# Exploring complex saddles and geometries through holography.

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- When studying the path integral for quantum gravity, we can often consider the complex analytic continuation of the metric, however some interesting puzzles can arise.
- A classic example comes from complexifying $S^{d+1}$: [Witten 2021].

$$ds^2 = \ell^2 \left[ \left( \frac{d\theta(u)}{du} \right)^2 du^2 + \cos^2\theta(u) d\Omega_d^2 \right]$$

When $\theta(u) = u$, $0 \leq u \leq \pi$, we have $S^{d+1}$, however when $\theta(u) = iu$, $-\infty \leq u \leq +\infty$, we have Lorentzian $dS_{d+1}$.

- If the universe started from nothing, i.e. $\cos^2\theta = 0$, we can have:

$$\theta = \left( n + \frac{1}{2} \right) \pi, \quad n \in \mathbb{Z}$$

and evolve into $\theta = iu$ as $u \to +\infty$. We thus have a family of complex geometries with initial conditions labeled by $n \in \mathbb{Z}$.

- More precisely, we complexified the action $\Psi \sim \exp[S_{(n)} + i\mathcal{I}(u)]$, where real part $S_{(n)} = (n + \frac{1}{2})S$, where $S$ is the de Sitter entropy. This implies we can increase the amplitude as $n \to \infty$.
- In principle we need to sum over all $n \in \mathbb{Z}$ when evaluating the path integral, however the proposal by Hartle-Hawking only needs $\theta(0) = \pm\frac{\pi}{2}$ or $n = 0, -1$.
- Kontsevich-Segal: For $\mathcal{M}^D$ with complex metric $g$, consider a p-form $A^{(p)}$ with field strength $dA^{(p)}$ and the action with $q = p + 1$:

$$I_q = \frac{1}{2q!}\int_M \mathrm{d}^D x \sqrt{\det g}\, g^{i_1 j_1} \cdots g^{i_q j_q} F_{i_1 i_2 \cdots i_q} F_{j_1 j_2 \cdots j_q}$$

the allowable $g$ is such that:

$$\mathrm{Re}\left(\sqrt{\det g}\, g^{i_1 j_1} \cdots g^{i_q j_q} F_{i_1 i_2 \cdots i_q} F_{j_1 j_2 \cdots j_q}\right) > 0, \quad 0 \le q \le D,$$

for all non-zero p+1 form $F^{(p+1)} = dA^{(p)}$. [Witten 2021].

## An Explicit Higher Spin $dS_3/CFT_2$ Correspondence

- Starting with pure $AdS_3$ gravity, we consider $SL(2,\mathbb{R})^2$ Chern-Simons gauge theory:

$$S = S_{CS}[A] - S_{CS}[\tilde{A}]\,, \quad S_{CS}[A] = \frac{k}{4\pi}\int \mathrm{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)$$

and CS level $k$ is related to $G_N$ and $\ell_{\mathrm{AdS}}$ via:

$$k = \frac{\ell_{\mathrm{AdS}}}{4G_N}\,.$$

- The independent gauge fields $A$ and $\tilde{A}$ take values in each copy of $SL(2,\mathbb{R})$ algebra: $[L_m, L_n] = (m-n)L_{m+n}$, $m,n = 0,\pm 1$

$$A = e^{-\rho L_0} a e^{\rho L_0} + L_0 d\rho\,, \quad \tilde{A} = e^{\rho L_0}\tilde{a} e^{-\rho L_0} - L_0 d\rho$$

$$a = a_+(x^+)dx^+\,, \quad \tilde{a} = \tilde{a}_-(x^-)dx^-$$

$a_+(x^+), \tilde{a}_-(x^-)$ are arbitrary functions of $x^\pm = t \pm \phi$ with $\phi \sim \phi + 2\pi$.

- For the $AdS_3$ BTZ black hole, it is given by:

$$a_+(x^+) = L_1 - \frac{2\pi\mathcal{L}^{\text{AdS}}}{k}L_{-1}\,, \quad \tilde{a}_-(x^-) = -L_{-1} + \frac{2\pi\mathcal{L}^{\text{AdS}}}{k}L_1\,,$$

$$\mathcal{L}_{\text{AdS}} = \frac{\ell^{\text{AdS}} r_+^2}{32\pi G_N} = \frac{k r_+^2}{8\pi}.$$

which yield the following metric:

$$\ell_{\text{AdS}}^{-2} ds^2 = d\rho^2 - \left(e^\rho - \frac{2\pi\mathcal{L}^{\text{AdS}}}{k}e^{-\rho}\right)\left(e^\rho - \frac{2\pi\mathcal{L}^{\text{AdS}}}{k}e^{-\rho}\right)dt^2$$

$$+ \left(e^\rho + \frac{2\pi\mathcal{L}^{\text{AdS}}}{k}e^{-\rho}\right)\left(e^\rho + \frac{2\pi\mathcal{L}^{\text{AdS}}}{k}e^{-\rho}\right)d\phi^2$$

and we can recover the usual BTZ black hole by coordinate change:

$$r = e^\rho + \frac{2\pi\mathcal{L}^{\text{AdS}}}{k}e^{-\rho}$$

- Taking $t \to it_{\mathrm{E}}$, the absence of conical singularity at the horizon $r_+$ demands following periodicity:

$$t_{\mathrm{E}} \sim t_{\mathrm{E}} + \beta^{\mathrm{AdS}}, \quad \beta^{\mathrm{AdS}} = \frac{2\pi}{r_+}$$

- The Chern-Simons gauge configurations can be classified by gauge invariant Wilson loop:

$$\mathcal{P}e^{\oint A} = \mathcal{P}e^{\oint dt_E A_{t_E}} = e^{-i\theta L_0} e^{\Omega} e^{i\theta L_0} \, .$$

The BTZ black hole corresponds to the eigenvalues of holonomy $\Omega$ equals $(+\pi, -\pi)$, however if we allow for the large gauge transformations, other values $(2\pi(n+\frac{1}{2}), -2\pi(n+\frac{1}{2}))$, $n \in \mathbb{Z}^+$ are also allowed.

We proposed an explicit $dS_3/CFT_2$ correspondence by suitable analytic continuation of $AdS_3/CFT_2$ one.

[Hikida, Nishioka, Takayanagi, Taki, 2022], [Chen, Hikida 2022], [Chen, Chen, Hikida 2022]

- We analytically continue the CS-theory to $G = SL(2,\mathbb{C})^2$:

$$S = S_{\rm CS}[A] - S_{\rm CS}[\bar{A}]\,, \quad S_{\rm CS}[A] = -\frac{\kappa}{4\pi}\int {\rm tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right)$$

$$\kappa = \frac{\ell}{4G_N}\,.$$

We have set $\ell_{\rm AdS} \to i\ell$ such that $k \to i\kappa$. Also $e^{-\rho} = e^{-(\tilde{\rho}+i\frac{\pi}{2})}$.

- We need to impose the complex conjugation as: $(L_0)^* = -L_0$, $(L_\pm)^* = L_\mp$. The gauge fields are now expressed as:

$$A = e^{-(\tilde{\rho}+\pi i/2)L_0} a e^{(\tilde{\rho}+\pi i/2)L_0} + L_0 d\tilde{\rho}\,, \quad \bar{A} = e^{(\tilde{\rho}+\pi i/2)L_0} \bar{a} e^{-(\tilde{\rho}+\pi i/2)L_0} - L_0 d\tilde{\rho}$$

$$a = a_+(x^+)dx^+\,, \quad \bar{a} = \bar{a}_-(x^-)dx^-\,.$$

where $x^\pm = it \pm \phi$, $\phi \sim \phi + 2\pi$.

- To obtain the $dS_3$ analogue of BTZ black hole, we can consider the following configuration:

$$a_+(x^+) = L_1 + \frac{2\pi\mathcal{L}}{\kappa}L_{-1}\,, \quad \bar{a}_-(x^-) = -L_{-1} - \frac{2\pi\mathcal{L}}{\kappa}L_1\,,$$

- Following the coordinate shift: $\tilde{\rho} \to \tilde{\rho} + \log\sqrt{\mathcal{L}/\kappa}$ and continuation $\tilde{\rho} = i\theta$, we have:

$$\ell^{-2}ds^2 = d\theta^2 - \frac{8\pi\mathcal{L}}{\kappa}\sin^2\theta dt^2 + \frac{8\pi\mathcal{L}}{\kappa}\cos^2\theta d\phi^2\,.$$

which can be mapped into $dS_3$ BTZ black hole metric via the coordinate and parameter transformations:

$$r = \sqrt{\frac{8\pi\mathcal{L}}{\kappa}}\cos\theta\,, \quad \mathcal{L} = \frac{\ell r_+^2}{32\pi G_N} = \frac{\kappa r_+^2}{8\pi}$$

- This allows us to obtain asymptotically $dS_3$ BTZ black hole geometry:

$$ds^2 = \ell^2 \left[ -(r_+^2 - r^2)dt^2 + \frac{1}{r_+^2 - r^2} dr^2 + r^2 d\phi^2 \right]$$

Here $\phi \sim \phi + 2\pi$ and the horizon is at $r_+ = \sqrt{1 - 8G_N E}$. Under $t \to -it_E$, the absence of conical singularity at horizon needs $t_E \sim t_E + 2\pi / r_+$. The Gibbons-Hawking entropy of $dS_3$ BTZ black hole is thus:

$$S_{\rm GH} = \frac{2\pi \ell r_+}{4G_N} = \frac{\pi \ell \sqrt{1 - 8G_N E}}{2G_N}$$

- We can define the holonomy matrix for the gauge field $A$ along the compactified time cycle as:

$$\mathcal{P} e^{\oint A} = \mathcal{P} e^{\oint dt_E A_+} = e^{-(i\theta + \pi i/2)L_0} e^{\Omega} e^{(i\theta + \pi i/2)L_0}$$

the eigenvalues for $\Omega$ are $(i\pi, -i\pi)$.

- Applying large gauge transformation, $\Omega$ now takes the values $(2\pi(n+\frac{1}{2}), -2\pi(n+\frac{1}{2}))$, $n \in \mathbb{Z}^+$, given by:

$$a = -\sqrt{\frac{2\pi\mathcal{L}}{\kappa}}\sigma_1(d\phi + (2n+1)dt_E)\,, \quad \bar{a} = -\sqrt{\frac{2\pi\mathcal{L}}{\kappa}}\sigma_1(d\phi - (2n+1)dt_E)$$

where $\sigma_1$ is a Pauli matrix. They generate the following metric:

$$ds^2 = \ell^2\left[d\theta^2 + \frac{8\pi(2n+1)^2\mathcal{L}}{\kappa}\sin^2\theta dt_E^2 + \frac{8\pi\mathcal{L}}{\kappa}\cos^2\theta d\phi^2\right]\,.$$

If we set $r = \sqrt{1-8G_N E}\cos\theta$ and $\frac{8\pi\mathcal{L}}{\kappa} = 1 - 8G_N E$.

- Each saddle point contribute to the Chern-Simons path integral with the action after Euclideanization:

$$-S\,(\equiv S_{\rm GH}^{(n)}) = 4\pi(2n+1)\sqrt{2\pi\kappa\mathcal{L}} = (2n+1)\frac{\pi\ell\sqrt{1-8G_N E}}{2G_N}$$

Naively, all these equivalent saddles need to be summed over, however they may lead to over-counting.

Using the explicit $dS_3/CFT_2$, we can select the correct saddle points from dual CFT. [Chen, Hikida, Taki, Uetoko, 2023]

- We start with the regularized action for Liouville theory:

$$S_{\rm L} = \frac{1}{4\pi}\int_D d^2\sigma[\partial_a\phi\partial_a\phi + 4\pi\mu e^{2b\phi}] + \frac{Q}{\pi}\oint_{\partial D}\phi d\theta + 2Q^2\ln R$$

- The vertex operator considered takes the form:

$$V_\alpha = e^{2\alpha\phi}, \quad h = \bar{h} = \alpha(Q-\alpha)$$

- While the background charge $Q$, central charge $c$ and coupling $b$ are related via:

$$c = 1 + 6Q^2 = 1 + 6(b + b^{-1})^2$$

and it can be related gravitational constant via:

$$c\,(\equiv ic^{(g)}) = i\cdot 6\kappa = i\frac{3\ell}{2G_N}$$

the classical limit $G_N \to 0$ corresponds to the large $c$ limit.

- For $dS_3$ BTZ black hole, it corresponds to the insertion of two heavy operators in such a limit:

$$b^{-2} = \frac{ic^{(g)}}{6} - \frac{13}{6} + \mathcal{O}((c^{(g)})^{-1}), \quad b \to 0$$

We also scale $\alpha = \eta/b$ and $\phi_c = 2b\phi$, $\lambda = \pi\mu b^2$ to obtain:

$$b^2 S_{\rm L} = \frac{1}{16\pi}\int_D d^2\sigma[\partial_a\phi_c\partial_a\phi_c + 16\lambda e^{\phi_c}] + \frac{1}{2\pi}\oint_{\partial D}\phi_c d\theta + 2\ln R + \mathcal{O}(b^2)$$

Here $\eta$ is kept fixed such that $2h = 2\alpha(Q-\alpha) = i\ell E$ and $1 - 2\eta = \sqrt{1 - 8G_N E}$. Need $0 < \eta < \frac{1}{2}$ for existence.

- The path integral for the two point function reduces to

$$\langle V_\alpha(z_1)V_\alpha(z_2)\rangle \equiv \int \mathcal{D}\phi_c e^{-S_{\rm L}} \exp\left(b^{-1}\alpha(\phi_c(z_1)+\phi_c(z_2))\right)$$

i. e. the operator insertions become $\delta$-sources in this heavy-limit.

- From the back-reacted action, we can deduce the E.O.M for $\phi_c$:

$$\partial\bar{\partial}\phi_c = 2\lambda e^{\phi_c} - 2\pi\eta[\delta^{(2)}(z-z_1) + \delta^{(2)}(z-z_2)]\,, \quad \lambda \equiv \pi\mu b^2$$

where near $z_{1,2}$, we set $\phi_c(z) \sim -4\eta|z - z_{1,2}|$, creating conical deficits on the CFT world sheet metric. We can easily generalize to higher point correlation functions.

- We have multiple allowed solutions $\phi_{c(n)} = \phi_{c(0)} + 2\pi n$, which yields the on-shell action for each $n$:

$$\begin{aligned} b^2\tilde{S}_{\mathrm{L}}^{(n)} &= 2\pi i(n+1/2)(1-2\eta) + (2\eta-1)\ln\lambda \\ &+ 4(\eta-\eta^2)\ln|z_{12}| + 2[(1-2\eta)\ln(1-2\eta) - (1-2\eta)]\,. \end{aligned}$$

where $\tilde{S}_{\mathrm{L}}^{(n)}$ is the modified action including the back reaction and in principle need to sum over all solutions labeled by $n$, which precisely corresponds to the CS monodromy.

- Happily the exact expression for the Liouville two point function is known: [DOZZ 1994, 1995].

$$\langle V_\alpha(z_1)V_\alpha(z_2)\rangle = |z_{12}|^{-4\alpha(Q-\alpha)}\frac{2\pi}{b^2}[\pi\mu\gamma(b^2)]^{(Q-2\alpha)/b}\gamma\left(\frac{2\alpha}{b}-1-\frac{1}{b^2}\right)\gamma(2b\alpha-b^2)\delta(0)$$

where $\gamma(x) = \Gamma(x)/\Gamma(1-x)$.

- Taking the semi-classical limit of the exact two point function: [Harlow, Maltz, Witten 2011].

$$\langle V_\alpha(z_1)V_\alpha(z_2)\rangle \sim \delta(0)|z_{12}|^{-4\eta(1-\eta)/b^2}\lambda^{(1-2\eta)/b^2}$$

$$\times\left(e^{-\pi i(1-2\eta)/b^2}-e^{\pi i(1-2\eta)/b^2}\right)\exp\left\{-\frac{2}{b^2}\left[(1-2\eta)\ln(1-2\eta)-(1-2\eta)\right]\right\}.$$

Here $\delta(0)$ comes from setting $\alpha=\alpha'$. In obtaining this limit, it is crucial that $\mathrm{Re}(b^{-2})<0$, i. e. consistent with our earlier relation between $b$ and $ic^{(g)}$ for dual CFT to de Sitter. This result can only be reproduced by $\tilde{S}_{\mathrm{L}}^{(n)}$ with $n=0,-1$, leading us to correct saddles.

- Taking the modulus of the two point function, where the $|z_{12}|$ dependence now cancels out, as $1/b^2 \sim ic^{(g)}/6$ is purely imaginary, we have:

$$|\langle V_\alpha(z_1)V_\alpha(z_2)\rangle| \sim \left| e^{\frac{\pi c^{(g)}}{6}\sqrt{1-8G_N E}} - e^{-\frac{\pi c^{(g)}}{6}\sqrt{1-8G_N E}} \right|$$

  The leading order contribution precisely reproduces the Gibbons-Hawking entropy $S_{\rm GH}$ of the corresponding dS black holes.

- This also implies that identification of the phase of the two point function:

$$\langle V_\alpha(z_1)V_\alpha(z_2)\rangle \sim \Psi \sim \exp\left(\frac{S_{\rm GH}}{2} + i\mathcal{I}\right) \Longrightarrow \mathcal{I} = \frac{c^{(g)}}{6}\log\lambda$$

  such that $\lambda$ is manifestly real. Consistent with the earlier CFT computation for the phase in [Hikida, Nishioka, Takayanagi, Taki, 2022]

It is also interesting to employ similar strategy to investigate the bulk geometries dual to higher point correlation functions of heavy operators.

- For three point function, we have: [DOZZ 1994, 1995]

$$\langle V_{\alpha_1}(z_1,\bar{z}_1)V_{\alpha_2}(z_2,\bar{z}_2)V_{\alpha_3}(z_3,\bar{z}_3)\rangle = \frac{C(\alpha_1,\alpha_2,\alpha_3)}{|z_{12}|^{2(h_1+h_2-h_3)}|z_{13}|^{2(h_1+h_3-h_2)}|z_{23}|^{2(h_2+h_3-h_1)}}$$

$$C(\alpha_1,\alpha_2,\alpha_3) = \left[\lambda\gamma(b^2)b^{-2b^2}\right]^{(Q-\sum_i \alpha_i)/b}$$

$$\times \frac{\Upsilon_b'(0)\Upsilon_b(2\alpha_1)\Upsilon_b(2\alpha_2)\Upsilon_b(2\alpha_3)}{\Upsilon_b(\sum_i \alpha_i - Q)\Upsilon_b(\alpha_1+\alpha_2-\alpha_3)\Upsilon_b(\alpha_2+\alpha_3-\alpha_1)\Upsilon_b(\alpha_3+\alpha_1-\alpha_2)}.$$

  where $\Upsilon_b(x)$ is the upsilon function and again take the large scaling limit $\alpha_i = \eta_i/b$, $b \to 0$ and $0 < \eta_i < \frac{1}{2}$ fixed for Seiberg bound.

- We can further divide the parameters into two classes:

$$\mathrm{I}: \sum_i \eta_i > 1, \qquad \mathrm{II}: \sum_i \eta_i < 1,\ \eta_i + \eta_j - \eta_k > 0.$$

  Class I comes from convergence of the path integral, while Class II requires the complex saddles to make senses.

- For Class I, DOZZ coefficient reduces in this limit to:

$$C(\alpha_1,\alpha_2,\alpha_3) \sim \lambda^{(1-\sum_i \eta_i)/b^2} \exp\left[\frac{1}{b^2}\left\{1-\sum_i \eta_i + F(2\eta_1) + F(2\eta_2) + F(2\eta_3) + F(0)\right.\right.$$

$$\left.\left. - F(\sum_i \eta_i - 1) - F(\eta_1+\eta_2-\eta_3) - F(\eta_2+\eta_3-\eta_1) - F(\eta_3+\eta_1-\eta_2)\right\}\right],$$

where $1/b^2$ is purely imaginary at leading order in $1/c$ expansion and $F(x)$ is a real function arising from the log $\Upsilon_b(x)$.

- The norm is thus:

$$|\langle V_{\alpha_1}(z_1,\bar{z}_1)V_{\alpha_2}(z_2,\bar{z}_2)V_{\alpha_3}(z_3,\bar{z}_3)\rangle|^2 \sim \mathcal{O}(1)$$

The obvious interpretation is that we cannot construct $S^2$ with three conical deficits given by $\eta_i\pi$ with $\sum_i \eta_i > 1$, hence no corresponding geometry.

- For Class II, using $\mathrm{Re}[(\sum_i \eta_i - 1)/b^2] > 0$, we obtain:

$$C(\alpha_1,\alpha_2,\alpha_3) \sim \left(e^{-\pi i\frac{1-\sum_i \eta_i}{b^2}} - e^{\pi i\frac{1-\sum_i \eta_i}{b^2}}\right)\lambda^{(1-\sum_i \eta_i)/b^2}$$
$$\times \exp\left[\frac{1}{b^2}\left\{F(2\eta_1)+F(2\eta_2)+F(2\eta_3)+F(0)-F\left(\sum_i \eta_i\right)\right.\right.$$
$$-F(\eta_1+\eta_2-\eta_3)-F(\eta_2+\eta_3-\eta_1)-F(\eta_3+\eta_1-\eta_2)$$
$$\left.\left.+2\left(1-\sum_i \eta_i\right)\log\left(1-\sum_i \eta_i\right)-2\left(1-\sum_i \eta_i\right)\right\}\right].$$

where again only two saddles contribute and its norm now yields:

$$|\langle V_{\alpha_1}(z_1,\bar{z}_1)V_{\alpha_2}(z_2,\bar{z}_2)V_{\alpha_3}(z_3,\bar{z}_3)\rangle|^2 \sim \exp\left[\frac{\pi c^{(g)}}{3}\left(1-\sum_i \eta_i\right)\right]$$

Most of the exponents cancel out due to the overall purely imaginary factor $\frac{1}{b^2}$.

- We propose the bulk geometry to be:

$$ds^2 = d\theta^2 + \cos^2\theta ds^2_{\text{con}}$$

where $ds^2_{\text{con}}$ denotes the metric of $S^2$ with three conical deficits [Umehara, Yamada 2000], where each deficit is created by the heavy vertex operator insertion with deficit angle $4\pi\eta_i$.

- The resultant volume is $(1 - \sum_i \eta_i) \geq 0$ fraction of $S^3$, reproducing the results from Liouville three point function in this limit.

So far we have only discussed the dS case, the situation with AdS is also interesting and somewhat puzzling.

- We can see this from the exact two point function but now with $\mathrm{Re}(b^{-2}) > 0$, the analytic continuation of $\Gamma$-function yields:

$$\langle V_\alpha(z_1)V_\alpha(z_2)\rangle \sim \frac{1}{e^{-\frac{\pi i}{b^2}(1-2\eta)} - e^{\frac{\pi i}{b^2}(1-2\eta)}}|z_{12}|^{-\frac{4}{b^2}\eta(1-\eta)}e^{-\frac{2}{b^2}[(1-2\eta)\ln(1-2\eta)-(1-2\eta)]}$$

after the expanding the denominator, it becomes an infinite series:

$$\frac{1}{e^{-\frac{\pi i}{b^2}(1-2\eta)} - e^{\frac{\pi i}{b^2}(1-2\eta)}} \sim e^{\pi i(1-2\eta)\frac{\ell_{\mathrm{AdS}}}{4G}}\sum_{n=0}^{\infty} e^{n\pi i(1-2\eta)\frac{\ell_{\mathrm{AdS}}}{2G}} .$$

- We can use holography dictionary to relate that $\mathcal{Z}_{\mathrm{AdS}} = \langle V_\alpha(z_1)V_\alpha(z_2)\rangle$, we have infinite many saddles:

$$\mathcal{Z}_{\mathrm{AdS}} = \sum_{n=0}^{\infty}\mathcal{Z}_n$$

$$\mathcal{Z}_n \sim e^{\frac{\ell_{\mathrm{AdS}}}{4G}(2n+1)\pi i(1-2\eta)}|z_{12}|^{-\frac{\ell_{\mathrm{AdS}}}{2G}\eta(1-\eta)}e^{-\frac{\ell_{\mathrm{AdS}}}{2G}[(1-2\eta)\ln(1-2\eta)-(1-2\eta)]} .$$

- It is interesting to consider the complexification of euclidean $AdS_3$:

$$ds^2 = \ell_{\mathrm{AdS}}^2 \left[ \left( \frac{d\theta(u)}{du} \right)^2 du^2 + \sinh^2 \theta(u) d\Sigma^2 \right]$$

  where $\theta(u)$ is a holomorphic function of $u$. If we consider geometries approach to $AdS_3$ as $u \to \infty$ and truncates at $u = 0$, thus need $\theta \to u, u \to \infty$ and $u = in\pi, u = 0$.

- The two geometries may be interpolated via:

$$\theta = n\pi i(1-u) \quad (0 \le u \le 1)\,, \quad \theta = (u-1) \quad (u > 1)$$

  while this yields $EAdS_3$ for $u > 1$, for $0 \le u \le 1$ the geometry becomes multiple wrapping over $S^3$ of imaginary radius $i\ell_{\mathrm{AdS}}$.

- This may seem somewhat unphysical, even though from Chern-Simons gauge theory, we can construct configuration with action $2\pi i n$, where $n$ labels $\pi_3$ of imaginary $S^3$.

We can also consider such an interpolation in the embedding space.

- Starting with the asymptotic geometry of Euclidean $\mathrm{AdS}_3$:

$$\tilde{X}_0^2 + X_1^2 + X_2^2 - X_3^2 = -\ell_{\mathrm{AdS}}^2$$

  We may want to interpolate it to Lorentzian $\mathrm{AdS}_3$ given by:

$$-X_0^2 + X_1^2 + X_2^2 - X_3^2 = -\ell_{\mathrm{AdS}}^2$$

  by setting $i\tilde{X}_0 = X_0$, however Lorentzian $\mathrm{AdS}_3$ has trivial bulk winding number and infinite volume and cannot be joined by large gauge transformations labeled by integers.

- Instead we may consider the interpolation $X_3 = i\tilde{X}_3$, such that:

$$\tilde{X}_0^2 + X_1^2 + X_2^2 + \tilde{X}_3^2 = -\ell_{\mathrm{AdS}}^2$$

  with $|X_3| \geq \ell_{\mathrm{AdS}}$, which yields $S^3$ of imaginary radius. We can thus glue the two geometries at $X_3 = i\tilde{X}_3 = \ell_{\mathrm{AdS}}$ and $X_1 = X_2 = \tilde{X}_0 = 0$.

An alternative approach to explore the interpolation of the complex geometries is using mini-superspace. [Chen, Hikida, Taki, Uetoko, 2024]

- Starting with Einstein-Hilbert action for positive cosmological constant and contact term:

$$I = -\frac{1}{16\pi G}\int d^3x\sqrt{g}\left(R - \frac{2}{\ell_{\rm dS}^2}\right) + I_{\rm bdy}\,. \quad I_{\rm CT} = \frac{1}{8\pi G}\int d^2x\sqrt{h}\sqrt{-\frac{1}{\ell_{\rm dS}^2}}$$

- Substituting the homogeneous metric ansatz:

$$ds^2 = \ell_{\rm dS}^2\left[N(\tau)^2 d\tau^2 + a(\tau)^2 d\Omega_2\right]$$

where $0 \leq \tau \leq 1$ and we can use the gauge redundancy to fix $N(\tau) = N$. The wave function of universe is now reduced to:

$$\Psi = \int_{\mathcal{C}} dN \int \mathcal{D}a\, e^{-I[a;N]-I_{\rm CT}}$$

$$I[a;N] = -\frac{\ell_{\rm dS}}{2G}\int_0^1 d\tau\, N\left(\frac{1}{N^2}\left(\frac{da}{d\tau}\right)^2 - a^2 + 1\right) + (\text{boundary contributions})$$

We thus need to evaluate it via saddle contributions and consider $\mathcal{C}$.

We reduce the problem into solving for $a(\tau)$ and fluctuations around it.

- If we impose the Dirichlet boundary conditions $a(0) = 0$ and $a(1) = a_1 > 0$, the solution to E.O.M for $a(\tau)$ is:

$$\bar{a}^{(N)}(\tau) = \frac{a_1}{\sin N} \sin(N\tau)$$

If we include the one-loop fluctuations around $\bar{a}^{(N)}(\tau)$, the path integral for $N$ becomes:

$$\Psi = \int_{\mathcal{C}} dN \left( \frac{1}{\sqrt{N} \sin N} \right)^{\frac{1}{2}} e^{-I[\bar{a}^{(N)};N]-I_{\mathrm{CT}}}$$

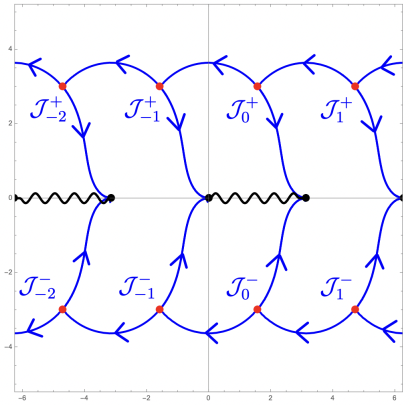
$$I[\bar{a}^{(N)}; N] = -\frac{\ell_{\mathrm{dS}}}{2G} \left( N + a_1^2 \cot N \right)$$

- To work out $\mathcal{C}$, we first consider the saddle for $N$ satisfying $\partial I[\hat{a}, N]/\partial N = 0$ or $\sin N = \pm a_1$, the solutions are $(m \in \mathbb{Z})$:

$$N_m^+ = \left( m + \frac{1}{2} \right) \pi + i \log \left( a_1 + \sqrt{a_1^2 - 1} \right)$$

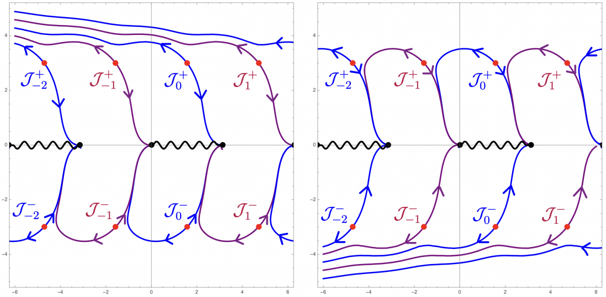
$$N_m^- = \left( m + \frac{1}{2} \right) \pi - i \log \left( a_1 + \sqrt{a_1^2 - 1} \right)$$

- It is interesting to note that in the complex $N$-plane, the steepest descent emanating from one saddle point can end up at another saddle point (red dots):




This implies that we have Stoke's lines specifically in 3-dimensions, and makes the identification of $\mathcal{C}$ difficult.

- Instead we can consider slight deformation of contours by deforming $\ell_{\text{dS}} \to \ell_{\text{dS}} \pm i\epsilon$:




We can identify the correct shift by interpreting it as sub-leading correction in:

$$c = -i\frac{3\ell_{\text{dS}}}{2G} + 13 + \mathcal{O}(G)\,.$$

such that it should be $\ell_{\text{dS}} \to \ell_{\text{dS}} + i\epsilon$.

- To complete the construction of the contour $\mathcal{C}$, notice that each saddle $N_m^{\pm}$:

$$\Psi_m^{\pm} \sim e^{\frac{(2m+1)\ell_{\rm dS}\pi}{4G}} (2a_1)^{\mp i \frac{\ell_{\rm dS}}{2G} \pm \frac{\epsilon}{2G}} \ .$$

  where in large $a_1$ limit, $\Psi_m^{-}$ in the lower half becomes suppressed. Since $N$ acts as time-direction, naively we should integrate along $i\mathbb{R}$ in the upper half or deforming it into $\mathcal{J}_{-1}^{+}$, however it implies only exponentially suppressed saddle $N_{-1}^{+}$ which is insufficient.

- This implies that we should also pick up additional saddles $N_0^{+}$ and the contour allowing this can be deformed into:

$$-\mathcal{J}_{-1}^{+} + \mathcal{J}_0^{-} + \mathcal{J}_0^{+} \ ,$$

  which goes around the branch cut and back to upper half plane. The contribution from $N_0^{-}$ in the lower half disappears in large $a_1$ limit.

- We may also impose Neumann boundary condition instead at $\tau = 0$, the action needs to be modified as:

$$I[a;N] = -\frac{\ell_{\rm dS}}{2G}\int_0^1 d\tau\, N\left(\frac{1}{N^2}\left(\frac{da}{d\tau}\right)^2 - a^2 + 1\right) - \frac{\ell_{\rm dS}}{GN}\, a\frac{da}{d\tau}\bigg|_{\tau=0}$$

- The solution to E.O.M. and the on-shell action are:

$$\bar{a}^{(N)}(\tau) = \frac{a_1}{\cos N}\cos(N\tau) + \frac{\sin(N(\tau-1))}{\cos N}$$
$$I[\bar{a}^{(N)};N] = -\frac{\ell_{\rm dS}}{2G}\left[N + \frac{2a_1}{\cos N} - (1+a_1^2)\tan N\right].$$

- However when extremising with respect to $N$:

$$\frac{\partial I[\bar{a}^{(N)};N]}{\partial N} = \frac{\ell_{\rm dS}}{2G}\frac{(\sin N - a_1)^2}{\cos^2 N}$$

such that $\sin N = a_1$ is now an inflexion point and the saddle point $n = -1$ needed to reproduce CFT result is absent.

We perform similar analysis for the negative cosmological constant case, here are the summary.

- The solution for Dirichlet boundary conditions yields:

$$\bar{a}^{(N)}(r) = \frac{a_1}{\sinh N}\sinh(Nr)\,.$$

where the fluctuation determinant around it can be evaluated analogous to yield:

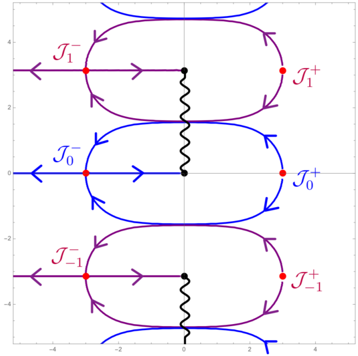
$$\mathcal{Z} = \int_{\mathcal{C}} dN \left(\frac{1}{\sqrt{N}\sinh N}\right)^{\frac{1}{2}} e^{-I[\bar{a}^{(N)};N]-I_{\rm CT}}$$

$$I[\bar{a}^{(N)};N] = -\frac{\ell_{\rm AdS}}{2G}\left[N + a_1^2\coth N\right]$$

- The one-loop corrected action yields the following saddle points:

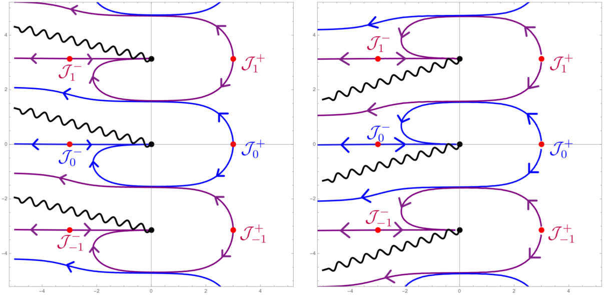
$$N_m^+ = \operatorname{arcsinh} a_1 + \pi i m \quad (m\in\mathbb{Z})\,,$$
$$N_m^- = -\operatorname{arcsinh} a_1 + \pi i m \quad (m\in\mathbb{Z})\,,$$

The saddle points and branch cuts are given here:




Here we again see that there are Stoke's phenomena where paths of steepest descend connecting saddle points, but moreover we have the thimble $\mathcal{J}_m^+$ crossing the branch cut. These make the identification of $\mathcal{C}$ difficult.

If we now deform the contour by shifting $\ell_{\mathrm{AdS}} \to \ell_{\mathrm{AdS}} \pm i\epsilon$, and choose the branch cuts so as to avoid the steepest descend paths, we obtain:




Here the individual saddle contribute as:

$$\mathcal{Z}_m^{\pm} \sim e^{\frac{im\pi\ell_{\mathrm{AdS}}}{2G}} (2a_1)^{\pm\frac{\ell_{\mathrm{AdS}}}{2G}}$$

for saddle labeled by $N_m^{\pm}$ respectively.

- For $\ell_{\rm AdS} \to \ell_{\rm AdS} + i\epsilon$, the final total contour is:

$$\mathcal{C} \to \sum_{m=0}^{\infty} \mathcal{J}_m^+ - \sum_{m=1}^{\infty} \mathcal{J}_m^- ,$$

which yields:

$$\mathcal{Z} \sim \sum_{m=0}^{\infty} e^{\frac{i\pi m(\ell_{\rm AdS}+i\epsilon)}{2G}} (2a_1)^{\frac{\ell_{\rm AdS}+i\epsilon}{2G}} = -\frac{2ie^{-\frac{i\pi\ell_{\rm AdS}}{4G}}}{\sin\left(\frac{\ell_{\rm AdS}\pi}{4G}\right)} (2a_1)^{\frac{\ell_{\rm AdS}}{2G}} .$$

which differs from the CFT result by an overall phase factor.

- For $\ell_{\rm AdS} \to \ell_{\rm AdS} - i\epsilon$, the final total contour is:

$$\mathcal{C} \to -\sum_{m=1}^{\infty} \mathcal{J}_{-m}^+ + \sum_{m=0}^{\infty} \mathcal{J}_{-m}^- .$$

which yields:

$$\mathcal{Z} \sim -\sum_{m=-\infty}^{0} e^{\frac{i\pi m(\ell_{\rm AdS}-i\epsilon)}{2G}} (2a_1)^{\frac{\ell_{\rm AdS}-i\epsilon}{2G}} = -\frac{2ie^{\frac{i\pi\ell_{\rm AdS}}{4G}}}{\sin\left(\frac{\ell_{\rm AdS}\pi}{4G}\right)} (2a_1)^{\frac{\ell_{\rm AdS}}{2G}} .$$

which again differs from the CFT result by an overall phase factor. The analysis allows us to simply pick the contour $\mathcal{C}$ to be $\mathbb{R}_+$.

- In this talk we demonstrate how holography may help us to explore the complex geometries arising from the gravitational path integral.
- We actually also studied higher point CFT correlation functions with heavy operator insertions which can be dual to Chern-Simons Wilson lines and compute their monodromy matrices.
- It would be interesting to extend the analysis presented to higher point CFT correlation functions and CFT on higher genus Riemann surfaces.
- Extensions to higher spin gravity theories and the dual $SL(N)$ Toda CFT may also be interesting.