A New Measure of Genuine Multipartite Entanglement

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Entanglement

- Consider a bipartite system $A \cup B$ in a state $\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$ with density matrices ρ_A and ρ_B. Such a state is called **separable** if it can be expressed as

$$\rho_{A\cup B} = \sum_i p_i \left(\rho_A^i \otimes \rho_B^i\right), \quad \sum_i p_i = 1, \quad p_i \geq 0.$$

  Otherwise it is called **entangled**.

- If a state has density matrix ρ, then
  - $\mathrm{Tr}(\rho^2) = 1 \iff$ pure state.
  - $\mathrm{Tr}(\rho^2) < 1 \iff$ mixed state.

- **Mixed state $\implies$ indication of a lost part of the system**

Entanglement

- Consider a bipartite pure state ρ_{AB}.
- *Entanglement entropy (EE)* is defined as the von Neumann entropy of the reduced density matrix $\rho_A = \mathrm{Tr}_B\ \rho_{AB}$

$$S(A) = -\mathrm{Tr}\ (\rho_A\ \log\rho_A) = -\sum_{\lambda_i} \lambda_i \log\lambda_i \quad \textit{and} \quad S(A) = S(B)$$

  where, λ_i are the eigenvalues of ρ_A.
- For a separable state: $|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$
  - eigen values of ρ_A or ρ_B are 0 and 1
  - $S(A) = 0 = S(B)$
- For an entangled state: Bell state $|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$$S(A) = \log\ 2$$

Motivation A

- What about (genuine) multipartite entanglement!
  - characterization and classification of states
  - performing quantum tasks
- How to measure it? **(simply)**

Multipartite entanglement

- Fully seperable states,

$$|\psi\rangle_{A_1,A_2\ldots A_N} = |\phi\rangle_{A_1} \otimes |\phi\rangle_{A_2} \otimes |\phi\rangle_{A_3} \cdots\cdots \otimes |\phi_N\rangle_{A_N}$$

- k-seperable states,

$$|\psi\rangle_{A_1,A_2\ldots A_N} = |\phi\rangle_{B_1} \otimes |\phi\rangle_{B_2} \otimes |\phi\rangle_{B_3} \cdots\cdots \otimes |\phi_N\rangle_{B_k}$$

  where $\cup_j B_j = \cup_i A_i$.

- **Genuine multipartite entanglement $\implies$ No seperability**

Tripartite entanglement

- Fully seperable states,

$$|\psi\rangle_{ABC} = |\phi\rangle_A \otimes |\phi\rangle_B \otimes |\phi\rangle_C$$

- Biseperable states,

$$\begin{aligned}
|\psi\rangle_{ABC} &= |\phi_{AB}\rangle \otimes |\phi_C\rangle \\
|\psi\rangle_{ABC} &= |\phi_A\rangle \otimes |\phi_{BC}\rangle \\
|\psi\rangle_{ABC} &= |\phi_{AC}\rangle \otimes |\phi_B\rangle
\end{aligned}$$

- **A state possess genuine tripartite entanglement iff it is not fully seperable or biseperable.**

Tripartite entanglement: classification

[**Dur**, **Vidal**, **Cirac** : **2000**]

- Four different classes are observed.
- **Fully Separable state:**
  - $|\psi\rangle_{ABC} = |000\rangle$
    - A, B and C are not entangled with each other.
- **Biseparable state:**
  - $|\psi\rangle_{ABC} = \frac{1}{\sqrt{2}}(|000\rangle + |110\rangle) = |Bell\rangle \otimes |0\rangle$
    - A and B are maximally entangled but C is not entangled with others.
- **GHZ state:**
  - $|\psi\rangle_{ABC} = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$
    - no residual bipartite entanglement between A and B if we trace out C.
- **Werner state:**
  - $|\psi\rangle_{ABC} = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$
    - maximum residual bipartite entanglement between A and B if we trace out C.

Genuine Multipartite Entanglement (GME) measure

[**Vedral**, **Plenio**, **Rippin**, **Knight** : **97**] [**Ma**, **Chen**, **Chen**, **Spengler**, **Gabriel**, **Huber** : **11**] [**Xie**, **Eberly** : **21**]

- A genuine multipartite entanglement measure, $\mathcal{R}$, is defined by the following properties:
  1. $\mathcal{R}$ should be zero for any fully separable state.
  2. $\mathcal{R}$ should be zero for any k-separable state.
  3. $\mathcal{R}$ should be strictly positive for all non-k-separable states.
  4. $\mathcal{R}$ should be invariant under local unitaries.
  5. $\mathcal{R}$ should be non-increasing on average under local operations and classical communication (LOCC).
  6. In 3 qubit case, $\mathcal{R}_{GHZ} > \mathcal{R}_{W}$ (weaker condition)
- Conditions (1), (2), (3), (4) and (5) are necessary for a measure to accurately characterize genuine multipartite entanglement.
- Condition (6) is motivated by the experimental evidence.
- A measure satisfying all the six conditions is known as a “proper” GME measure.

Use Entanglement Entropy !!!

[**Nielsen**, **Kempe** : **00**] [**Gadde**, **Krishna**, **Sharma** : **23**]

- Can we combine entanglement entropy of various subsystems and define a multipartite entanglement measure?
- Examples are,

$$I(A:B) = S(A) + S(B) - S(AB)$$
$$I_3(A:B:C) = S(A) + S(B) + S(C) - S(AB) - S(BC) - S(AC) + S(ABC)$$

- Consider the following states,

$$|\psi\rangle_{ABC} = \frac{1}{\sqrt{3}}\left(|000\rangle + \sqrt{2}|111\rangle\right)$$
$$|\phi\rangle_{ABC} = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right)$$

- Each of the reduced density matrices have exactly the same spectrum.
- Impossible to distinguish $|\psi\rangle$ and $|\phi\rangle$ utilizing any combination of entanglement entropy.

[**Dutta**, **Faulkner** : **19**]

- *Purification:* A bipartite quantum system $A \cup B$ in a mixed state ρ_{AB} is purified by embedding the system $A \cup B$ in a larger tripartite system $A \cup B \cup C$.
- Purification is not unique $\implies$ problems with *entanglement of purification*
- *Canonical purification:* Purification by doubling the Hilbert space $\mathcal{H}_{AB}$ to $\mathcal{H}_{ABA^*B^*}$.

$$\rho_{AB} \qquad\qquad |\sqrt{\rho_{AB}}>_{AA^*BB^*}$$

$$\rho_{AB} = Tr_{A^*B^*}(|\sqrt{\rho_{AB}}><\sqrt{\rho_{AB}}|)$$

Reflected Entropy

[**Dutta, Faulkner** : **19**]

- Reflected entropy is defined as,

$$S_R(A:B) = S(AA^*).$$

- $A \cup B$ in a pure state : $S_R(A:B) = 2S(A) = 2S(B)$

- The reflected entropy satisfy the bound,

$$\min\{2S(A), 2S(B)\} \geq S_R(A:B) \geq I(A:B).$$

  where $I(A:B) = S(A) + S(B) - S(AB)$.

- The lower bound comes from the strong subadditivity of A, A^* and B,

$$I(A:BB^*) \geq I(A:B)$$

- The upper bound comes from the positivity of the mutual information,

$$I(A:A^*) \geq 0, \qquad I(B:B^*) \geq 0$$

- **Reflected entropy distinguishes the isospectral states.** [**JKB, Giataganas, Mondal, Wen** : **23**]

Markov gap and problems

[Akers, Rath : 19][Hayden, Parrikar, Sorce : 21]

- The Markov gap is defined as

$$h(A:B) = S_R(A:B) - I(A:B).$$

- Following the lower bound of reflected entropy, $h(A,B) \geq 0$.
- Example:
  - Bell state $\implies$ $h(A:B) = 0$
  - Biseparable state $\implies$ $h(A:B) = 0$
  - W state $\implies$ $h(A:B) = .57$
  - GHZ state $\implies$ $h(A:B) = 0$!!!

Latent entropy (L-entropy)

[**JKB**, **Malvimat**, **Yoon** : **24**]

- The bipartite *Latent Entropy* for two parties is defined as
$$\ell_{AB} = \min\{2S(A), 2S(B)\} - S_R(A:B).$$
- ℓ_{AB} is strictly positive following the upper bound of the reflected entropy.
- Multipartite L-entropy is defined as the geometric mean of bipartite L-entropy for all possible bipartitions.
$$\ell_{A_1 A_2 \cdots A_n} = \left(\prod_{i<j} \ell_{A_i A_j} \right)^{\frac{2}{n(n-1)}} .$$
- The bound on the bipartite L-entropy is,
$$0 \leq \ell_{A_i A_j} \leq \mathrm{Min}\{2\log[d_{A_i}], 2\log[d_{A_j}], \log[d_{\overline{A_i A_j}}]\}$$
where d_A is the dimension of the Hilbert space of A.
- Spin off: We define generalized Markov gap as,
$$h_{A_1 A_2 \cdots A_n} = \left(\prod_{i<j} h_{A_i A_j} \right)^{\frac{2}{n(n-1)}} .$$

[**JKB**, **Malvimat**, **Yoon** : **24**]

- **Fully seperable state:** $|\psi\rangle_{ABC} = |000\rangle$
  - $S_R(A:B) = S_R(B:C) = S_R(A:C) = S(A) = S(B) = S(C) = 0$
  - $\ell_{ABC} = 0 \Longrightarrow$ (condition 1)
- **Biseparable state:** $|\psi\rangle_{ABC} = \frac{1}{\sqrt{2}}(|000\rangle + |110\rangle) = |Bell\rangle \otimes |0\rangle$
  - $S_R(A:B) = 2S(A) = 2S(B)$
  - $S_R(A:C) = S_R(B:C) = S(C) = 0$
  - $\ell_{ABC} = 0 \Longrightarrow$ (condition 2)

L-entropy

[**JKB, Malvimat, Yoon : 24**]

- **GHZ state:** $|\psi\rangle_{ABC} = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$
  - $S_R(A:B) = S_R(B:C) = S_R(A:C) = S(A) = S(B) = S(C) = 1$
  - maximum tripartite L-entropy, $\ell_{ABC} = 1 \implies$ (condition 3)
- **W state:** $|\psi\rangle_{ABC} = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$
  - $S_R(A:B) = S_R(B:C) = S_R(A:C) = 1.49$
  - $S(A) = S(B) = S(C) = 0.92$
  - $\ell_{ABC} = 0.35 \implies$ (condition 3)
- $\ell_{A_iA_j}$ is invariant under local unitaries $\implies$ (condition 4)
- $\ell_{A_iA_j}$ is non-increasing under LOCC $\implies$ (condition 5)
- $\ell_{ABC}(GHZ) > \ell_{ABC}(W) \implies$ (condition 6)
- **Multipartite L-entropy is a measure of proper genuine multipartite entanglement.**
- For four-party system, the multipartite L-entropy obtains the maximum value for *cluster* states.

Motivation B

- Identification of special class of states !!!
- Does this identification agrees well with the existing QIT results ???

k-uniform states

[**Scott** : **04**][**Facchi**, **Florio**, **Marzolino**, **Parisi**, **Pascazio** : **10**]

- A n-party pure state is said to be k-uniform iff any k-party reduced density matrix is maximally mixed.
- k-uniform states are very important in the context of multipartite entanglement.
- For any n-party state, $k \leq \lfloor \frac{n}{2} \rfloor$
- L-entropy maximizes for 2-uniform states where $n \geq 5$. [**JKB**, **Malvimat**, **Yoon** : **24**]

2-uniform states

[JKB, Malvimat, Yoon : 24]

Table 2. Existence of 2-uniform states of N subsystems with $d \geq 2$ level

N / d	$2 \leq N \leq 3$	4	5	$N \geq 6$
$d = 2$	–	?	b	a
$d = 3$	–	a	a	a
$d = 4$	–	a	a	a
$d = 5$	–	a	a	a
$d = 6$	–	?	a	a
$d = 7$	–	a	a	a
$d = 8$	–	a	a	a
$d = 9$	–	a	a	a
$d = 10$	–	a	?	a
$d \geq 11$	–	a	a	a

[Pang, Zhang, Lin, Zhang : 19]

- d is the dimension and N is the number of parties.
- Using an optimization procedure on L-entropy, we found various 2-uniform states.

Motivation B.2: 3-uniform states

Table 3. Existence of 3-uniform states of N subsystems with $d \geq 2$ levels

N d	$2 \leq N \leq 5$	6	7	8	9	10	11	$N \geq 12$
$d = 2$	–	?	?	a	b	a	a	a
$d = 3$	–	?	?	a	a	a	a	a
$d = 4$	–	a	?	a	a	a	a	a
$d = 5$	–	a	?	a	a	a	a	a
$d > 6$ is a prime power	–	a	a	a	a	a	a	a
$d \geq 6$ is not a prime power	–	?	?	a	?	?	?	a

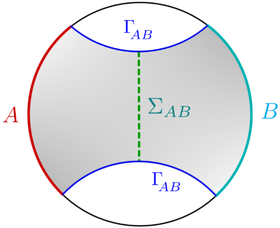
[Pang, Zhang, Lin, Zhang : 19]

- Can we find a variant of L-entropy to detect 3-uniform state?
- **More ambitious goal: find k-generalized L-entropy to establish a gradation of k-uniform state.**

Motivation C

- Holography
  - structure of entanglement in holographic states
  - geometric description
  - multipartite entanglement in presence of black holes

- Entanglement wedge cross section (EWCS) : codimension two surface with minimized area dividing the wedge for $A \cup B$.

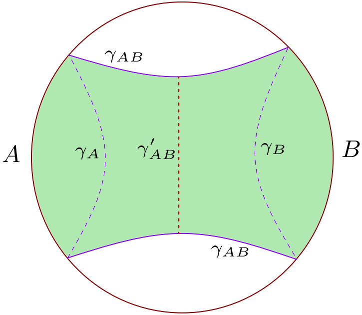


- For a disconnected wedge, $EWCS = 0$
- Reflected entropy is twice the area of EWCS [Dutta, Faulkner: 19]

$$S_R(A:B) = 2\mathcal{A}\left[\Sigma_{AB}\right]$$

Holographic L-entropy

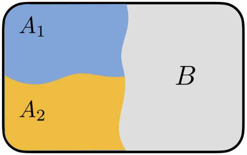
[**JKB**, **Malvimat**, **Yoon** : **24**]



- The holographic bipartite L-entropy is given as,

$$\ell_{AB} = \frac{\text{Min}\{\text{Area}\,[\gamma_A]\,,\text{Area}\,[\gamma_B]\} - \text{Area}\,[\Sigma_{AB}]}{2G}$$

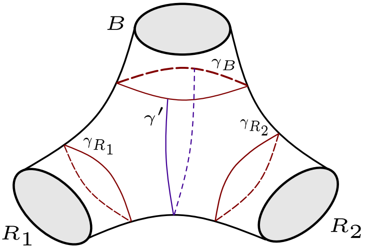
Black hole evaporation: Page curve



- Consider the multiboundary wormhole model with three boundaries
- Here one boundary will be considered as the black hole (B) and the other two as the two radiation regions (R_1 and R_2).
- Black hole evaporation procedure can be obtained by decreasing the size of B.

Black hole evaporation: Page curve

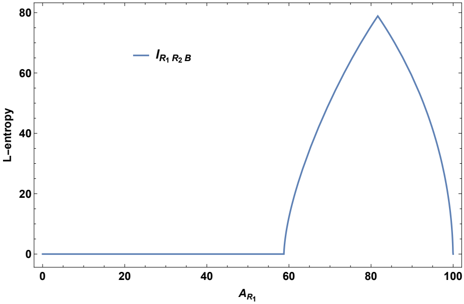
[Akers, Engelhardt, Harlow : 19] [JKB, Malvimat, Yoon : 24]



- Consider the bipartition R_1 and R_2.
- EWCS, $\Sigma_{AB} = \text{Min}_{\mathcal{A}}\{\gamma_{R_1}, \gamma', \gamma_{R_2}\}$
- $\ell_{R_1 R_2} = 2\text{Min}\{\mathcal{A}\left[\gamma_{R_1}\right], \mathcal{A}\left[\gamma_{R_2}\right]\} - 2\mathcal{A}\left[\Sigma_{AB}\right]$

Page curve

[JKB, Malvimat, Yoon : 24]

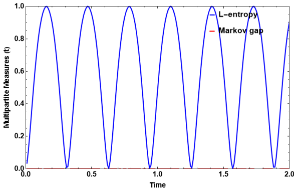


- No multipartite entanglement until the Page time.
- It reaches the maximum when $\mathcal{A}\left[\gamma_{R_1}\right] = \mathcal{A}\left[\gamma_B\right] = \mathcal{A}\left[\gamma_{R_2}\right]$
- $\ell^{max}_{R_1 B R_2} = \mathcal{A}\left[\gamma_{R_1}\right] = \mathcal{A}\left[\gamma_B\right] = \mathcal{A}\left[\gamma_{R_2}\right] \Longrightarrow$ all the degrees of freedom are involved in constructing the multipartite entanglement.
- Complete evaporation of the black hole makes the system bipartite and the multipartite entanglement vanishes.

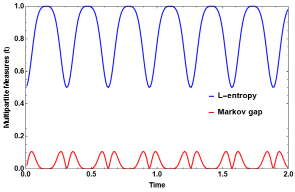
Application

[JKB, Malvimat, Yoon : 24]

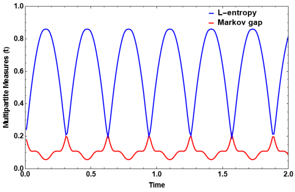
- Nearest neighbors Ising model $H = J_y \sum_i \sigma_y^i \sigma_y^{i+1}$: 8-party L-entropy



(a) Initial state fully separable

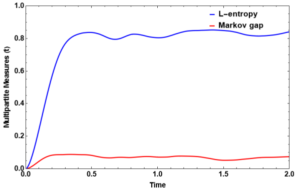


(b) Initial state is $|GHZ\rangle_8$

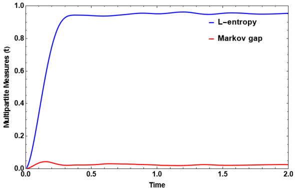


(c) Initial state is $|W\rangle_8$

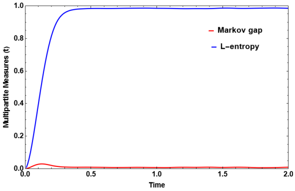
- SYK $H = \sum_{i<j<k<l} J_{ijkl} \chi^i \chi^j \chi^k \chi^l$: n-party L-entropy



(a) $n = 6$



(b) $n = 8$



(c) $n = 10$

Wrap up

Conclusions

- Propose a new measure for genuine multipartite entanglement using the reflected entropy
- It maximizes for GHZ in three party, cluster in four party and 2-uniform states in $n \geq 5$ party.
- Obtain the Page curve for black hole evaporation process
- Explore the measure in the context of random states and multipartite states at a finite temperature. (Not in this talk)

Future

- Find higher uniform states utilizing generalized L-entropy
- L-entropy in generic multiboundary wormholes
- Approach some specific sectors of multipartite entanglement

Thank you