Gravity and Lattice

- a step towards regularization of gravity -

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(Related work: S. Aoki, HF, N.Kan, M. Koshino and Y. Matsuki, 2304.13954)

When I first met Aoki-san

In 2001, I was an M1 student at YITP,

Aoki-san gave an intensive lecture at Kyoto U.:

特別講義 " 格子ゲージ理論入門 "

青木 慎也 氏 (筑波大学物理学系)

１１月５日（月）１０：００～１６：００

１１月６日（火）１０：００～１６：００

１１月７日（水）１０：００～１２：００

This lecture was so impressive and interesting to affect my decision of choosing lattice gauge theory for my master thesis.

My first paper with Aoki-san

In 2004 when I was a DC2 student, I joined the JLQCD collaboration, and worked on lattice QCD with a fixed topology, which helped the large-scale simulation of overlap fermion in 2007.

PRL **98,** 172001 (2007) PHYSICAL REVIEW LETTERS week ending 27 APRIL 2007

Two-Flavor Lattice-QCD Simulation in the ϵ Regime with Exact Chiral Symmetry

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(JLQCD Collaboration)

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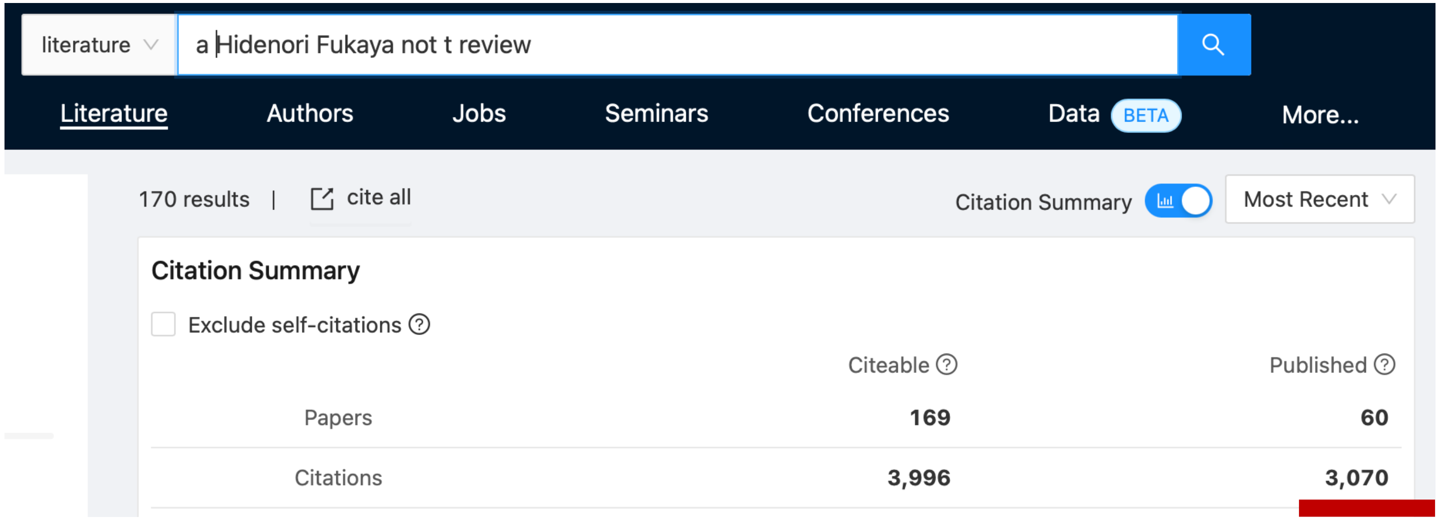
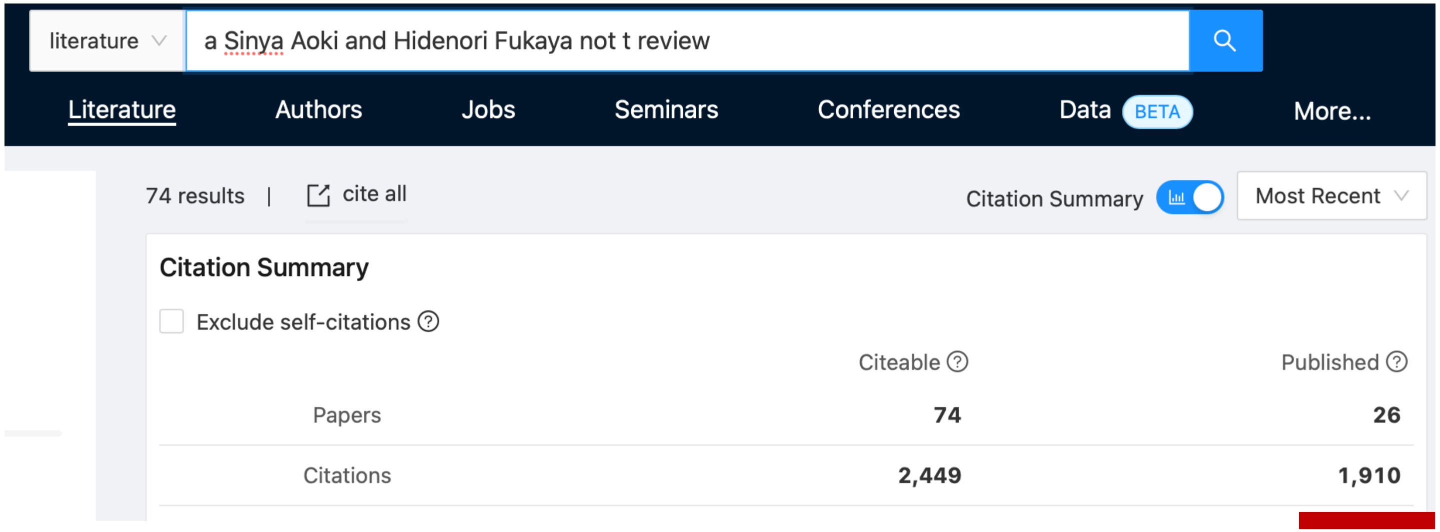
(Received 2 February 2007; published 24 April 2007)

We perform lattice simulations of two-flavor QCD using Neuberger's overlap fermion, with which the exact chiral symmetry is realized at finite lattice spacings. The ϵ regime is reached by decreasing the light

My Sinya-Aoki-dependency rate

at 11:10 May 21 2025





～62%.

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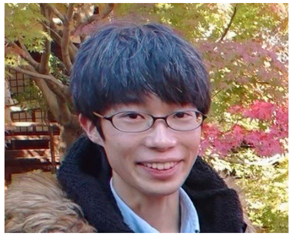

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(Related work: S. Aoki, HF, N.Kan, M. Koshino and Y. Matsuki, 2304.13954)

What is (quantum) gravity?

Gravity = String theory

Gravity = CFT (AdS/CFT Mardacena 1997)

Gravity = Matrix model

(IKKT 1996, BFSS 1996, SYK 2015…)

Gravity = Gradient (smearing) flow (S. Aoki et al.)

… Many possibilities?

Einstein's equivalence principle tells us

Gravity = Anything (at least locally).

Gravity = Lattice ?

Anything is gravity.

Einstein's equivalence principle:

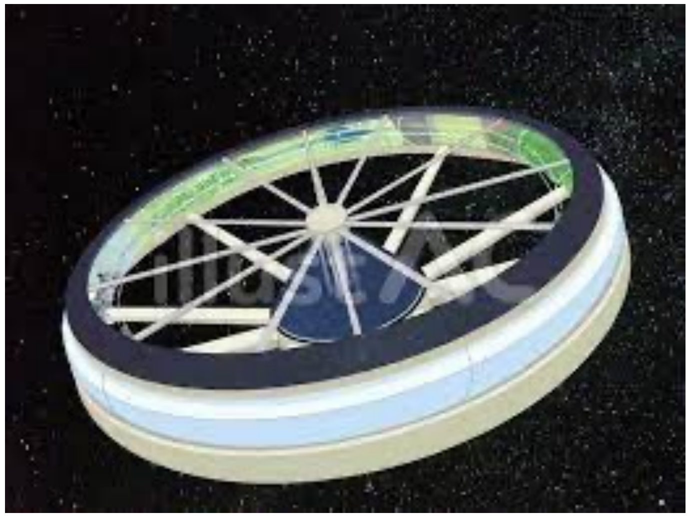
inertial mass = gravitational mass

Any acceleration = gravity (at least locally).

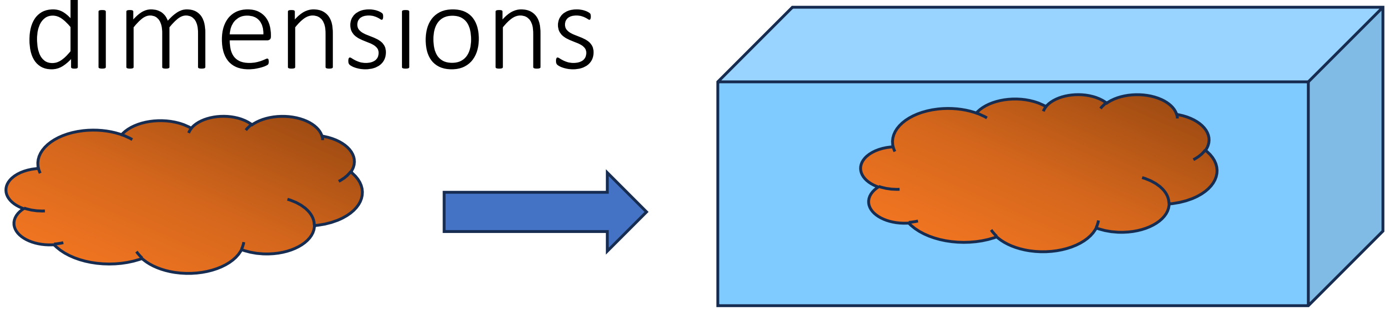
Cf.) centrifugal force
in space colony.

Constraint force

= gravitational force



"Global" equivalence principle in higher dimensions



Nash's embedding theorem [1956] tells

- Any Riemannian manifold Y can be isometrically embedded into a finite-dimensional flat Euclidean space X=R^n.

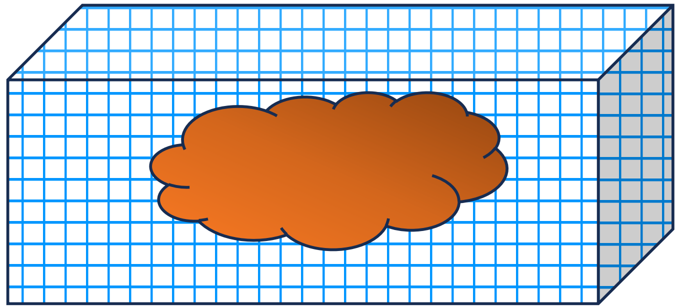
$$x^\mu = x^\mu(y^1, \cdots, y^n)$$

- The metric on Y is induced by the embedding function:

$$g_{ij} = \sum_{\mu\nu} \delta_{\mu\nu} \frac{\partial x^\mu}{\partial y^i} \frac{\partial x^\nu}{\partial y^j}.$$

which is unique up to general covariance (so is vielbein).

Lattice regularization of higher dim. Euclidean space as gravity



Nash's theorem "tells" any gravitational field can be described in a flat Euclidean space.

1. Prepare a higher dim. square lattice.
2. Put the target curved manifold as a defect and constrain fields on it.
3. Fluctuate the submanifold quantum mechanically with a Boltzmann weight of the Einstein Hilbert action.
4. Then the effective theory may describe quantum gravity.

Grand unification?

Higher dim. Euclidean square lattice theory

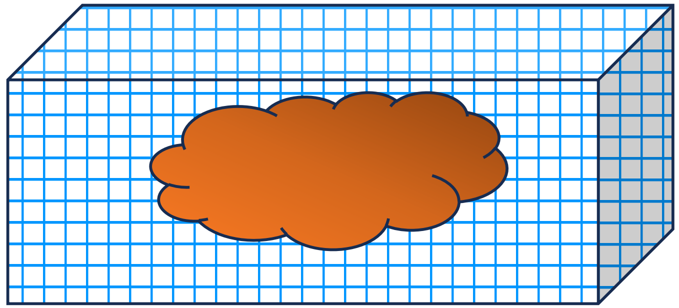
QCD and Electro-Weak + defect source (axion?)

Hadron physics

Gravity

The effective theories of hadrons and gravity do not need to be renormalizable.

A lot of problems (of course)



1. Our universe is not Euclidean but Lorentzian!
2. Higher dim. gauge theories are not renormalizable!
3. Can we realize theory with intrinsic curvature only ?
4. Why don't you try chiral gauge theory first?

…

But let us start with the easiest example:

QFT with a classically fixed gravitational background

In this talk, we will see this can be put on a square lattice:

by curved domain-wall fermions.

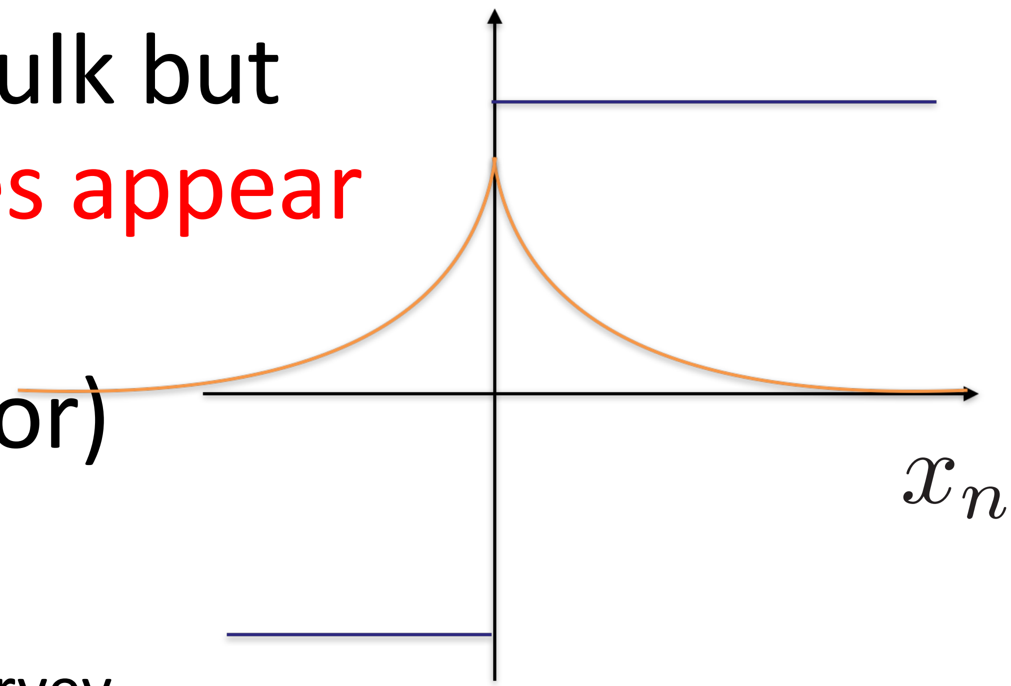
Contents

Flat domain-wall fermion

[Jackiw-Rebbi 1976, Callan-Harvery 1985, Kaplan 1992]

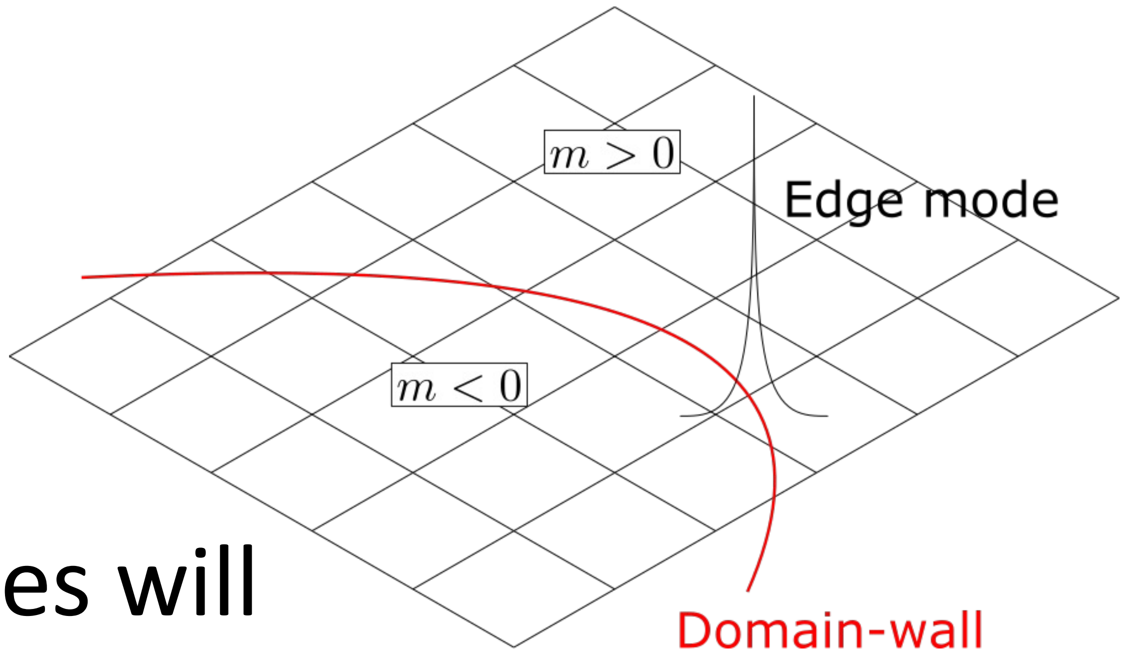
$$S = \int d^n x \bar{\psi}(D + \mathrm{sgn}(x_n)M)\psi$$

Massive fermion in bulk but massless chiral modes appear at domain-wall.
(~ topological insulator)



Note : anomaly inflow was first discussed by Callan and Harvey 1985 using domain-wall fermion.

Curved domain-wall fermion



Massless chiral modes will

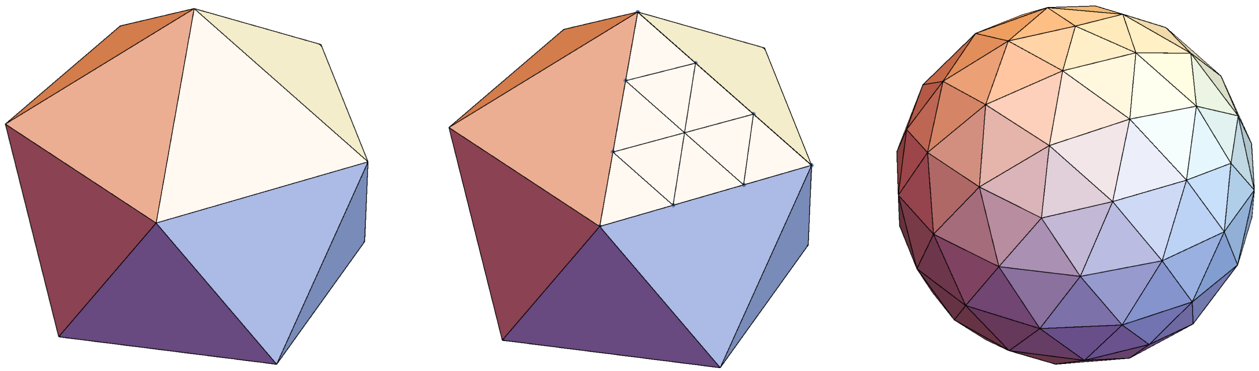
1) be constrained into the curved sub-manifold,
2) accelerate by the constraint,
3) feel gravity due to the equivalence principle.

Compared to previous works

The triangular lattice was popular to describe curved space-time

[Ambjorn et al. 2001, Brower et al. 2015, 2017].

But

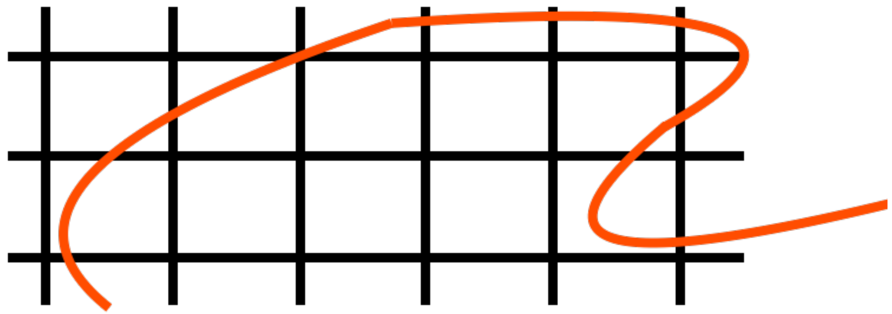


Figures by brower et al. 2015

1) continuum limit is nontrivial: many parameters(links, # of sites, angles...) to tune.

2) (rotational) symmetry needs counter terms.

Our new approach on square lattices



1) Curved space is a submanifold of higher-dimensional flat space=square lattice,
2) No gravitational degrees of freedom assigned to sites nor links (gravity appears as an effective fields),
3) Only one parameter (lattice spacing) is tuned to the continuum limit,
4) rotational symmetry is automatically recovered due to 90-degree rotations of square lattices.

Curved domain-wall fermion (in continuum)

[S.Aoki, HF 2022]

$$H = \bar{\gamma}\left(\sum_{i=1}^{n+1}\gamma^i\frac{\partial}{\partial x^i} + m\mathrm{sign}(f)\right) = \bar{\gamma}\,(D + m\mathrm{sign}(f))$$

$$\gamma^a = -\sigma_2\otimes\tilde{\gamma}^a,\ \ \gamma^{n+1} = \sigma_1\otimes 1,\ \ \bar{\gamma} = \sigma_3\otimes 1$$

$$\{\tilde{\gamma}^a, \tilde{\gamma}^b\} = 2\delta^{a,b},\ \ (a, b = 1, \cdots, n)$$

where a smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ determines where Y is located by $Y = \{f = 0\}$. We will show below

1) the edge localized modes generally exist,
2) they are "chiral" eigenstates of $\gamma_{\mathrm{normal}} = \boldsymbol{n}\cdot\boldsymbol{\gamma}$
3) they feel gravity through the induced spin connection.

Induced vielbein

Let us perform a local Lorentz transformation to rewrite the coordinate $(y^1, \cdots, y^n, t)$

Coordinate for Y

Normal direction to domain-wall

so that the vielbein on Y are given by $\dfrac{\partial x^\mu}{\partial y^i}$ and

$$e_{n+1} = \frac{1}{||\mathrm{grad}(f)||}\mathrm{grad}(f)$$

$$\mathrm{grad}(f) = \sum_I g^{IJ} \frac{\partial f}{\partial x^I} \frac{\partial}{\partial x^J}$$

Note that

$$e_j(f) = e_j^I \frac{\partial f}{\partial x^I} = 0 \qquad j = 1, 2, \cdots n$$

Induced spin connection on the edge

The Dirac operator becomes (after rescaling $\psi = \left(g^{IJ}\frac{\partial f}{\partial x^I}\frac{\partial f}{\partial x^J}\right)^{\frac{1}{4}}\psi'$)

$$\bar{\gamma}\left(D' + m\mathrm{sign}(f)\right) = \bar{\gamma}\gamma^a\left(e_a + \frac{1}{4}\sum_{bc}\omega_{bc,a}(e_Y)\gamma^b\gamma^c\right)$$

→ Spin connection on Y

$$+\bar{\gamma}\gamma^{n+1}\left(e_{n+1} - \frac{1}{2}\mathrm{tr}h + \frac{1}{4}e_{n+1}\left(\log(g^{IJ}\frac{\partial f}{\partial x^I}\frac{\partial f}{\partial x^J})\right) + m\gamma_{n+1}\mathrm{sign}(f)\right)$$

Covariant derivative in the normal direction + mass term
= Constraint force

to which we generally have chiral edge modes

$$\psi = \underbrace{\left(\frac{\partial f}{\partial t}\right)^{\frac{1}{2}}}_{=const.}\exp\left[-m|t|\left\{1 + \frac{1}{8m|t|}\int_0^t dt'\mathrm{tr}h(y,t')\right\}\right]v_+ \otimes \chi(y)$$

h : extrinsic (mean) curvature

at least, approximately
when m is large enough.

$$\gamma_{n+1}v_+ = +v_+$$

The effective Dirac operator

The effective Dirac operator on $\chi(y)$ leads to

$$iD^Y|_{t=0} = i\sum_{a=1}^{n}\tilde{\gamma}^a\left(e_a + \frac{1}{4}\sum_{bc}\omega_{bc,a}\tilde{\gamma}^b\tilde{\gamma}^c\right)\Bigg|_{t=0}$$

(intrinsic) Spin connection on Y

The "gravity" on the domain-wall Y is encoded in the spectrum of the Dirac operator.

Lattice discretization of higher-dim. flat Euclidean space = Lattice regularization of edge-QFT on a curved space.

Contents

S^2 domain-wall fermion in continuous R^3

Let us consider a Shamir domain-wall Dirac operator in $\mathbb{R}^3$

$$D = \sum_{i=1}^{3} \sigma^i \left(\frac{\partial}{\partial x^i} - iA_i(x) \right) + m(x)$$

U(1) gauge field

where

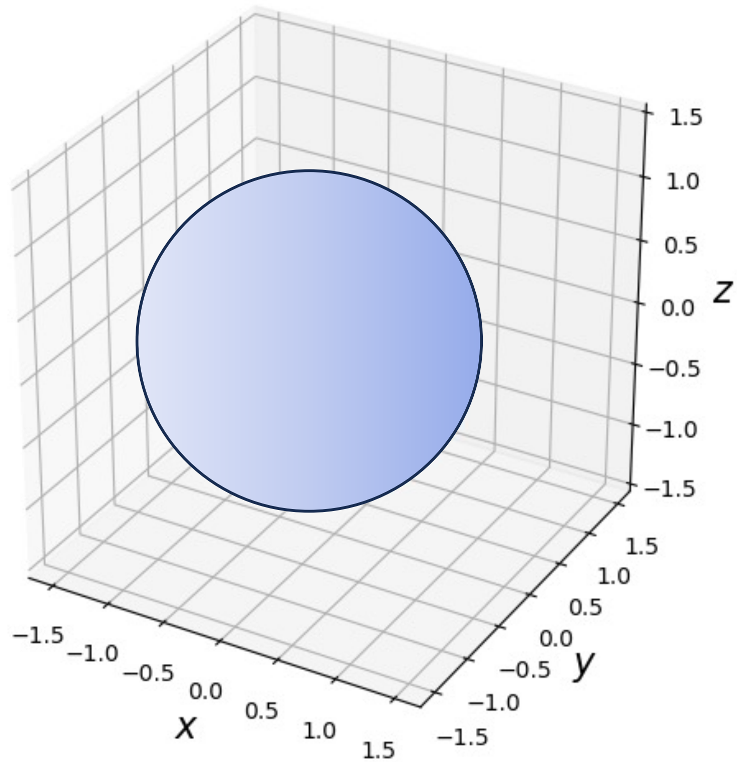
$$m(x) = \begin{cases} -m & \text{for } |x| \leq r_0 \\ +\infty & \text{otherwise} \end{cases},$$

which is equivalent to impose

the boundary condtion

$$\sigma_r \psi(x) = +\psi(x) \quad \text{at } |x| = r_0.$$

$$\sigma_r = \frac{\sum_i \sigma^i x_i}{|x|},$$



S^2 domain-wall edge modes

In polar coordinate,

$$D = \sigma_r \left(\frac{\partial}{\partial r} + \frac{1}{r} - \frac{r_0}{r} iD^{S^2} \right) - m,$$

where (after Local Lorentz transformation)

$$iD^{S^2} = -\frac{1}{r_0}\sigma_3 \left[\sigma_1 \left(\frac{\partial}{\partial \theta} - i\hat{A}_\theta \right) + \frac{\sigma_2}{\sin\theta} \left(\frac{\partial}{\partial \phi} - i\hat{A}_\phi + \frac{i}{2} - \frac{i\cos\theta}{2}\sigma_3 \right) \right].$$

Spin connection

This has edge-localized solutions = Weyl fermion appears!

$$\psi_+^e = \frac{1}{r}\exp[-m(r_0 - r)]P_+ \chi(\theta, \phi), \quad P_+ = \frac{1+\sigma_r}{2}$$

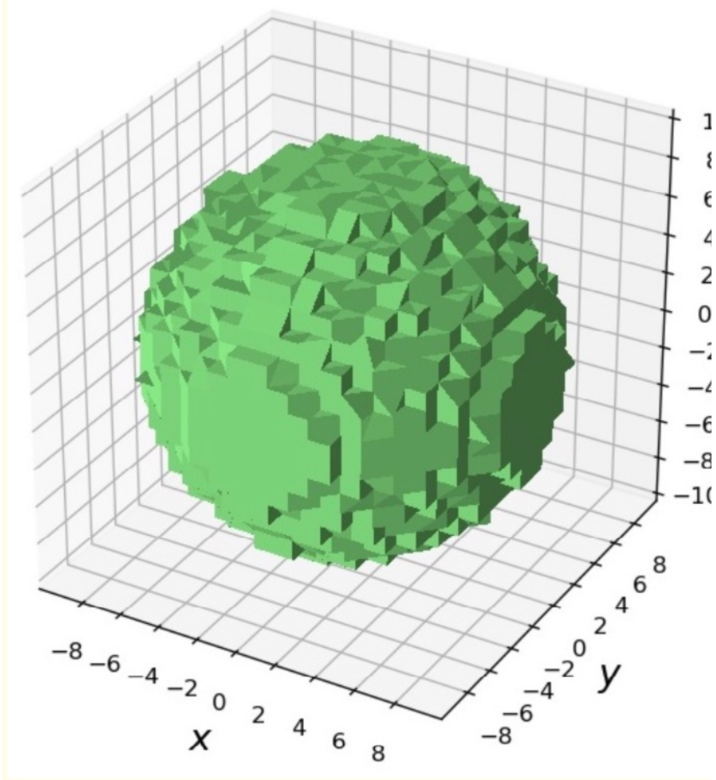
S^2 domain-wall fermion on a lattice

Let us consider the lattice domain-wall Dirac operator

$$D_{DW} = \sum_{i=1}^{3} \sigma^i \frac{\nabla_i - \nabla_i^\dagger}{2} + \sum \frac{1}{2} \nabla_i \nabla_i^\dagger - m,$$

and put Shamir's boundary condition:

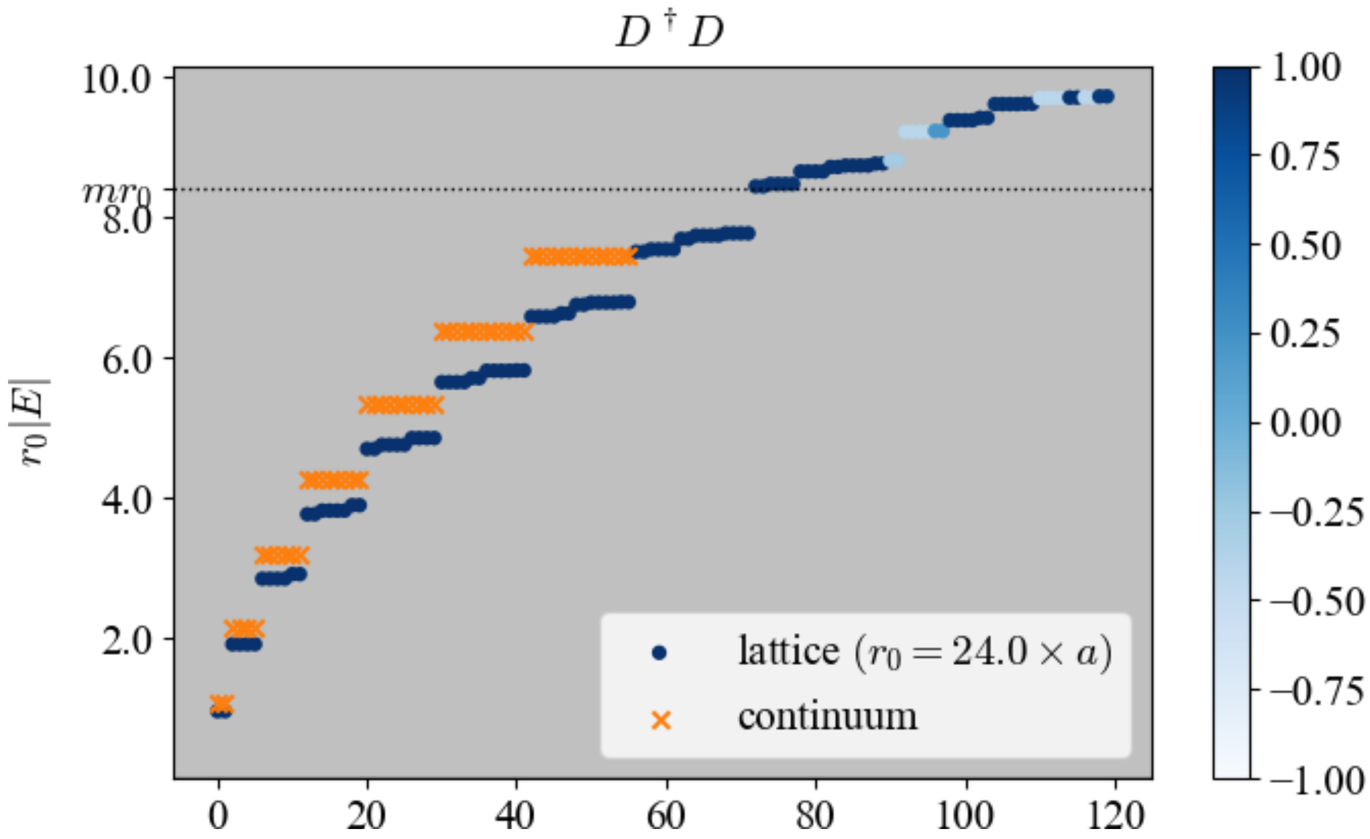
$$m(x) = \begin{cases} -m & \text{for } |x| \leq r_0 \\ +\infty & \text{otherwise} \end{cases},$$



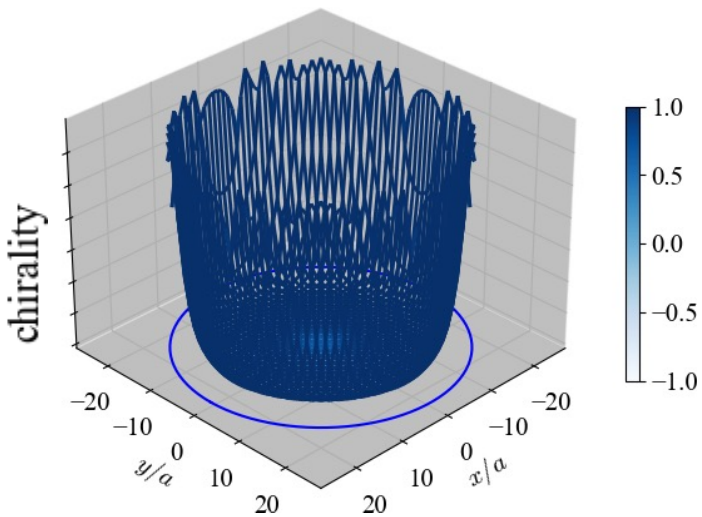
We numerically solve the eigenproblem of

$$D_{DW}^\dagger D_{DW} \qquad D_{DW} D_{DW}^\dagger$$

Free Dirac spectrum and chirality



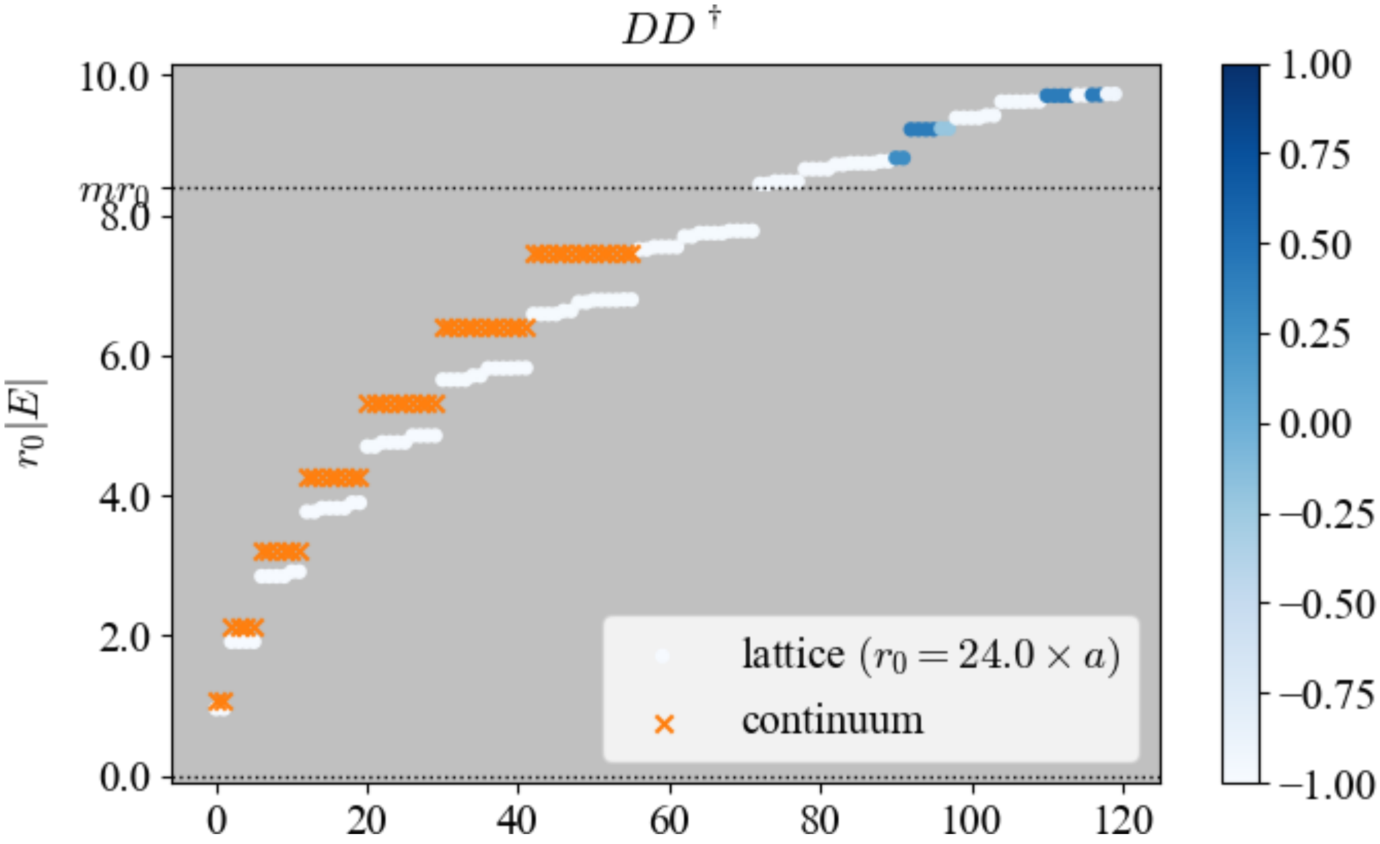
Eigenfunction amplitude distribution



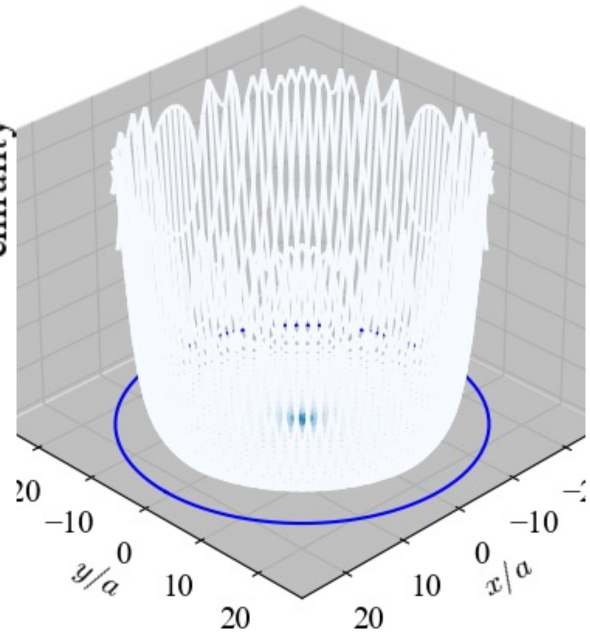
We find edge-localized modes, which

1) are (almost) chiral $\langle \sigma_r \rangle \sim +1$,
2) have a gap from zero (as a gravitational effect),
3) agree well with the continuum theory.

Free Dirac spectrum and chirality



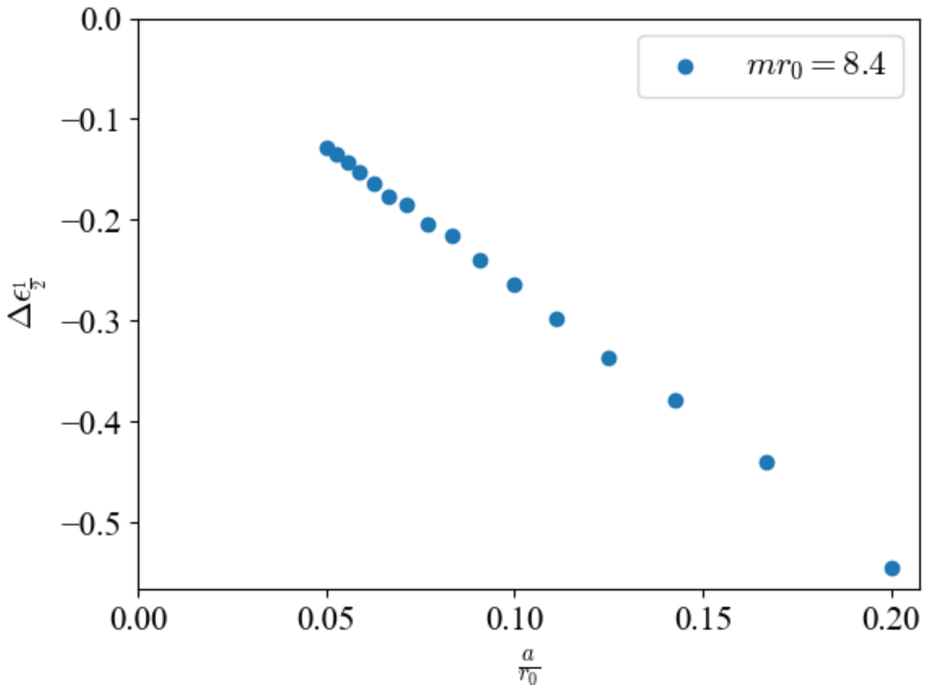
Eigenfunction amplitude distribution



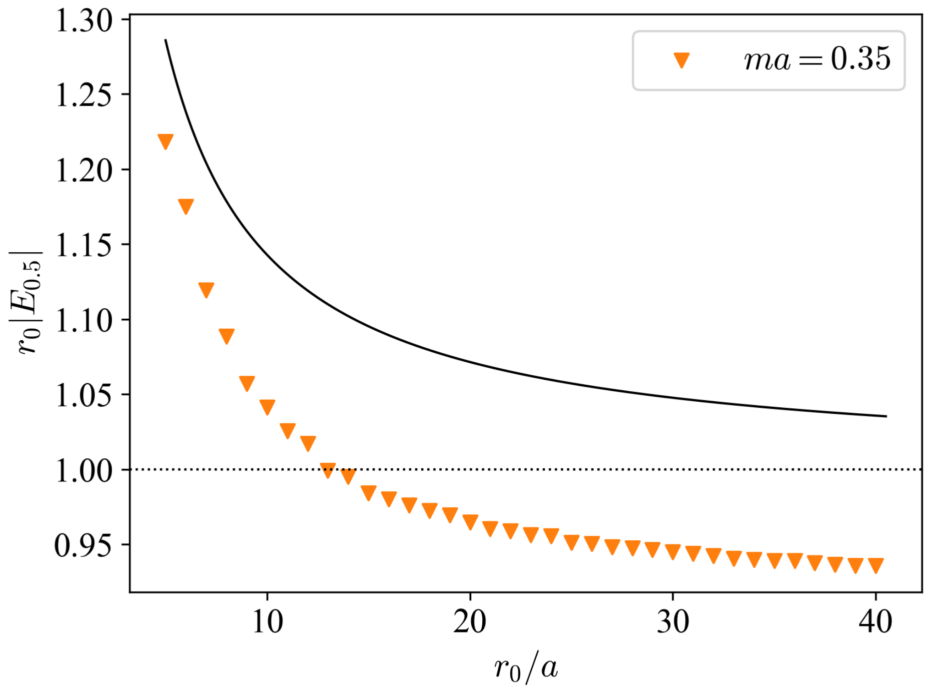
We find edge-localized modes, which

1) are (almost) chiral $\langle \sigma_r \rangle \sim -1$,
2) have a gap from zero (as a gravitational effect),
3) agree well with the continuum theory.

Continuum and large volume limits

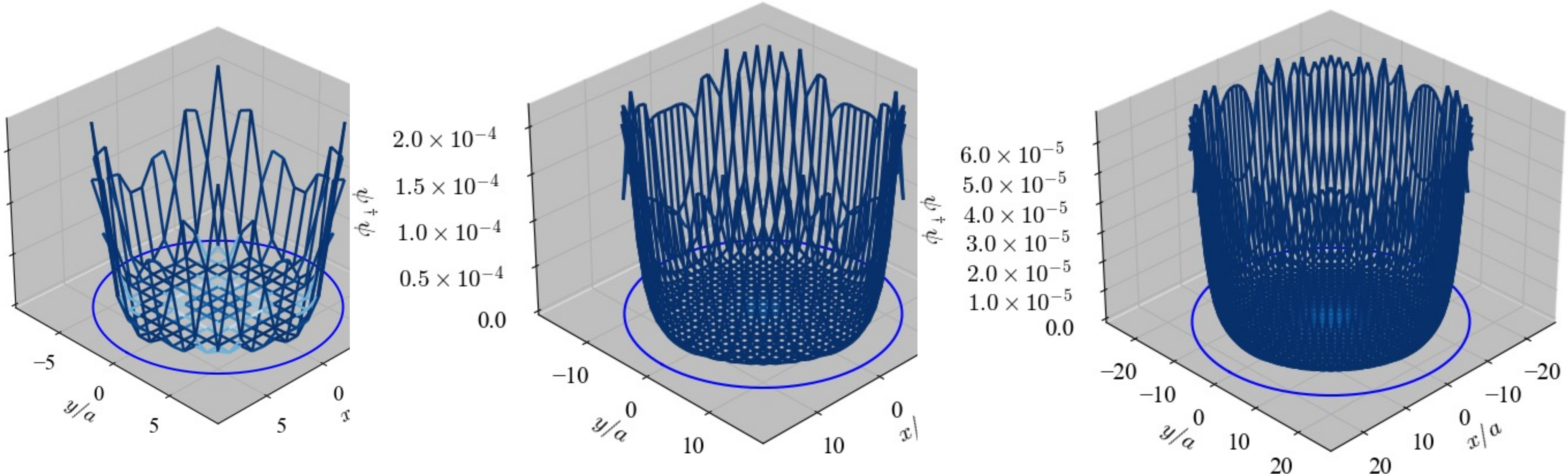


Deviation from continuum 1st eigenvalue

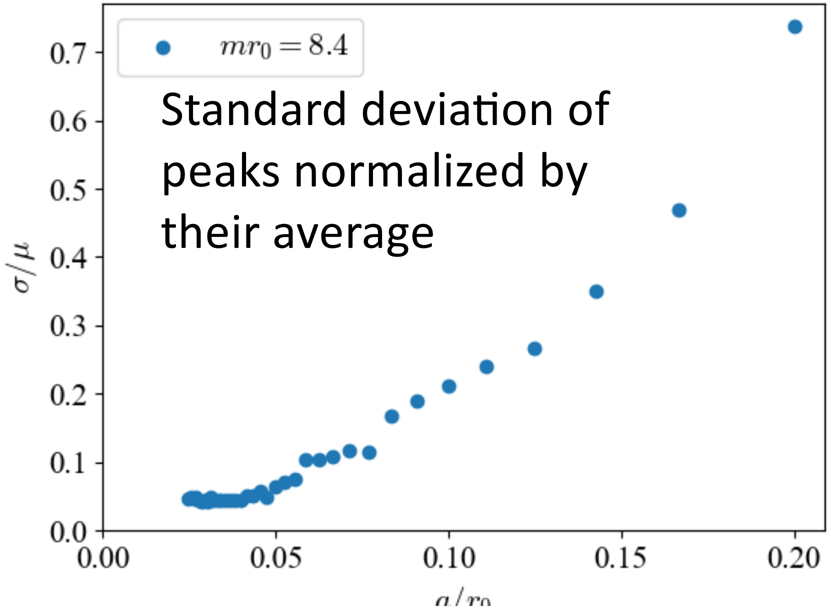


Systematics due to finite lattice spacing and finite volume are well under control.

Recovery of rotational symmetry

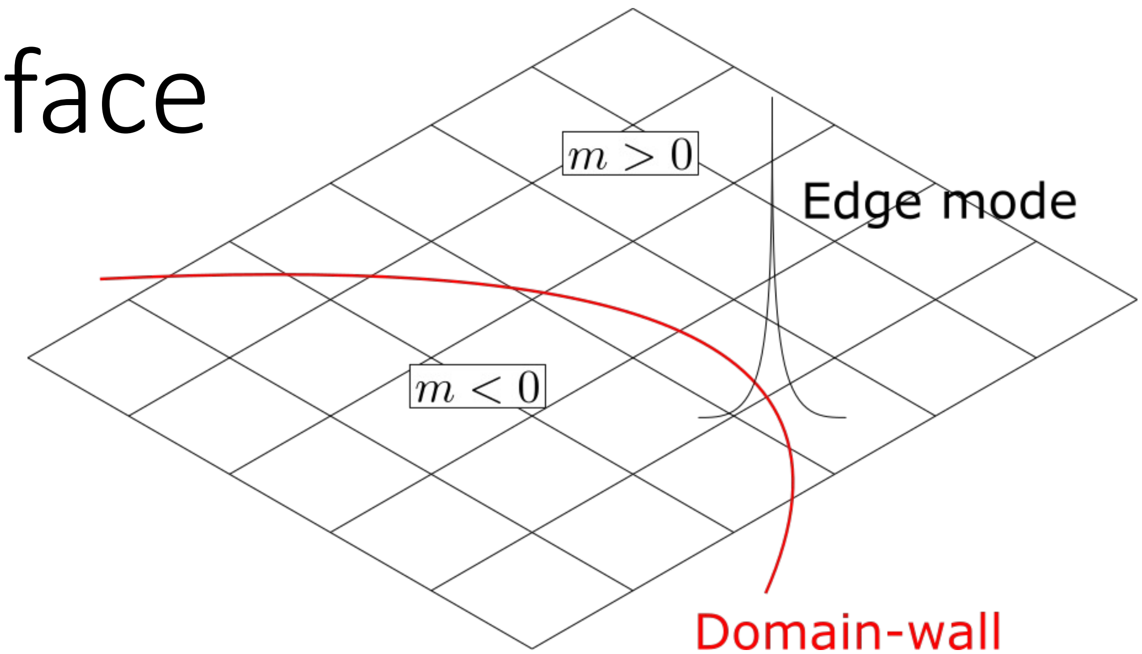


The zigzag peaks become rotational-symmetric towards the continuum limit.



Contents

Curved domain-wall fermion
~ topological insulator
with curved surface

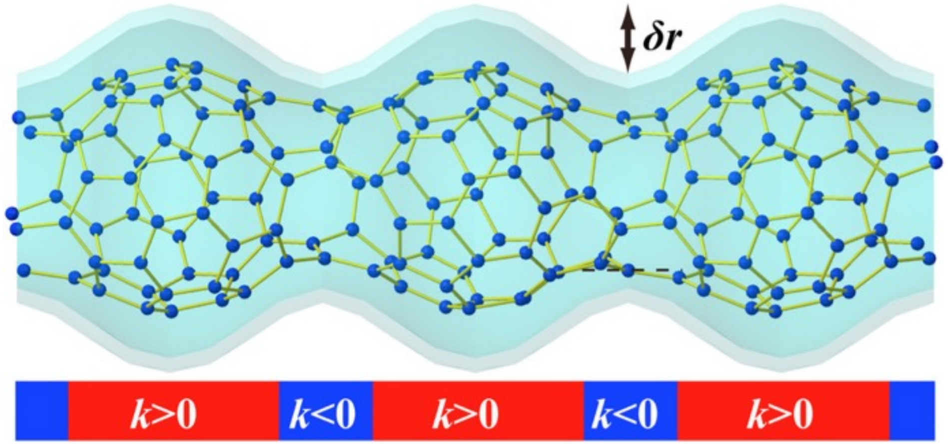


Can surface electrons on topological insulators feel gravity?

Some positive answers from condensed matter physics : Parente et al. 2011, Imura et al. 2012.

“Gravity” in condensed matters

Onoe et al. 2012 detected a “curvature” of C_{60} peanut-shell-shaped polymer through photoemission spectroscopy of the surface electrons.



The result shows that electrons follow

$$H = -\frac{\hbar^2}{2m^*}\left[\frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\partial_j\right) + (h^2 - k)\right]$$

which is reflected to the density of states.

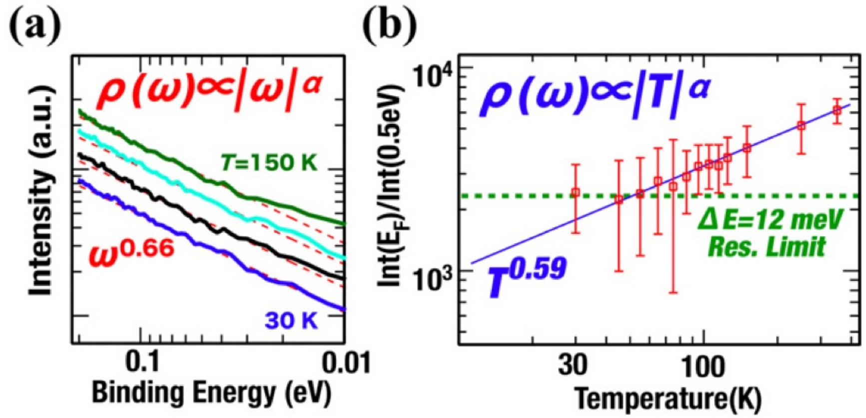
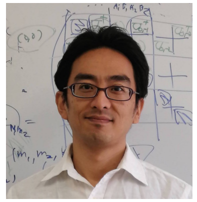
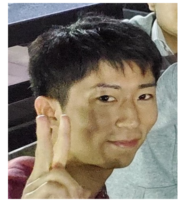


Fig. 3: (Color online) The power-law dependence of the PES spectral function shown in fig. 2 on the binding energy (a) and temperature (b).

Monopole in topological insulator

S. Aoki, HF, N. Kan, M. Koshino and Y. Matsuki (U. Osaka), 2304.13954

Topological insulator = negative mass fermion

= θ=π vacuum.

Witten effect : a magnetic monopole becomes dyon with electric charge $q = \frac{\theta}{2\pi}$

What is the microscopic mechanism ?

We find curved domain-wall fermion describes it.

Numerical analysis on a 3D flat lattice

We consider 3D Dirac Hamiltonian with spherical domain-wall.

L=24,32, 48 lattices

Domain-wall radius r0=(3/8)L

Monopole put at (L/2,L/2,L/2)

Anti-monopole at (L/2,L/2,1/2)

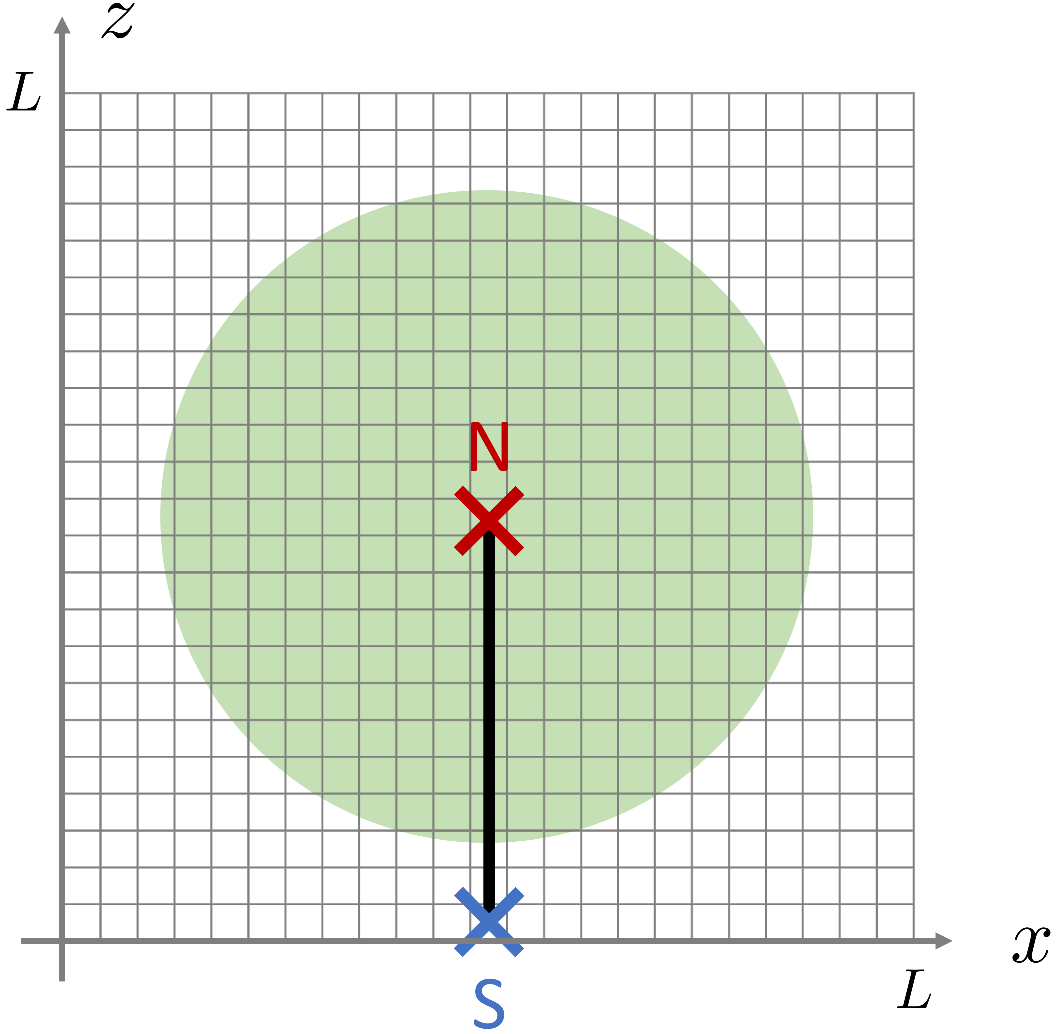
m(r<r0) = -14/(L+1)

m(r>r0) = +14/(L+1)

Open boundary condition at

x_i = 0 or L (m=∞ outside)

Monopole charge n=0,1,-2



Near-zero eigenvalues/functions

Circle: lattice results

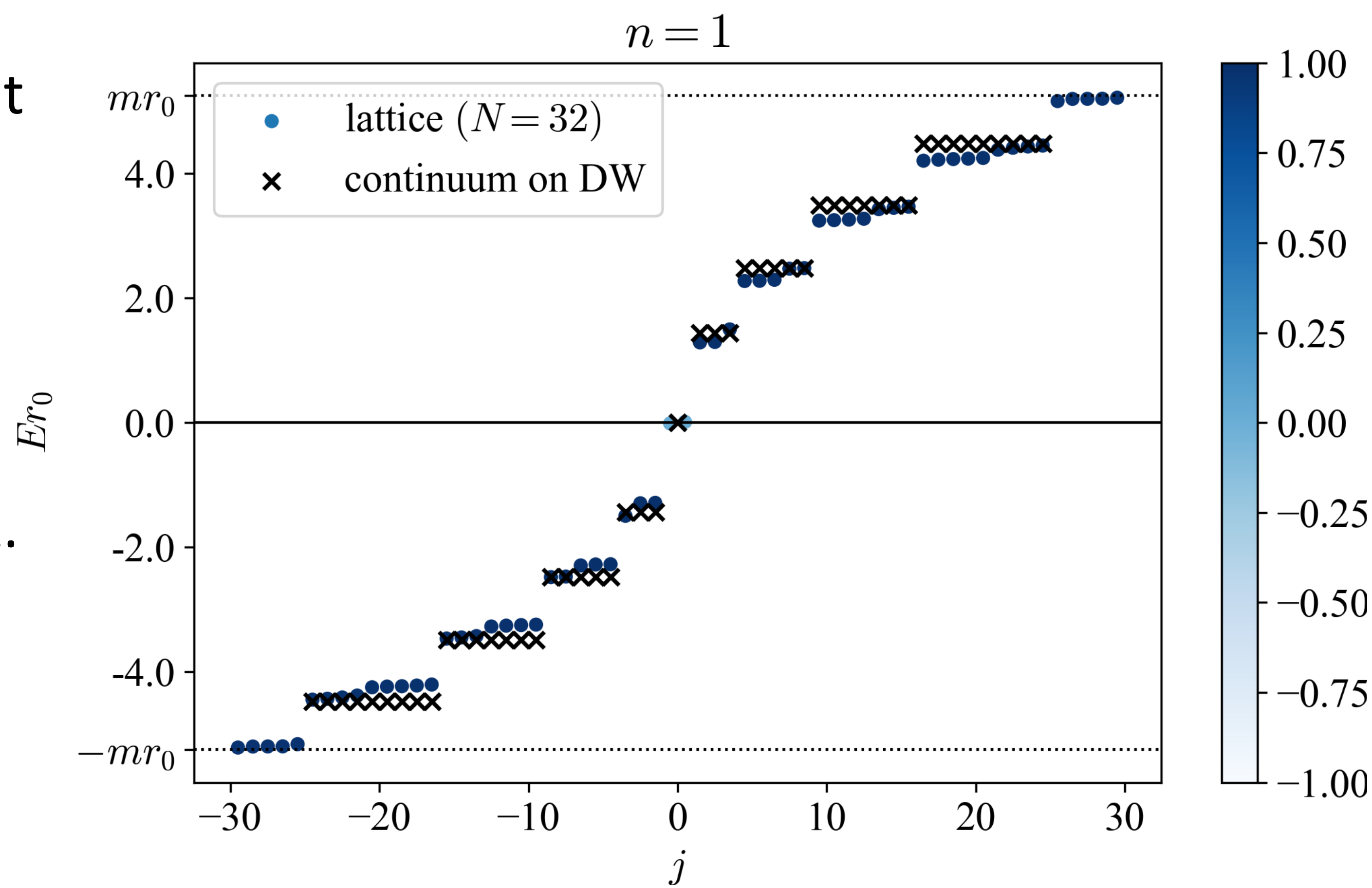
Crosses: analytic result on DW

Gradation: chirality.

$$\langle \sigma_1 \otimes \sigma_r \rangle_k$$

Monopole charge n=1.

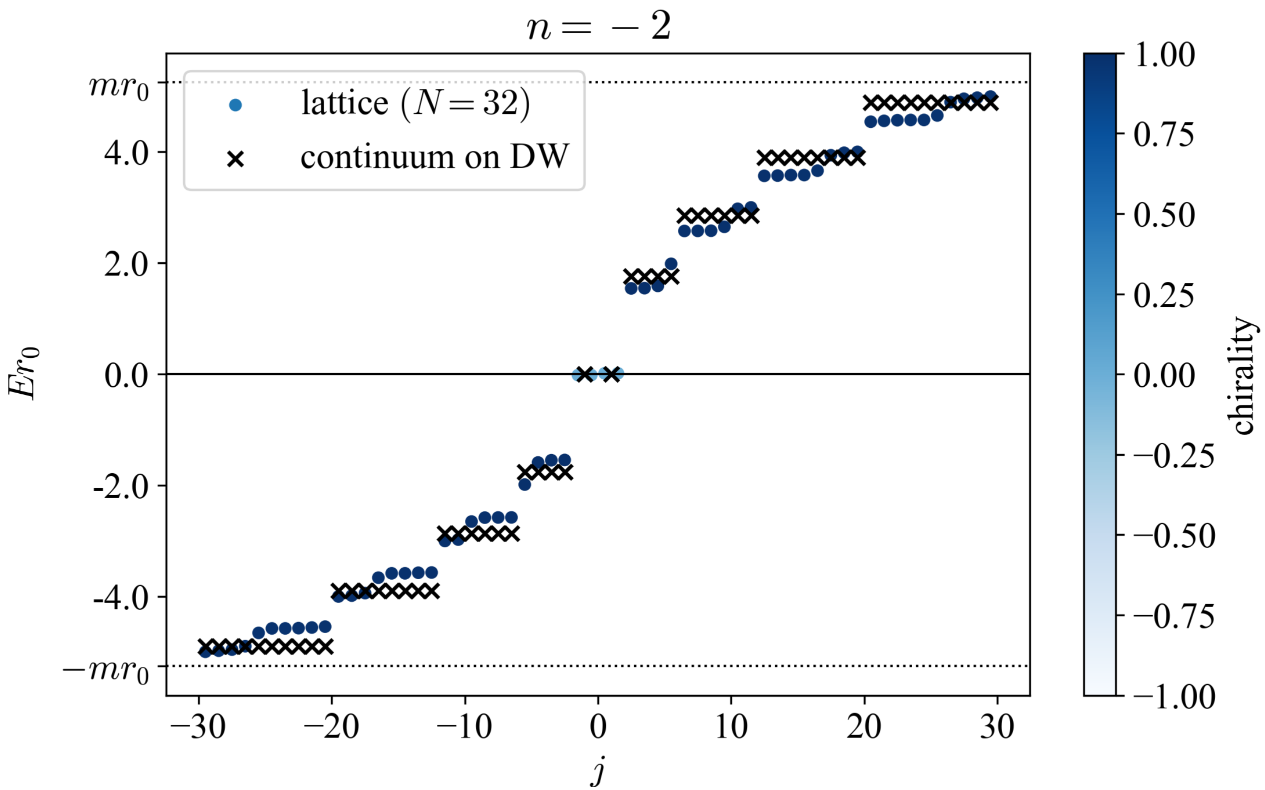
We have two zero modes, while the continuum prediction is one.



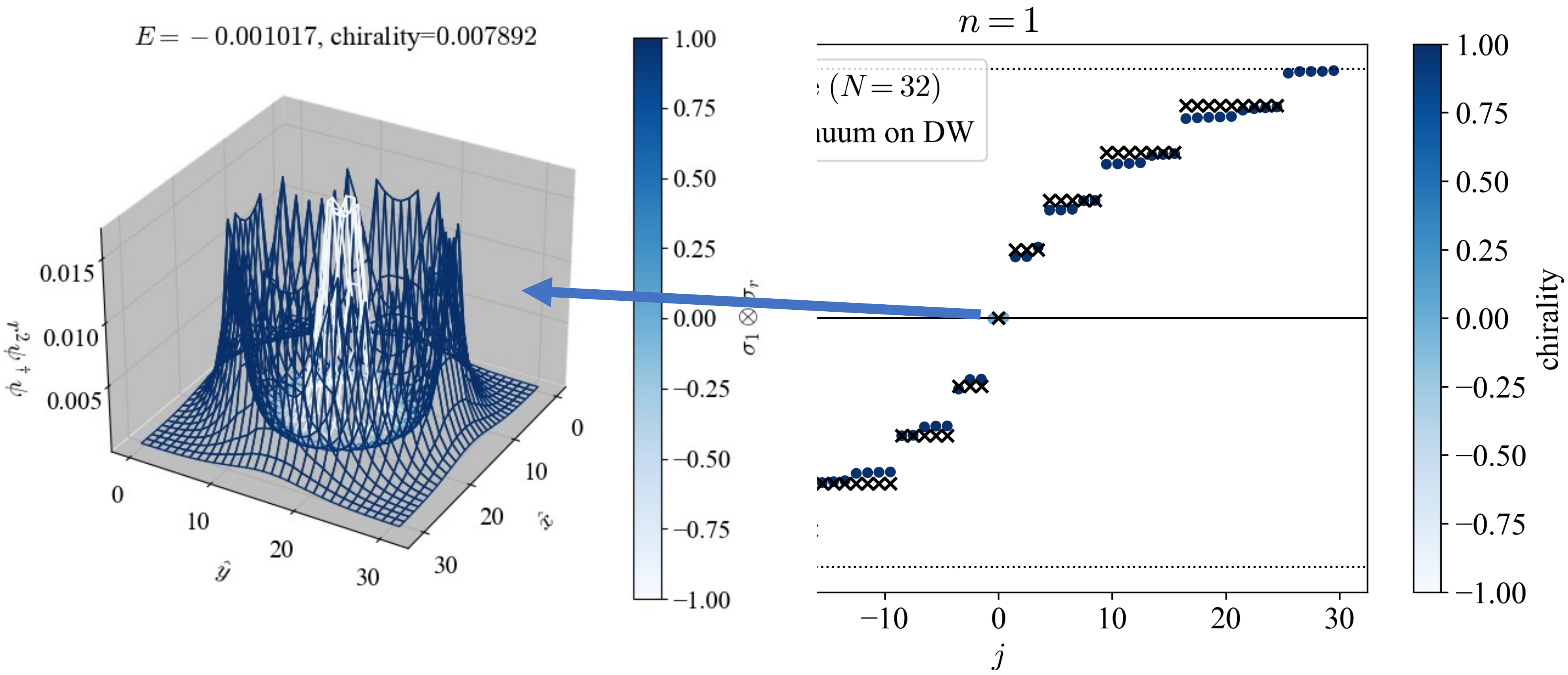
Near-zero eigenvalues/functions

n=-2

We have four zero modes, while the continuum prediction is two.

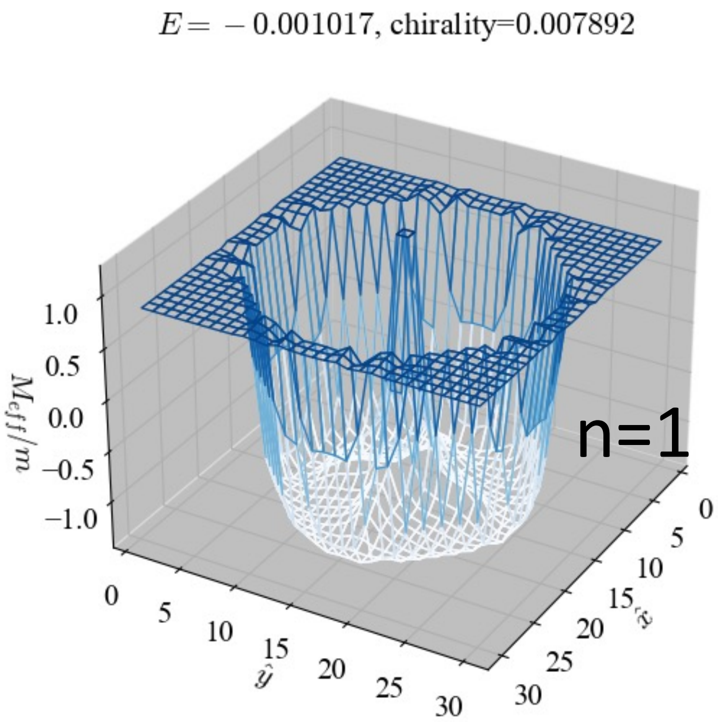


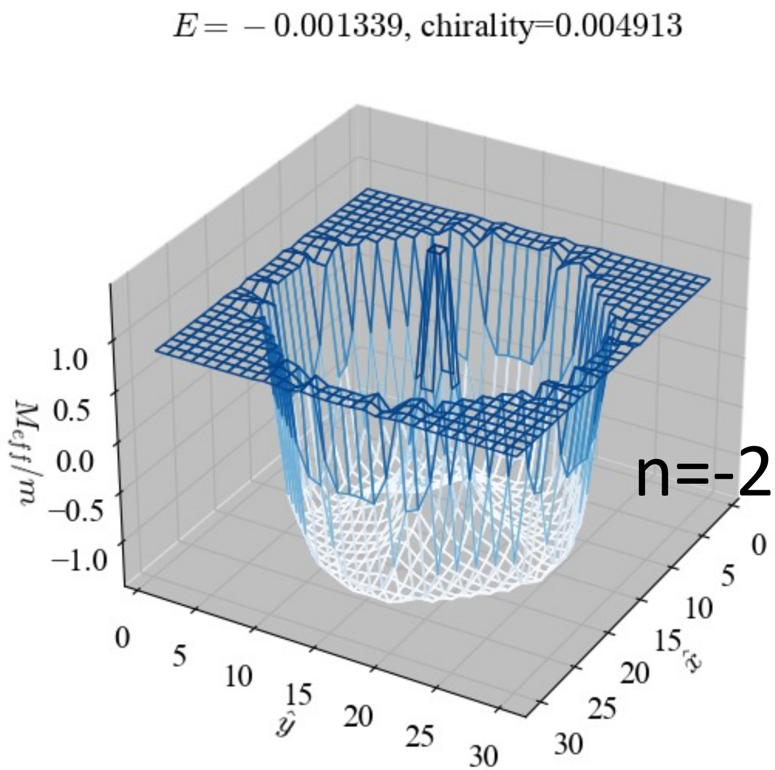
Near-zero eigenvalues/functions



Effective mass at z=(L+1)/2 slice

$$\frac{\phi_0(\boldsymbol{x})^\dagger \left[-\sum_{i=1,2,3} \frac{1}{2}\nabla_i^f \nabla_i^b + m\right] \phi_0(\boldsymbol{x})}{\phi_0(\boldsymbol{x})^\dagger \phi_0(\boldsymbol{x})}$$

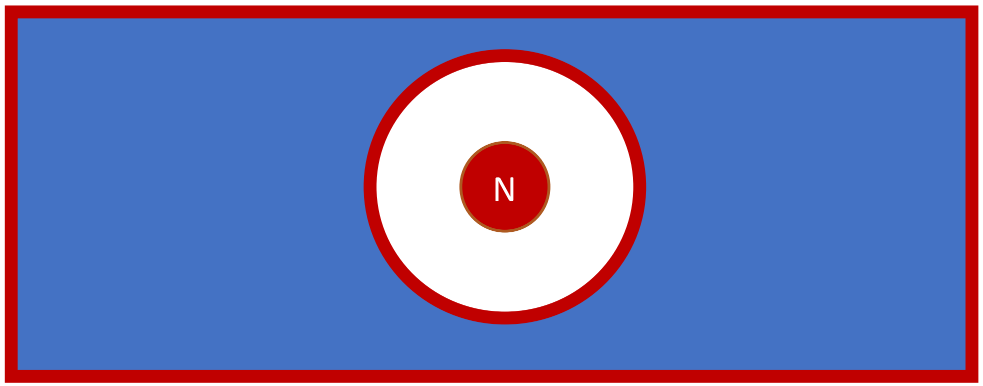


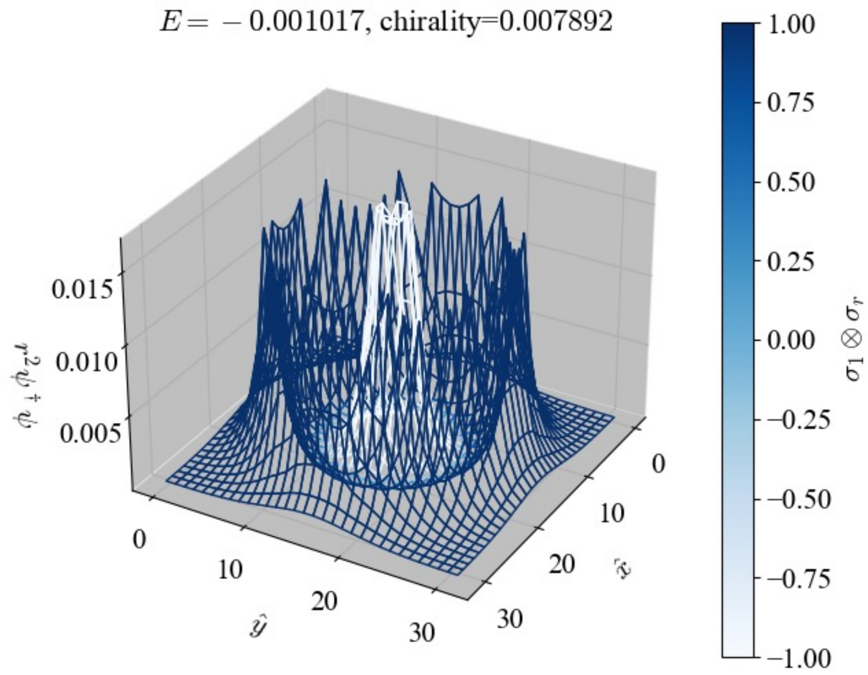


A small spherical domain-wall is created near the monopole, which captures n zero modes.

Gravity of the S^2 domain-wall

A magnetic monopole locally gives a positive mass shift to turn its neighbor into a normal insulator.





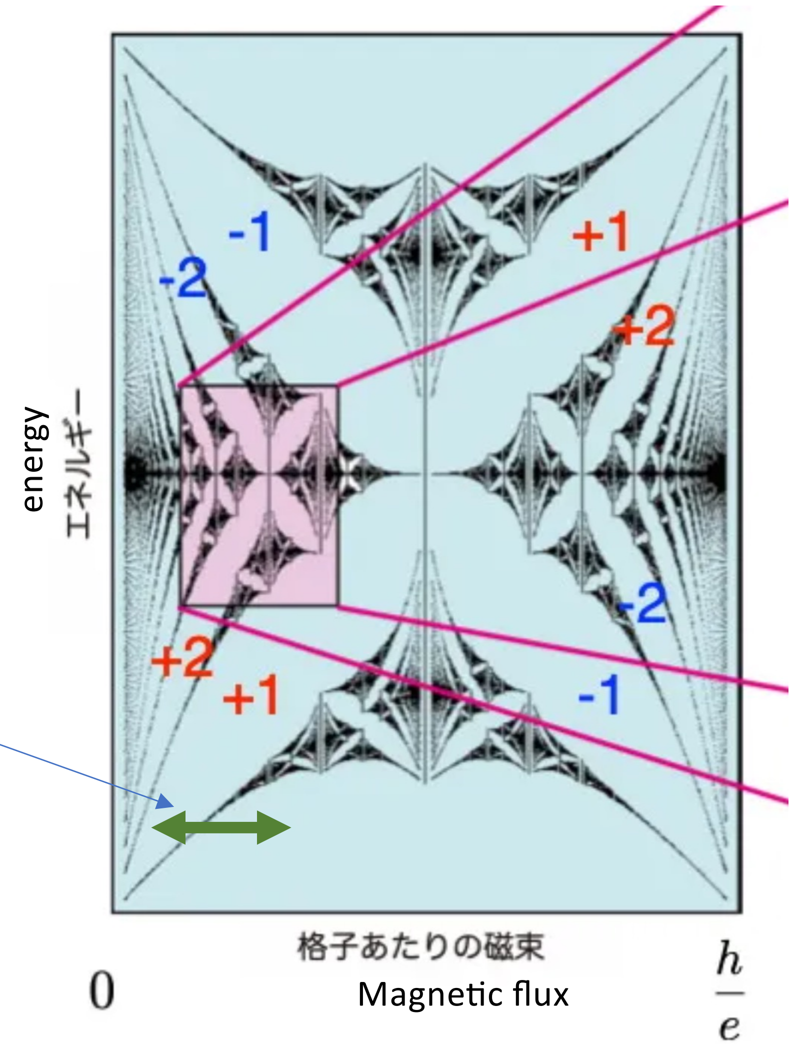
Gravity is so strong that only zero modes are captured.

Is this well-known?

Changing topological phase by Magnetic field is observed in quantum Hall systems.

Our system corresponds to here.

From webpage of Hatugai lab, U. Tsukuba

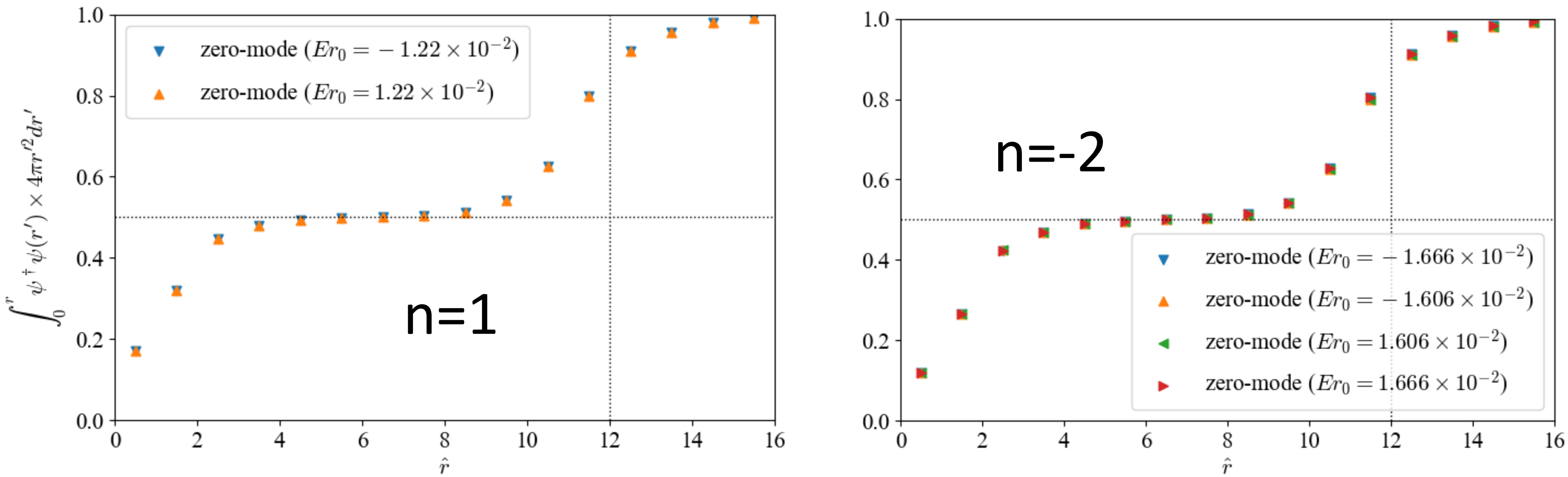


Mixing with surface zero modes makes the amplitude around monopole $\sim$ 1/2

Cumulative distribution

$$C_k(r) = \int_{|\boldsymbol{x}|<r} d^3x \phi_k(\boldsymbol{x})^\dagger \phi_k(\boldsymbol{x})$$

Saturates to 1/2 until r $\sim$ 0.8 r0



This explains why the electric charge is fractional.

Re-interpretation of the Maxwell theory

Original description by Witten

$$\partial_\mu F^{\mu\nu} = -\frac{\theta}{8\pi^2}\partial_\mu \tilde{F}^{\mu\nu} \qquad q_e = \int d^3x \nabla \cdot \boldsymbol{E} = -\frac{\theta}{4\pi^2}\int d^3x \nabla \cdot \boldsymbol{B} = -\frac{\theta q_m}{2\pi}$$

Our reinterpretation = θ term has a defect.

$$\partial_\mu F^{\mu\nu} = -\frac{1}{8\pi^2}\partial_\mu \left[\theta(r)\tilde{F}^{\mu\nu}\right] \quad q_e = \int d^3x \nabla \cdot \boldsymbol{E} = -\frac{1}{4\pi^2}\int d^3x \nabla\theta(r) \cdot \boldsymbol{B}$$

$$= -\frac{1}{4\pi^2}\int d^3x \pi\delta(r - r_1)\boldsymbol{e}_r \cdot \boldsymbol{B} = -\frac{\theta q_m}{2\pi}$$

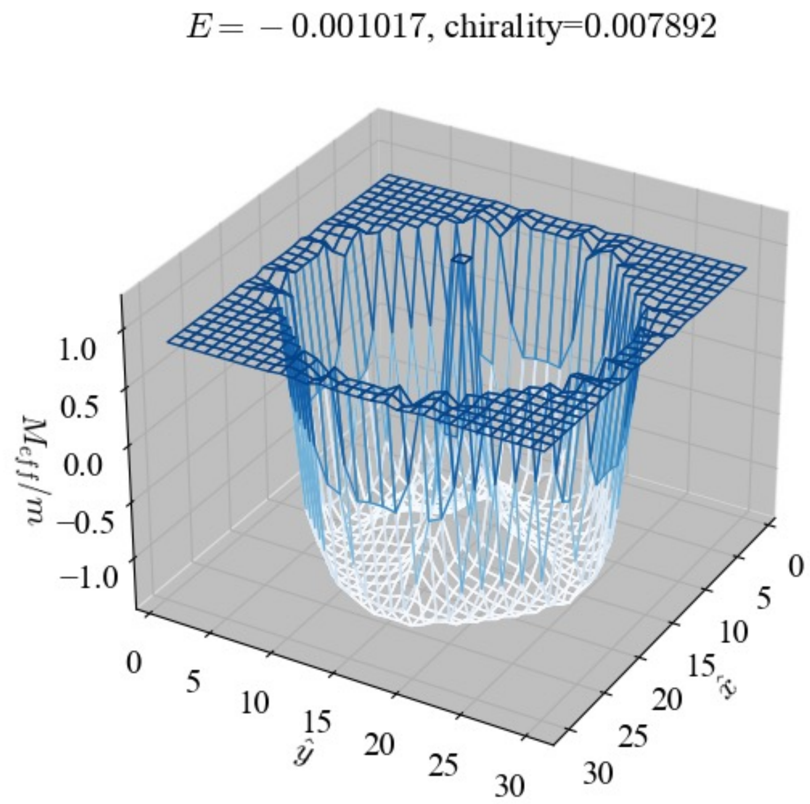
The result is the same!

But our case does not require a true monopole with $\nabla \cdot \boldsymbol{B} \neq 0$

Monopole-massless electron scattering

[Callan 1982, Rubakov 1982]

Our observation suggests that monopole makes fermions very massive.



Dimensional regularization or whatever would fail to keep the chiral symmetry in the presence of the monopole.

Summary

1. Einstein’s equivalence principle and Nash’s theorem tells anything = gravity. “Anything” can be a lattice gauge theory.
2. We find massless edge-localized fermion on curved domain-walls on a square lattice feels gravity through the induced spin connection.
3. Witten effect is explained by the electron zero modes on a small spherical domain-wall created by the strong magnetic field of the monopole.

Backup slides

Creation/annihilation of the domain-walls

So far, we only quantized fermion and treated the gravity and gauge fields as background.

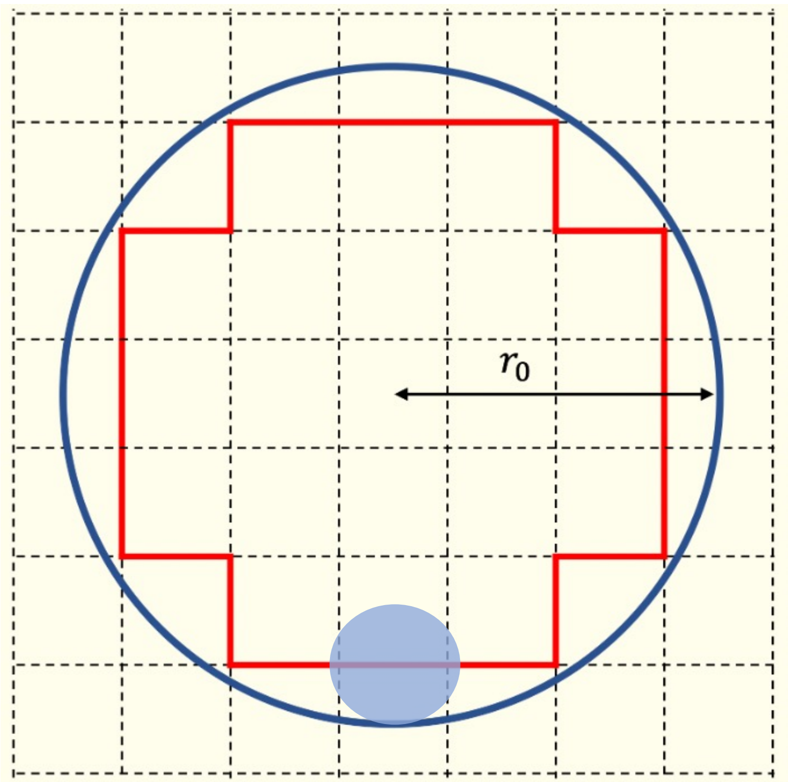
However, we have seen

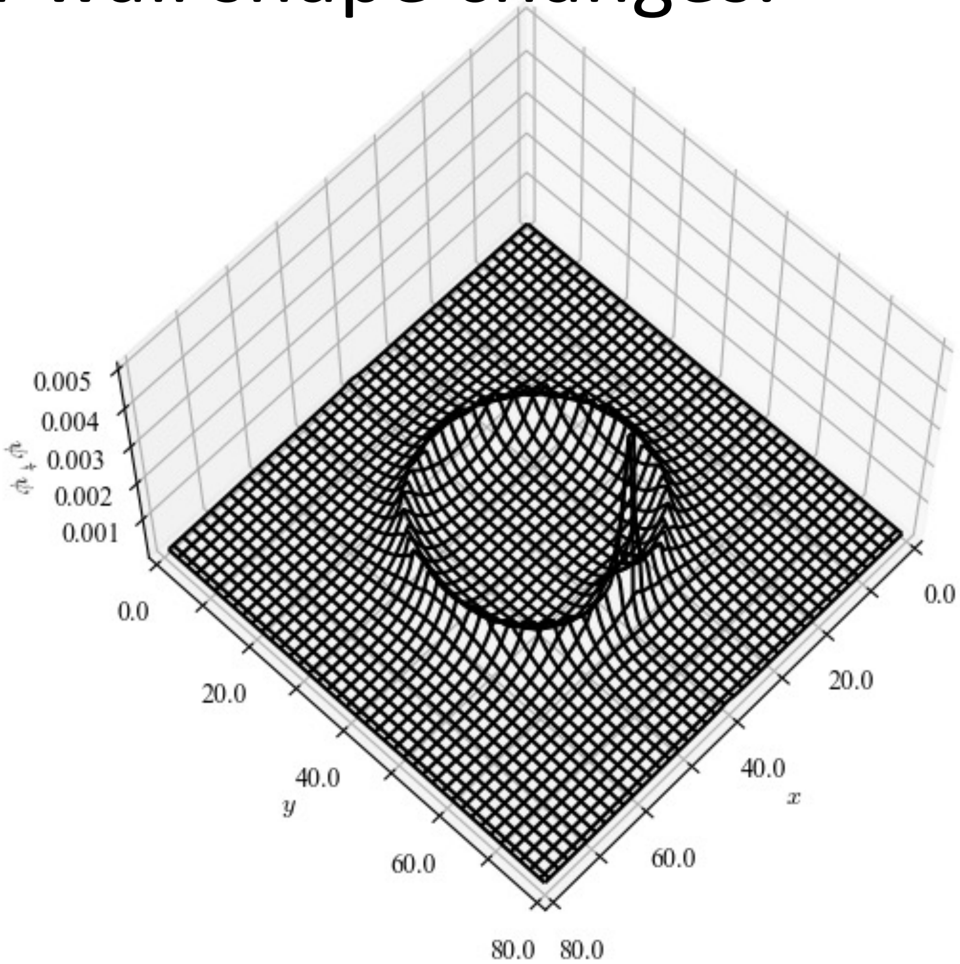
1) creation/ annihilation of DW is automatically implemented,

2) metric – U(1) gauge interaction is encoded as the effective interaction through the Wilson term.

Fluctuation of the domain-walls

Let us move the magnetic flux attached to the domain-wall -> domain-wall shape changes.





Quantum gravity?

$$H = \frac{1}{a}\sigma_3 \left(\sum_{i=1,2} \left[\sigma_i \frac{\nabla_i - \nabla_i^\dagger}{2} + \underbrace{\frac{1}{2}\nabla_i \nabla_i^\dagger \right] + \epsilon a m}_{=: m_{\rm eff}} \right)$$

If we treat the sign of mass term ϵ as a dynamical variable (like Ising spin) and perform a path integral,

$$Z = \sum_{\epsilon(x)=\pm 1} \det(H(\epsilon)) \exp(-S_{\rm Ising}(\epsilon))$$

The effective theory may be "some" quantum gravity described by the induced spin connections (metric and vielbein are effective fields).