# Aspects of Non-perturbative Hilbert Spaces

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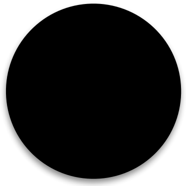
M.M: 2410.20662
M.M, S.Ruan, S.Shibuya, K.Yano: To appear
K.Okuyama, M.M, S.Mori: Work in progress

# Finite Black Hole Entropy

**Bekenstein-Hawking Entropy:** [Bekenstein (1973)][Hawking (1975)]
BH has finite entropy given by

$$S_{\mathrm{BH}} = \frac{\mathrm{Area[Horizon]}}{4G_N}$$

Black hole horizon



- This implies that the number of BH microstates is finite, and therefore the spectrum is discrete

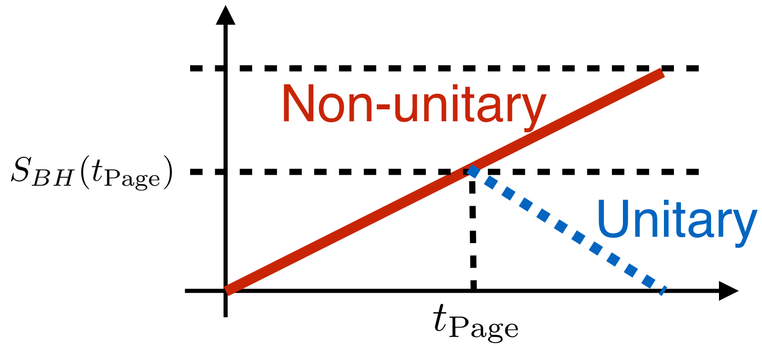

BH Spectrum

- Thus perturbative Hilbert space which contains infinite local degrees of freedom, must be modified non-perturbatively

# Perturbative Hilbert Space & Failure

**Black hole information paradox:** [Hawking (1974)]
Hawking radiation entangled with the BH, appears to have much larger entropy than allowed; more than BH entropy


Entropy of the Hawking radiation

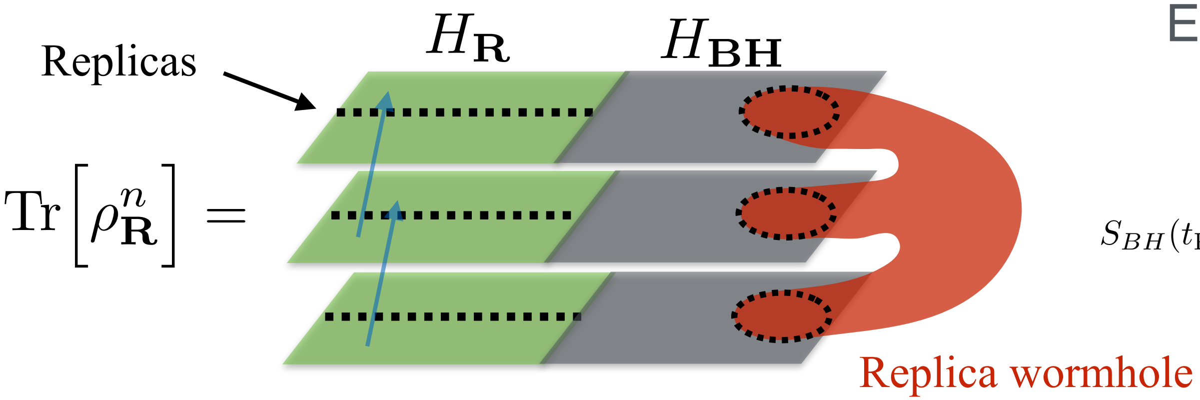


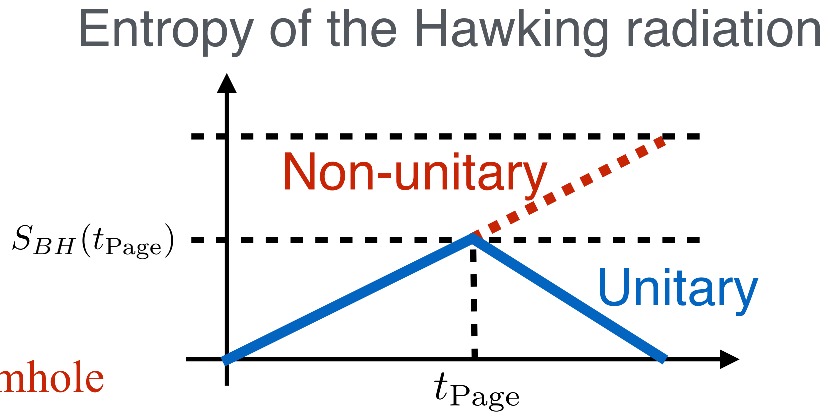

This is the sharpest example where the perturbative Hilbert space breaks down badly

**Question: How can we incorporate the effects of non-perturbative Hilbert space?**

# Fix: Bulk Euclidean Wormholes

**Discreteness from Euclidean Wormholes:**
Adding Euclidean wormholes to the gravitational path-integral was found to give unitary Page curve, resolving the paradox for the Hawking radiation entropy




- Capture discreteness of the spectrum
- Justified for low-dimensional gravity on AdS

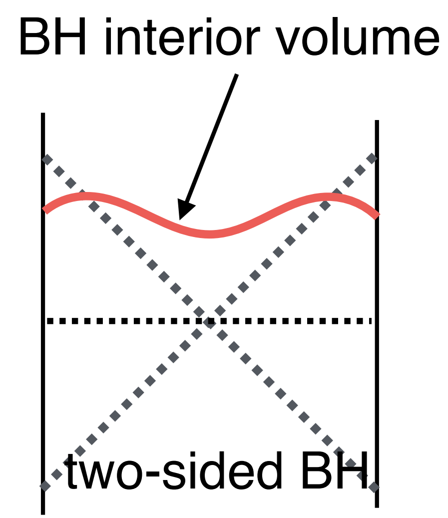
[Penington (2019)][Almheiri, Engelhardt, Markolf, Maxfield (2019)][Almheiri, Mahajan, Maldacena, Zhao (2019)]
[Penignton, Shenker, Stanford, Yang(2019)][Almheiri, Hartman, Maldacena, Shaghoulian, Tajdini (2019)]

# Non-perturbative Hilbert space for Black Hole Interior

**Ultimate goals**:

- Physics at the horizon and interior
- Infalling observer experience?
- Mechanism for information restoration?

**Perturbative Hilbert space for BH interior**:
We can canonically quantize gravity, variables like timeshift and interior volume are continuous




**Similar to information paradox, this is not consistent with the finite BH entropy**

# Two-sided black hole

**Dual to Thermofield double state:** [Maldacena (2001)]

$$|\mathrm{TFD}\rangle = \frac{1}{\sqrt{Z(\beta)}}\sum_i e^{-\beta E_i/2}|E_i^L\rangle|E_i^R\rangle$$

- Two sides are connected via smooth horizon and non-traversable wormhole

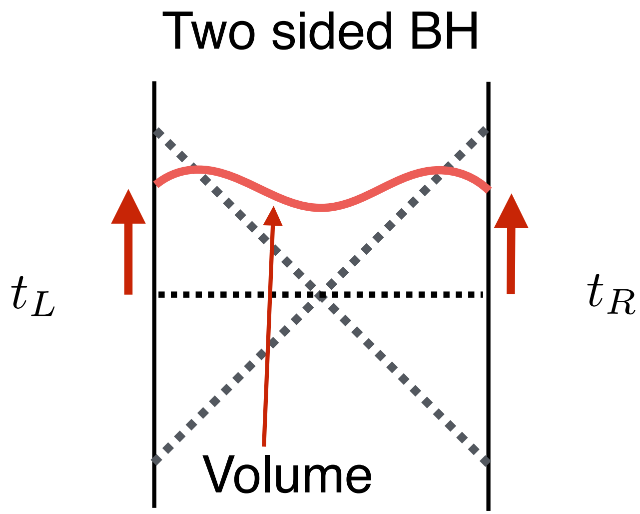
- The state evolves non-trivially under

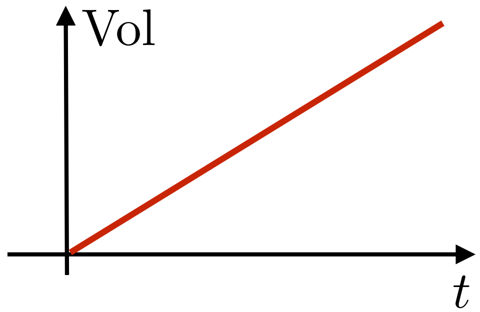
$$H = H^L + H^R$$

$$|\mathrm{TFD}(t)\rangle = \frac{1}{\sqrt{Z(\beta)}}\sum_i e^{-(\beta+4it)E_i/2}|E_i^L\rangle|E_i^R\rangle$$

- Classically, it has eternally linearly growing interior volume

  [Susskind (2013)][Susskind, Stanford (2014)]




**Both continuity and unboundedness are problems!**

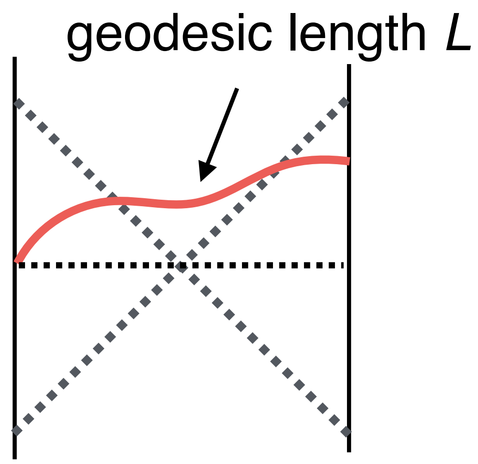
# Length state without wormholes

In JT gravity, we know the quantum state corresponding to fixed geodesic length state in perturbative Hilbert space

- Hamiltonian

$$H = \frac{P^2}{2} + 2e^{-L}$$

$P = \dot{L}$ is the conjugate momentum




- Overlap between the energy and the length eigenstate

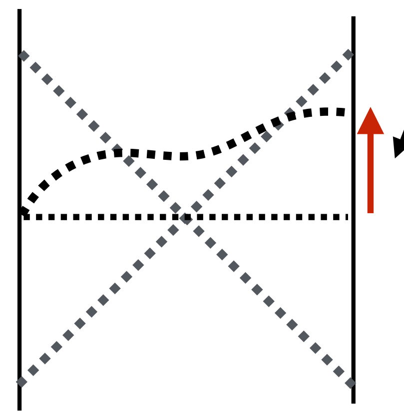
$$\langle l|E\rangle = e^{-S_0/2} 2^{3/2} K_{i2\sqrt{E}}(2e^{-l/2})$$

Here *E* denotes twice the single sided energy

# Fixed timeshift state without wormholes

In JT gravity, we know the quantum state corresponding to fixed timeshift state in perturbative Hilbert space

- Hamiltonian is the canonical momentum of $\delta$




- Overlap between the energy and the timeshift eigenstate

$$\langle E|\delta\rangle = \frac{e^{-i\delta E}}{\sqrt{e^{S_0} D_{\text{Disk}}(E)}}$$

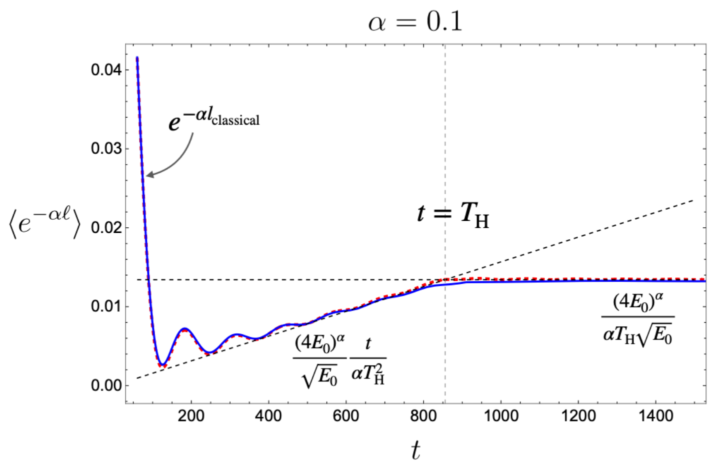
# What we will do in this talk

[1] : **Constructing length / timeshift states in non-perturbative Hilbert space**

[2] : **New probes for non-perturbative length and timeshift**

- Generating function for the length and timeshift display dip-ramp-plateau behavior

$$\langle e^{-\alpha l}\rangle_t = \int dl P(l,t) e^{-\alpha l}$$



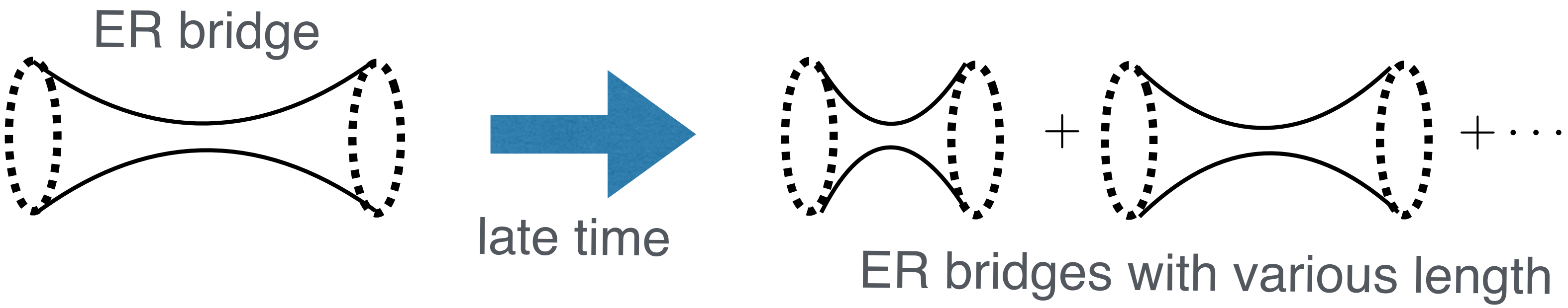

- Timescale is the same as the spectral form factor
- These are probes of microscopic chaotic spectrum

# What we will do in this talk

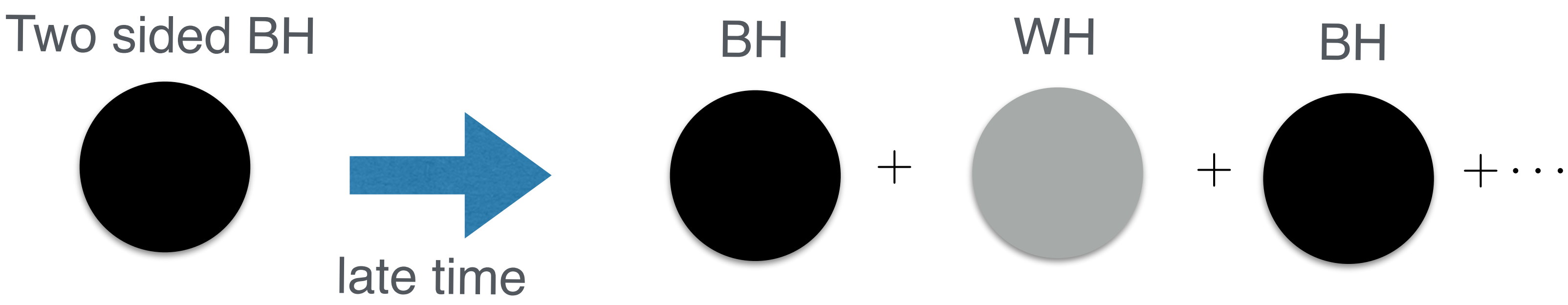
[3] : **Non-perturbative observable**

- We construct non-perturbative linear operators for the length and the timeshift
- We keep track of the TFD time evolution and find:

**Old BH → Uniform superposition of arbitrary length ER bridges**




**Old BH→ Uniform superposition of BH and WH**



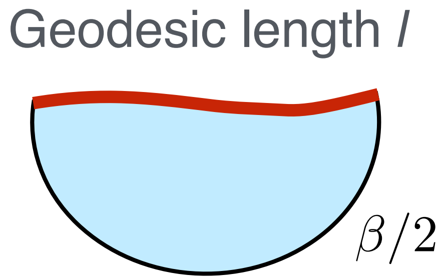
# Construction of
# Non-perturbative length &
# timeshift States

# Bulk quantum state from Hartle-Hawking prescription

**Generalized Hartle-Hawking prescription:**
The bulk wavefunction is given by the sum over all possible geometries

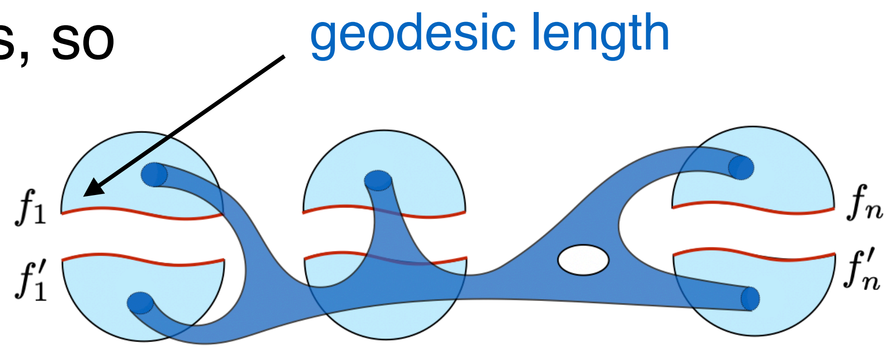
- We seek for non-perturbative quantum state $|l\rangle$ which satisfies

$$\mathbb{E}\left[\langle l|\mathrm{TFD}(\beta/2)\rangle\right] =$$




with all possible wormhole corrections, so

$$\mathbb{E}\left[\langle l_1|\mathrm{TFD}(\beta_1/2)\rangle\langle\mathrm{TFD}(\beta_2/2)|l_2\rangle\cdots\right] = \sum_{\mathrm{Wormholes}}$$



# Geodesic length over-complete basis

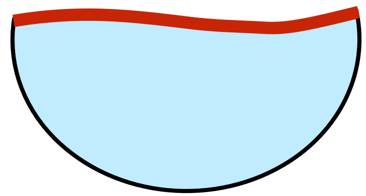
**Fixed geodesic length state is given in JT by** [Iliesiu et.al. (2024)]

$$|l\rangle = \sum_{|E_i - E_0| < \Delta E'/2} \psi_{\rm Disk}(E_i, l)|E_i\rangle$$

Microcanonical

JT gravity wave function with fixed length geodesic and energy

Microscopic energy eigenstate (Two-sided energy)




- The wavefunction is

$$\langle l|E\rangle = e^{-S_0/2} 2^{3/2} K_{i2\sqrt{E}}(2e^{-l/2})$$

- We take the microcanonical ensemble and the Hilbert space dimension is finite

$$N' = e^{S_0} D(E_0)\Delta E'$$

# Timeshift

**Timeshift $\hat{\delta}$ :**
Average of time at the left and right boundaries, conjugate to the Hamiltonian

- Positive timeshift state corresponds to BH (expanding interior), while negative timeshift state corresponds to WH (contracting interior)

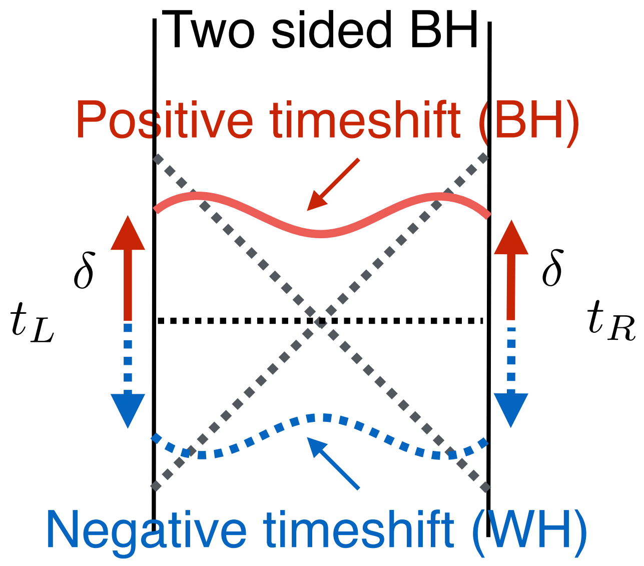
$$|\delta\rangle = \sum_{|E_i - E_0| < \Delta E'/2} \psi_{\mathrm{Disk},\delta}(E_i, \delta) |E_i\rangle$$

Microcanonical

Microscopic energy eigenstate

Wavefunction

$$\psi_{\mathrm{Disk},\delta}(E, \delta) = \frac{e^{-i\delta E}}{\sqrt{e^{S_0} D_{\mathrm{Disk}}(E)}}$$



# Non-perturbative Overlaps and Probe of Chaos

# Length and Timeshift distribution

**Overlaps**:
If length/timeshift states are orthogonal, the following overlaps define probability distribution for length/timeshift

geodesic length *L*

$$P(l,t) = |\langle \mathrm{TFD}(t)|l\rangle|^2 \; = \frac{e^{2S_0}}{Z}\int dE_1 \int dE_2 \; e^{-i(E_1-E_2)t}\psi_{E_1}(\ell)\psi_{E_2}(\ell)\langle D(E_1)D(E_2)\rangle$$

timeshift *δ*

$$P(\delta,t) = |\langle \mathrm{TFD}(t)|\delta\rangle|^2 = \frac{e^{2S_0}}{Z^2}\int dE_1 \int dE_2 \; e^{-i(E_1-E_2)(t-\delta)}\langle D(E_1)D(E_2)\rangle$$

We compute these overlaps by using sine-kernel for density of state two point function, corresponding to all genus contributions

$$\langle D(E_i)D(E_j)\rangle \approx D_{\mathrm{Disk}}(E_i)D_{\mathrm{Disk}}(E_j) + e^{-S_0}\delta(E_i-E_j)D_{\mathrm{Disk}}(E_i) - e^{-2S_0}\frac{\sin^2(\pi e^{S_0}D_{\mathrm{Disk}}(E_i)(E_i-E_j))}{\pi^2(E_i-E_j)^2}$$

# Length distribution

**Length**:
At large *l* and sufficiently large *ΔE*, we can approximate

$$\frac{\langle \mathrm{TFD}(t)|l\rangle\langle l|\mathrm{TFD}(t)\rangle}{\langle l|l\rangle} \to \frac{2}{\Delta E \Delta E'}\left[\frac{\sin^2\left((t_l - t)\frac{\Delta E}{2}\right)}{\left(t_l - t\right)^2} + \frac{\sin^2\left((t_l + t)\frac{\Delta E}{2}\right)}{\left(t_l + t\right)^2}\right]$$

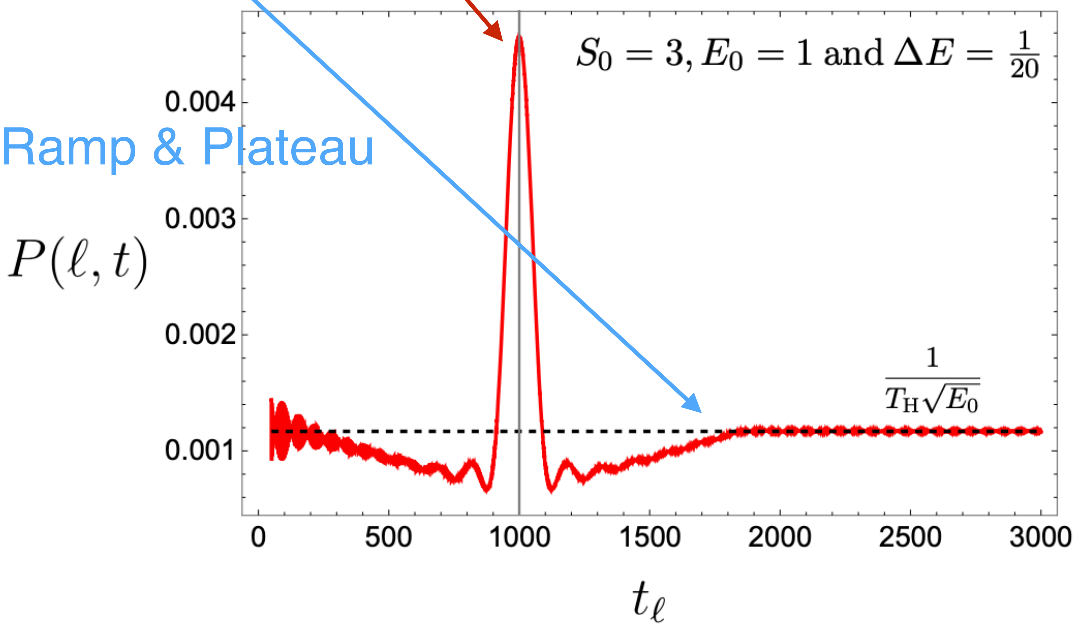
$$+\frac{\pi}{T_H^2 \Delta E'}\left[\mathrm{Min}\left[T_H,\ |t_l - t|\right] + \mathrm{Min}\left[T_H,\ |t_l + t|\right]\right]$$

Classical peak

Ramp & Plateau

Classical relation between time and the length :

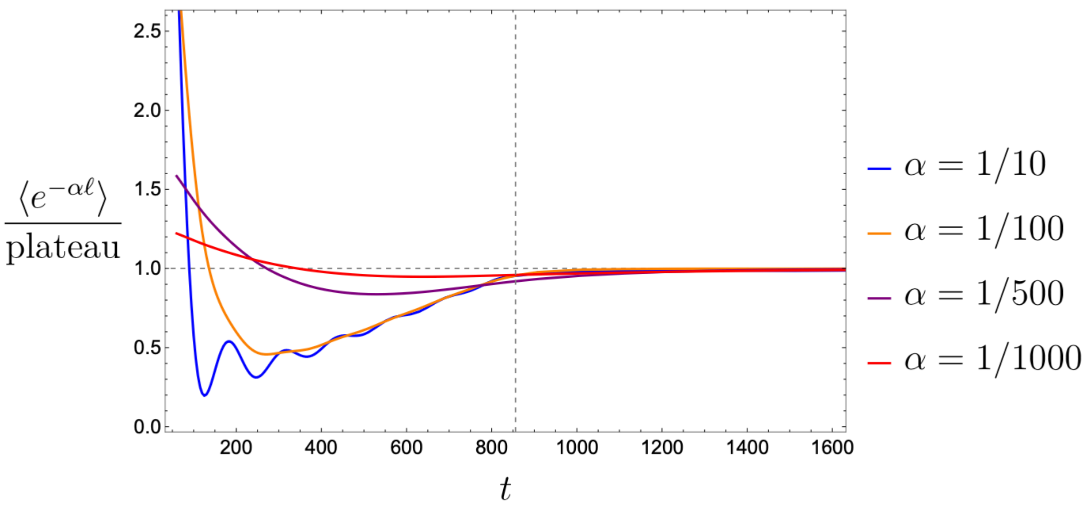
$$t_l := \frac{l + \log(4E_0)}{2\sqrt{E_0}}$$

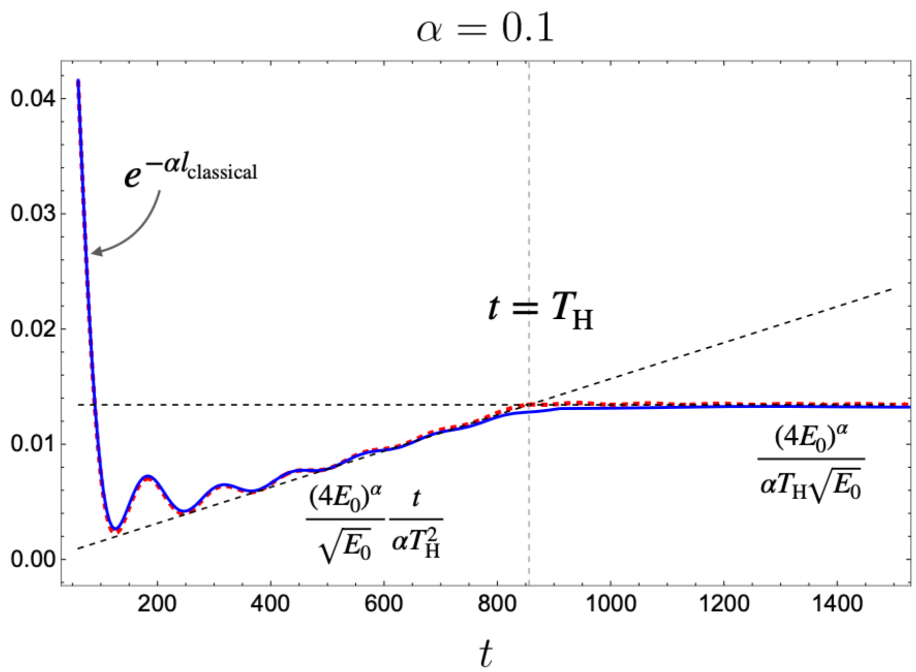


# Length Generating Function

**Dip-ramp-plateau behavior**:
Generating function exhibits dip-ramp-plateau behavior, reaching plateau at the Heisenberg time

$$\langle e^{-\alpha l} \rangle_t = \int dl P(l,t) e^{-\alpha l}$$




- As we take smaller $\alpha$, it diverges and the ramp disappears, i.e. early exponential decay is followed immediately by the plateau

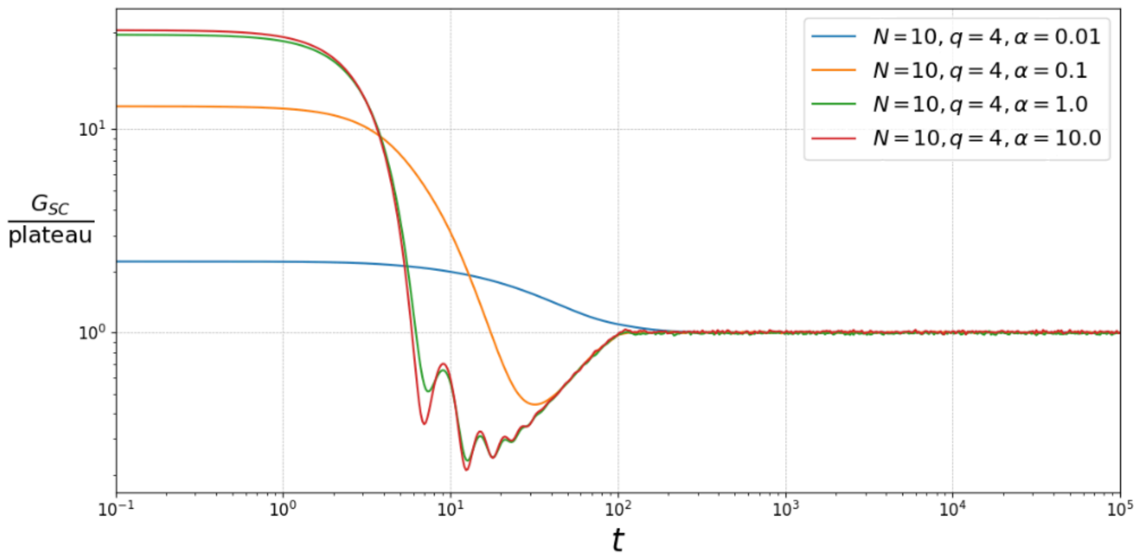
# Probe for Chaotic Spectrum

**Generating function as probe for chaotic spectrum**:
The length generating function can be written as

$$\langle e^{-\alpha l}\rangle_t \sim \sum_{E_1,E_2} \frac{\alpha \cos\left((E_1-E_2)t\right)}{(E_1-E_2)^2+2(E_1+E_2)\alpha^2}$$

We can apply this quantity in any system to probe “internal length”

- We studied SYK model and indeed find the dip-ramp-plateau behavior

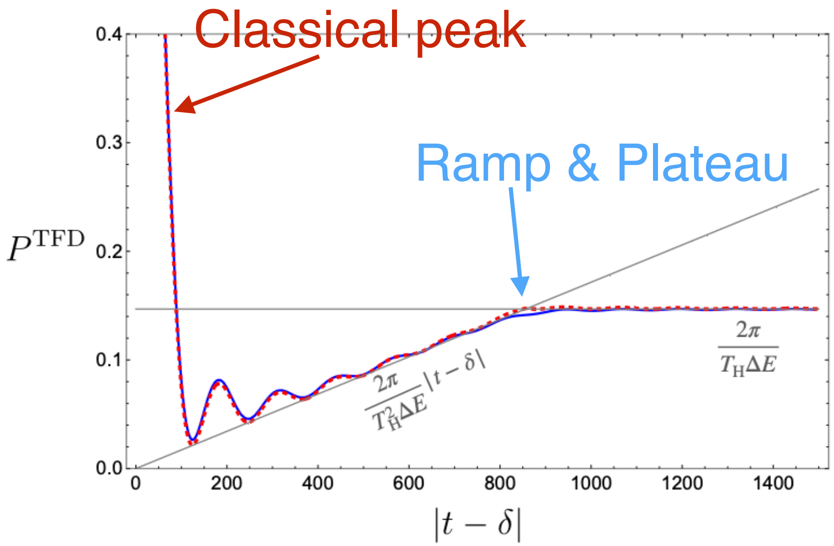


# Timeshift distribution

**Timeshift**:
It is directly related the spectral form factor

$$P^{\mathrm{TFD}}(\delta, t) := \langle \mathrm{TFD}(t)|\delta\rangle\langle\delta|\mathrm{TFD}(t)\rangle = \frac{|Z(0+i(t-\delta))|^2}{Z(0)^2} = \mathrm{SFF}\left(|t-\delta|\right)$$

- Because the timeshift state is also microcanonical TFD state



# Timeshift Generating Function

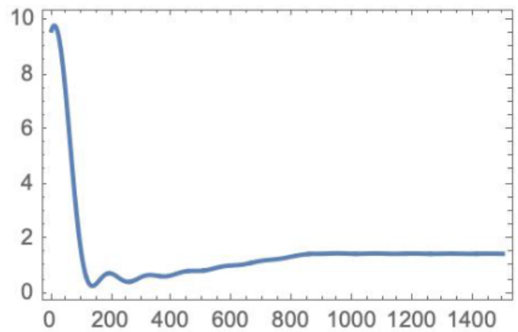
**Positive and Negative part**:
We divide into two parts;

timeshift $\delta$

$$\langle e^{-\alpha\delta}\rangle_{+,t} = \int_0^{\infty} d\delta P(\delta,t)e^{-\alpha\delta} \qquad \langle e^{\alpha\delta}\rangle_{-,t} = \int_{-\infty}^{0} d\delta P(\delta,t)e^{\alpha\delta}$$

We again find the dip-ramp-plateau behavior, reaching plateau at the Heisenberg time

$$\langle e^{-\alpha\delta}\rangle_{+,t}$$




**Spectral representation**:
For any system

$$\langle e^{\mp\alpha\delta}\rangle_{\pm,t} := \sum_{E_i,E_j} e^{i(E_i-E_j)t}\frac{1}{\alpha \pm i(E_i-E_j)}$$

# Non-perturbative Length and Timeshift Operators

# Non-orthogonality

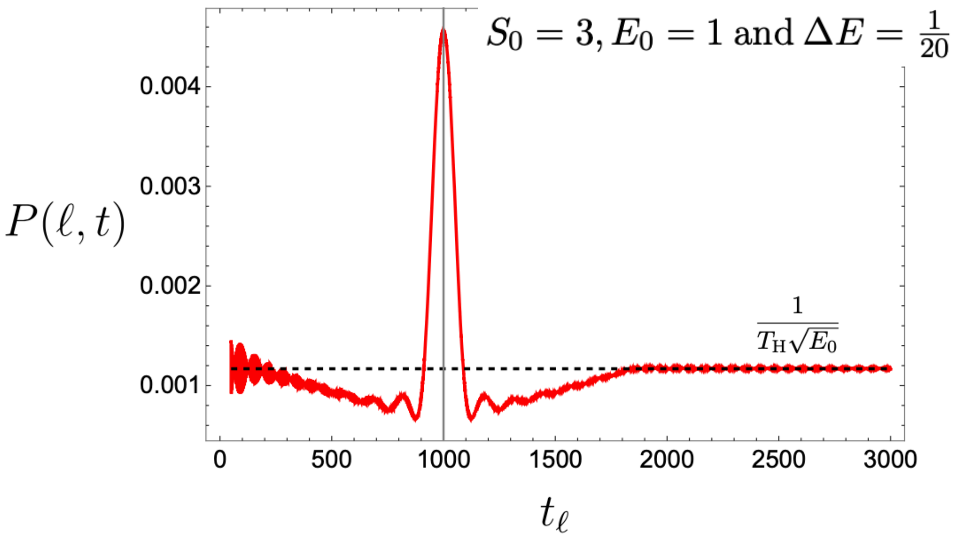
**Non-zero Overlap for all states**:
The overlap becomes constant for large $|t - t_l|$ or $|t - \delta|$
This implies that

- Length states cannot be eigenstates of an Hermitian operator. For example no Hermitian operator like

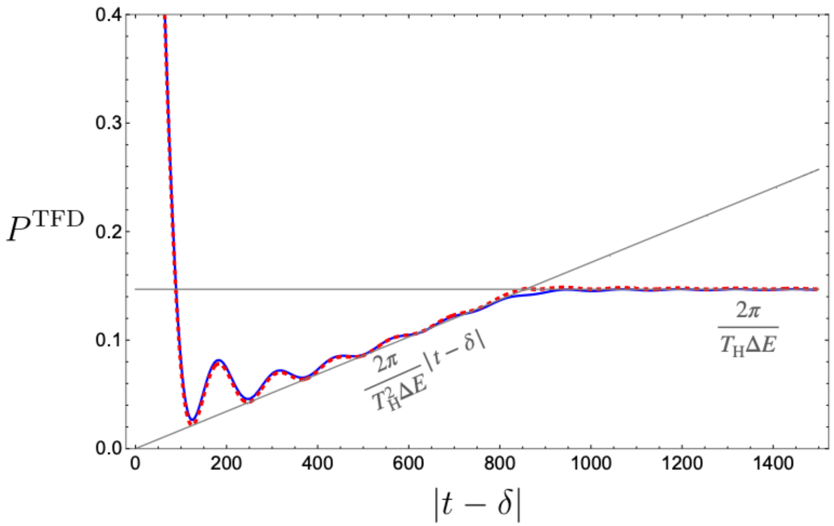
$$\hat{l}|l\rangle = l|l\rangle$$

- Thus P(l,t) and P(δ,t) are not probability distribution

geodesic length *L*




timeshift *δ*



# Non-orthogonality makes naive resolution of identity ill-defined

[MM] [MM, Ruan, Shibuya, Yano (To appear)]

The length states give the resolution of identity on the disk

$$\int_{-\infty}^{\infty} dl \ |l\rangle\langle l| = e^{S_0} \int_{E_0-\Delta E/2}^{E_0+\Delta E/2} dE \ D_{\mathrm{Disk}}(E)|E\rangle\langle E| = \mathbb{I} \qquad \text{Disk}$$

However, with Euclidean wormholes

$$\int_{-\infty}^{\infty} dl \ |l\rangle\langle l| \neq \mathbb{I} \qquad \text{Wormholes}$$

The left hand side is actually divergent for TFD state

$$\langle \mathrm{TFD}| \Big[ \int_{-\infty}^{\infty} dl |l\rangle\langle l| \Big] |\mathrm{TFD}\rangle = \infty$$

# Construction of baby-universe corrected length state [MM]

We have non-orthogonal length state

$$|l_1\rangle,\ |l_2\rangle,\ |l_3\rangle,\cdots\ (l_1 < l_2 < l_3 < \cdots)$$

We identify corrected length states by removing shorter length states by Gram-Schmidt procedure

$$|l_1\rangle^{NP} = |l_1\rangle,\ |l_2\rangle^{NP} = \frac{|l_2\rangle - |l_1\rangle\langle l_1|l_2\rangle}{\sqrt{1-|\langle l_1|l_2\rangle|^2}},\ \cdots$$

Continuing this process many times, we will reach

$$|l_{N+1}\rangle = \sum_{i=1}^{N} c_i |l_i\rangle^{NP} = \sum_{i=1}^{N} d_i |l_i\rangle$$

in terms of shorter wormhole states

for dimension *N* of the microcanonical window

# Orthogonalization via replica trick

This Gram-Schmidt procedure seems hard to perform, but we can do so by considering this manifestly positive operator

$$\hat{S}[m] := \sum_{i=1}^{m} |l_i\rangle\langle l_i|$$

whose nonzero positive eigenstates are spanned by

$$|l_1\rangle^{NP},\ |l_2\rangle^{NP}, \cdots, |l_m\rangle^{NP}\ (l_1 < l_2 < \cdots < l_m)$$

Thus the projector onto this subspace can be obtained via

$$\hat{P}[m] = \lim_{n\to+0} \hat{S}[m]^n$$

Then we obtain

$$\hat{P}[m] - \hat{P}[m-1] = |l_m\rangle^{NP}\langle l_m|^{NP}$$

# Non-perturbative length operator

The corrected length operator is now given by

$$\hat{L}^{NP} = \sum_{i=1}^{N} l_i |l_i\rangle^{NP} \langle l_i|^{NP}$$

- The spectrum is unchanged, except it terminates at $i$=$N$

Length probability distribution is conveniently written as (assuming continuity)

$$D(l) = \text{Tr}\left[\frac{d}{dl}\hat{P}[l]\right]$$

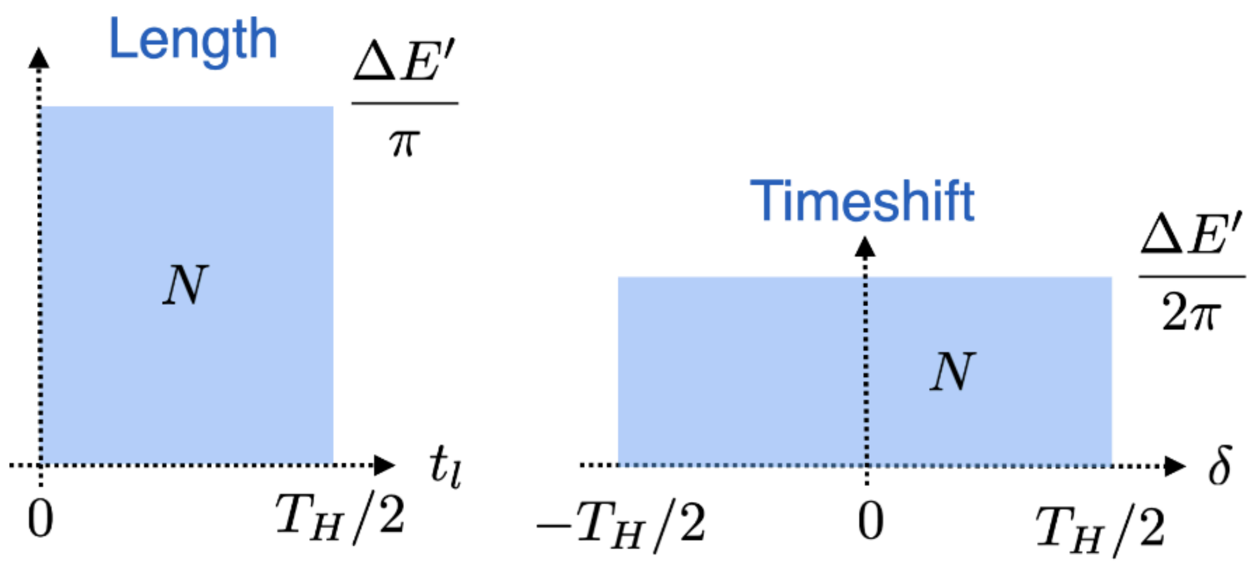
Length probability distribution $P(l)$ on TFD state is conveniently written as

$$P[l] = \langle \text{TFD}(t)|\frac{d}{dl}\hat{P}[l]|\text{TFD}(t)\rangle$$

# Length/timeshift Spectrum and Probability

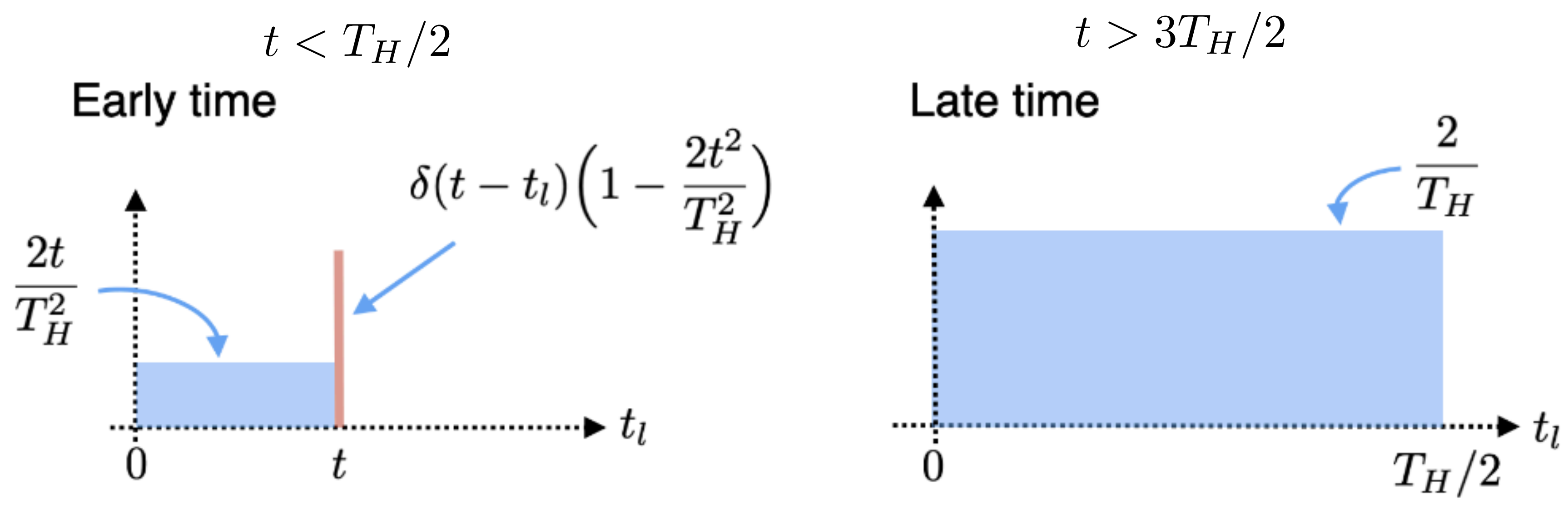
We consider perturbative expansion in terms of $t_l$ and $\delta$ up to second order. The results are already highly non-trivial

**Spectrum**:
Density of states turns out to be uniform but terminates at Heisenberg time (in unit of $t_l$ and $\delta$)



# Length Probability

**Probability distribution**:
**Early time**: Classical peak + constant probability to have shorter interior length. Classical linear growth, and small variance
**Late time**: Uniform probability, no peak. Saturation but large variance

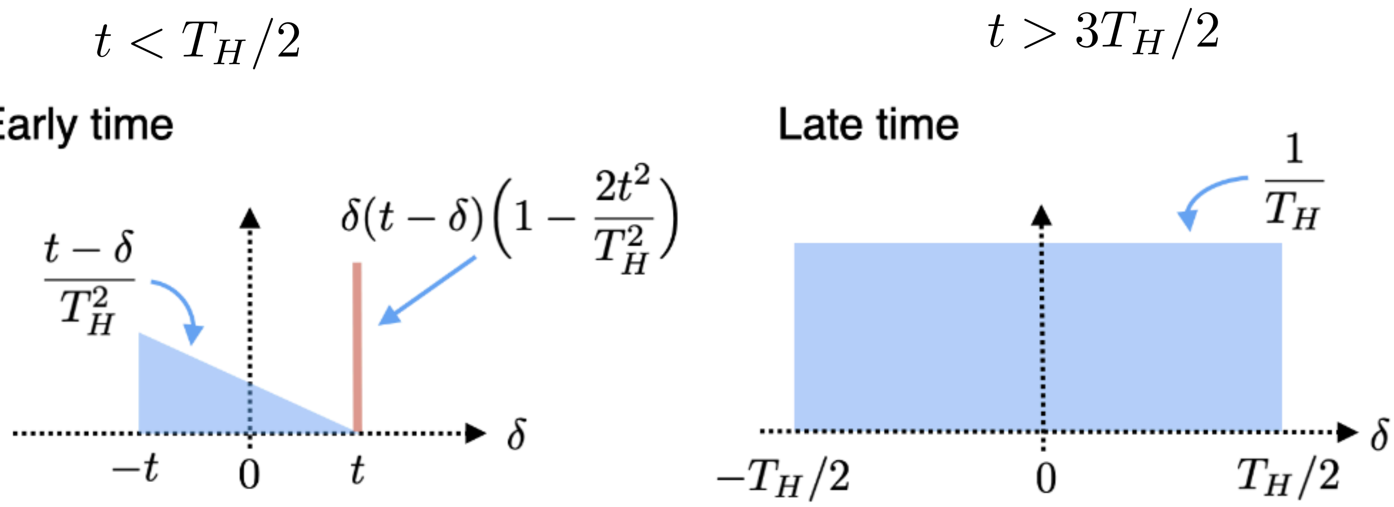



$$\mathbb{E}[t_l]^{(0+1)} = t - \frac{t^3}{T_H^2}, \quad \sqrt{\mathbb{E}\left[(t_l - \mathbb{E}[t_l])^2\right]^{(0+1)}} = \frac{t^2}{T_H}\sqrt{\frac{2}{3} - \frac{t^2}{T_H^2}}$$

This result is similar but slightly different from [Stanford, Yang (2022)]

# Timeshift Probability

**Probability distribution**:
**Early time**: Classical peak + constant probability to have smaller timeshift absolute value. Classical linear growth, and small variance
**Late time**: Uniform probability, no peak. Saturation but large variance



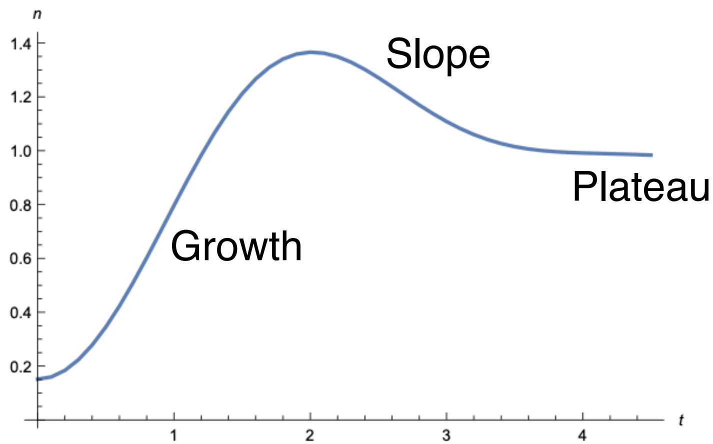

- In particular, it is equally possible to have BH and WH at late time (Susskind's grey hole)

$$P^{\mathrm{BH}}(t) = P^{\mathrm{WH}}(t) = \frac{1}{2}$$

# Length in DSSYK [Okuyama, MM, Mori, work in progress]

We can play the same game for the ETH matrix version of the DSSYK [Jafferis et.al. (2022)]

- DSSYK itself has too large multi-boundary correlation
- Instead consider the matrix model with the same leading density of states
- The behavior of the length displays growth-slope-plateau behavior similar to the spread complexity

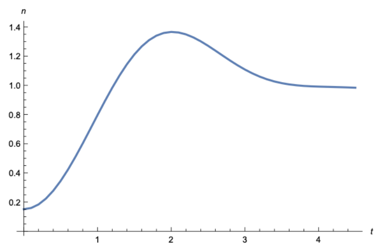


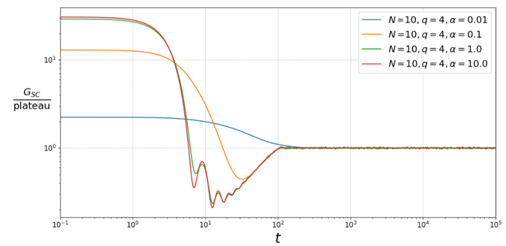

Time vs length $n$: $N$=3

# Summary and Future Directions

## [Summary]

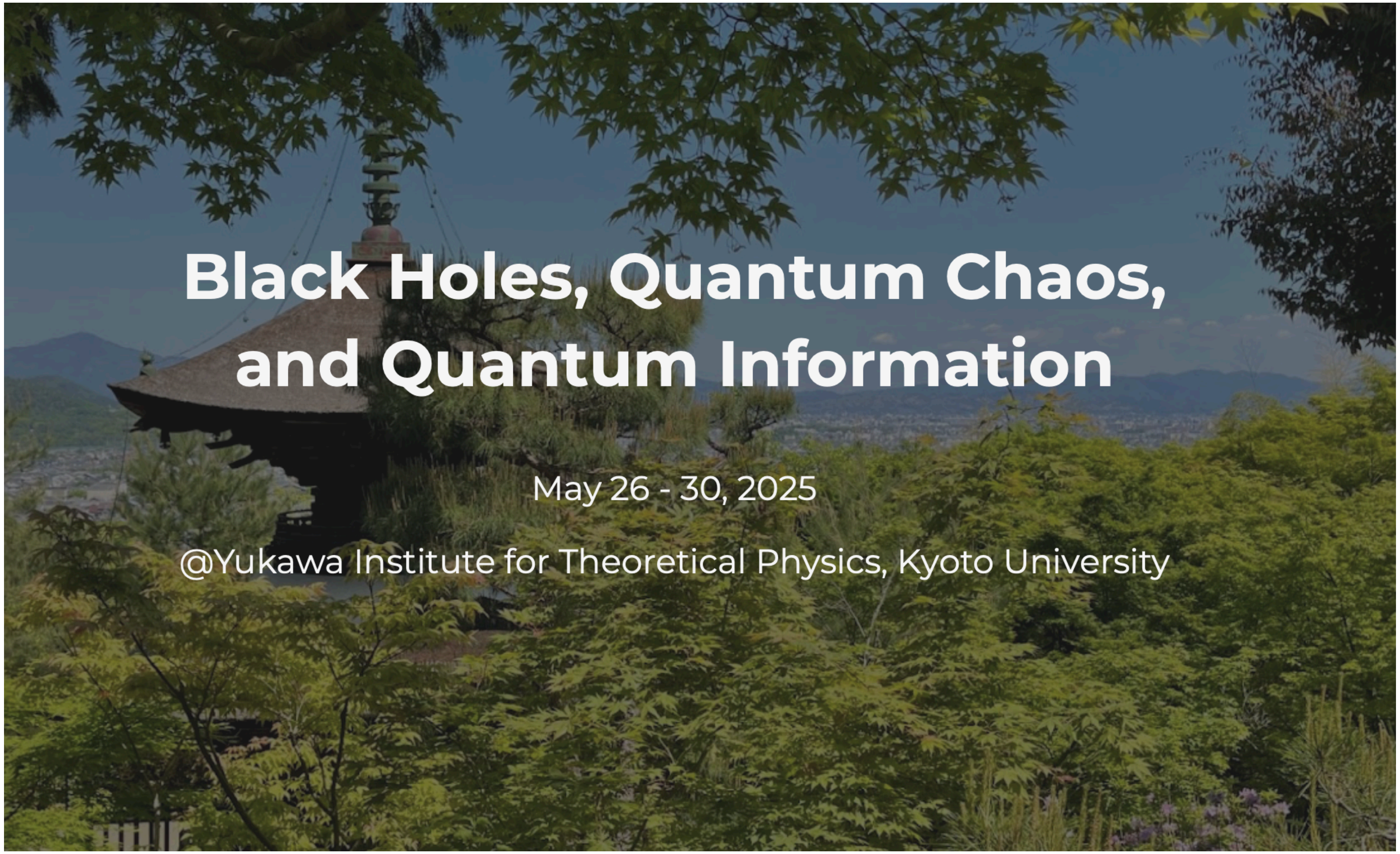
- We constructed non-perturbative length and timeshift states and operators, seeing growth-slope-plateau behavior
- Proposed new quantities probing “interior length” in arbitrary systems, seeing dip-ramp-plateau behavior




## [Future directions]

- Non-perturbative Hilbert space in de Sitter space
- Generalization to “bulk complexity measures” examples like Wheeler-de-Witt action [MM, Ruan, Shibuya, Yano (Work in progress)]




## Invited Speakers

Chi-ming Chang (Tsinghua)

Yiming Chen (Stanford)

Felix Haehl (Southampton)*

Veronika Hubeny (UC Davis)

Luca Iliesiu (UC Berkeley)

Seok Kim (Seoul National)

Shiraz Minwalla (Tata)*

Beatrix Muehlmann (IAS)

Kazumi Okuyama (Shinshu)*

Mukund Rangamani (UC Davis)

Ying Zhao (MIT)

**Period: May 26-30 2025**

**Registration will be open soon**

**Check YITP webpage**

## Organizers

Masamichi Miyaji (YITP), Tokiro Numasawa (University of Tokyo), Tadashi Takayanagi (YITP), Kotaro Tamaoka (Nihon), Zhenbin Yang (Tsinghua)

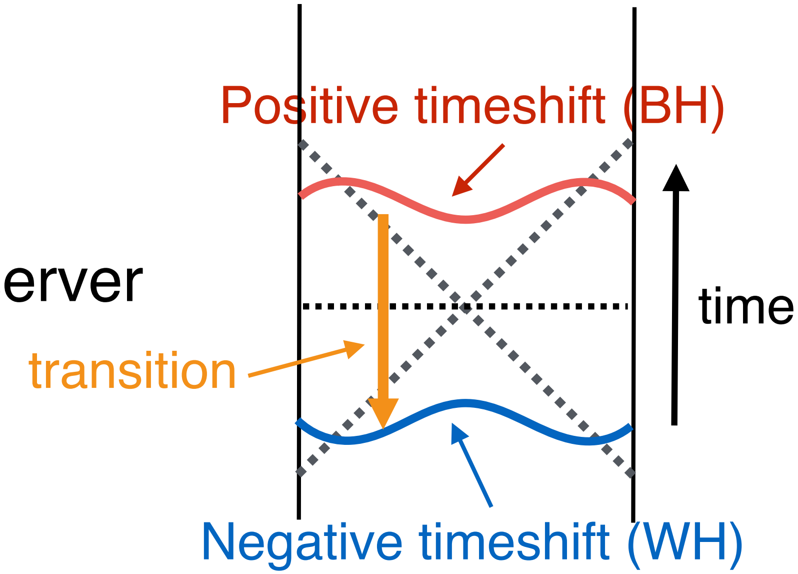
# Appendix

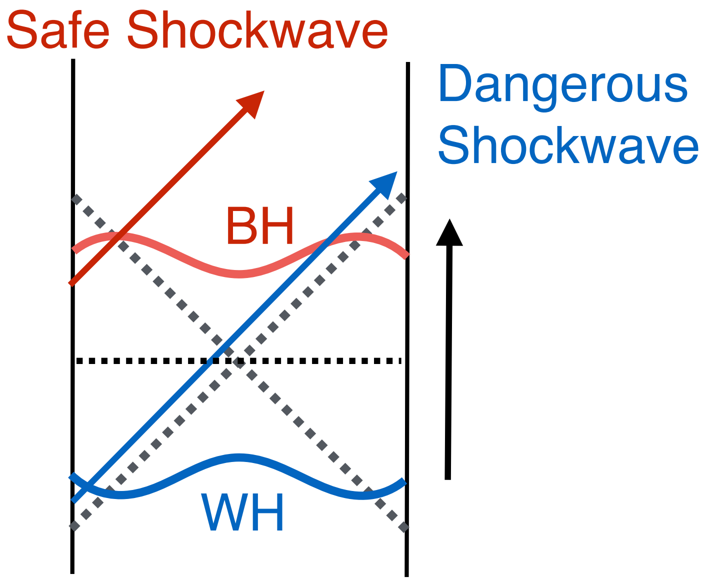
# Applications to Bulk Physics?

**Transition from BH to white hole(WH)** [Stanford, Yang (2022)]
We can investigate the transition probability from BH into WH

- BH can get *younger*, by emitting baby universes

- Further investigation may uncover observer experience in the interior, like *firewall*



# Spectral Complexity and Length?

**Limit**:
Taking small α gives the spectral complexity [Gabor, Iliesiu, Mezei (2021)]

$$\langle l\rangle_t \overset{?}{=} -\lim_{\alpha\to 0}\frac{d\langle e^{-\alpha l}\rangle_t}{d\alpha} \sim \underbrace{\boxed{\sum_{E_1,E_2}\frac{1-\cos\left((E_1-E_2)t\right)}{(E_1-E_2)^2}}}_{\text{Spectral complexity}} + \underbrace{\boxed{O\left(\frac{1}{\alpha^2 T_H}\right)}}_{\text{Divergence}}$$

However, there are several problems relating the length and the spectral complexity

- Length states are not orthogonal to each other
- Divergent in *α*
- Qualitatively different from finite α (only classical + plateau)

These suggest that the *interior length can be probed well only when α is sufficiently large*

# Timeshift?

**Pathological Limit**:
If we consider the natural definition

$$\langle\delta\rangle := -\lim_{\alpha\to+0}\frac{d}{d\alpha}\left(\langle e^{-\alpha\delta}\rangle_{+,t} + \langle e^{\alpha\delta}\rangle_{-,t}\right)$$

we arrive at

$$\langle\delta\rangle = 0$$

This is not reproducing the classical early time behavior $\langle\delta\rangle \sim t$

- Again this suggests the above limit does not lead to faithful bulk description, and highly dependent on regularization scheme
- Only when α is sufficiently large, we can use it to probe the bulk