Thermofield Theory of Large N Matrix Models

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With Junjie Zheng and Xianlong Liu:

1. 2501.15421 Thermofield Theory of Large N Matrix Models
2. 2109.13381 Dynamical Symmetry and the Thermofield State at Large N(with JZ XL and J.Yoon
3. 2304.11767 Symmetries and the Hilbert Space of Large N Extended States (with JZ XL

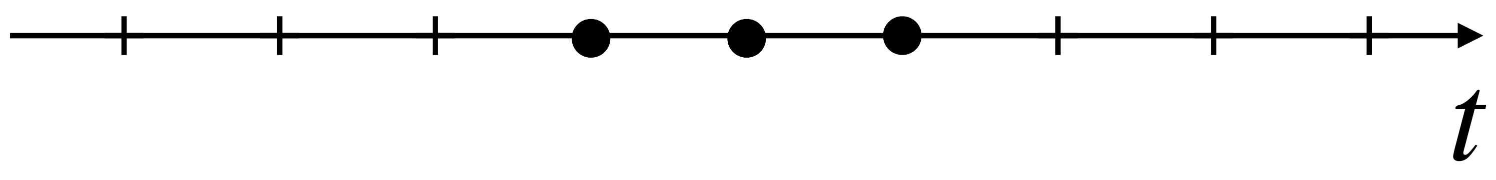
Large N Multi-Matrix Model

- Multi-matrix models: BMN, BFSS , KS lattice
- Numerical methods: Monte-Carlo, Bootstrap, Variational, Collective
- Collective/Master field method:(Large N)
    - [de Mello Koch, AJ, Liu, Mathaba & Rodrigues, 21']
    - [Mathaba, Mulokwe & Rodrigues, 23']

Large N : Schwinger-Dyson (SD) Equations

$$\Box\,\Phi(C) = \sum_{C=C_1+C_2} \Phi(C_1)\Phi(C_2) \qquad \mathrm{Tr}\left(M_{a_1}(t_1)M_{a_2}(t_3)\cdots\right) = \Phi(C)$$

- Traces (Wilson) loop variables
- Obey a closed set of Nonlinear Eqs
- One can solve initial value problem $\Phi(t_1, t_2, \cdots)$



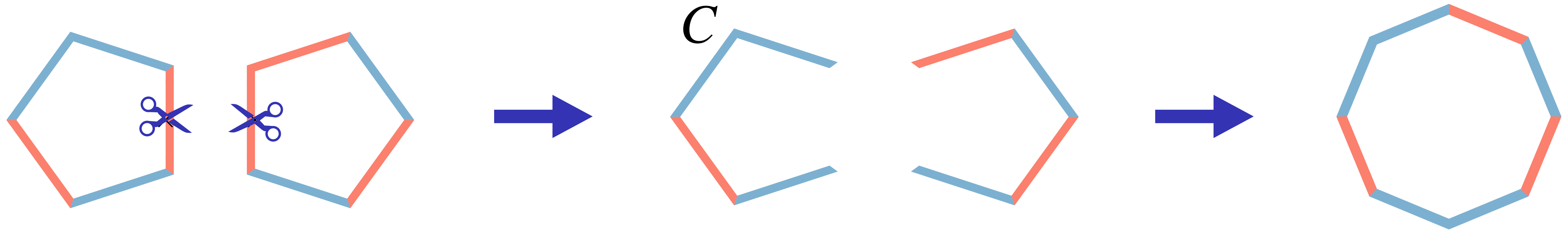
$$\phi(c) = \mathrm{Tr}\left(M_1(0)M_2(0)\cdots\right)$$

- Collective Hamiltonian :t=0

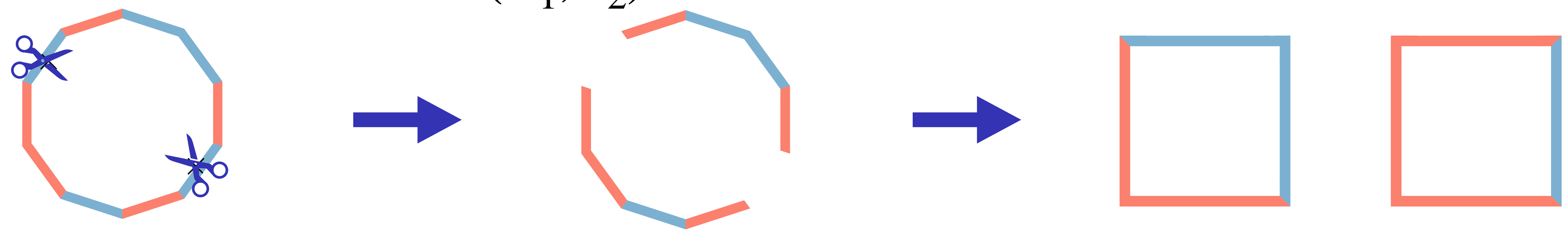
$$H^{\text{col}} = \frac{1}{2}\sum_{c_1,c_2} \hat{\Pi}(c_1)\Omega(c_1,c_2)\hat{\Pi}(c_2) + V^{\text{col}}$$

Collective Hamiltonian

Loop Joining $\Omega(C_1, C_2) = \sum_{C} j(C_1, C_2; C)\Phi(C)$



Loop Splitting $\omega(C) = \sum_{(C_1,C_2)} p(C; C_1, C_2)\Phi(C_1)\Phi(C_2)$



Collective potential $$V_{\rm col}[\Phi] = N^2 \left(\frac{1}{8}\omega\Omega^{-1}\omega + V[\Phi] \right)$$

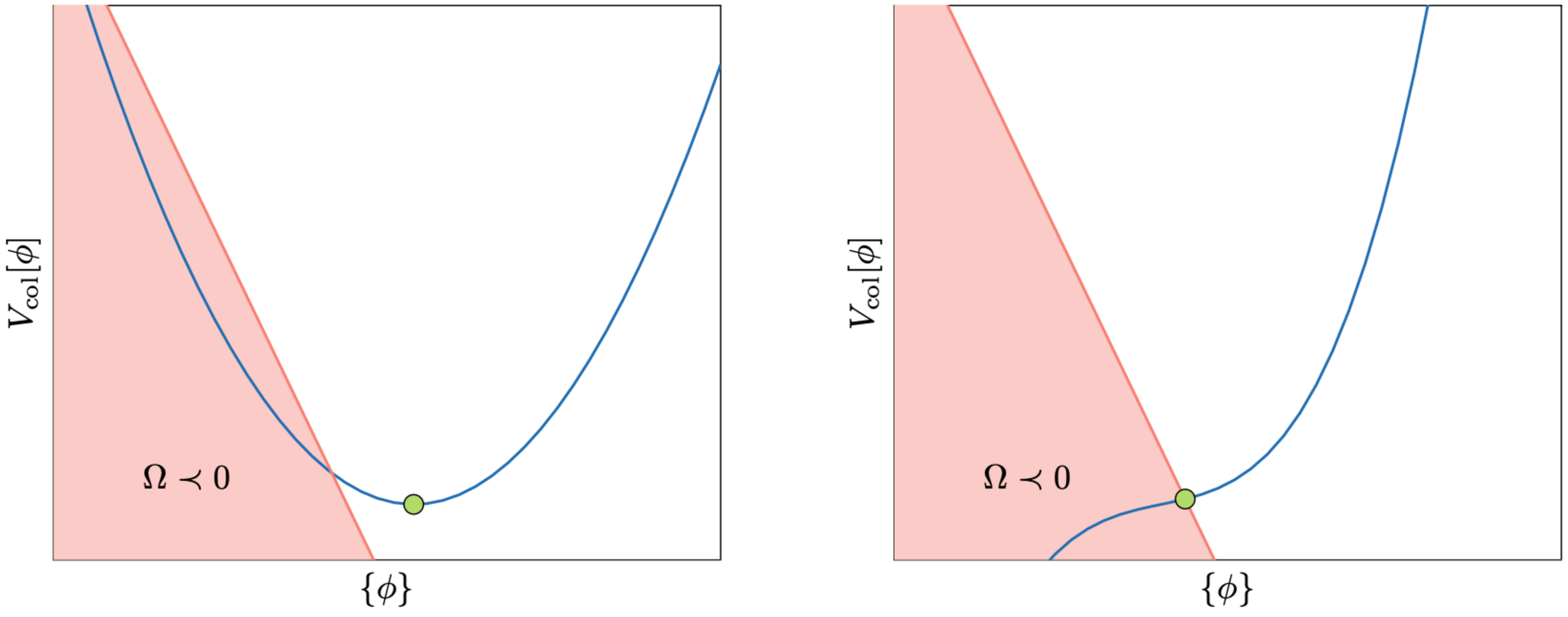
Numerical Optimization (Large N Bootstrap): $V_{\rm col}$, numerically solved [d.M.Koch, A.J., Liu, Mathaba and

Collective Potential:Minimization at Large N

$$V_{\text{col}} = \frac{N^2}{2} \sum_{c_1,c_2} \omega(c_1)\Omega^{-1}(c_1,c_2)\omega(c_2) + V$$

$$V = N^2 \sum_c g_c \phi(c)$$ Interaction:linear

- Constrained minimization
- Positivity: $\Omega > 0$
- $\phi(c) = \phi(M_a)$



(a) The case when the positivity is irrelevant. (b) The case when the positivity of Ω is relevant.

Loop Truncation

- l : length of a loop $\quad \mathrm{Tr}(\underbrace{MM\cdots M}_{l})$
- $\bar{l}$: maximum length of loops
- $\bar{\mathcal{N}}$: size of truncation
- Ω : $\bar{\mathcal{N}}$ by $\bar{\mathcal{N}}$ matrix
- $\phi(c)$ with $l \leq 2\bar{l} - 2$
- Master field minimization : M_1 M_2 ..

$\bar{l}$	$\bar{\mathcal{N}}$	$\mathcal{N}_{\bar{l}}$
4	9	15
6	15	37
8	23	93
10	37	261
12	57	801
14	93	2615
16	153	8923
18	261	31237

Collective Hamiltonian : Expansion in 1/N

$$H^{\text{col}} = \frac{1}{2}\sum_{c_1,c_2} \hat{\Pi}(c_1)\Omega(c_1,c_2)\hat{\Pi}(c_2) + V^{\text{col}}$$

$$[\hat{\eta}, \hat{\Pi}] = i$$

- $\phi(c) = \phi_0(c) + \hat{\eta}(c)$ $\quad$ $H = \mathcal{E} + H_2 + \frac{1}{N}H_3 + \frac{1}{N^2}H_4 + \cdots$

- N^0 order:

$$H_2 = \frac{1}{2}\sum_{c_1,c_2} \pi(c_1)\underline{\Omega_0(c_1,c_2)}\pi(c_2) + \frac{1}{2}\sum_{c_1,c_2} \eta(c_1)\underline{V_0(c_1,c_2)}\eta(c_2)$$

From ϕ_0

- Vertices :known

$$H_n = \sum_{i_1,i_2,\cdots,i_n} V_0(i_1,i_2,\cdots,i_n)\eta_{i_1}\eta_{i_2}\cdots\eta_{i_n}$$

$\uparrow$

Finite Temperature

- T< T_c   Spectrum O(1): Sequence of oscillators:Gives the Free Energy
- AdS gas

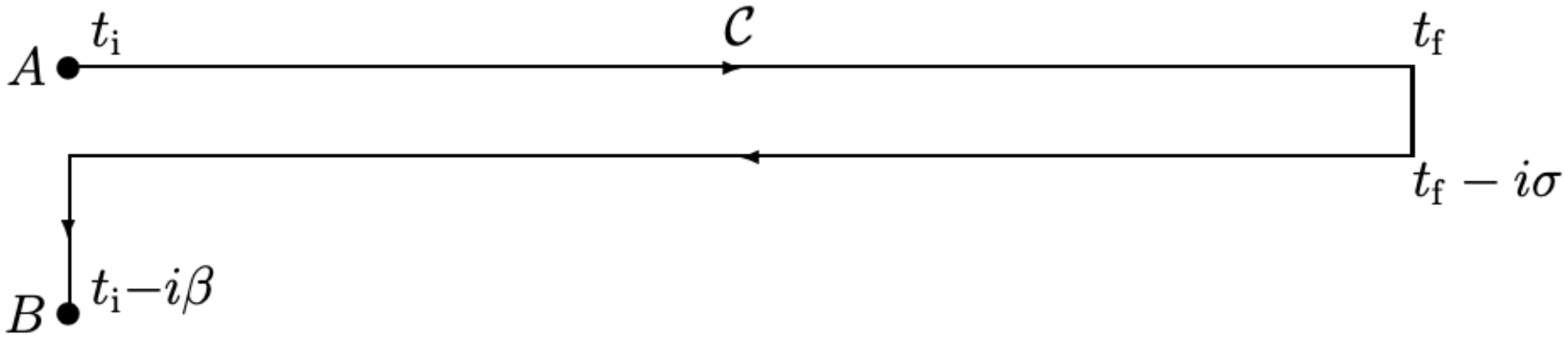
- Deconfinement;Phase Transition T_C

- T>T_c :unconfined :N^2
- [Cho,Gabai ,Sandor,Yin 24']Thermal bootstrap$Z(\beta) = \mathrm{Tr}\, e^{-\beta H}$

Real Time Simulation:Thermofield Double

- Thermal State $|0(\beta)\rangle$ ⟺ Hartle-Hawking State : Two-sided BH

- Schwinger-Keldysh contour



- Real-time evolution $\partial_t : H_- = H - \tilde{H}$

- Imaginary-time evolution $\partial_\tau : H_+ = H + \tilde{H}$

- Thermal state

$$H_- |0(\beta)\rangle = 0$$

Thermal State(contnd)

$$|0(\beta)\rangle = e^{-\beta H_+/4}|I\rangle \qquad |I\rangle = \sum_n |n, \tilde{n}\rangle$$

$$|0(\beta)\rangle = \sum_{\{i\},\{j\}} e^{-\beta \mathscr{E}_{\{i\},\{j\}}/2} \left(A^\dagger_{i_1,j_1} A^\dagger_{i_2,j_2} \cdots \right) \left(\tilde{A}^\dagger_{i_1,j_1} \tilde{A}^\dagger_{i_2,j_2} \cdots \right) |0,\tilde{0}\rangle$$

- Ungauged model: Gauged in doubled
- High temperature phase $T > T_c$
- $\mathscr{E}(\beta) = \frac{1}{2}\langle 0(\beta)|H_+|0(\beta)\rangle$ $\qquad F(\beta) = N^2 F_0(\beta) + F_1(\beta) + \frac{1}{N^2}F_2 + \cdots$

Thermal State

- Non-uniqueness: many solutions
- I: $$H_- |\Psi\rangle = 0$$
- Additional constraints: Kubo-Martin-Schwinger (KMS) conditions
- II: $$e^{-\beta H_+/4} \left(A - \tilde{A}^\dagger\right) e^{\beta H_+/4} |0(\beta)\rangle = 0$$
- For any A and $\tilde{A}^\dagger$ in the Hilbert space
- Question: How to Implement I and II Non-perturbatively

Deformations:(of H+)

- Single trace deformation [Maldacena & Qi, 18']

$$H'_+ = H_+ + \mu \sum_j O_j \tilde{O}_j$$

- Overlap $\langle \text{gs}(\mu) \,|\, 0(\beta) \rangle \sim 1$ studied in SYK

- Bogolyubov

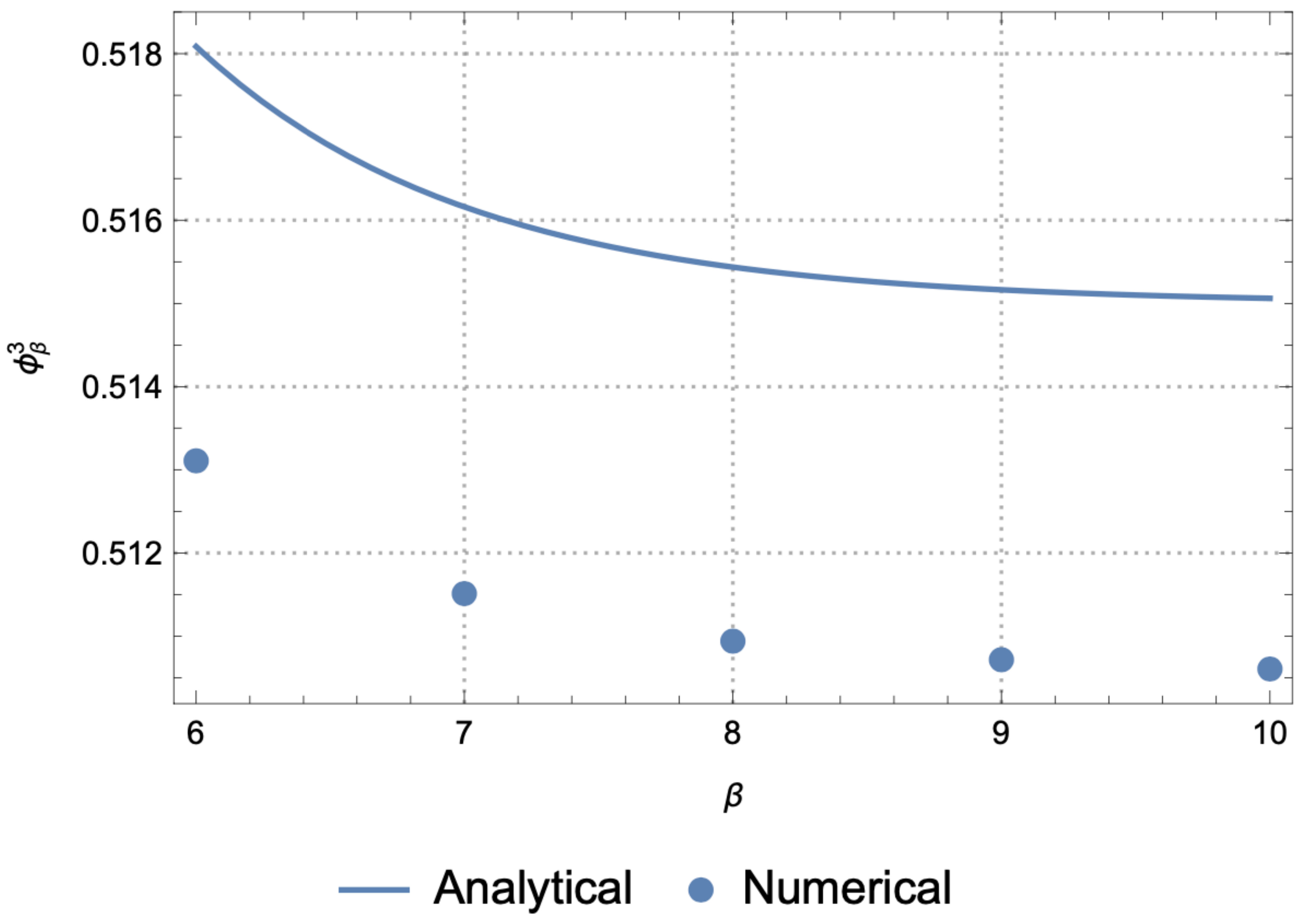
$$\mathcal{E}_\beta = e^{-iG_2} \mathcal{E} e^{iG_2}$$

$$\begin{aligned}\mathcal{E}_\beta =& \frac{1}{2}\Big[\cosh(2\theta_\beta)\mathcal{E} + \sinh(2\theta_\beta)\,\mathrm{tr}(\Pi_1\Pi_2 - M_1M_2)\Big] \\ &+ \frac{g_4}{8}\Big[\big(3+\cosh(4\theta_\beta)\big)\big(\,\mathrm{tr}\big(M_1^4\big) + \mathrm{tr}\big(M_2^4\big)\big) - 4\sinh(4\theta_\beta)\big(\,\mathrm{tr}\big(M_1^3M_2\big) + \mathrm{tr}\big(M_1M_2^3\big)\big) \\ &+ \big(\cosh(4\theta_\beta) - 1\big)\big(4\,\mathrm{tr}\big(M_1^2M_2^2\big) + 2\,\mathrm{tr}(M_1M_2M_1M_2)\big)\Big]\,.\end{aligned}$$

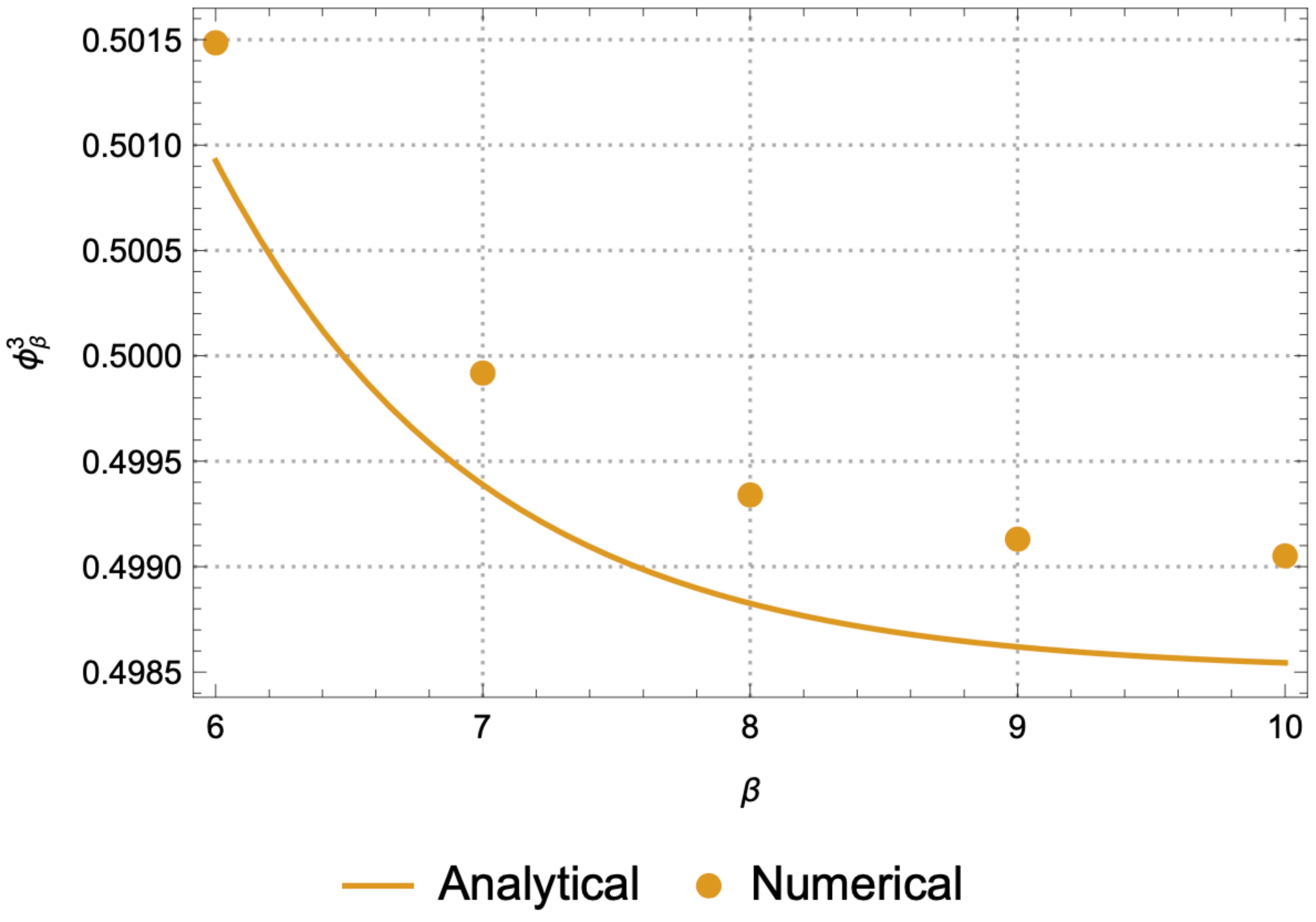
Deformation

Weak Coupling / All Temperature

$g_4 = -0.01$



$g_4 = 0.001$



Approach to Solving H_- :

- $H_- = H - \tilde{H}$: unbounded
- H_-^{col} : likewise
- Minimization $H_-^{\text{col}} = 0$
- Symmetries :
  - "Bogolyubov symmetry" $[G, H_-] = 0$
  - Exchange symmetry $\{M\} \leftrightarrow \{\tilde{M}\}$

Clasification

- Loops (under $\{M\} \leftrightarrow \{\tilde{M}\}$)
- Sym/Antisym $\phi(c) = \{\phi(a), \phi(s)\}$
- H_{-}^{col} is Antisymmetric
- And $V_{-}^{\text{col}}\left(\phi(a), \phi(s)\right) = 0$ For $\phi(a) = 0$
- Need to solve $\bar{V}\left(\phi(s)\right) \equiv \partial_a V_{-}^{\text{col}}\left(\phi(a) = 0, \phi(s)\right) = 0$
- To accomplish $H_{-}^{\text{col}} = 0$

Bogolyubov Symmetry:free case

$$H_- = \frac{1}{2}\left(P^2 + M^2\right) - \frac{1}{2}\left(\tilde{P}^2 + \tilde{M}^2\right) \qquad G_2 = P\tilde{M} - \tilde{P}M \qquad [G_2, H_-] = 0$$

- Bogolyubov symmetry $A_\theta = \cosh\theta\, A - \sinh\theta\, \tilde{A}^\dagger$ Unruh
- Thermal loops $\langle \phi(c) \rangle_\beta = \phi_\beta(c)$
- Bogolyubov transform: thermal loops $\phi_\beta(c) = \sum_{c'} W(c, c') \phi_{\mathrm{gs}}(c')$
- Bogolyubov matrix : e.g.

$$W_2 = \begin{bmatrix} \cosh^2(2\theta) & \sinh(4\theta) & \sinh^2(2\theta) \\ \frac{1}{2}\sinh(4\theta) & \cosh(4\theta) & \frac{1}{2}\sinh(4\theta) \\ \sinh^2(2\theta) & \sinh(4\theta) & \cosh^2(2\theta) \end{bmatrix}$$

  - Example $l = 2$

Interacting Theory

- Bogolyubov symmetry : not known
- Perturbation theory : gM^3

$$G = G_2 + \frac{g_3}{\sqrt{N}} G_3 + \frac{g_4}{N} G_4 + \cdots$$

$$\hat{G}_f^{(3)} = f\Big(\operatorname{tr}\big(M_1^2 \Pi_2\big) + \operatorname{tr}\big(M_2^2 \Pi_1\big) + 2\operatorname{tr}\big(\Pi_1^2 \Pi_2\big) + 2\operatorname{tr}\big(\Pi_2^2 \Pi_1\big) \\ - \operatorname{tr}(M_1 M_2 \Pi_1) - \operatorname{tr}(M_2 M_1 \Pi_1) - \operatorname{tr}(M_1 M_1 \Pi_2) - \operatorname{tr}(M_1 M_2 \Pi_2)\Big) .$$

- Singularities at Higher order :On-shell symmetry

Vector models:Analytical solution [AJ, Yoon ,Liu & Zheng, 22']

- Interaction $$g\left(\left(\varphi\cdot\varphi\right)^2-\left(\tilde{\varphi}\cdot\tilde{\varphi}\right)^2\right)$$
- Bi-locals $$\Phi(\vec{x},\vec{y})=\begin{pmatrix}\varphi\cdot\varphi & \varphi\cdot\tilde{\varphi}\\ \tilde{\varphi}\cdot\varphi & \tilde{\varphi}\cdot\tilde{\varphi}\end{pmatrix}(\vec{x},\vec{y})$$

$$\Phi_f(\vec{x},\vec{y})=\int\frac{\mathrm{d}^d\vec{k}}{(2\pi)^d}\frac{\mathrm{e}^{\mathrm{i}\,\vec{k}\cdot(\vec{x}-\vec{y})}}{2\omega_f(\vec{k})}\begin{pmatrix}\mathrm{ch}\,f(\vec{k}) & \mathrm{sh}\,f(\vec{k})\\ \mathrm{sh}\,f(\vec{k}) & \mathrm{ch}\,f(\vec{k})\end{pmatrix}$$

- Gap equation $$\omega_f^2-\omega^2=g\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\cosh f(\vec{k})}{2\omega_f(\vec{k})}$$
- Free parameter f - inverse temperature β relation : $$\tanh f=e^{-\beta\omega_f/2}$$
- Exact on-shell symmetry of H_coll : Nonlinear Bogolyubov

Matrix Thermofield

H_- Collective

- Gauging: $U\{M\}U^\dagger \qquad U\{\tilde{M}\}U^\dagger$
- Loops in doubled set $\phi(c) = \mathrm{Tr}\left(M^{n_1}\tilde{M}^{\tilde{n}_1}M^{n_2}\tilde{M}^{\tilde{n}_2}\cdots\right)$
- Collective representation

$$H_-^{\text{col}} = \frac{1}{2}\sum_{c,c'}\Pi(c)\Omega_-(c,c')\Pi(c') + V_-$$

Antisymmetric

- Joining

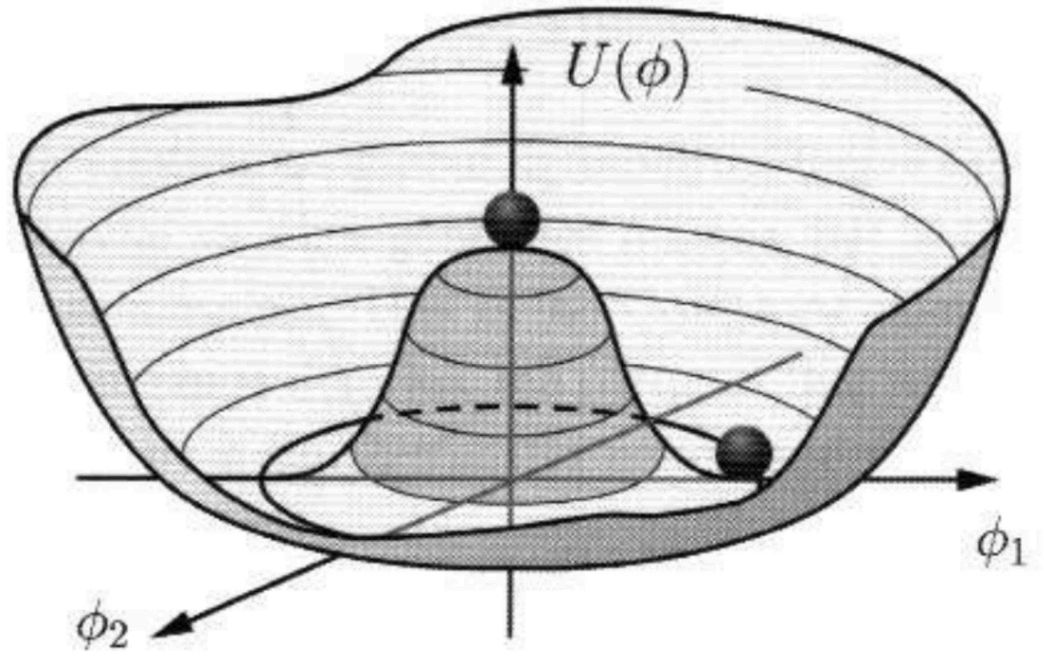
$$\Omega_-(c,c') = \frac{\partial\phi(c)}{\partial M}\frac{\partial\phi(c')}{\partial M} - \frac{\partial\phi(c)}{\partial\tilde{M}}\frac{\partial\phi(c')}{\partial\tilde{M}}$$

Antisymmetric

Numerical Optimization

- a : anti-symmetric loops; s : symmetric loops $M \leftrightarrow \tilde{M}$
- Expected values $\langle \phi(a) \rangle = 0$ $\langle \phi(s) \rangle \neq 0$
- Only needs to solve $\bar{V}\left(\phi(s)\right) \equiv \partial_a V_{-}^{\text{col}}\left(\phi(a) = 0, \phi(s)\right) = 0$
- Loss function $L = \bar{V}\left(\phi(s)\right) \Omega_{+} \bar{V}\left(\phi(s)\right) + \left(\phi_4 - \phi_4^0\right)^2 + \text{constraints}$
- Subject to positivity: Ω_{+}

- Caricature:



- Observed: degenerate thermal loop values (under exchange)

$$\phi_3 = \phi_5 \qquad \phi_{10} = \phi_{15} \qquad \cdots$$

- Ground state is not unique — one-parameter family of solutions

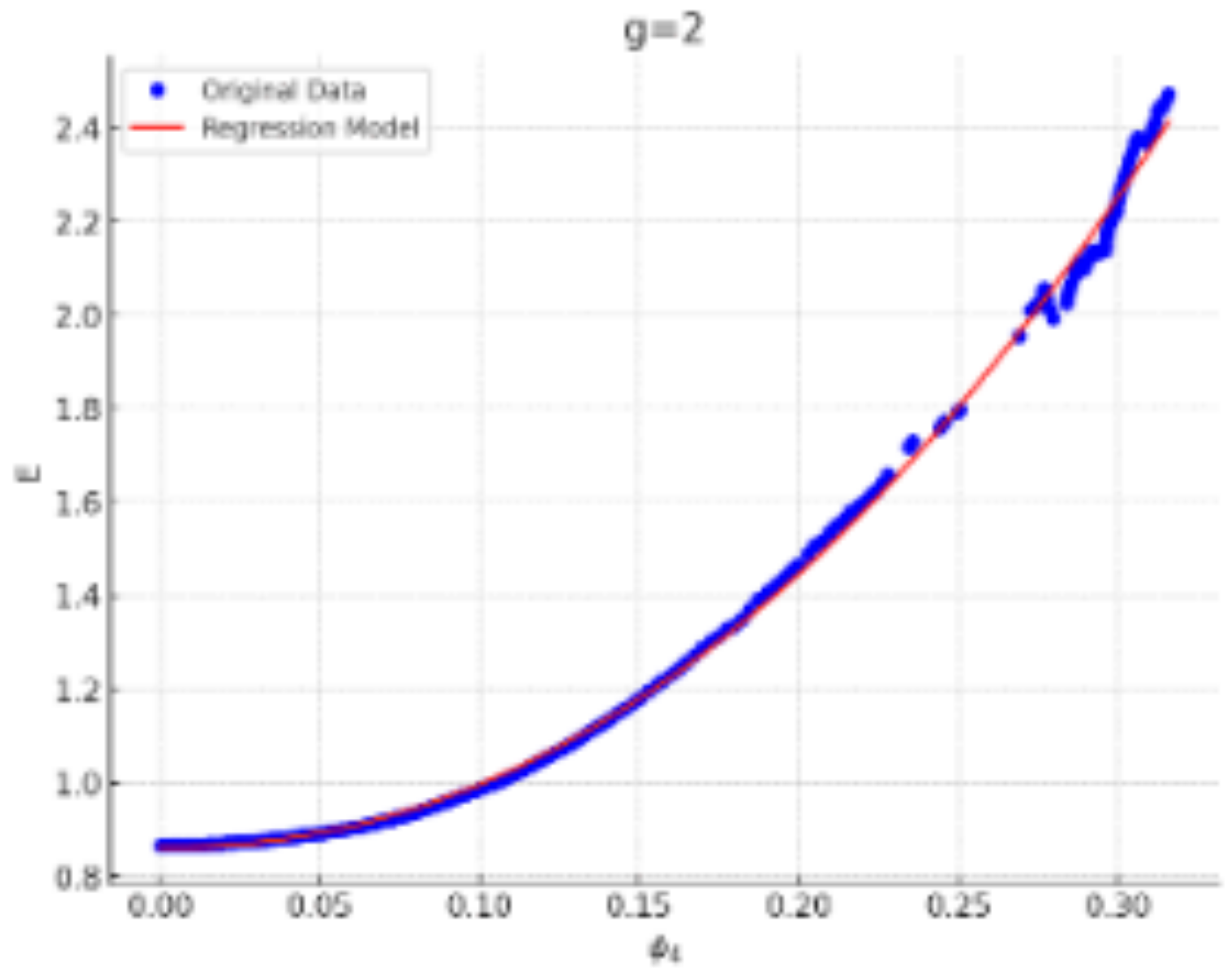
$$\phi_f$$

- Fix one variable ϕ_4^0 :unique Solution

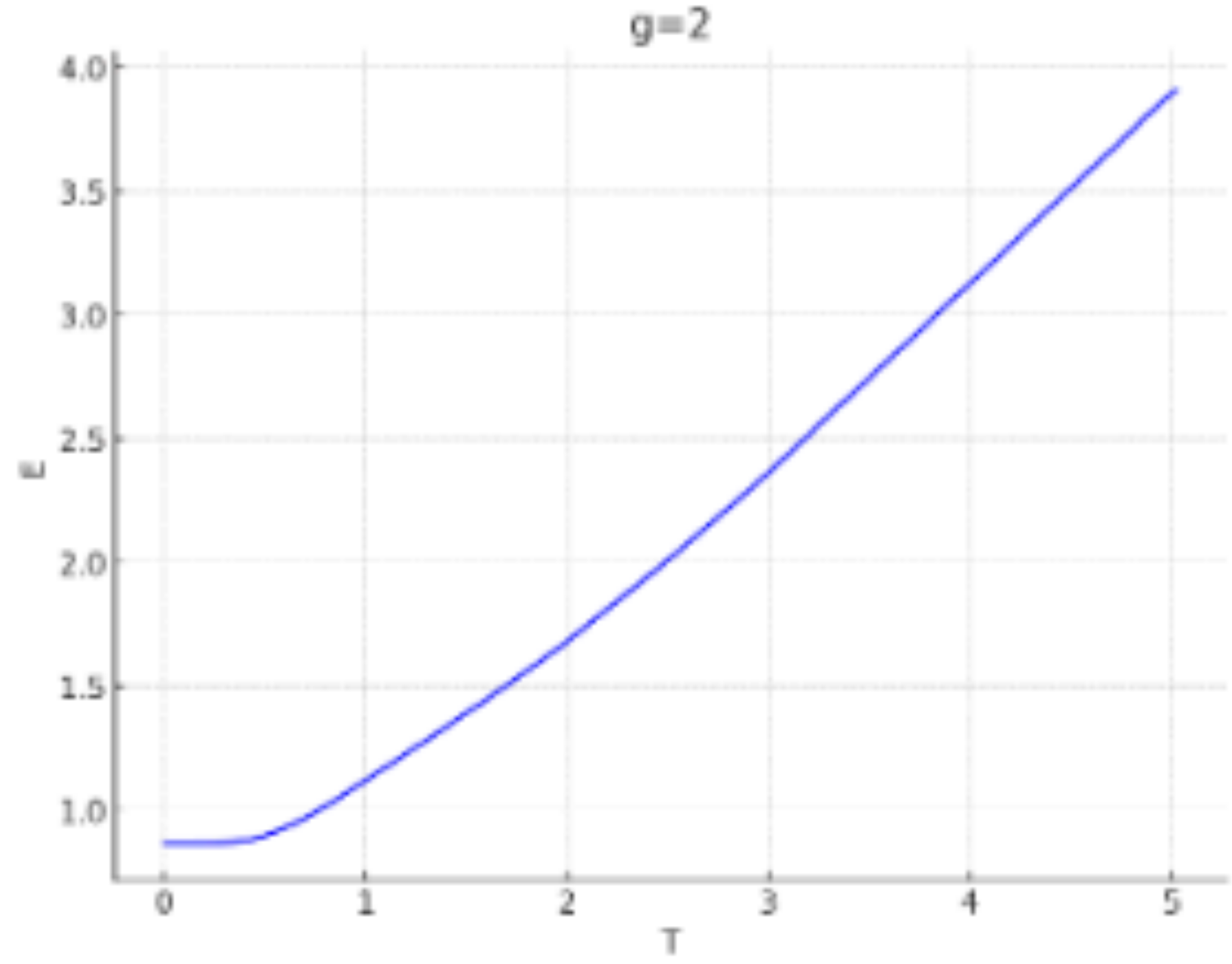
Numerical Optimization

- Observed degeneracy (in numerical optimization): $\phi \to \phi_f$

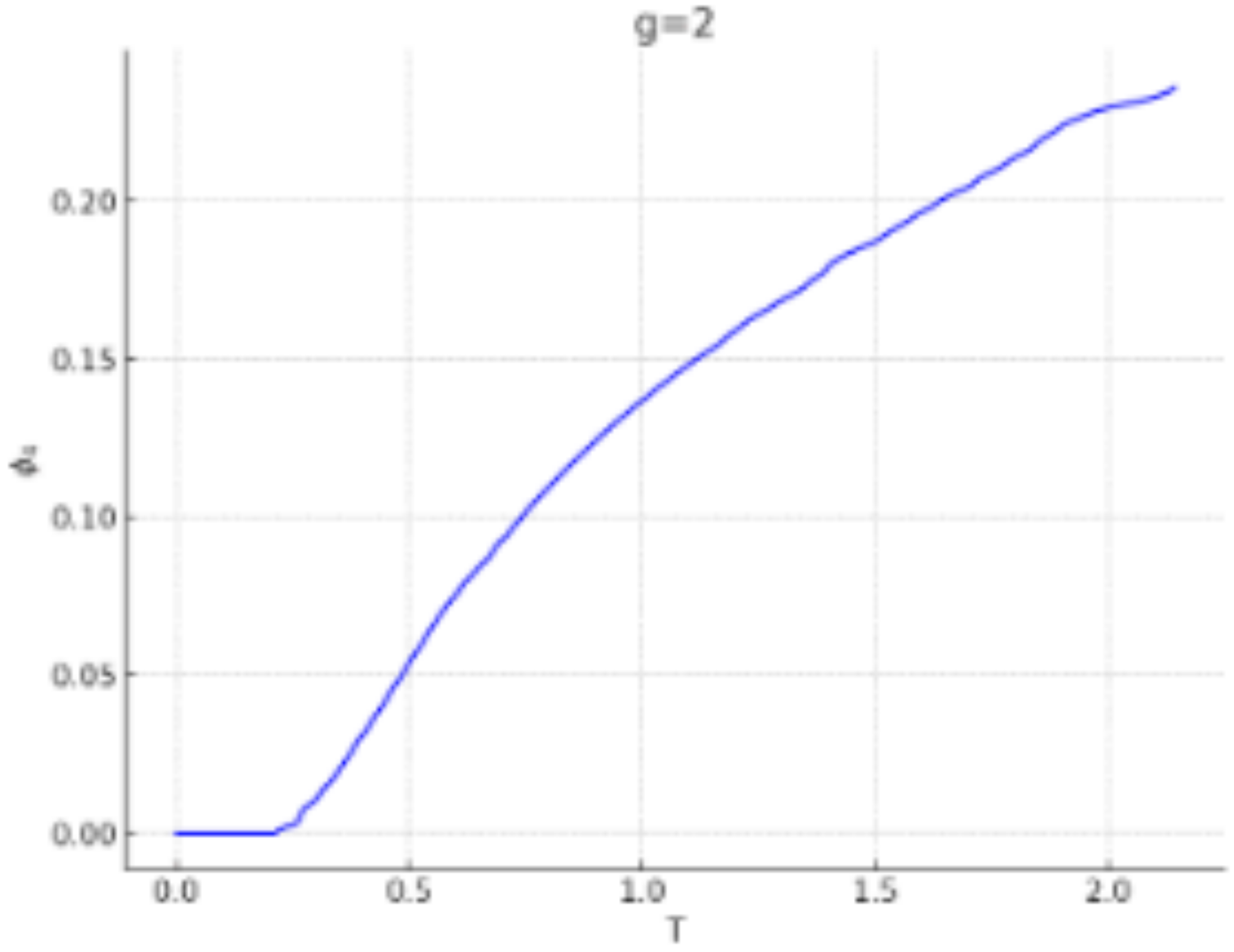
$E(f) = H_+^{\text{col}} \qquad \phi_4 = \phi_4(f) \qquad \Rightarrow \qquad E = E(\phi_4)$



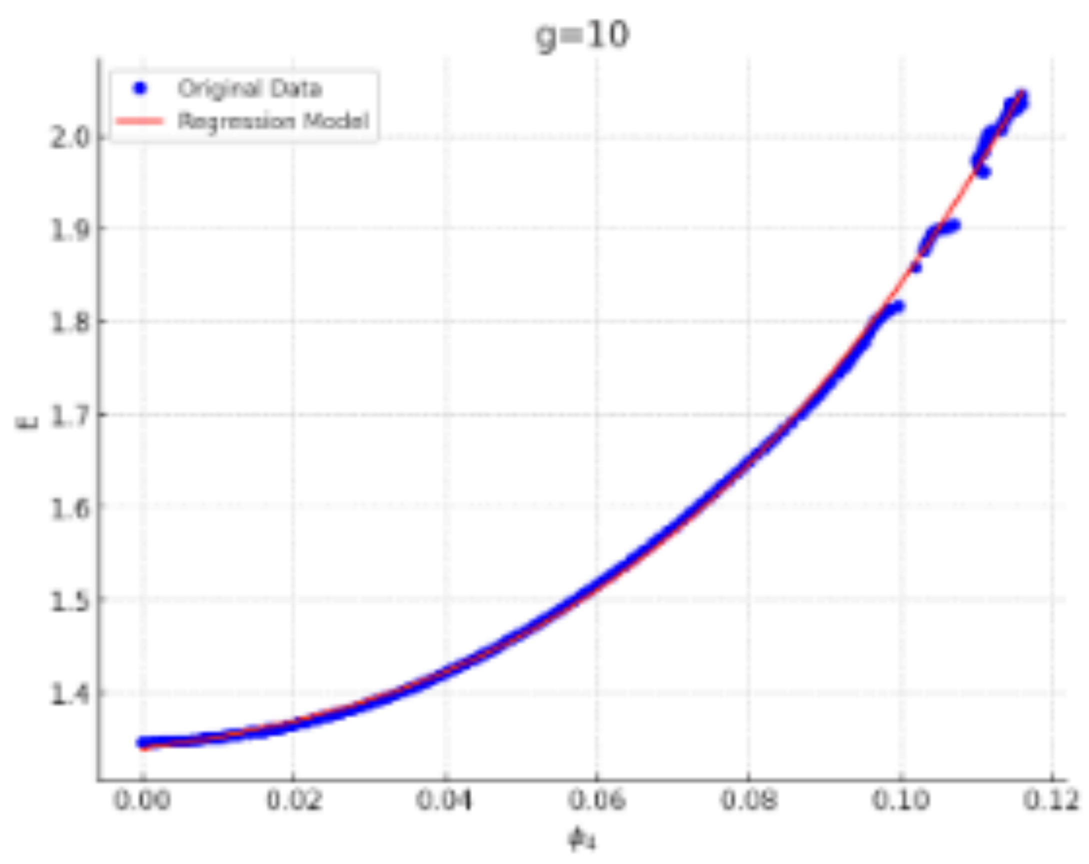
(a) E vs ϕ_4, $g = 2$



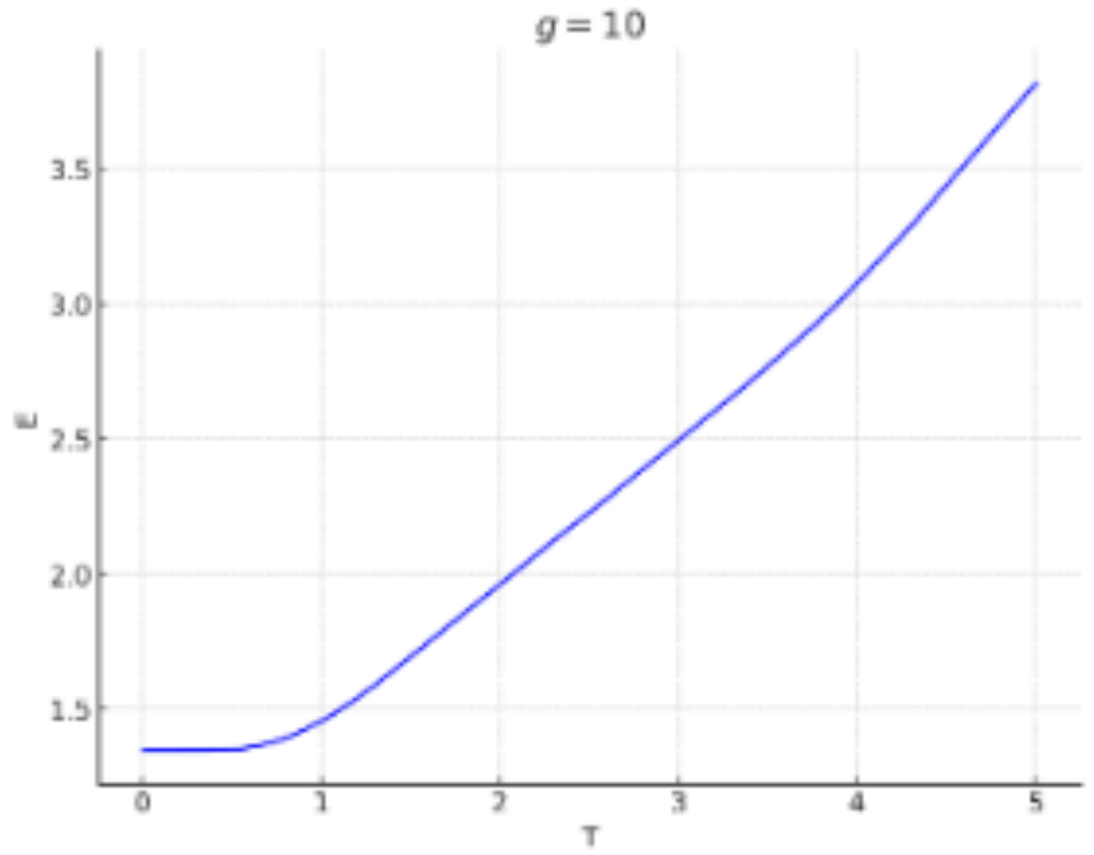
(b) E vs T, g=2



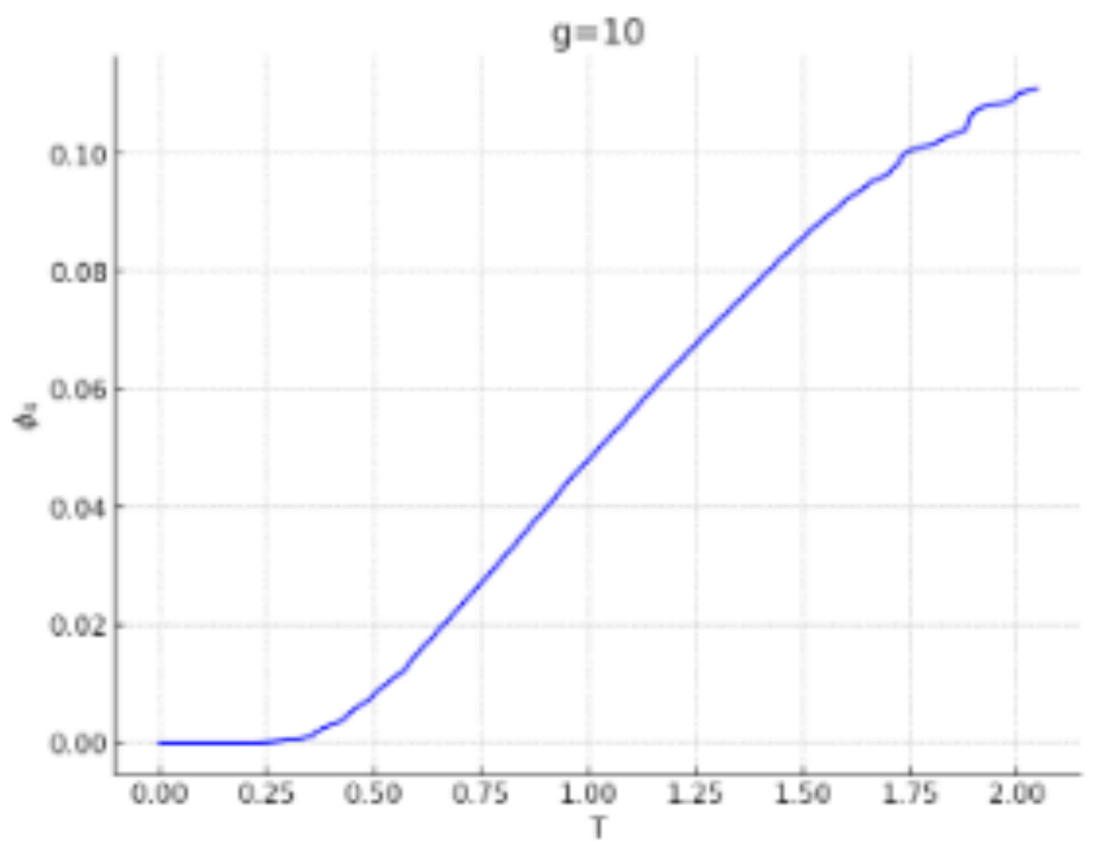
(c) ϕ_4 vs T, g=2



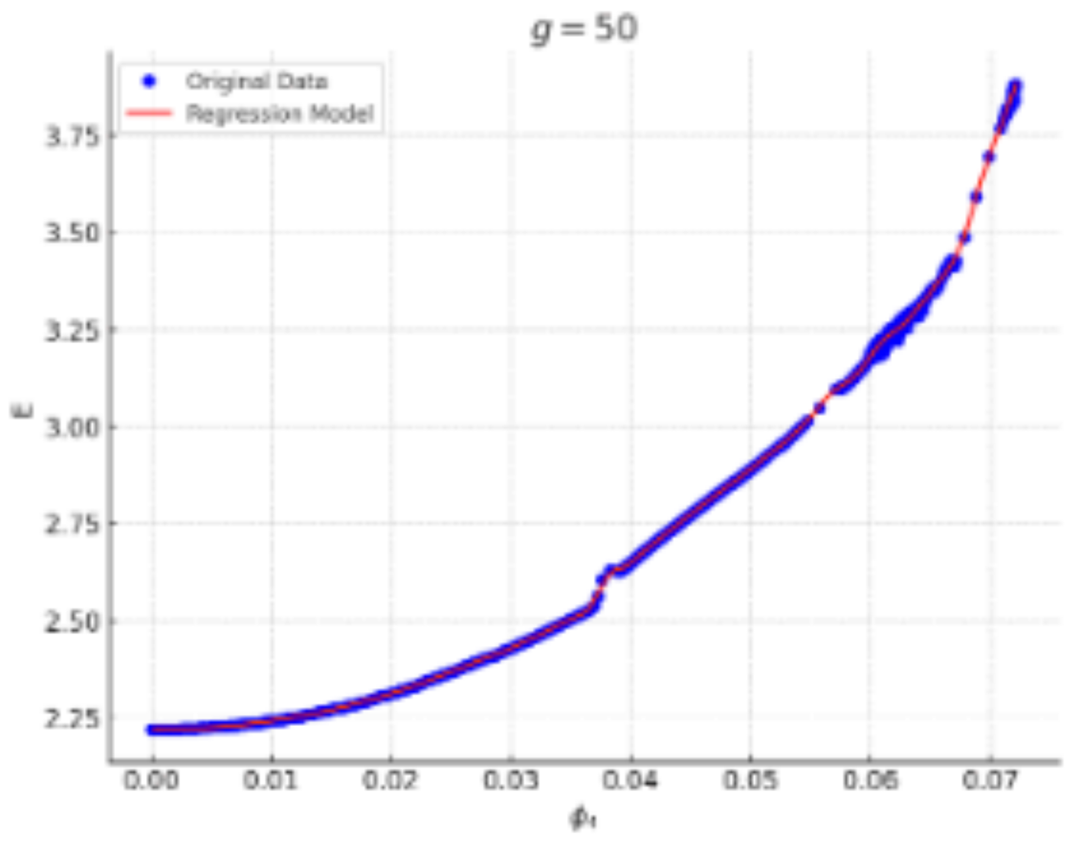
(a) E vs ϕ_4, $g = 10$



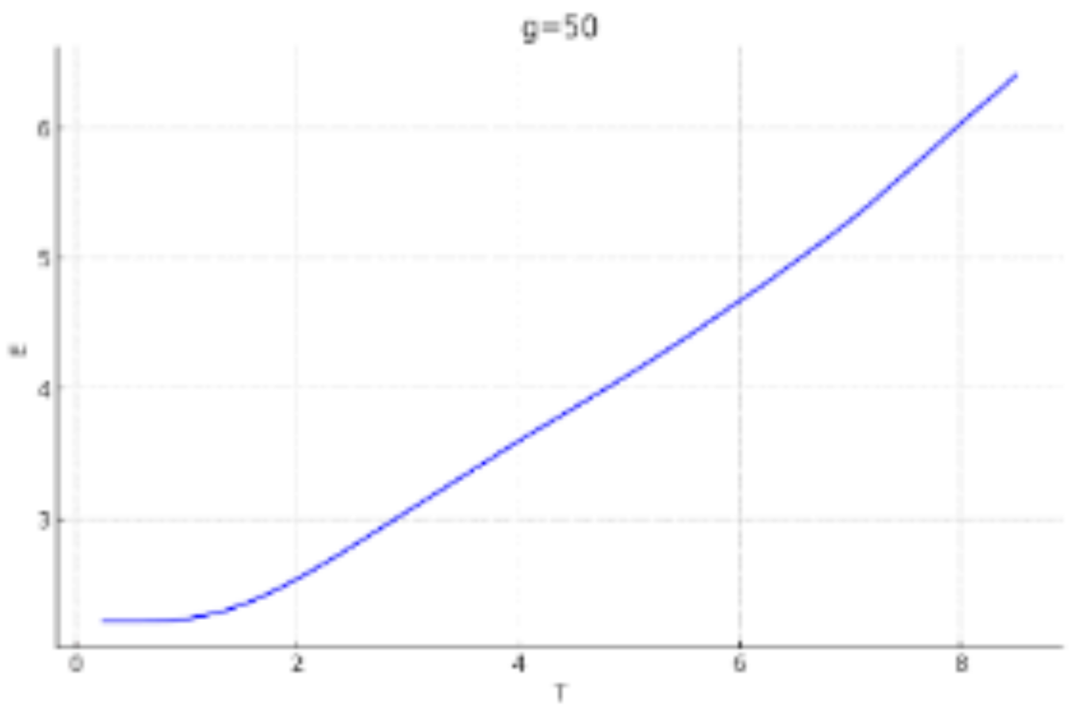
(b) E vs T, $g = 10$



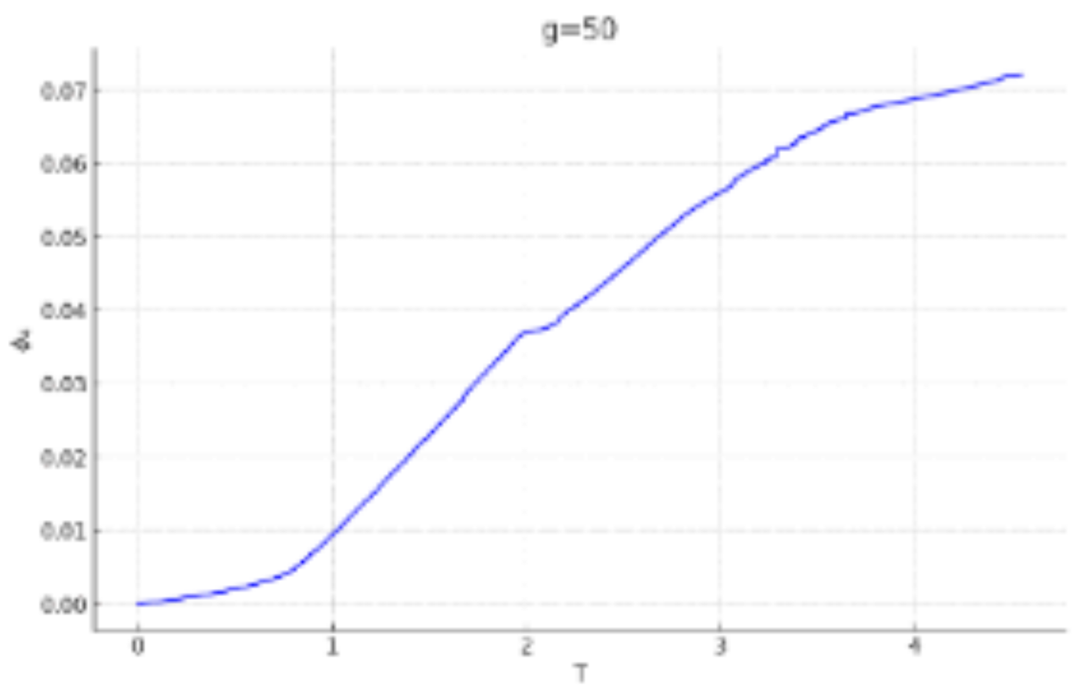
(c) ϕ_4 vs T, $g = 10$



(a) E vs ϕ_4, $g = 50$



(b) E vs T, $g = 50$



(c) ϕ_4 vs T, $g = 50$

O(1)

- Spectrum , fluctuations

$$H_-^{\text{col}} = NE + H_2 + \frac{1}{N} H_3 + \cdots \qquad \phi = \phi_f + \hat{\eta}$$

- O(1) $$H_2 = \frac{1}{2}\hat{\Pi}\Omega_f\hat{\Pi} + \frac{1}{2}\hat{\eta}V_f\hat{\eta} \qquad \Omega_f = \Omega_-(\phi_f)$$

- number of Lagrange multipliers (zero modes)
- Solve for Normal Modes

- The eigenvalue problem / spectrum

$$(\omega_\alpha^2 \Omega_f - V_f) \cdot X_\alpha = 0$$

- ω_α : frequencies; X_α : normal basis
- Thermal propagators

$$iD_{c,c'} = \langle \eta(c)\eta(c') \rangle = iJ\,\mathrm{diag}(d)J^T$$

- Diagonal : normal mode in pairs $\{\pm m_\alpha\}$ + zero mode m_0

$$d^{\pm}(t) = \frac{e^{\mp i\omega_f|t|}}{\pm 2m_f\omega_f} \qquad d_0 = \frac{1}{2m_0\omega_0}$$

Measuring the Temperature :KMS Conditions

- KMS conditions on matrices $M(t - i\beta/2) = \tilde{M}(t) \qquad \tilde{M}(t - i\beta/2) = M(t)$
- and correlation functions

$$D_{M^2,M^2}(t - i\beta/2) = D_{\tilde{M}^2,M^2}(t) \qquad D_{M\tilde{M},M\tilde{M}}(t - i\beta/2) = D_{\tilde{M}M,M\tilde{M}}(t)$$

$$\cdots$$

- Parameter: f - β relation

$$\sum_\alpha \frac{1 - e^{-\omega_\alpha \beta/2}}{2 m_\alpha \omega_\alpha} J^2_{\phi\alpha} = 0$$

- Many other KMS conditions : all satisfied (?)

Alternative :β Minimization

- Loss function $L = \bar{V}\left(\phi(s)\right)\Omega_{+}\bar{V}\left(\phi(s)\right) + \left(\sum_{\alpha}\frac{1-e^{-\omega_{\alpha}\beta/2}}{2m_{\alpha}\omega_{\alpha}}J_{\phi\alpha}^{2}\right)^{2}$ + constraints
- Specifies the (inverse) temperature β
- Enforced a KMS condition:Unique solution for thermal loops
- Summary: Thermal Optimization on the (real)time SK contour:Thermodynamics +Construction of the Thermal State

Comment on:Symmetries and Large Operators

$$\hat{H}_2 = \frac{1}{2}\mathrm{Tr}[\pi^{\mathrm{T}}K\pi + \eta^{\mathrm{T}}V\eta]$$ zero modes $$\mathrm{Tr}[Ku_k] = 0 \quad \mathrm{Tr}[Vu_k] = 0$$

The symmetry operators instead have Large $\mathcal{O}(\sqrt{N})$ terms:

$$H_+ = \sqrt{N}H_+^1 + H_+^2 + \ldots \qquad G_f = \sqrt{N}G_f^1 + G_f^2 + \ldots$$

- The zero modes are in one-to-one correspondence with the leading order of the symmetry operat is implicit)

$$H_{+,1} = \mathrm{Tr}[c_1 u^{\mathrm{T}}\eta] \qquad \Longleftrightarrow \qquad 0 = [\hat{H}_2, H_+^1] = \mathrm{Tr}[c_1 K u]$$

$$G_{f,1} = \mathrm{Tr}[c_2 v^{\mathrm{T}}\pi] \qquad \Longleftrightarrow \qquad 0 = [\hat{H}_2, G_f^1] = \mathrm{Tr}[c_2 V v]$$

- An infinite re-summation is needed to compute the symmetry transformation:
- Large N Matrix models also feature large operators of $\mathcal{O}(N)$
- And in numerical simulation a Single Zero mode was identified

Conclusions

- Large N Schwinger- Kelydish Optimization
- Implemented in:Multi-Matrix models
- Thermal State $$\Psi_\beta[\eta] = \exp\left(-\frac{1}{2}\,\mathrm{Tr}[\eta^{\mathrm{T}}\mathcal{G}^{-1}\eta]\right)$$
- Hilbert Space:(L-R)+Zero mode(Goldstone mode related to f):
- ‘Large’ Symmetry operators /Witten

Thank you!