Universality of Heavy Operators in Matrix Models

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[2507.21207] with A. Guerrieri and P. Vieira

AdS/CFT Primer

- In a CFT, the natural objects to consider are correlation function of local operators. If it is a holographic theory, like $\mathcal{N} = 4$ Super Yang-Mills, there is a dictionary between the local operators and objects in the bulk.
- First, we have the **Light** operators, whose scaling dimensions are $\Delta \sim N^0$ (for instance, single trace operators like $\operatorname{tr} XXZZ$). These correspond to massless fields and string states in the bulk.
- Then there are heavier operators whose scaling dimension are $\Delta \sim N$. These correspond to branes in the bulk (for instance, $\det Z$)
- When the dimension $\Delta \sim N^2 \sim \frac{1}{G_N}$ – we call these **Huge** operators – there will be strong gravitational backreaction in the bulk and the geometry will no longer be $AdS_5 \times S^5$. (for example $\det Z^N$)

Huge operators

- If we insert a typical huge operator, then by the Eigenstate Thermalization Hypothesis, it is well approximated by a thermal state. The bulk dual of it should be a black-hole spacetime.
- Classically, black-holes have no hair
- So, if we compute $\langle \texttt{Huge Huge Light} \rangle$ correlators for generic huge operators, it should be insensitive to the details of the huge operator.
- We don't know how to compute such correlators in the CFT at strong coupling for generic huge operators.
- We can either (i) look at very special huge operators, where we have some analytic control, or (ii) ask these questions in a simpler toy model

The $\frac{1}{2}$-BPS sector of $\mathcal{N} = 4$ Super Yang-Mills

- In the $\frac{1}{2}$-BPS sector of $\mathcal{N} = 4$ SYM, we can hope to make some progress.
- Local $\frac{1}{2}$-BPS operators are given by gauge invariants made out of a single adjoint scalar like $\operatorname{tr} Z^4, (\operatorname{tr} Z^2)^2$ etc. (I'll denote the six scalars as $X, Y, Z, \bar{X}, \bar{Y}, \bar{Z}$).
- The advantage of working within this sector is that two and three point functions are protected by non-renormalization theorems which imply that they are coupling independent.
- In the free theory, we simply need to do some wick contractions to compute these correlators.

Two point functions of $\frac{1}{2}$-BPS operators

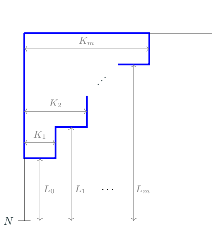
Let O_1 and O_2 be two $\frac{1}{2}$-BPS operators. Their two point function is given by

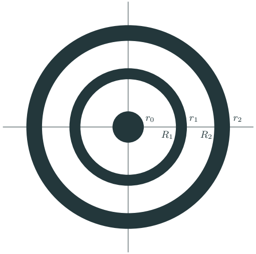
$$\langle O_1 O_2 \rangle = \int dZ\, d\bar{Z}\; e^{-N \frac{1}{\kappa} \operatorname{tr} Z \bar{Z}} O_1(Z) O_2(\bar{Z})$$

where $\kappa = \frac{n_1 . n_2}{|x_1 - x_2|^2}$ is the propagator between the two operators.

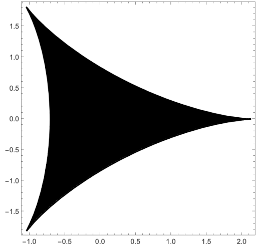
We can evaluate this integral by saddle point at large N.

Schur Polynomials $O(Z) = \chi_R(Z)$



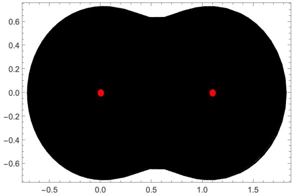


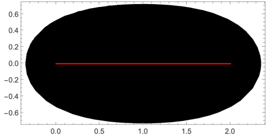
Exponentials $O(Z) = \exp[N \operatorname{tr} V(Z)]$



Coherent States

$$O_J(Z) = \int dU \exp[N \operatorname{tr} JUZU^\dagger]$$





[Kazakov-HM-Vieira'24 and Anempodistov-Holguin-Kazakov-HM '25]

The Hoppe Model

The background densities in the $\frac{1}{2}$-BPS sector are very non-universal. They have many feautures and details that depend on the huge operators.

We're interested in toy models for universality. A simple modification of the matrix integral we considered above is to add a commutator squared interaction term– this is called the Hoppe model

$$\int dZ\, d\bar{Z}\; e^{-N\,\mathrm{tr}\left(Z\bar{Z}+\lambda[Z,\bar{Z}]^2\right)}$$

This is a very simple model, that is solvable at large N without huge operator insertations [Hoppe '89, Kazakov-Kostov-Nekrasov '98].

At strong coupling, we will see that the model exhibits universality for a large class of huge operators.

Hoppe with a Huge Source

Let us write $Z = \frac{X+iY}{\sqrt{2}}$ and rewrite this Complex 1MM as a Hermitian 2MM

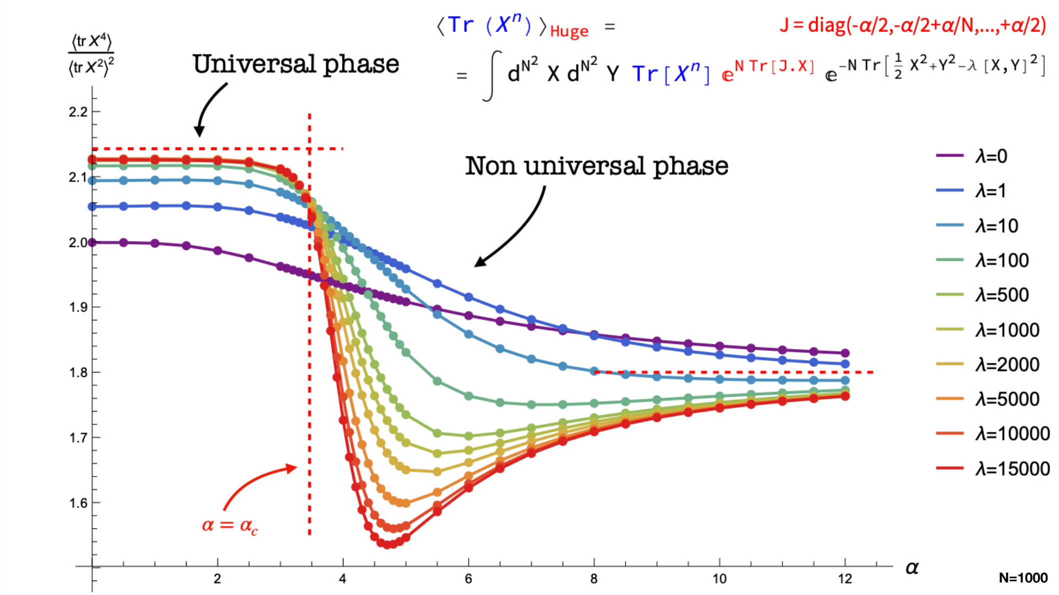
$$\int dX\, dY\ e^{-N\,\mathrm{tr}\left(\frac{X^2}{2}+\frac{Y^2}{2}-\lambda[X,Y]^2\right)}$$

We want to deform this by inserting a huge operator $O_{\mathsf{huge}}(X,Y)$.

As an example, we can add a source for the matrix X

$$O_{\mathsf{huge}}(X) = \exp[N\,\mathrm{tr}\,JX]$$

with a uniform linear source $J = \mathrm{diag}(-\frac{\alpha}{2}, \ldots - \frac{\alpha}{2} + \frac{j\alpha}{N}, \ldots, \frac{\alpha}{2})$



$\langle \mathrm{Tr}\,(X^n)\rangle_{\mathrm{Huge}} =$
J = diag(-α/2,-α/2+α/N,...,+α/2)
$= \int d^{N^2}X\, d^{N^2}Y\, \mathrm{Tr}[X^n]\, e^{N\,\mathrm{Tr}[J.X]}\, e^{-N\,\mathrm{Tr}[\frac{1}{2}X^2+Y^2-\lambda[X,Y]^2]}$
$\frac{\langle \mathrm{tr}\, X^4\rangle}{\langle \mathrm{tr}\, X^2\rangle^2}$
Universal phase
Non universal phase
$\alpha = \alpha_c$
α
λ=0
λ=1
λ=10
λ=100
λ=500
λ=1000
λ=2000
λ=5000
λ=10000
λ=15000
N=1000

Hoppe with a Huge Source

Let us now see how to solve this. We have

$$\int dY \underbrace{\left(dU \prod_i dx_i \, \Delta(x)^2\right)}_{\texttt{from } X=UxU^\dagger} \exp\left[N \operatorname{tr}\left(-\frac{x^2}{2} \underbrace{-\frac{Y^2}{2} + \lambda[x,Y]^2}_{\texttt{gaussian in Y}} + JUxU^\dagger\right)\right]$$

where $\Delta(x) = \det_{1 \le i,j \le N} x_i^{j-1} = \prod_{i<j}(x_i - x_j)$.

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The unitary integral can be done exactly using the HCIZ formula,

$$\int dU\, e^{N \operatorname{tr} AUBU^\dagger} = \frac{\det_{ij} e^{N a_i b_j}}{\Delta(a)\Delta(b)}$$

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$$\int \prod_i dx_i \underbrace{\prod_{i>j} \frac{(x_i - x_j)^2}{1 + 2\lambda(x_i - x_j)^2}}_{\Delta_\lambda(x)^2} e^{-\frac{N}{2}\sum x_i^2} \times \left(\frac{\det_{nm}(e^{N x_n J_m})}{\Delta(x)} = e^{-\frac{N\alpha}{2} \operatorname{tr} X} \frac{\Delta(e^{\alpha x})}{\Delta(x)} \right)$$

Hoppe at Large N and Large λ

At **large** N, we can write down eigenvalue saddle point equations

$$-x + 2\fint dy\, \frac{\rho(y)}{(x-y)(1+2\lambda(x-y)^2)} + \int \rho(y)\left(-\frac{\alpha}{2} + \frac{\alpha e^{\alpha x}}{e^{\alpha x} - e^{\alpha y}} - \frac{1}{x-y}\right) = 0$$

At **strong coupling**, the principal value kernel above is the derivative of a δ-function,

$$2\fint dy\, \frac{\rho(y)}{(x-y)(1+2\lambda(x-y)^2)} = -\frac{2\pi}{\sqrt{2\lambda}}\rho'(x) + \mathcal{O}\left(\frac{1}{\lambda}\right)$$

Let us drop the source by setting $\alpha = 0$. Then, we have

$$\rho'(x) = -\frac{\sqrt{2\lambda}}{2\pi}\, x$$

$$\Rightarrow \rho(x) = \frac{3}{4L^3}(L^2 - x^2), \quad L = L_{\texttt{vacuum}} \equiv \frac{(3\pi)^{\frac{1}{3}}}{(2\lambda)^{\frac{1}{6}}}$$

In particular, this gives us $\frac{\langle \operatorname{tr} X^4\rangle}{\langle \operatorname{tr} X^2\rangle^2} = \frac{15}{7}$, which corresponds to the plateau in the plot above.

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The strong coupling density for the vacuum was also studied in [Berenstein-Hanada-Hartnoll '08]

Hoppe with a Huge Source at Strong Coupling

$$2\fint dy\,\frac{\rho(y)}{(x-y)(1+2\lambda(x-y)^2)}+\int\rho(y)\left(-\frac{\alpha}{2}+\frac{\alpha e^{\alpha x}}{e^{\alpha x}-e^{\alpha y}}-\frac{1}{x-y}\right)=x$$

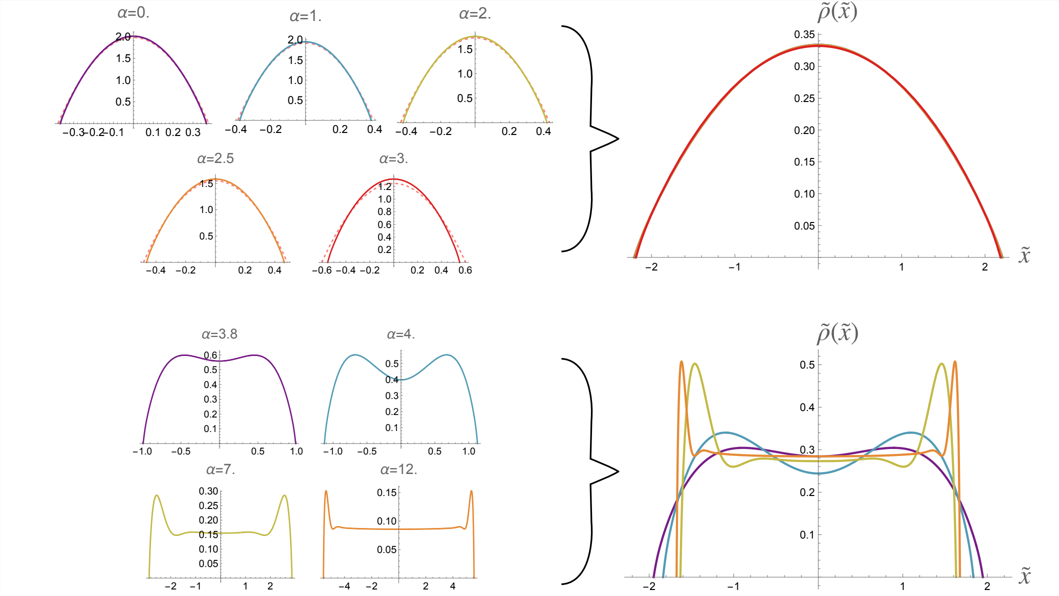
Now, let us plug in a parabolic ansatz for $\rho(x)$

$$\fint_{-L}^{L}dy\left(\frac{2}{(x-y)(1+2\lambda(x-y)^2)}-\frac{\alpha}{2}+\frac{\alpha e^{\alpha x}}{e^{\alpha x}-e^{\alpha y}}-\frac{1}{x-y}\right)\frac{L^2-y^2}{4L^3/3}=$$
$$\left(\frac{\alpha^2}{12}+\frac{L_{\text{vacuum}}^3}{L^3}\right)x+O\left(\frac{1}{\sqrt{\lambda}}\right)$$

So, we solve the SPEs if

$$L(\alpha)=L_{\text{vacuum}}\times\frac{1}{\left(1-\frac{\alpha^2}{12}\right)^{\frac{1}{3}}}\propto\lambda^{-\frac{1}{6}}$$

So, the shape of the density and all normalized moments are universal for $\alpha<\sqrt{12}$



α=0.
α=1.
α=2.
α=2.5
α=3.
α=3.8
α=4.
α=7.
α=12.
ρ̃(x̃)
x̃

More General Huge Operators

For a more general source $O_{\mathsf{huge}} = \exp[N \operatorname{tr} JX]$, we need to solve the following SPEs

$$-x + 2 \fint dy \, \frac{\rho(y)}{(x-y)(1+2\lambda(x-y)^2)} + \frac{\partial}{\partial x} \log I(\{x_i\}, \{j_k\}) = 0$$

where $I(\{a_i\}, \{b_i\}) = \int dU \exp[N \operatorname{tr} U A U^\dagger B]$.

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In order to solve this, we can use a nice equivalence between large N unitary HCIZ integrals and a 1D fluid flow [Matytsin '93]

$$\frac{1}{N^2} \log I(a,b) \approx \text{Action of} \left(\rho(a) \; \boxed{\longrightarrow \quad \frac{\partial f}{\partial t} + f \frac{\partial f}{\partial x} = 0 \quad \longrightarrow} \; \sigma(b) \right)$$

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Plugging this into the SPE, we obtain

$$\fint dy \, \rho(y, t=0) \left(\frac{1}{x-y} - \frac{4\lambda(x-y)}{1+2\lambda(x-y)^2} \right) + v(x, t=0) = 0$$

We know the density of the source parameters $\{j_i\}$ at $t=1$ and we have this non-trivial boundary condition at $t=0$.

More General Huge Operators

This fluid is *integrable* and we have the following conservation laws [Kazakov-HM-Vieira '24, Matytsin '93]

$$\frac{1}{m+1}\oint \frac{dx}{2\pi i}\, x^n G_+(x)^{m+1} = \frac{-1}{n+1}\oint \frac{dx}{2\pi i}\, x^m G_-(x)^{n+1}$$

where

$$G_+(x) = x + v(x,0) + i\pi\rho(x,0)\ , \qquad G_-(x) = x - v(x,1) - i\pi\rho(x,1)$$

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At the end of the day, we find that there still exists a universal phase, where the distribution is

$$\rho(x) = \frac{L^2 - (x-x_0)^2}{4L^3/3}$$

where

$$x_0 = \operatorname{tr} J\ , \qquad L = L_{\text{vacuum}} \times \frac{1}{\left(1 - \operatorname{tr} J^2 + (\operatorname{tr} J)^2\right)^{\frac{1}{3}}}$$

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We can also consider more general things like Schur polynomials $\chi_R(X)$, $\exp[N \operatorname{tr} V(X)]$, etc. In the paper, we show that for all these cases, there is a universal parabola regime.

Light Probes involving both X and Y

So far, we have been studying eigenvalue density of only X. What about probes with both X's and Y's?

For instance, we would like to evaluate things like

$$\int dX\, dY\; e^{-S_{\text{hoppe}}(X,Y)}\, O_{\mathsf{huge}}(X)\ \operatorname{tr}\left(X^{a_1}Y^{b_1}X^{a_2}Y^{b_2}\dots X^{a_n}Y^{b_n}\right)$$

At strong coupling, we can show that the matrices commute at leading order. So,

$$\begin{aligned}\langle \operatorname{tr}\left(X^{a_1}Y^{b_1}X^{a_2}Y^{b_2}\dots X^{a_n}Y^{b_n}\right)\rangle &= \langle \operatorname{tr} X^{\Sigma a_i}Y^{\Sigma b_i}\rangle \\ &= \int \rho(x,y) x^{\Sigma a_i} y^{\Sigma b_i}\end{aligned}$$

Once we know the density $\rho(x)$, we can uplift it to a 2D density as follows

$$\rho(x,y) = \frac{2}{\pi}\left(\frac{\lambda}{8\pi^2}\right)^{\frac{1}{4}}\sqrt{\rho(x) - \left(\frac{\lambda}{8\pi^2}\right)^{\frac{1}{2}} y^2}$$

Light Probes involving both X and Y

Consider for instance

$$\left\langle \mathrm{Tr}(X^{n-k} Y X^k Y) \right\rangle = \frac{1}{\mathcal{Z}} \int dx_1 \ldots dx_N \sum_{i,j} \frac{x_i^{n-k} x_j^k}{1 + 2\lambda(x_i - x_j)^2} \Delta_\lambda(x)^2 \exp\left(- \frac{N}{2} \sum_{i=1}^{N} x_i^2 \right) O_{\text{huge}}(x)$$

$$= \int dx \int dx' \frac{x^{n-k} x'^k \rho(x) \rho(x')}{1 + 2\lambda(x - x')^2}$$

At strong coupling, the denominator becomes a delta function $\delta(x - x')$

$$\left\langle \mathrm{Tr}(X^{n-k} Y X^k Y) \right\rangle \simeq \frac{\pi}{\sqrt{2\lambda}} \int dx\, x^n \rho(x)^2$$

So the order of X and Y does not matter. In a similar way, one can show that

$$\langle \mathrm{Tr}(X^n Y^m) \rangle = C_{m/2} \left(\frac{\pi}{\sqrt{2\lambda}} \right)^{m/2} \int dx\; x^n \rho(x)^{1+m/2} = \int dx\, dy\, x^n y^m \rho(x, y)$$

where

$$\rho(x, y) = \frac{2}{\pi} \left(\frac{\lambda}{8\pi^2} \right)^{\frac{1}{4}} \sqrt{\rho(x) - \left(\frac{\lambda}{8\pi^2} \right)^{\frac{1}{2}} y^2}$$

Sources for both X and Y

So far, we have seen universality of heavy operators which involve only the X matrix. We would like to verify that this holds for heavy operators involving both matrices. For instance with

$$O_{\text{huge}}(X,Y) = \exp[N\,\mathrm{tr}\,(J_X X + J_Y Y)]$$

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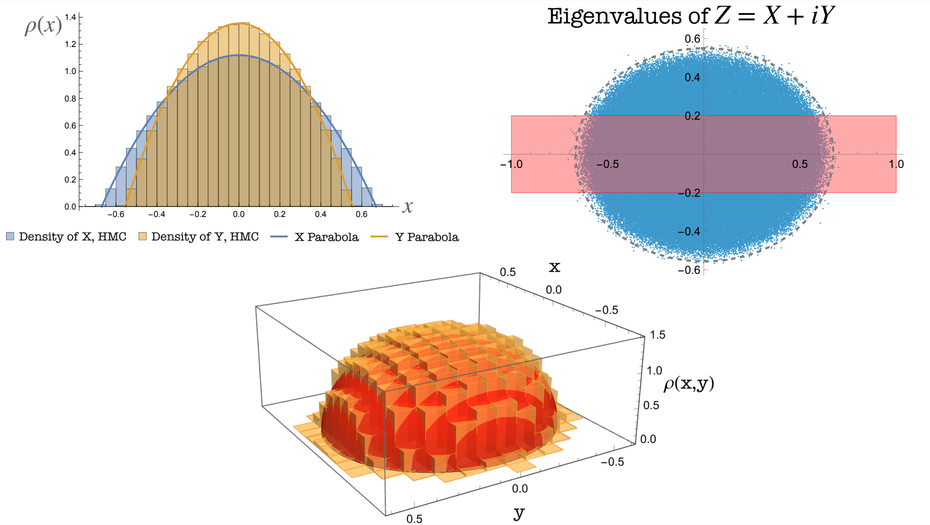
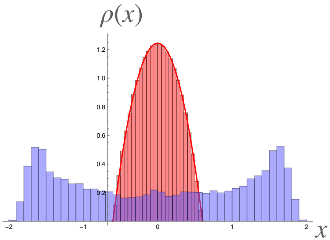
We have no analytic predictions, but if the distributions are parabolas, we can find their support by measuring the moments in HMC and noting that $\langle \operatorname{tr} X^2 \rangle = \frac{L_X^2}{5}$ and $\langle \operatorname{tr} Y^2 \rangle = \frac{L_Y^2}{5}$

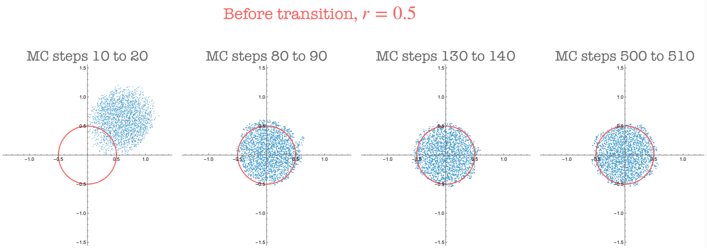
For instance, we can take the eigenvalues of J_X and J_Y to be uniformly distributed inside a rectangle,

$$J_X = \mathrm{diag}(j_X)\ , \qquad J_Y = U^\dagger \mathrm{diag}(j_Y) U\ , \qquad (j_X, j_Y) \in \left[-\frac{\alpha}{2}, \frac{\alpha}{2}\right] \times \left[-\frac{\beta}{2}, \frac{\beta}{2}\right]$$

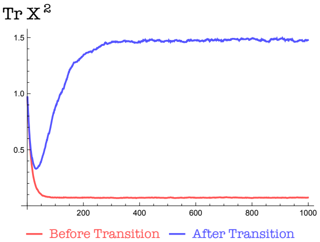




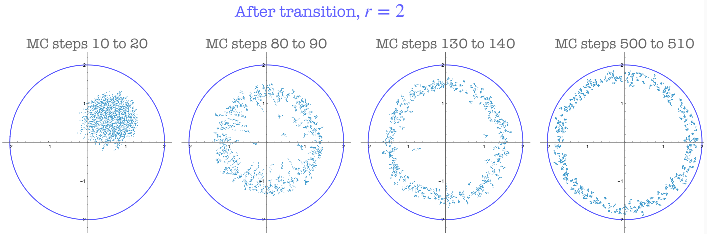
ρ(x)
x



Before transition, r = 0.5
MC steps 10 to 20
MC steps 80 to 90
MC steps 130 to 140
MC steps 500 to 510



Tr X²
Before Transition
After Transition



After transition, r = 2
MC steps 10 to 20
MC steps 80 to 90
MC steps 130 to 140
MC steps 500 to 510

Abelianization

Not every huge operator has a black-hole phase. For instance, multi-trace huge operator like $(\operatorname{tr} X^p)^{\alpha N^2/p}$ or Schur polynomials with a single very long row have very different physics: With these insertions one eigenvalue decouples from the others such that we have a symmetry breaking $U(N) \to U(1) \times U(N-1)$. This is what we call *Abelianization*

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To illustrate this, consider a 1MM with a polynomial potential $V(X) = \sum_{m=2}^{q} \frac{t_m}{m} \operatorname{tr} X^m$ with the huge operator insertion $(\operatorname{tr} X^p)^{\alpha N^2/p}$. The SPEs are

$$-t_2 x_i - \cdots - t_q x_i^{q-1} + \alpha N \frac{x_i^{p-1}}{x_1^p + \sum_{j=2}^{N} x_j^p} + \frac{1}{N} \sum_{j \neq i} \frac{1}{x_i - x_j} = 0 \,, \qquad i = 1, \ldots, N \,.$$

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Let us consider the latter where $x_1 \gg 1$ and the other x_i are $O(1)$. Then,

$$i = 1: \qquad -t_q x_1^{q-1} + \frac{\alpha N}{x_1} \;\Rightarrow\; x_1 = \left(\frac{\alpha N}{t_q}\right)^{1/q}$$

$$i = 2, \ldots, N: \qquad -t_2 x_i - \cdots - t_q x_i^{q-1} + \frac{1}{N} \sum_{j \neq i,1} \frac{1}{x_i - x_j} = 0 \,,$$

Abelianization

For the Abelianization saddle, $x_1 \sim N^{1/q}$. The contribution to the effective action from the insertion of the huge operator scales as

$$\frac{\alpha N^2}{p} \log \mathrm{Tr} X^p \simeq \frac{\alpha N^2}{p} \log \frac{x_1^p}{N} \sim \alpha \left(\frac{p}{q} - 1 \right) N^2 \log N$$

When $p > q$, i.e. when the huge operator's power is bigger than the degree of the potential, the Abelian saddle has a logarithmic divergence. Therefore, it dominates over the usual saddle whose action scales as $O(N^2)$.

While we illustrated this for the 1MM, the same phenomenon occurs in multi-matrix models, including the Hoppe model.

Comments about Higher Dimensional Matrix Models

- There is an interesting three matrix model which reduces to the Hoppe model when one of the matrices is integrated out

$$\int dX\, dY\, dZ\, \exp\left[-N\,\mathrm{tr}\left(\tfrac{1}{2}X^2 + \tfrac{1}{2}Y^2 + \tfrac{1}{2}Z^2 + \sqrt{2\lambda}Z[X,Y]\right)\right]$$

- At strong coupling all three matrices commute with each other and we can define a joint density $\rho(x,y,z)$ which turns out to be a uniform distribution inside a ball,

$$\rho(x,y,z) = \frac{3}{4\pi}\frac{1}{L^3}\,, \qquad x^2 + y^2 + z^2 \le L^2 \qquad L = L_{\mathrm{vacuum}} = \frac{(3\pi)^{\frac{1}{3}}}{(2\lambda)^{\frac{1}{6}}}$$

- If we insert a heavy operator like $O_{\mathrm{huge}}(X)$ into this integral, the density $\rho(x,y,z)$ will still be the uniform distribution but now inside a squashed ball. This is a simple generalization of the universality we find to a higher dimensions.

Higher Dimensional Matrix Models

- What about bosonic Yang-Mills matrix models with $D > 2$?

$$\int DX_\mu \ \exp\left[N\,\mathrm{tr}\left(-\frac{X_\mu X^\mu}{2} + \frac{\lambda}{2}[X_\mu, X_\nu][X^\mu, X^\nu]\right)\right]$$

- The behaviour of this model is very different from the $D = 2$ case. The matrices no longer commute even at strong coupling! All matrix elements scale as $\lambda^{-\frac{1}{4}}$ and there is no separation of scales between eigenvalues and off-diagonal modes.
- More interesting case is the polarized IKKT matrix integral (Komasu-Martina-Penedones-Vuigner-Zhao '24, Hartnoll-Liu '24). The fermions increase the effective repulsions between the bosonic modes and therefore, this model (with the added mass term) should also be commuting. Does it exhibit huge operator universality?

Conclusions and Future Directions

- The Hoppe model at $\lambda = 0$ shows up as the matrix integral which computes $\frac{1}{2}$-BPS correlators. Here, different huge operators give different densities and probe correlators. This is what we expect from LLM geometries.

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- Is there a holographic dual for the Hoppe model? Perhaps in the topological B-model?