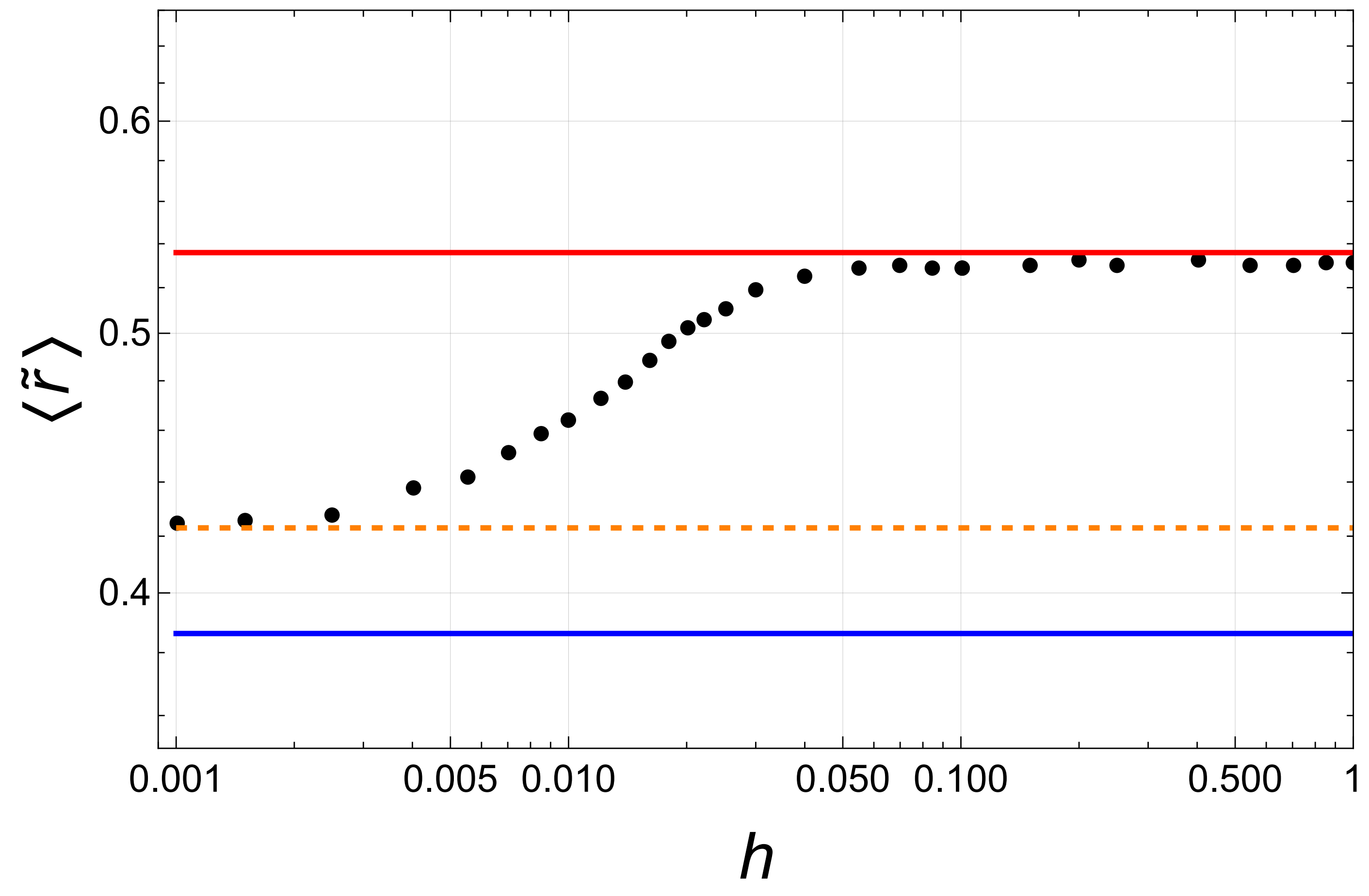
$\langle \bar{r} \rangle$ parameter of $H_r(h)$

One can compute the so-called r-parameter; a tool for detecting correlations in the energy spectrum. Defining the nearest-neighbor energy-spacings $s_n := E_{n+1} - E_n$ for an ordered energy spectrum $\{E_n\}_{n=1}^{d}$, the ratios

$$\bar{r}_n := \min\left\{ \frac{s_n}{s_{n-1}}, \frac{s_{n-1}}{s_n} \right\}$$

can be used to define the r-parameter; the average of these ratios: $\langle \bar{r} \rangle = \frac{1}{d-1} \sum_{n=1}^{d-1} \bar{r}_n$

The r-parameter of $H_r(h)$ as a function of h for 100 realizations with $d = 1000$:



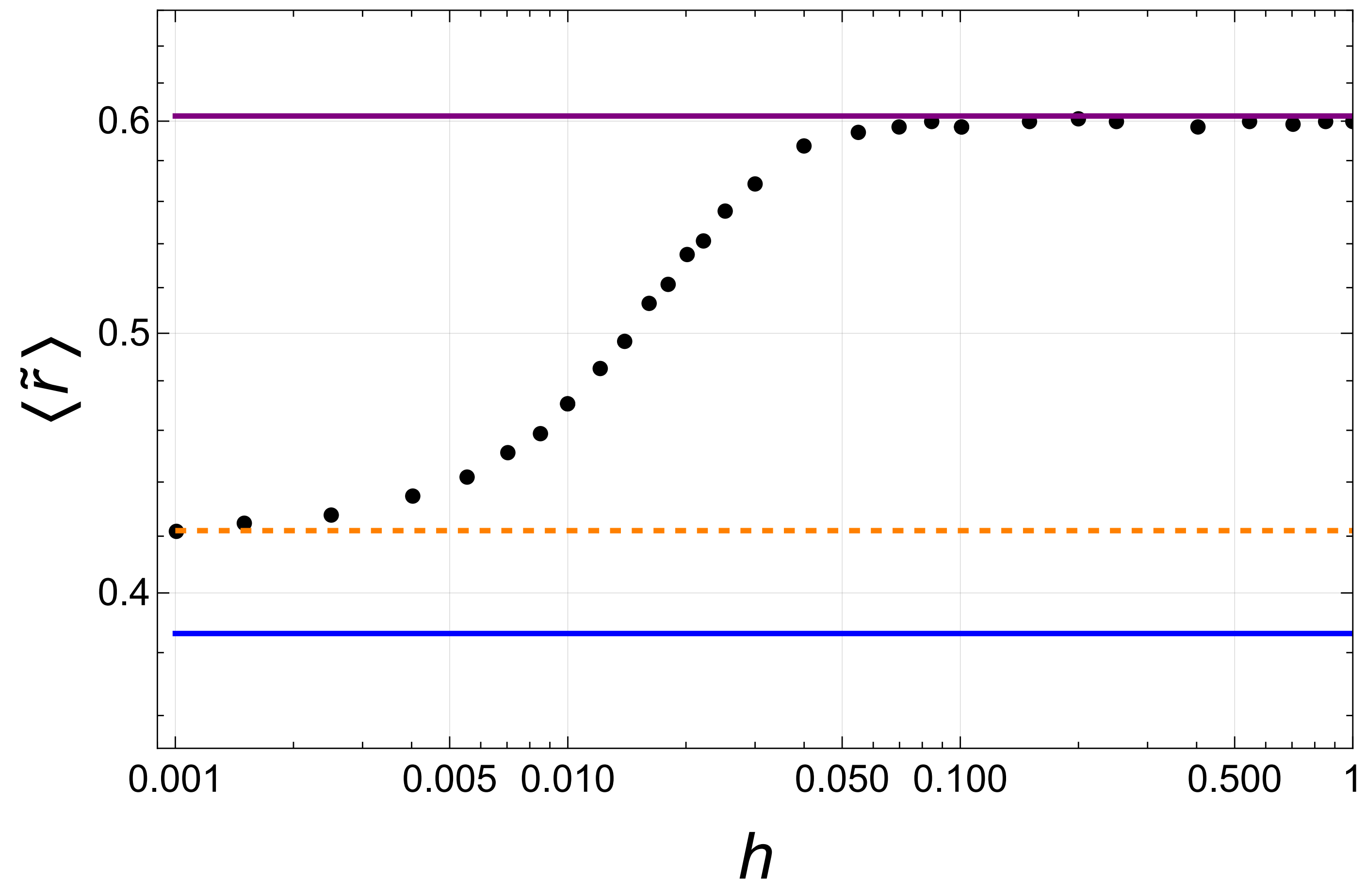
$\tilde{\beta} = 1$ (GOE)

$\langle \bar{r} \rangle_{GUE} \approx 0.603$

$\langle \bar{r} \rangle_{GOE} \approx 0.536$

$\lim_{h \to 0} \langle \bar{r}(h) \rangle \approx 0.42$

$\langle \bar{r} \rangle_{Poisson} \approx 0.386$



$\tilde{\beta} = 2$ (GUE)

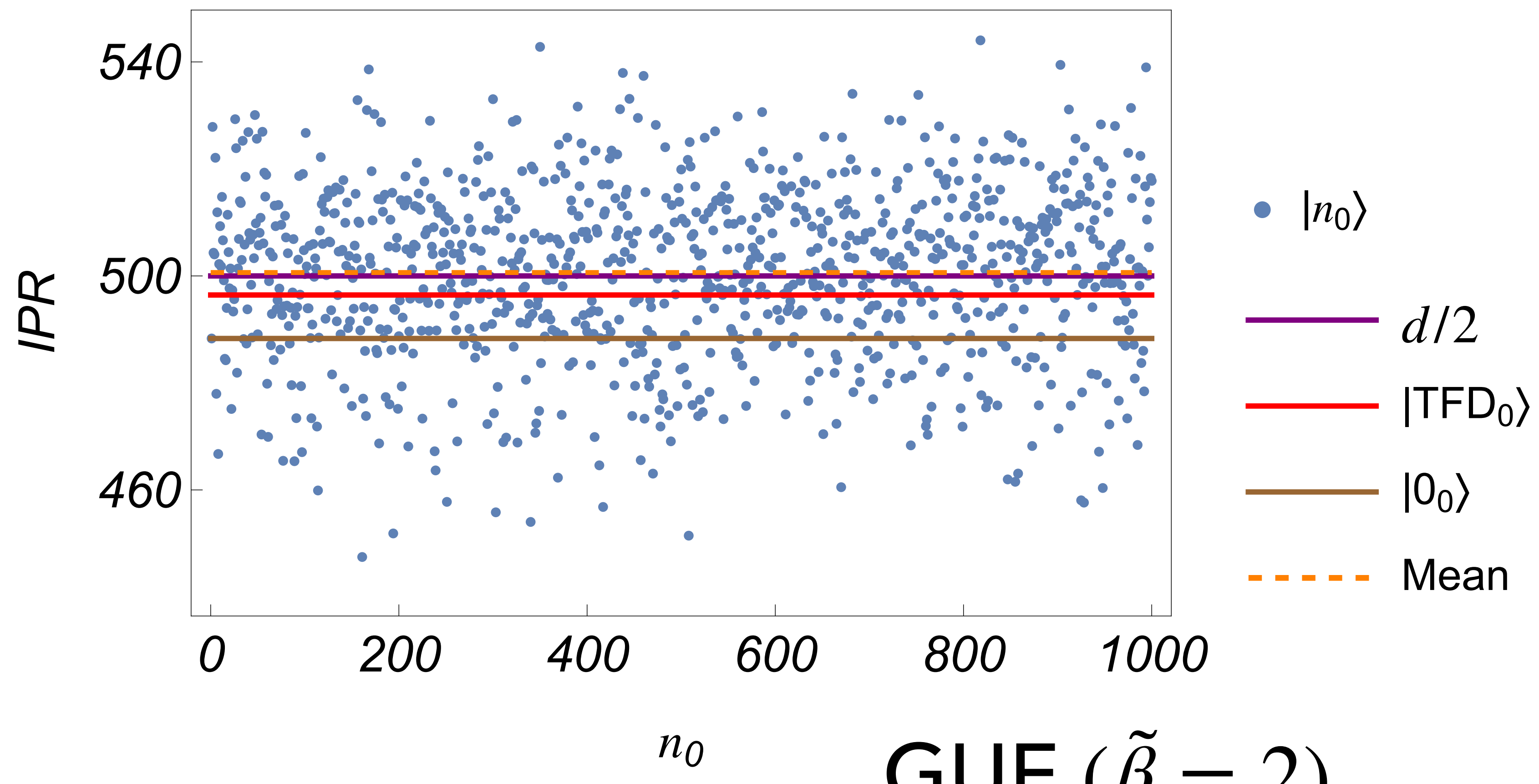
Random Matrix Quenches

This quench protocol provides a way to study the evolution of states that are not directly constructed from the eigenstates of the evolving Hamiltonian. Time evolution is implemented by the **post-quench** Hamiltonian H

$H_0|n_0\rangle = E_n^0|n_0\rangle$ Pre-quench $t<0$ $t=0$ $t>0$ Post-quench t $H|n\rangle = E_n|n\rangle$

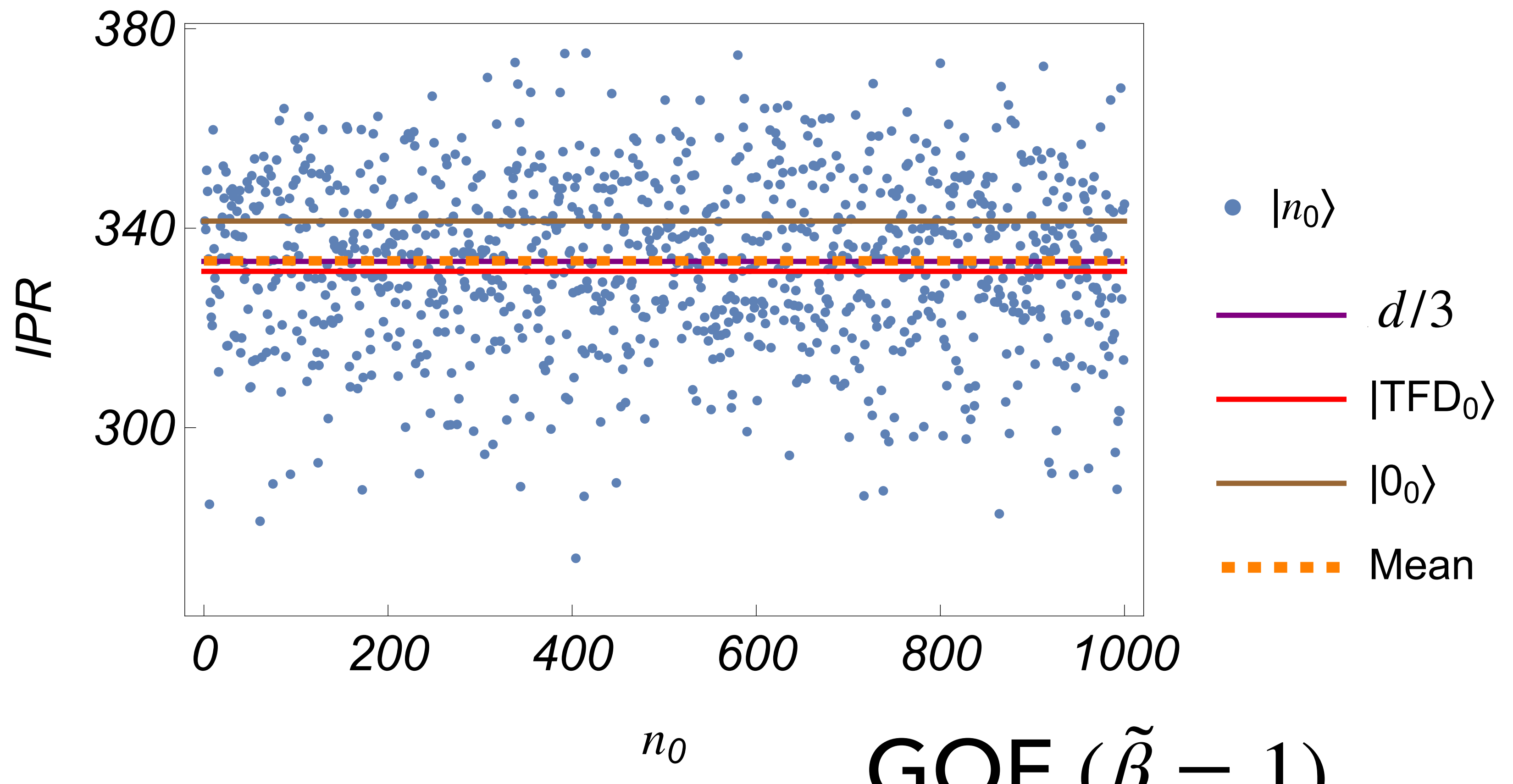
GOE $(\tilde{\beta}=1)$, GUE $(\tilde{\beta}=2)$ $\ni$ $H_0 := H_r(-1) = \begin{pmatrix} A & -B \\ -B^\dagger & C \end{pmatrix}$ Sudden quench $H := H_r(1) = \begin{pmatrix} A & B \\ B^\dagger & C \end{pmatrix} \in$ GOE $(\tilde{\beta}=1)$, GUE $(\tilde{\beta}=2)$

The eigenstates of the pre-and post-quench Hamiltonians are completely random with respect to each other, as can be verified by computing the **inverse participation ratio** IPR$(|n_0\rangle)$,



GUE $(\tilde{\beta}=2)$

$$\mathrm{IPR}(|\psi_0\rangle) := \frac{1}{\sum_{n\geq 0}|\langle n|\psi_0\rangle|^4}$$



GOE $(\tilde{\beta}=1)$

Choices of Initial States $|\psi_0\rangle$

Our goal is to study the evolution of the generalized spread complexities in such a quench protocol for different choices of the initial state $|\psi_0\rangle$: pre-quench TFD state $|\text{TFD}_0(\beta=0)\rangle$, pre-quench ground state $|0_0\rangle$ and post-quench TFD state $|\text{TFD}(\beta=0)\rangle$:

1. For a realization of $H_r(\pm 1)$, find $|\psi_0\rangle$ and tridiagonalize $H_r(+1)$ to find $H_r^{\mathbf{K}}(+1)$.

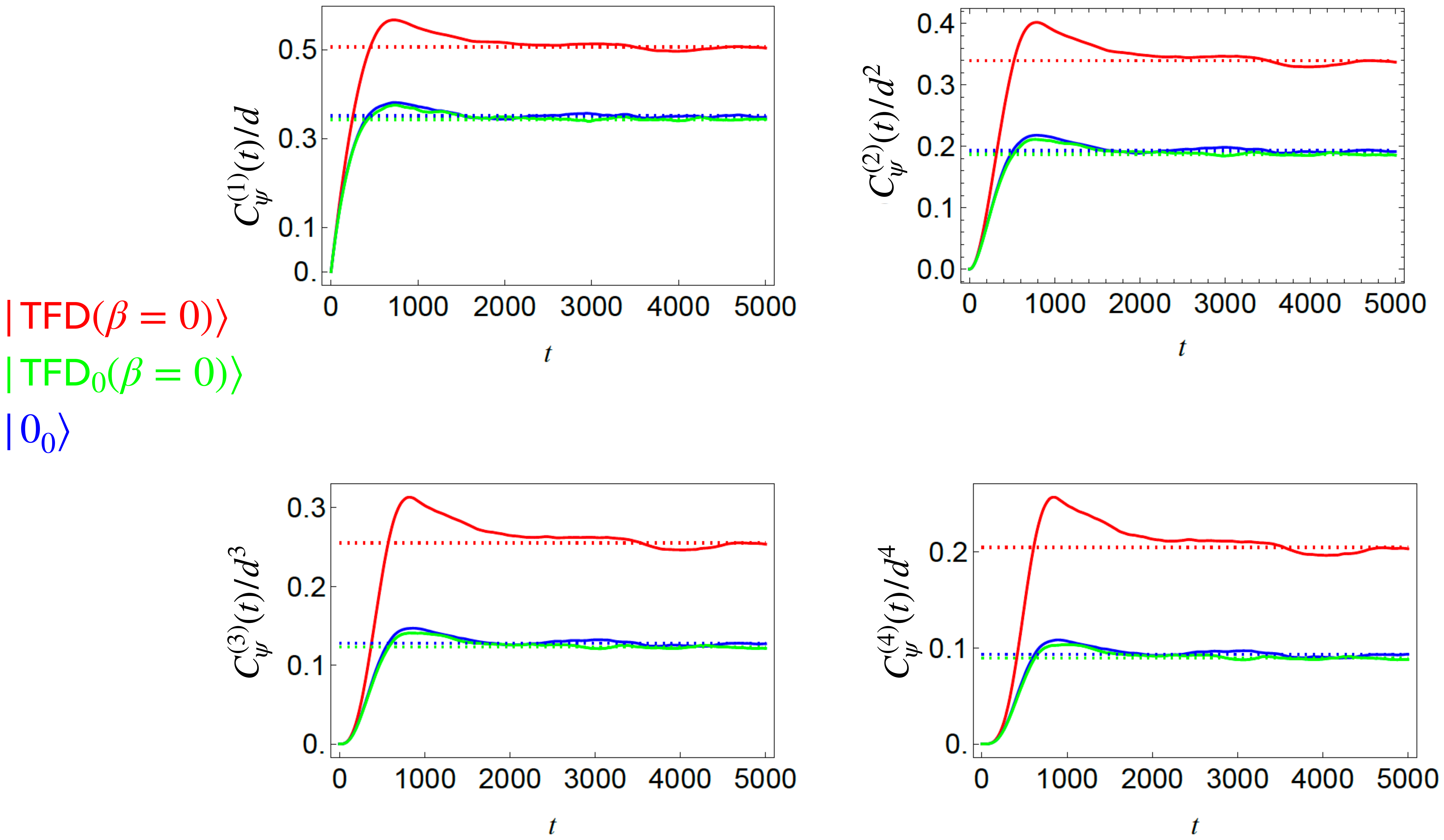
2. Compute $C_\psi^{(m)}(t) = \langle\psi_0|\hat{n}_\psi^{(m)}(t)|\psi_0\rangle$ by expressing $|\psi_0\rangle$ and $\hat{n}_\psi^{(m)}(t)$ in the Krylov basis:

$$C_\psi^{(m)}(t) = \underbrace{(1,0,\ldots,0)}_{\langle K_0|} . e^{itH_r^{\mathbf{K}}(1)} . \underbrace{\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2^m & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d^m \end{pmatrix}}_{\langle K_n|\hat{n}_\psi^{(m)}|K_m\rangle} . e^{-itH_r^{\mathbf{K}}(1)} . \underbrace{(1,0,\ldots,0)^T}_{|K_0\rangle} \quad , \qquad H_r^{\mathbf{K}}(+1) = \begin{pmatrix} a_0 & b_1 & 0 & 0 & \cdots \\ b_1 & a_1 & b_2 & 0 & \cdots \\ 0 & b_2 & a_2 & \ddots & \cdots \\ \vdots & \vdots & \ddots & \ddots & b_{d-1} \\ 0 & 0 & 0 & b_{d-1} & a_{d-1} \end{pmatrix}$$

(Contains info. about $|\psi_0\rangle$ and $H_r(+1)$)

GOE Random Matrix Quenches

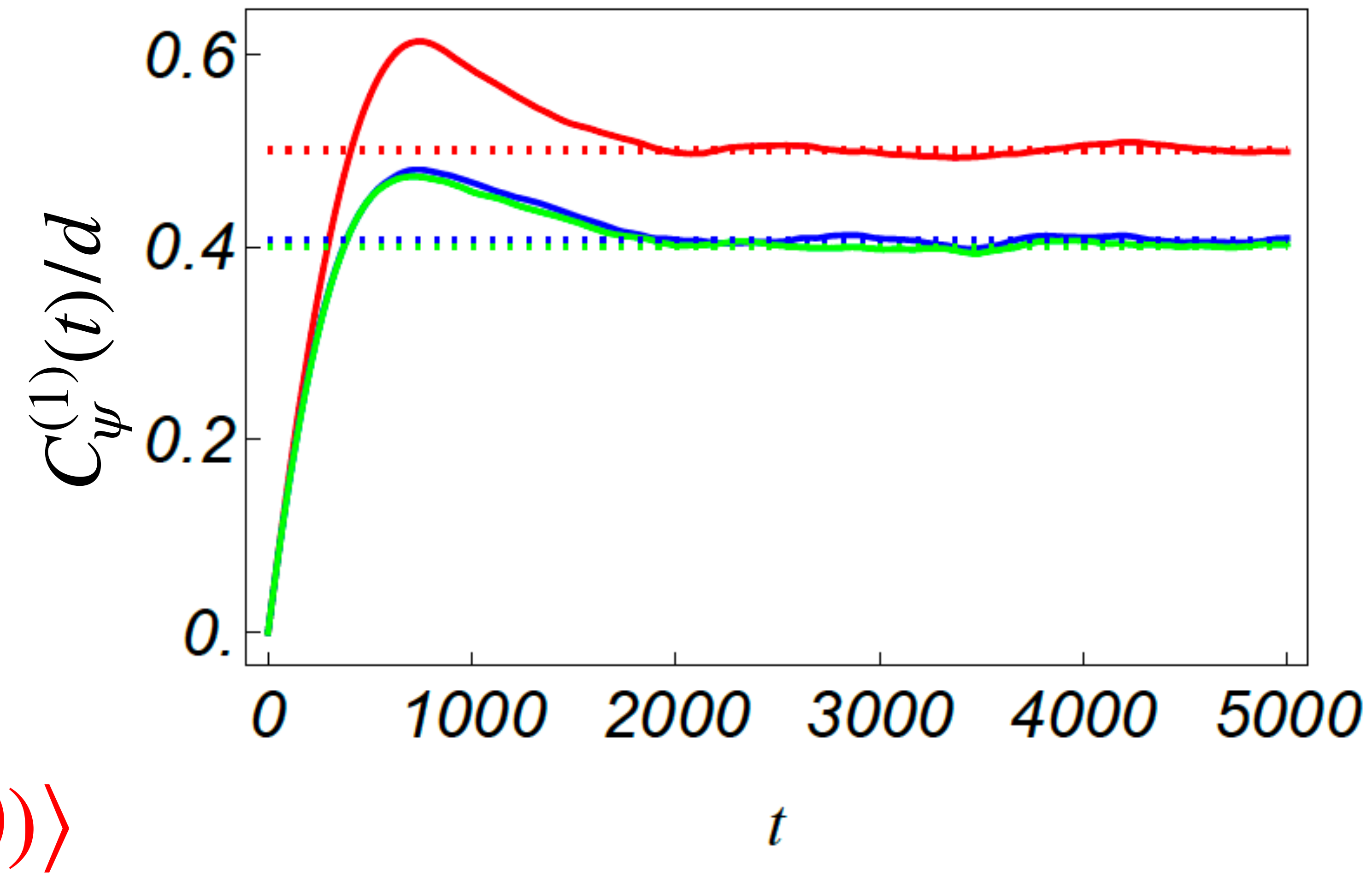
For the GOE case: (e.g. numerically averaging over 4 realizations of $H_r(\pm 1)$ with $\tilde{\beta} = 1$ and $N = 1000$)

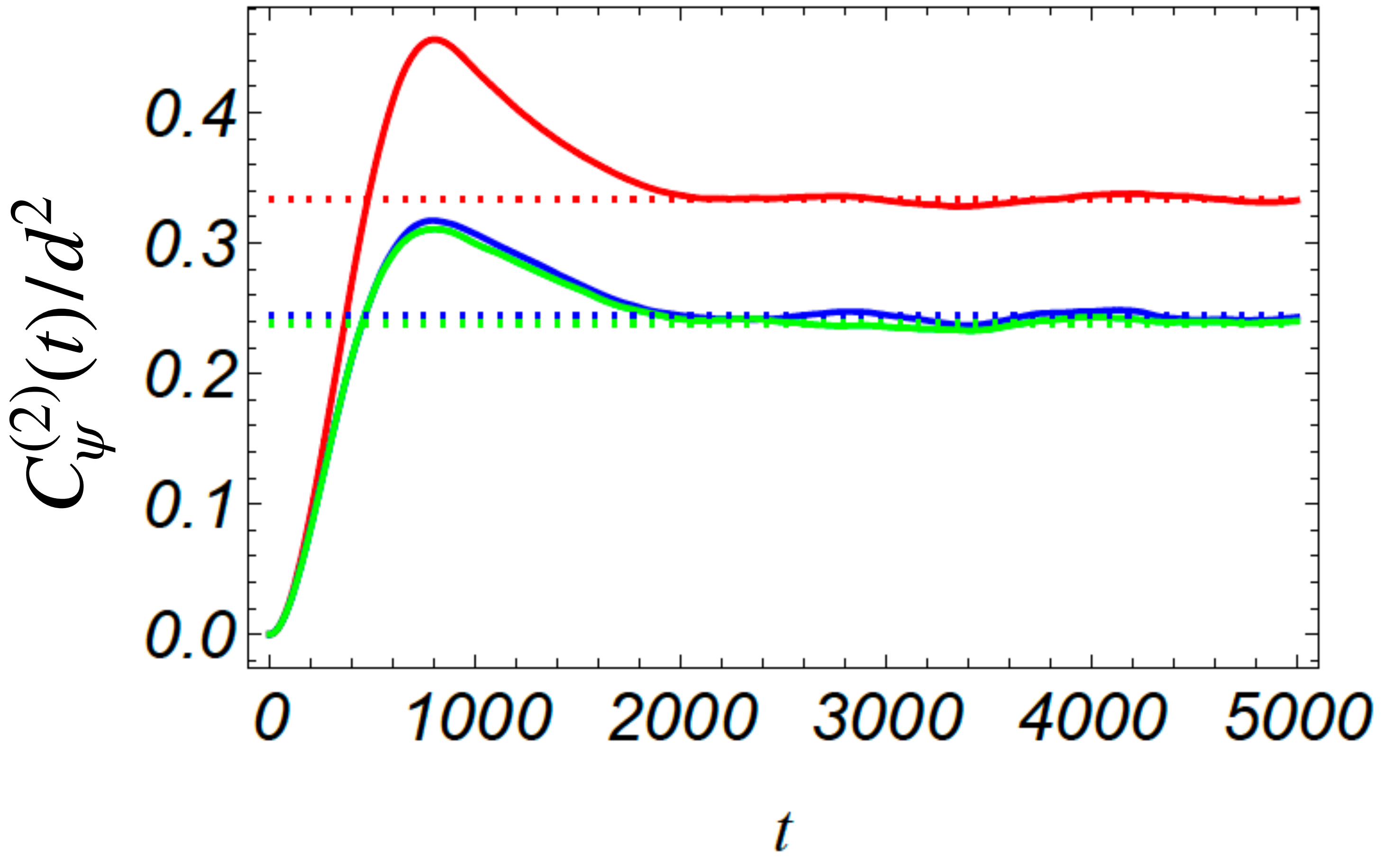


N.B. Averaging over more realizations will smoothen the behavior after the slope (reduce oscillations), but unlikely to change the behavior drastically.

GUE Random Matrix Quenches

Similar situation for GUE: (e.g. numerical average over 4 realizations of $H_r(\pm 1)$ with $\tilde{\beta} = 2$ and $N = 1000$)

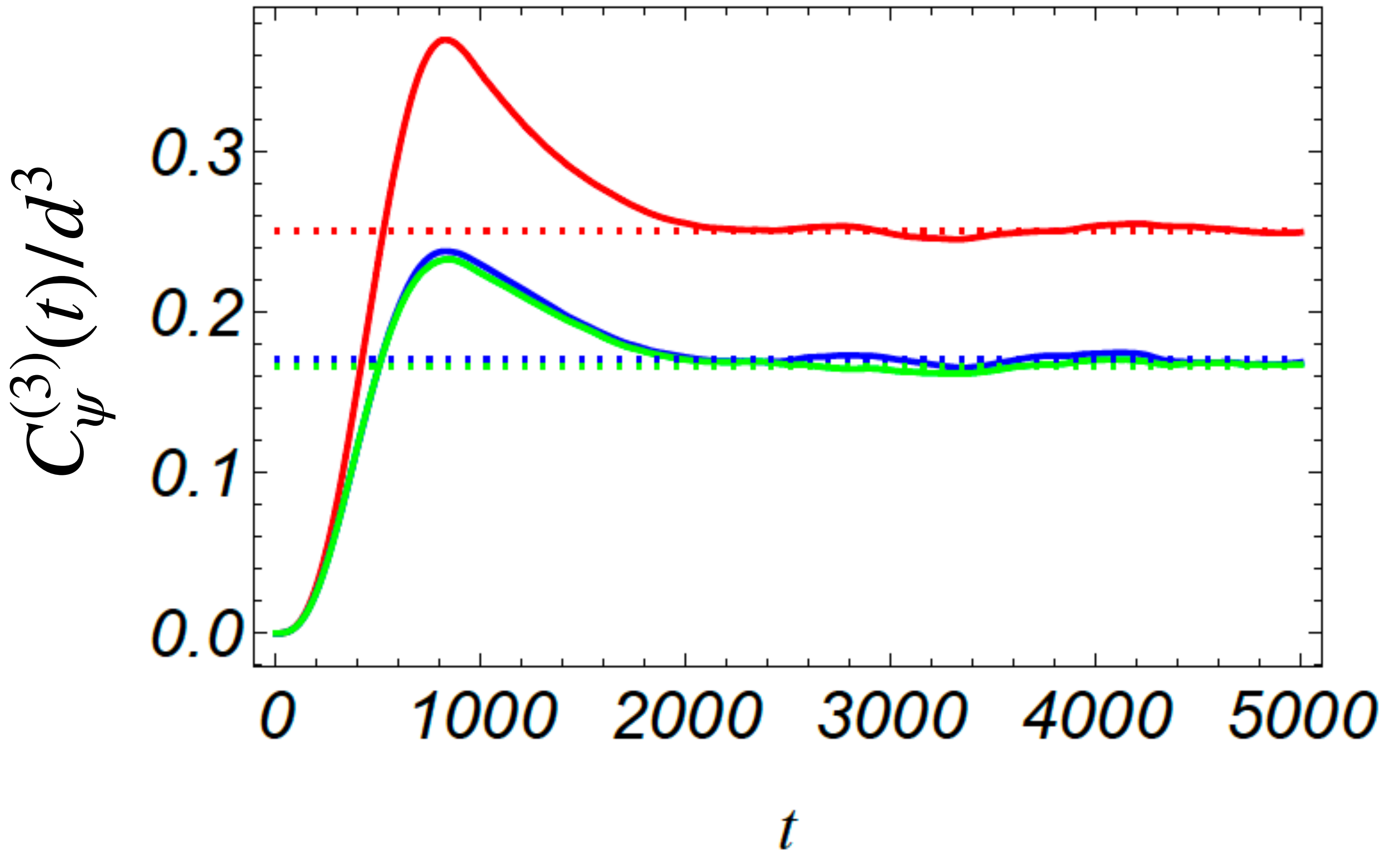


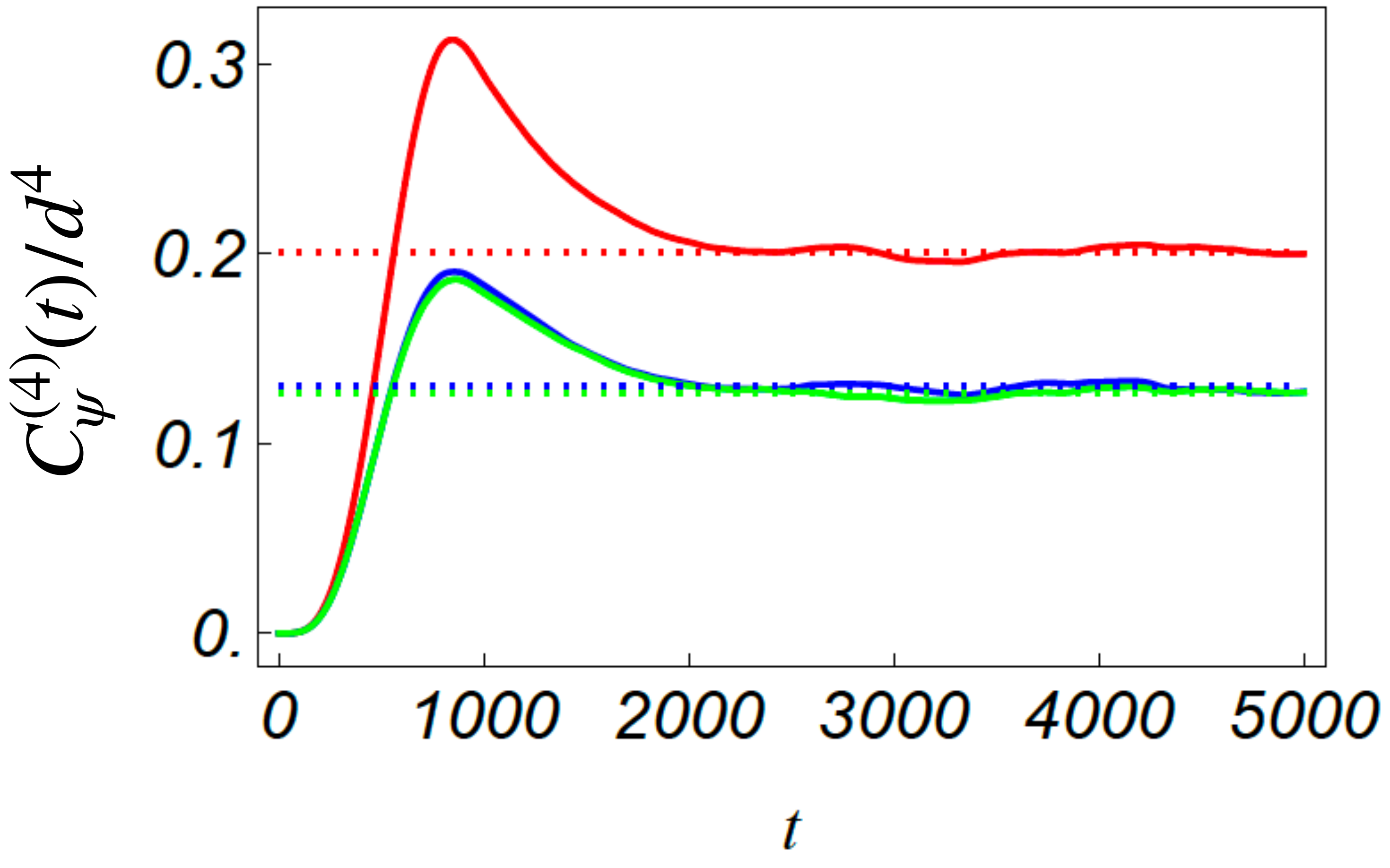


$|\text{TFD}(\beta = 0)\rangle$

$|\text{TFD}_0(\beta = 0)\rangle$

$|0_0\rangle$

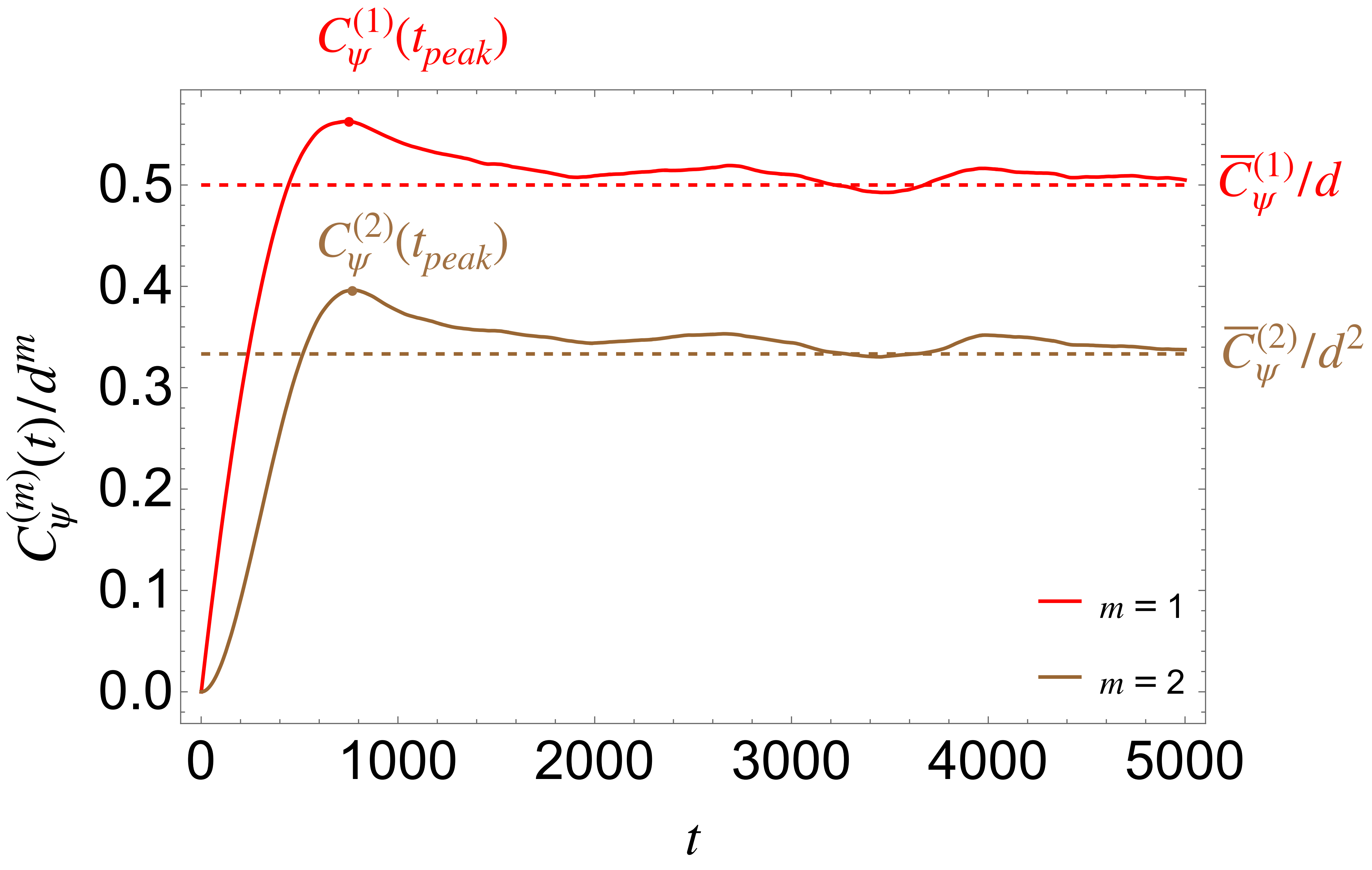




N.B. Averaging over more realizations will smoothen the behavior after the slope (reduce oscillations), but unlikely to change the behavior drastically.

Quantifying the Peak

One way to quantify the peak in $C_\psi^{(m)}(t)$ is by introducing the "peak parameter" $P_\psi^{(m)}$:



Example for $H_r(\pm 1) \in$ GOE, $|\psi_0\rangle = |\text{TFD}(\beta = 0)\rangle$ and for $d = 1000$.

$$P_\psi^{(m)} := \frac{C_\psi^{(m)}(t_{peak}) - \overline{C}_\psi^{(m)}}{C_\psi^{(m)}(t_{peak})}$$

where $\overline{C}_\psi^{(m)}$ is the infinite time-average of $C_\psi^{(m)}(t)$.

- If the peak exists, then $C_\psi^{(m)}(t_{peak}) \geq \overline{C}_\psi^{(m)}$ and $1 > P_\psi^{(m)} \geq 0$.

The Peak in the Continuum Limit

In the continuum limit, we assume that the discrete Krylov basis index n can be mapped to a continuous coordinate, $n \mapsto x = \epsilon n$, where ϵ is a small parameter (lattice spacing). Assuming a smooth dependence on n, $a_n \mapsto a(x_n)$, $b_n \mapsto b(x_n)$ and $\phi_n(t) \mapsto \phi(x_n, t)$.

- In this case, the recurrence relation for $\phi_n(t)$ and Schrödinger equation become first-order differential equations.

- In the continuum limit [Fu, Pal, Pal, Kim (2024)]

$$C_\psi^{(m)}(t) = \int \mathrm{d}E\mathrm{d}\omega\, J_m(\omega)\rho_0(E,\omega)\rho_0(E,\omega)e^{i\omega t}$$

$$J_m(\omega) = \frac{2}{\epsilon^{m-1}}\int_0^{y(\epsilon L)} \mathrm{d}y\, x^m(y)\, b(y)\, \cos(\omega y) \qquad \mathrm{d}y(x) = \mathrm{d}x/(2\epsilon b(x)) \qquad \rho_0(E_i, E_j) = \langle E_i | \psi_0\rangle\langle\psi_0 | E_j\rangle$$

$$\omega = E_i - E_j \;, E = (E_i + E_j)/2$$

- For Hamiltonians belonging to $\tilde{\beta}$-ensembles, and for $|\psi_0\rangle = (1,0,\ldots,0)^T$ or $|\mathrm{TFD}(\beta = 0)\rangle$, the Lanczos behave like randomly distributed variables, with ensemble average (in the large-d limit) given by:

$$\langle a(x)\rangle = 0 \qquad \langle b(x)\rangle = \sqrt{1 - \frac{x}{\epsilon d}}$$

The Peak in the Continuum Limit

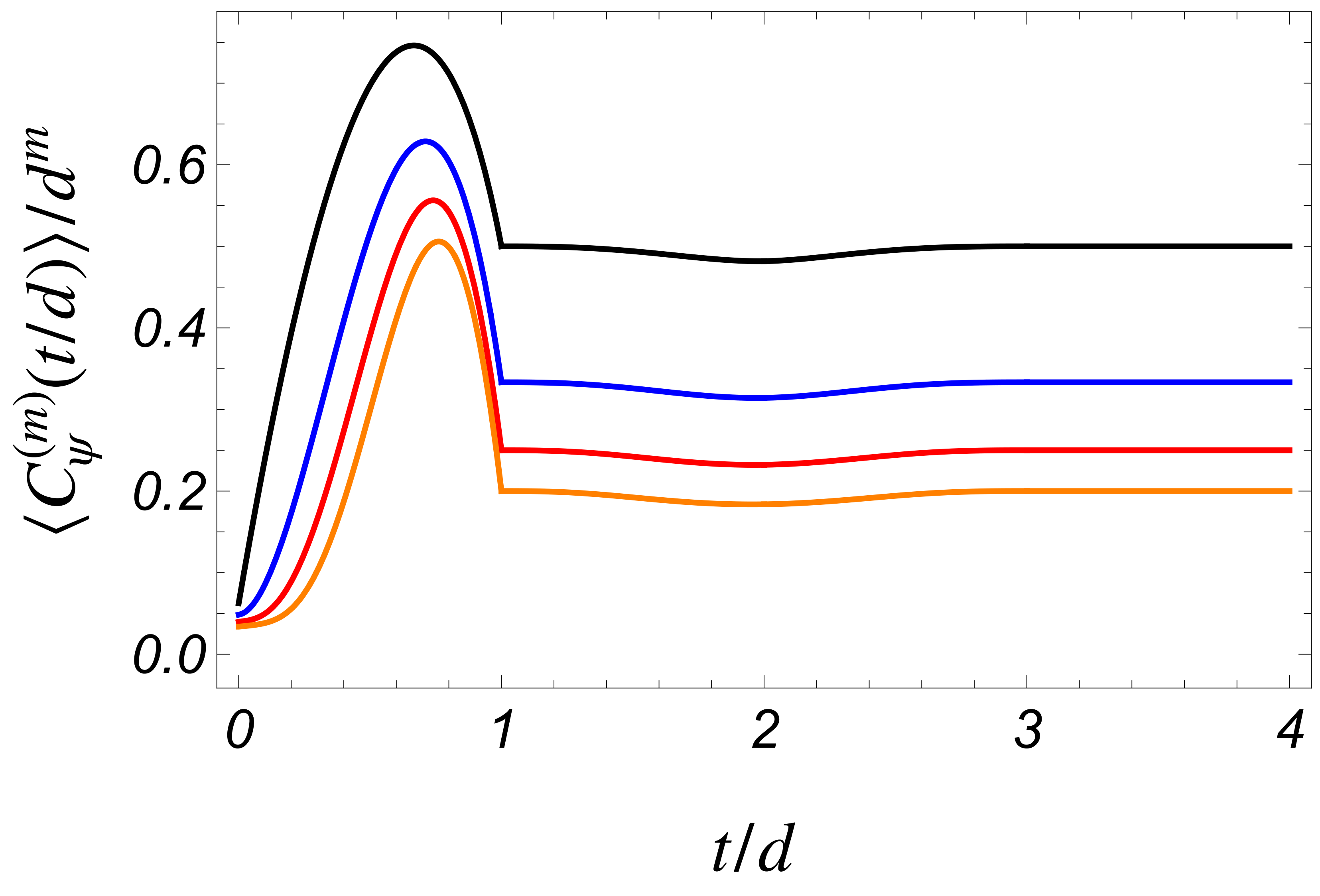
Taking the GUE ensemble average of $C_\psi^{(m)}(t)$, and using the fact that:

(Wigner semicircle law)

$$\langle \rho(E_i)\rho(E_j)\rangle = \langle \rho(E)\rangle \delta(\omega) + \langle \rho(E_i)\rangle\langle \rho(E_i)\rangle \left(1 - \frac{\sin^2(\pi\langle\rho(E)\rangle\omega)}{(\pi\langle\rho(E)\rangle\omega)^2}\right)$$

$$\langle \rho(E)\rangle = \frac{d}{2\pi}\sqrt{4-E^2}$$

One can find $C_\psi^{(m)}(t)$ at different time intervals (for the $\beta = 0$ TFD state), in particular around the peak



[Fu, Pal, Pal, Kim (2024)]

$$\langle C_\psi^{(1)}(v)\rangle = \frac{7\pi + 640v - 960v^2 + (320+20\pi)v^3 - 15\pi v^4 + 3\pi v^5}{2(160 + 5\pi - 160v + 15\pi v^2 - 5\pi v^3)}, \quad \text{for} \quad v = t/d < 1\,,$$

$$\langle C_\psi^{(2)}(v)\rangle = \frac{(19\pi/7) + 640v^2 - 1280v^3 + (800+10\pi)v^4 - (160+12\pi)v^5 + 5\pi v^6 - (5\pi/7)v^7}{160 + 5\pi - 160v + 15\pi v^2 - 5\pi v^3}, \quad \text{for} \quad v = t/d < 1\,,$$

Order of Complexity (m)	Parameter P_m	Ratio $\frac{P_{m+1}}{P_m}$
$m = 1$	0.33	—
$m = 2$	0.47	1.42
$m = 3$	0.55	1.17
$m = 4$	0.60	1.09

Summary

- Quantum quenches within the framework of RMTs allows for a systematic study of Krylov state complexity and its generalizations for a variety of initial states in systems that transition from one chaotic phase to another.

- Higher-order generalized Krylov state complexities are more sensitive to the presence of the peak.

- Building up on recent works, and using the continuum limit approximation and in the large matrix dimension limit, we can understand the origin of the peak due to the density-pair energy correlations = spectral rigidity.

Open Questions and Future Directions

- We need to refine our working definition of quantum chaos. “Scrambling is necessary but not sufficient for chaos” [Dowling, Kos, Modi (2023)]. What about the initial state/operator dependence? An interplay of quantum versions of ergodic hierarchies [Gesteau (2023), Ouseph et al. (2023)], free probability theory [Voiculescu (1985)] and Krylov subspace methods could open the way to understand these questions.

- Recent efforts have **matched** the wormhole (Einstein-Rosen bridge) length in Jackiw-Teitelboim (JT) gravity, with the Krylov state complexity of chord states in the triple-scaling limit of the double-scaled Sachdev-Ye-Kitaev (DSSYK) model ([Rabinovici et al. (2023)]). Very recently, the late-time saturation of Krylov state complexity was studied in this same context [Balasubramanian et al. (2024)]. Do other holographic complexity proposals (complexity=anything) correspond to different generalizations of Krylov complexity? What about complexity in de Sitter space?

- Another defining feature of holographic (and computational) complexity is the switchback effect. Is Krylov state complexity (and its higher-order generalizations) sensitive to this phenomenon?

- “Krylov state complexity is not a measure of distance between states” [Aguilar-Gutierrez, Rolph (2023)]. Yet, its time average is related to an upper bound on Nielsen complexity [Craps, Evnin, Pascuzzi (2023)]. What is the precise connection between Krylov state complexity (and its higher-order generalizations) and these measures of quantum complexity?

Thank you!

Additional Slides

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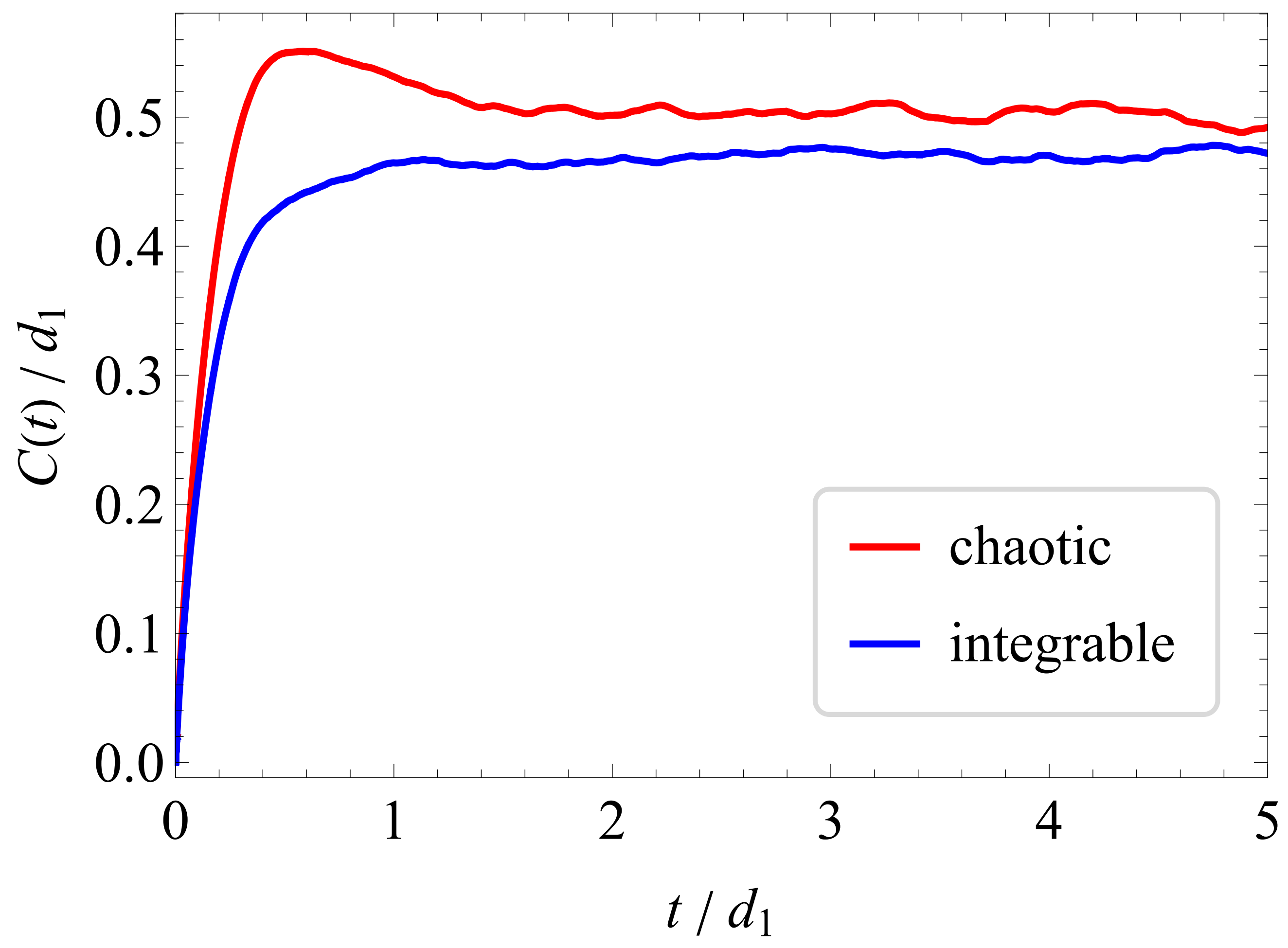
Characteristic Features of Krylov state complexity

- TFD state at $\beta \to 0$

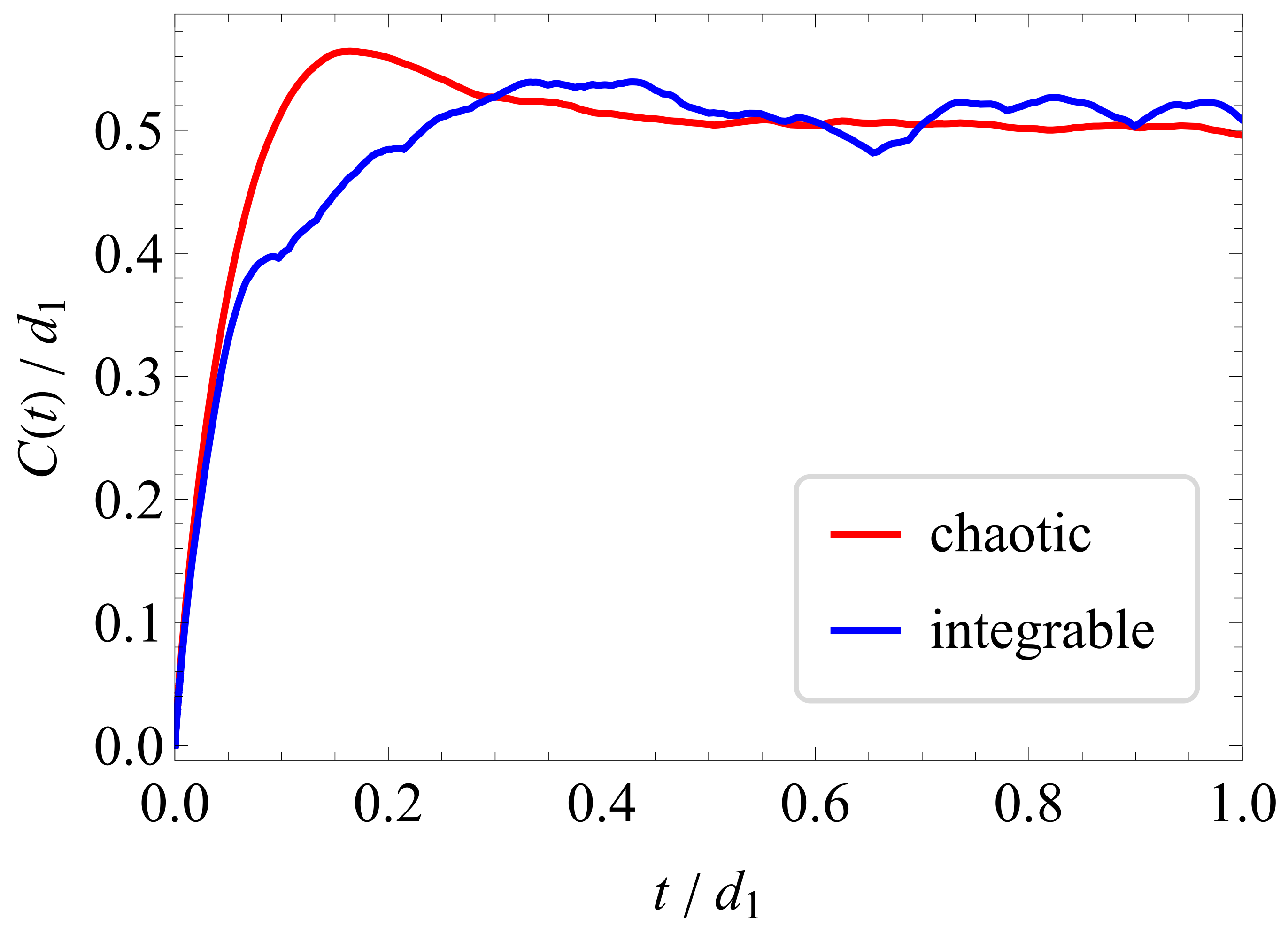
- NNN deformation of XXZ

($N = 15$ sites , even parity sector and $M_z = -5/2$)



- MFI

($N = 12$ sites, even parity sector)



- TFD state for MFI in its chaotic phase

($N = 12$ sites, even parity sector)