# On the landscape of minimal strings

**Victor A. Rodriguez**
UC Santa Barbara

Matrix Model for Superstring/M-theory, YITP workshop ✧ Dec 1, 2025

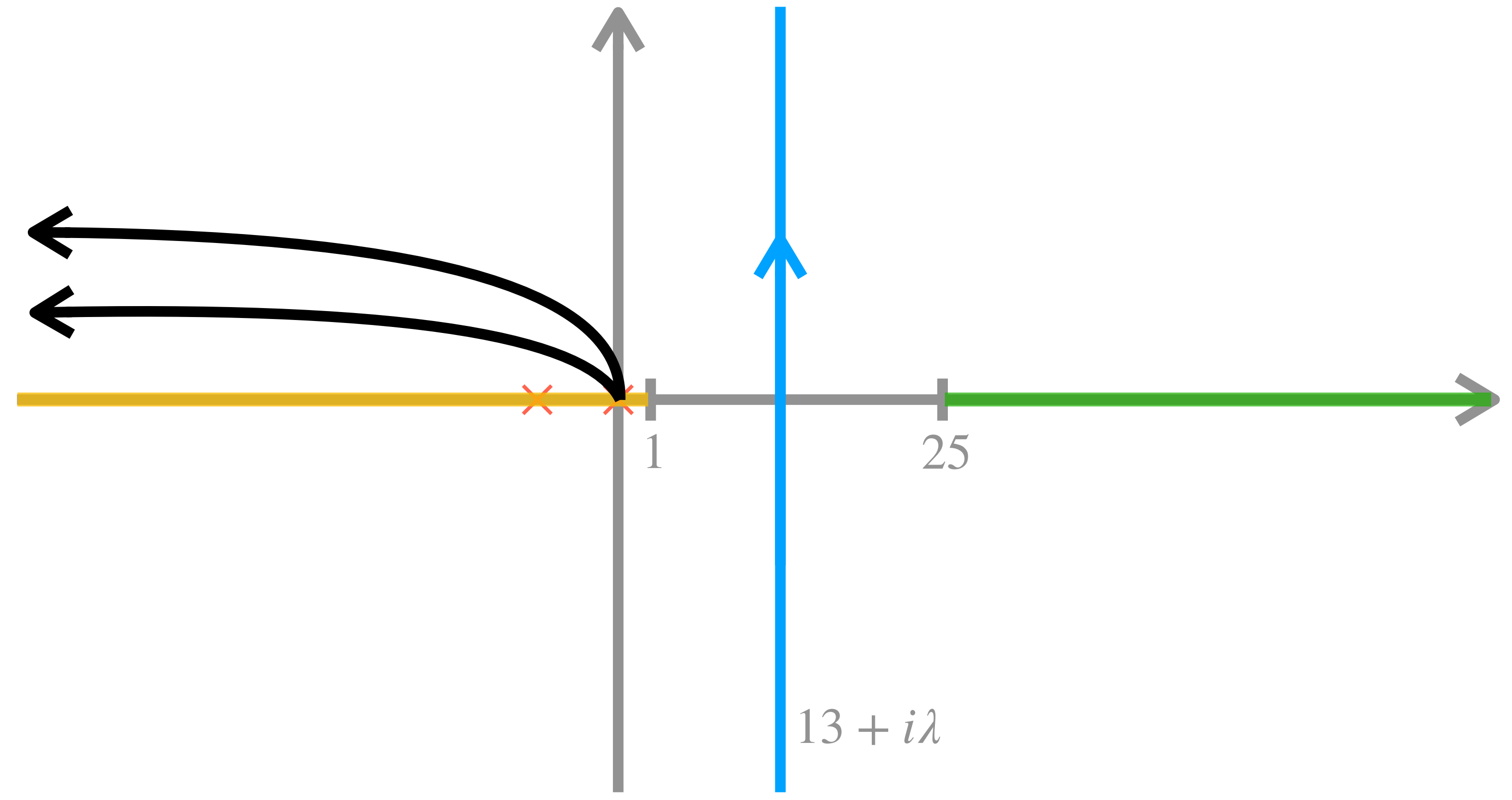
based on [2410.09179 + wip]
w/ S. **Collier**, L. **Eberhardt**, B. **Mühlmann**
and M. **Usatyuk**, Z. **Wang**

UNIVERSITY OF CALIFORNIA President's Postdoctoral Fellowship Program

# Main messages for today

1. Overview of the minimal string landscape

    - known + unknown




2. Learn what are the Virasoro minimal string & Complex Liouville string

    - Very rich physics: stringy and gravitational

3. Application: D-instantons, and the gluing formula

    - In general difficult, and require SFT. Not today! Maybe apply to higher d

# Outline

- Intro to low-d strings + main setting w/ Collier-Eberhard-Muhlmann-**VAR**
- 3 main classes of examples
  - The Virasoro minimal string (VMS) [2309.10846]
  - The complex Liouville string ($\mathbb{C}$LS) [2409.17246, 2409.18759, 2410.07345, 2410.09179]
  - ADE minimal string (A/D/E MS)
- Worldsheet boundaries and the gluing formula [2410.09179] + WIP Collier-Eberhardt-**VAR**
- Discussion

# Setting: Low-d string theory

Theoretical laboratories for fundamental string theory and quantum gravity.

Simpler than higher-dimensional strings (no susy, no $\infty$ tower of states, …)
but still exhibit

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but still exhibit

- holographic and stringy dualities (opportunity to derive them!)
  [reviews: Klebanov; Ginsparg Moore; Polchinski; Nakayama; Anninos Mühlmann; …]
- non-perturbative effects (D-instantons)
  [Balthazar **VAR** Yin; Sen; Eniceicu Mahajan Murdia Sen;…]
- time-dependent phenomena (rolling tachyons)
  [review: Sen ’04; **VAR;** Balthazar Chu Kutasov; Cho Mazel Yin; Sen; Itzhaki …]

# Worldsheet string theory

- The 2d worldsheet theory

  (bosonic, no susy)

Rules of the game: $\left( \text{2d CFT} \right) + \text{b,c ghosts}$ $\qquad c_{\text{total}} = 0$

$c = 26$ $\qquad c_{\text{gh}} = -26$

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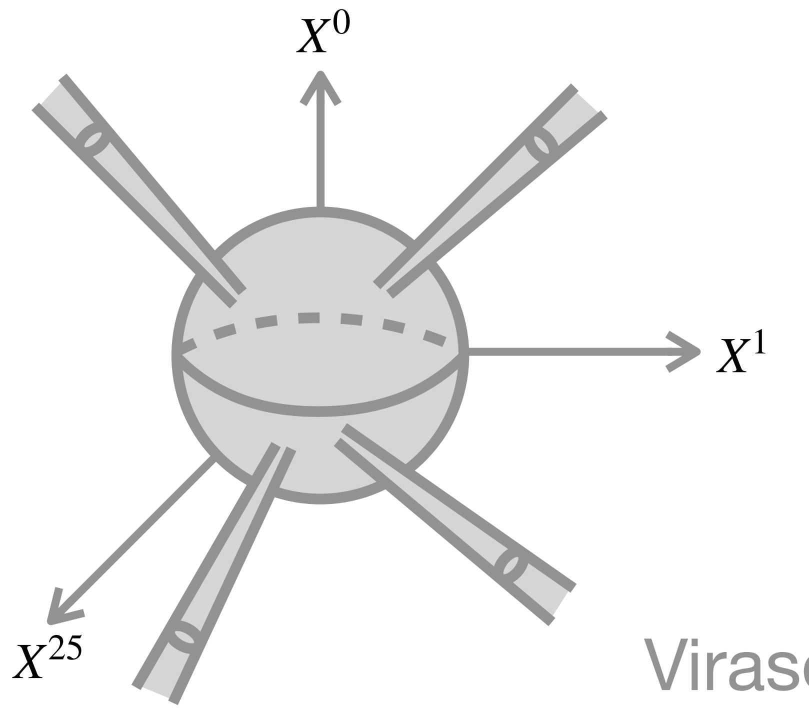
- 26 free bosons
  ($c = 1$ each)

"26d bosonic string theory"




Virasoro-Shapiro amplitude

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"26d bosonic string theory"

( + free fermions → 10d superstrings )
insanely rich landscape
stringy dualities, holographic duality, …

# Worldsheet string theory

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$c = 26 \qquad c_{\text{gh}} = -26$

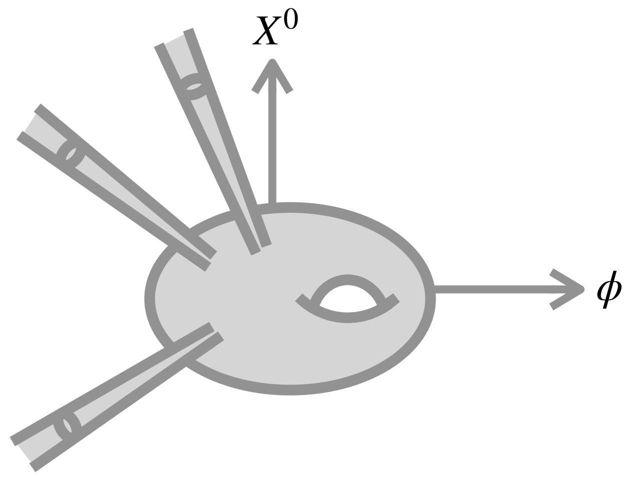
- timelike free boson ($c = 1$) + Liouville CFT ($c = 25$)

more interesting CFT!

[…, Balthazar-**VAR**-Yin 17-22']

"$c = 1$, or 2d string theory"

(dual to Matrix QM)



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[…, Balthazar-**VAR**-Yin 17-22']

- timelike free boson ($c = 1$) + Liouville CFT ($c = 25$) — "$c = 1$, or 2d string theory"
- $(p', p)$ min. model ($c < 1$) + Liouville CFT ($c > 25$) — "minimal" string theory (2d quantum gravity)

# Worldsheet string theory

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Low-d strings:

- timelike free boson ($c = 1$) + Liouville CFT ($c = 25$) — "$c = 1$, or 2d string theory"
- $(p', p)$ min. model ($c < 1$) + Liouville CFT ($c > 25$) — "minimal" string theory (2d quantum gravity)

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[VAR 23']
[Collier-Eberhardt-Mühlmann-VAR 23']

Newer ones:

- timelike Liouv. CFT ($c \leq 1$) + Liouville CFT ($c \geq 25$)   "Virasoro minimal string"

[Collier-Eberhardt-Mühlmann-VAR 24']

- Liouville CFT $c$ + Liouville CFT $c^*$   "complex Liouville string"
  (2d quantum gravity)

*insight:
slightly non-unitary

# “Minimal” strings

- Main paradigm: holographic duality with double-scaled matrix integrals

| Worldsheet CFT: | “Matter” CFT | + | Liouville CFT | + | b, c ghosts |
|---|---|---|---|---|---|
| | $c_m \leq 1$ | | $c_L = 26 - c_m$ | | $c_{\text{gh}} = -26$ |

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$$\Updownarrow$$

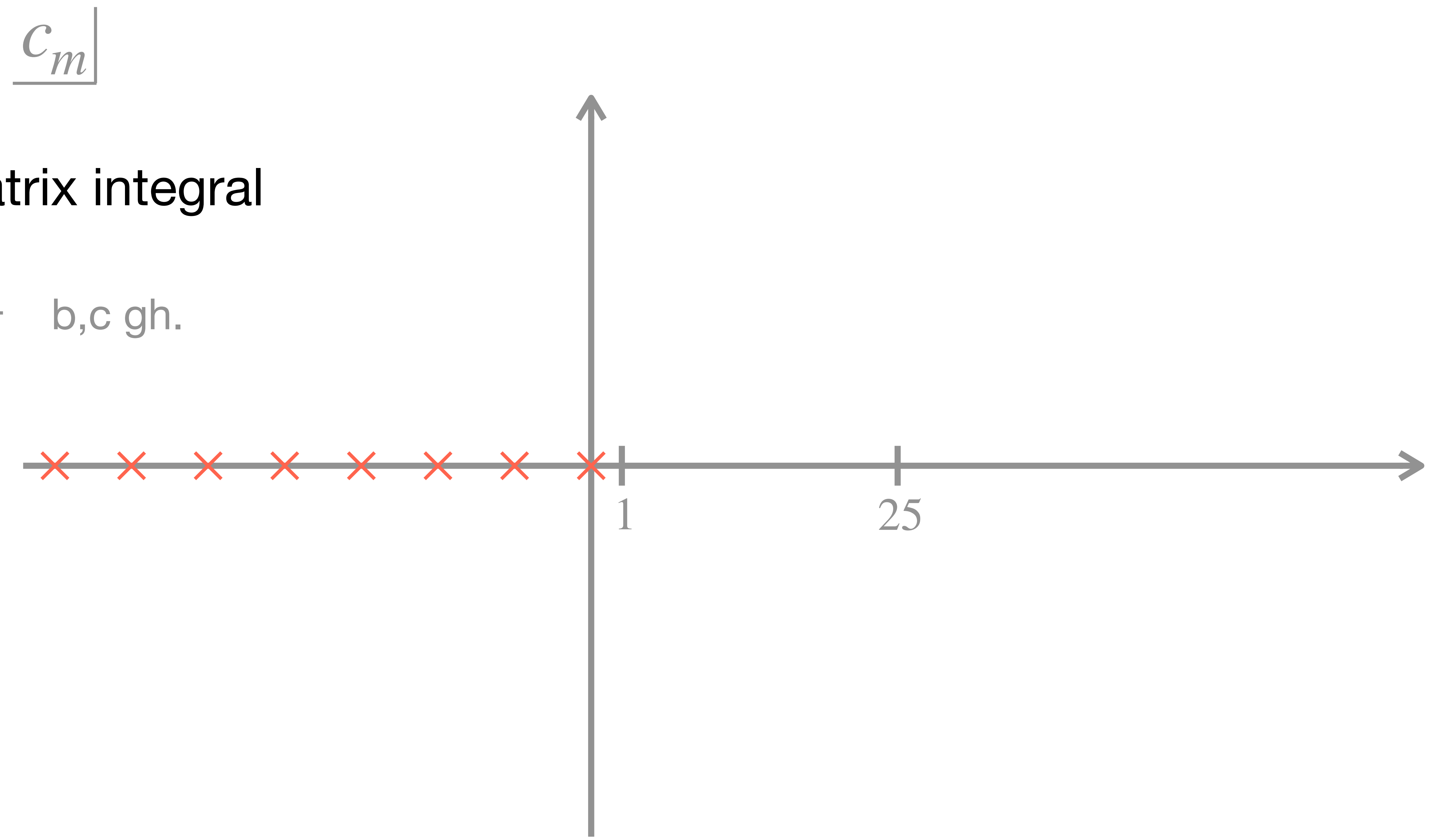
double-scaled matrix integral:

$$\int_{\mathbb{R}^{N\times N}} \mathrm{d}M \; \mathrm{e}^{-NV(M)}$$

# “Minimal” strings

$(2,p)$ minimal string ↔ **1**-matrix integral

$(2,p)$ min. model + Liouv. CFT + b,c gh.



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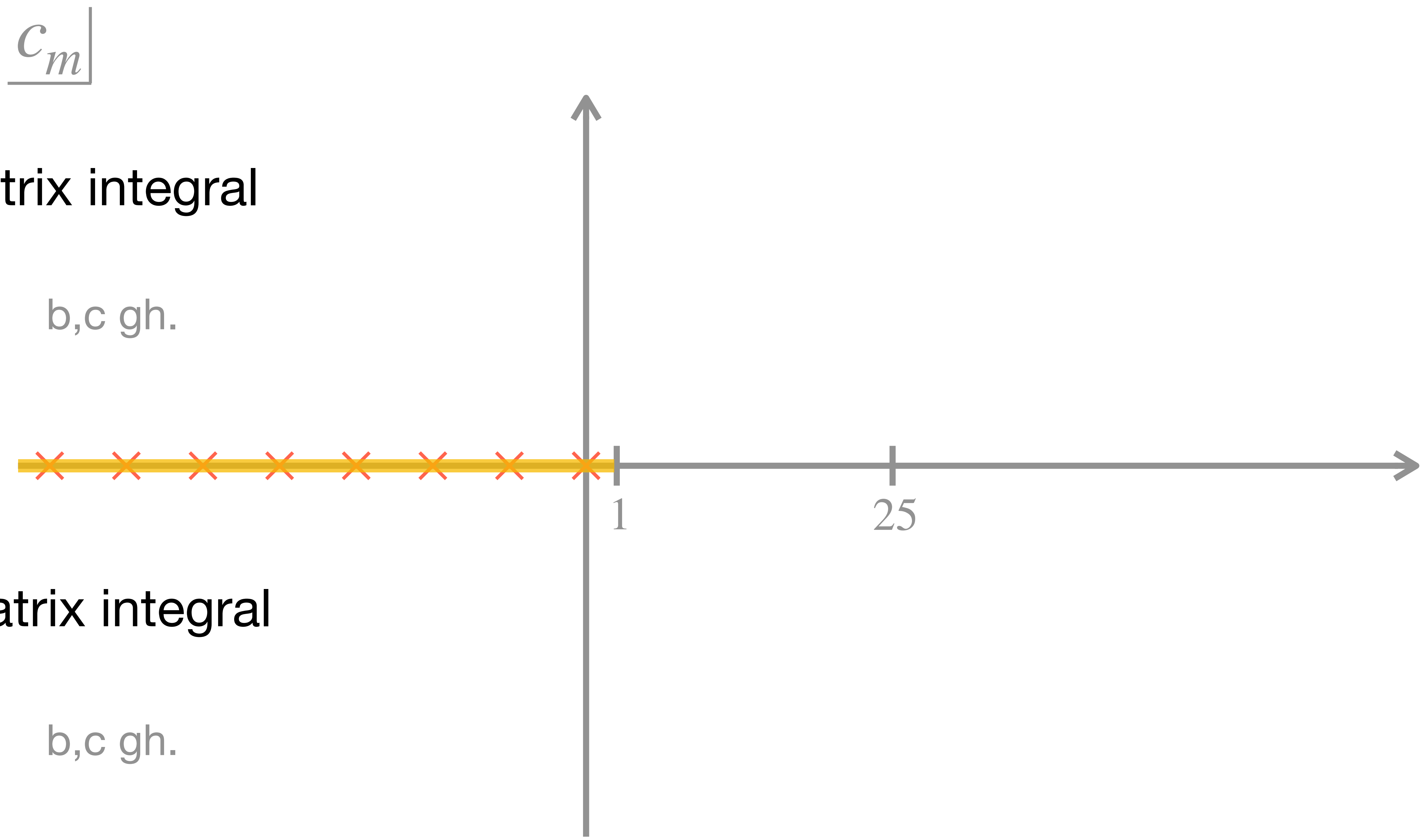
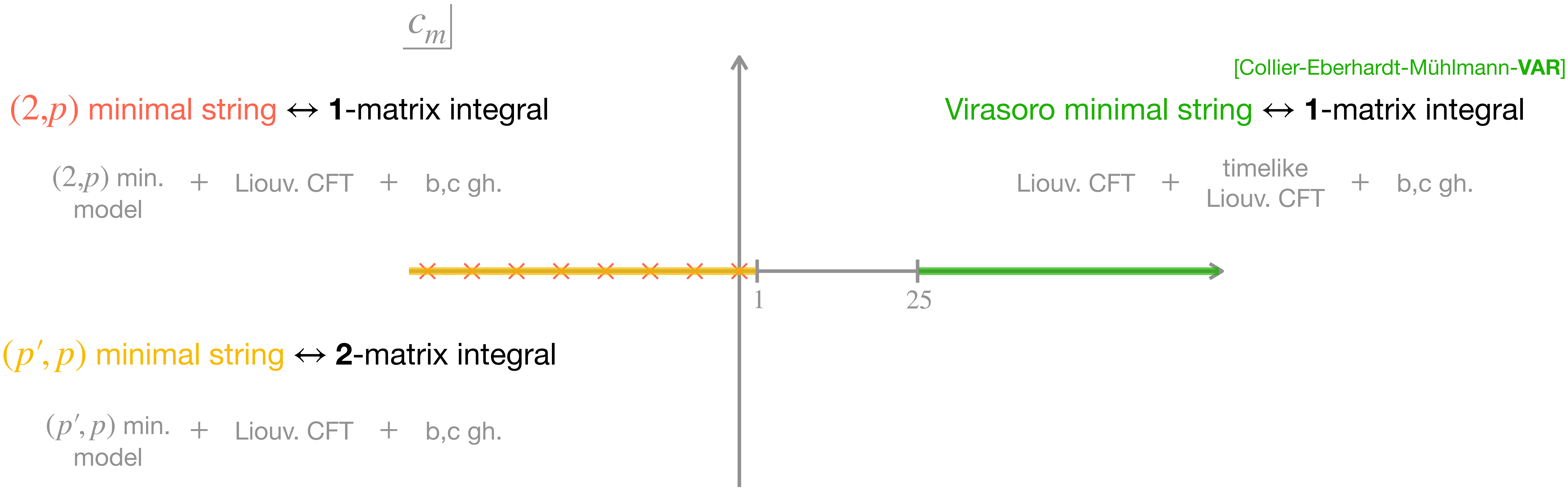



$(p',p)$ minimal string ↔ **2**-matrix integral

$(p',p)$ min. model + Liouv. CFT + b,c gh.

# “Minimal” strings



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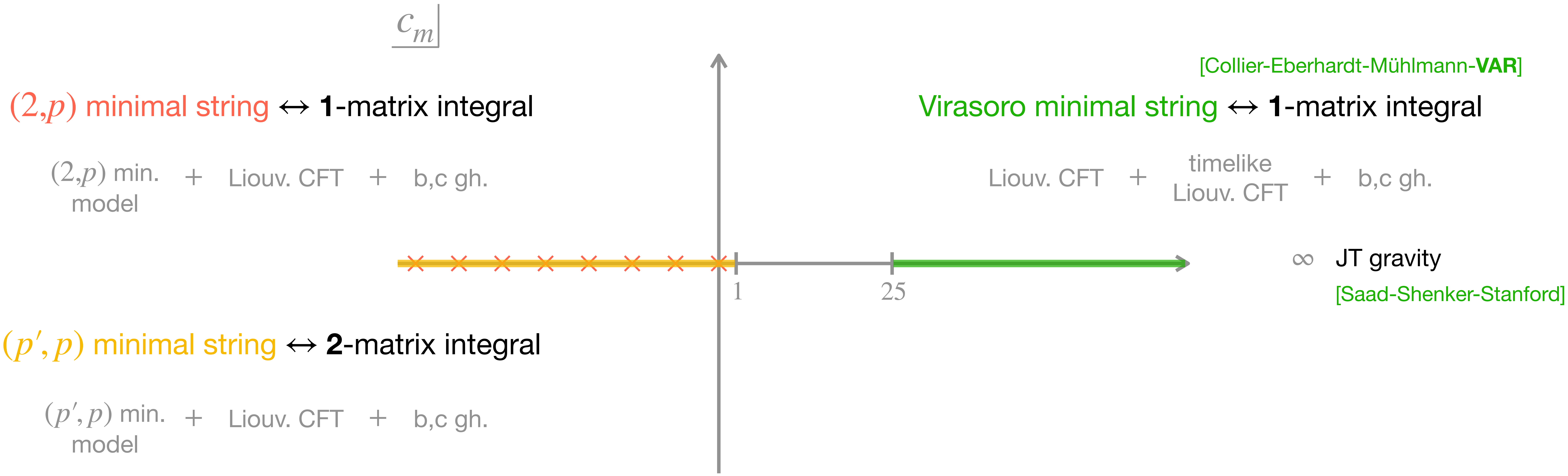

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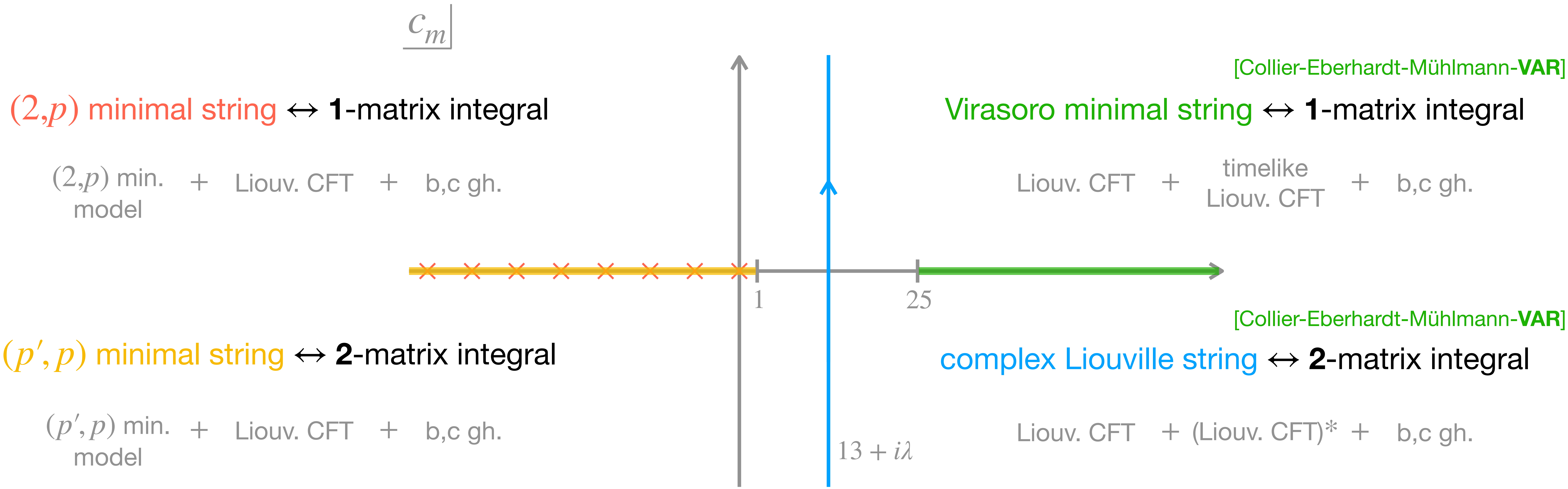

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[Collier-Eberhardt-Mühlmann-**VAR**]

$(2,p)$ minimal string ↔ **1**-matrix integral

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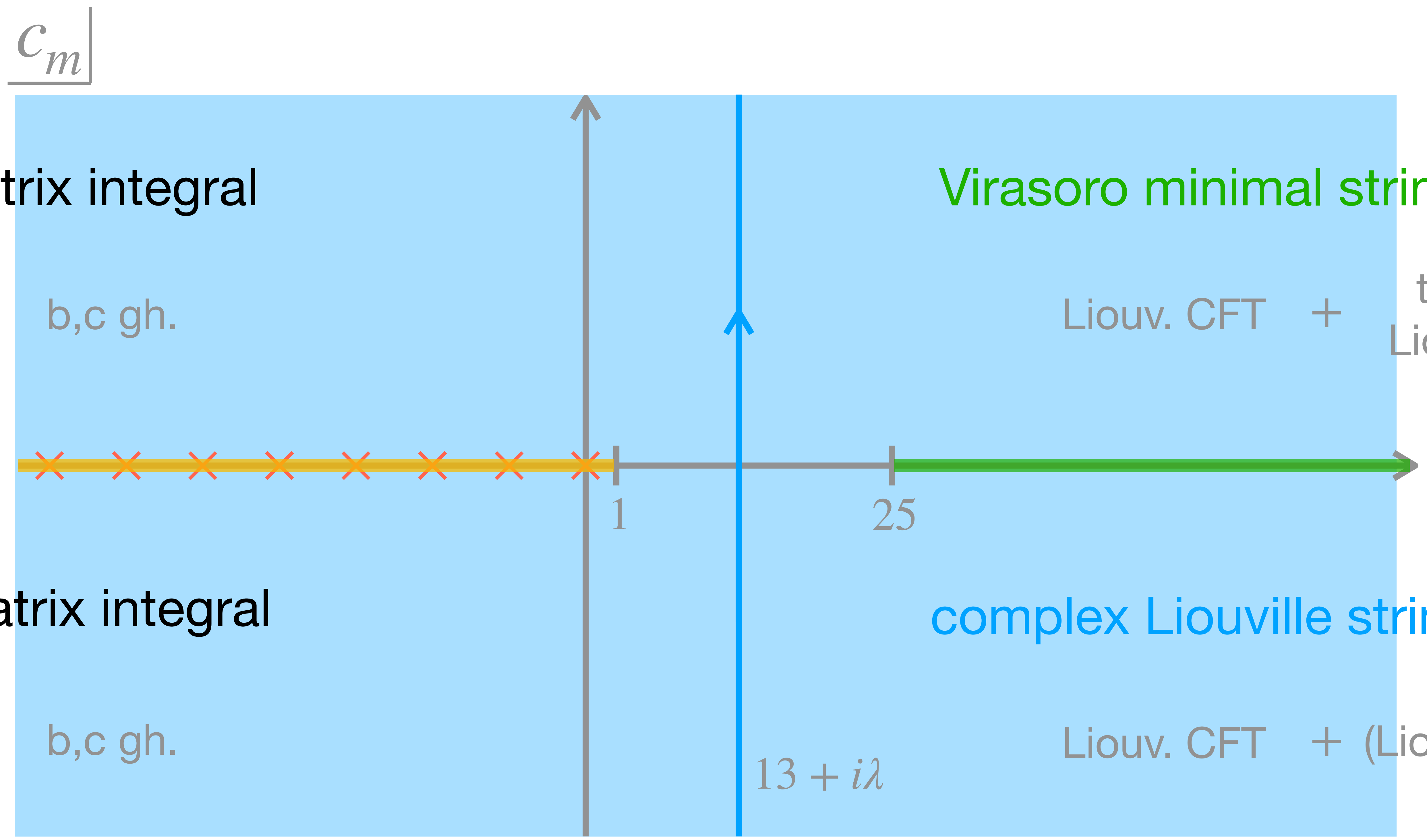
Virasoro minimal string ↔ **1**-matrix integral

Liouv. CFT + timelike Liouv. CFT + b,c gh.




[Collier-Eberhardt-Mühlmann-**VAR**]

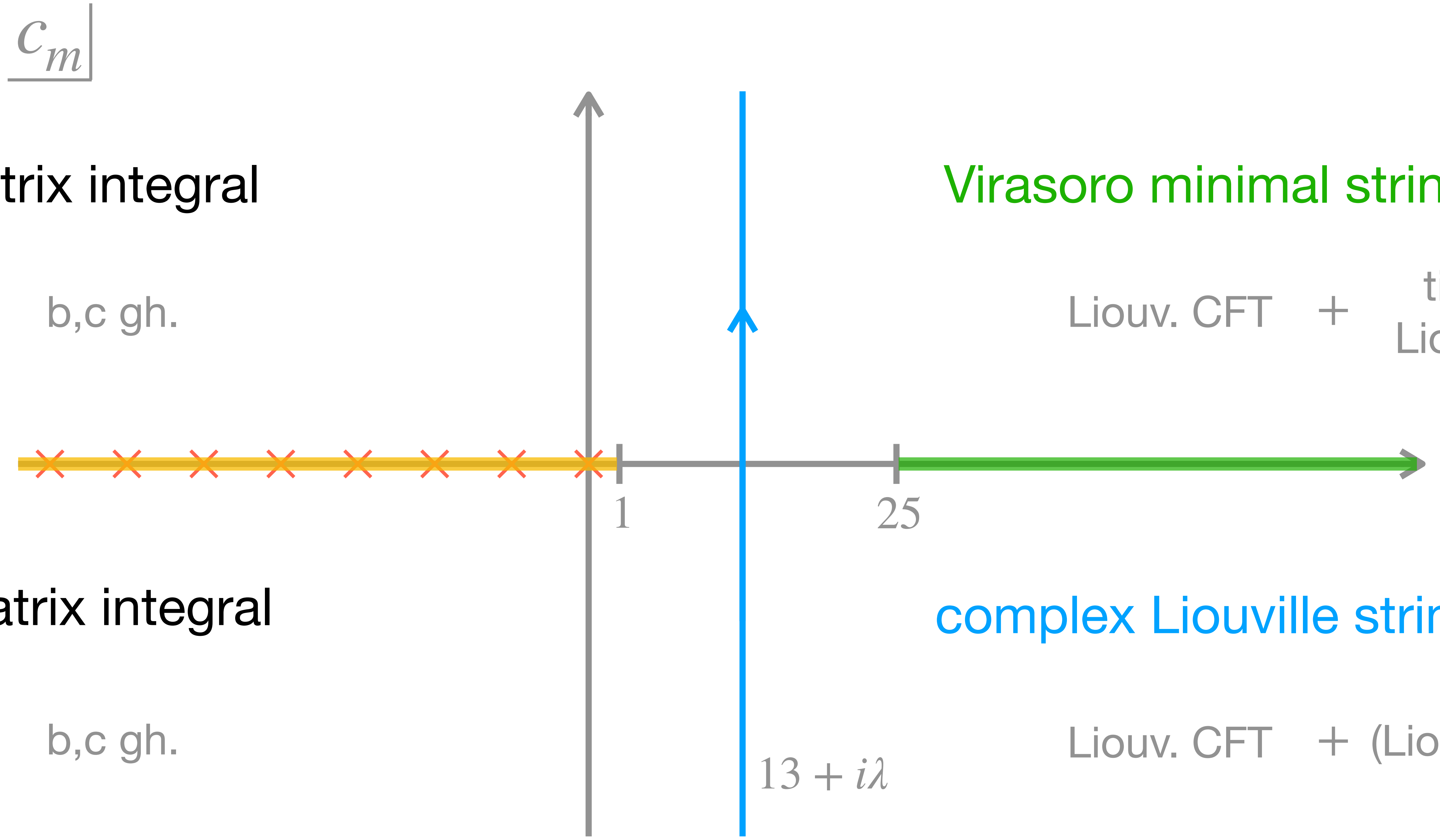
$(p',p)$ minimal string ↔ **2**-matrix integral

$(p',p)$ min. model + Liouv. CFT + b,c gh.

complex Liouville string ↔ **2**-matrix integral

Liouv. CFT + (Liouv. CFT)* + b,c gh.

# “Minimal” strings

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# “Minimal” strings

[**VAR**-Usatyuk-Wang]

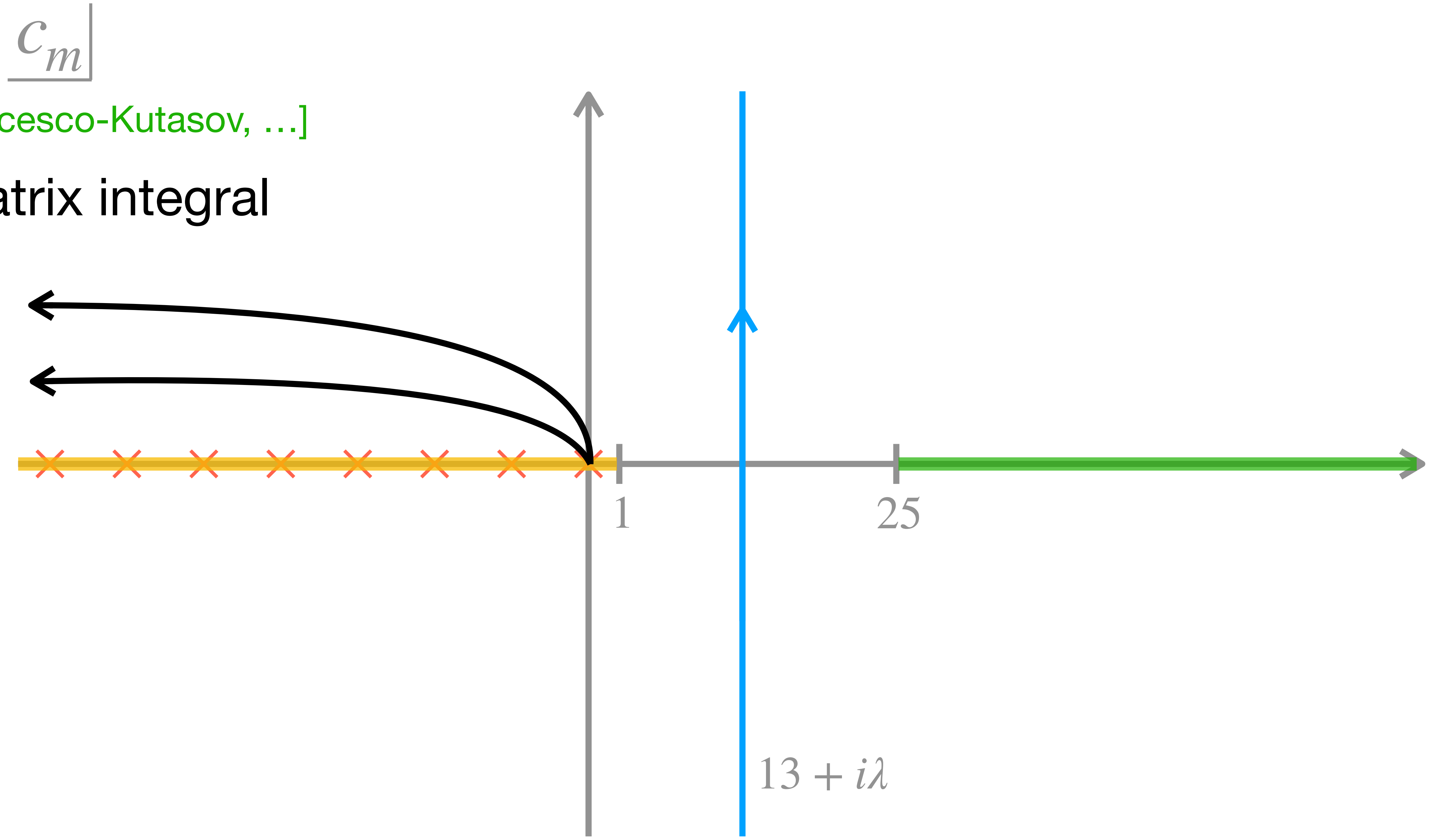
[diFrancesco-Kutasov, …]

D-series minimal string ↔ **4**-matrix integral

$(p, p')$ D-min. model + Liouv. CFT + b,c gh.

E-series ↔ **4**-matrix integral

$(p, p')$ E-min. model + Liouv. CFT + b,c gh.



# Minimal strings

$$\begin{matrix} \text{matter CFT} \\ c = c_{\mathrm{m}} \end{matrix} \otimes \begin{matrix} \text{Liouville CFT} \\ c = 26 - c_{\mathrm{m}} \end{matrix} \otimes \begin{matrix} \mathfrak{bc}\text{-ghosts} \\ c = -26 \end{matrix}$$

| string theory | matter CFT | #-matrices | solvable? |
|---|---|---|---|
| Minimal String (MS/AMS) [3–10] | A-series minimal model | 2 | TR |
| VMS [11] | timelike Liouville | 1 | TR |
| JT gravity [12] | $\mathrm{AMS}_{2,p\to\infty}$ or $\mathrm{VMS}_{b\to 0}$ | 1 | TR |
| CLS [13] | complex-Liouville | 2 | TR |
| $c = 1$ [14] | timelike boson | $\infty$ | FF |
| DMS, EMS [15, 16] | D,E-series minimal model | 4(+?) | $\times$ |

Let's first describe the CFT ingredients

# "Minimal" strings — WHY

- Explore/chart the landscape of minimal string theories
  (much simpler than the landscape descending from 10d superstrings)

- *Re-interpretation* of dualities as theories of 2d quantum gravity on the ws
  matrix integral = random average over dual quantum systems [Saad-Shenker-Stanford]

- What's possible to describe with this string theory machinery?
  (new applications for string theory)
  2d, and 3d quantum gravity with dS spacetimes

- Low-d string theory → dualities can be tested to a very high degree
  (derive them)

- Develop computational tools applicable to many/all strings (higher-d)
  (technical development)

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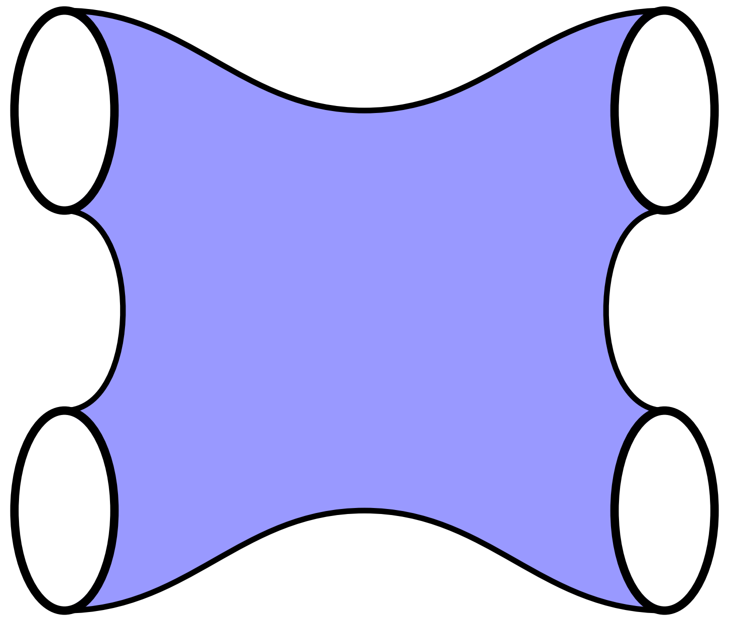
# The worldsheet description

## VMS, $\mathbb{C}\mathcal{LS}$, ADE MS



# Minimal strings

| | | | | | |
|---|---|---|---|---|---|
| VMS: | Timelike Liouville CFT<br>$c > 25$ | + | Liouville CFT<br>$\hat{c} < 1$ | + | b, c ghosts<br>$c_{\mathrm{gh}} = -26$ |
| $\mathbb{C}$LS: | (Liouville CFT)*<br>$c_- = 13 - i\lambda$ | + | Liouville CFT<br>$c_+ = 13 + i\lambda$ | + | b, c ghosts<br>$c_{\mathrm{gh}} = -26$ |
| ADE MS: | ADE min. model<br>$c_- = 13 - i\lambda$ | + | Liouville CFT<br>$c_+ = 13 + i\lambda$ | + | b, c ghosts<br>$c_{\mathrm{gh}} = -26$ |

Let's first describe the CFT ingredients

# 1. Liouville CFT

Liouville CFT *bootstrap*

$$c = 1 + 6\,(b + b^{-1})^2 \geq 25, \quad b \in (0,1]$$

non-compact, unitary CFT

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   scalar Vir primaries $V_P$

$$h_P = \bar{h}_P = \frac{c-1}{24} + P^2$$

$P \in \mathbb{R}_{\geq 0}$ Liouv. momentum

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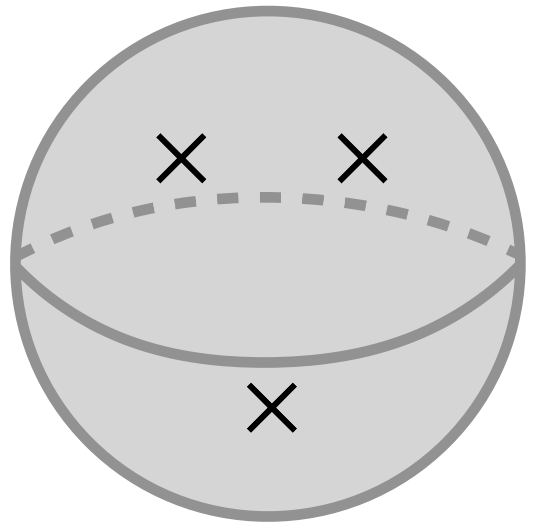
2. Three-point coefficients

[Dorn Otto Zamolodchikov^2, Teschner]



$$= \langle V_{P_1} V_{P_2} V_{P_3} \rangle^{(b)} = C_b(P_1, P_2, P_3)$$

$$= \frac{\Gamma_b(2Q)\Gamma_b(\frac{Q}{2} \pm iP_1 \pm iP_2 \pm iP_3)}{\sqrt{2}\Gamma_b(Q)^3 \prod_{k=1}^{3} \Gamma_b(Q \pm 2iP_k)}$$

# 1. Liouville CFT

## Liouville CFT *bootstrap*

$c = 1 + 6\,(b + b^{-1})^2 \geq 25, \quad b \in (0,1]$    non-compact, unitary CFT

$\Gamma_b(x)$ is like $\Gamma(x)$

$$\Gamma_b(z+b) = \frac{\sqrt{2\pi} b^{bz-\frac{1}{2}}}{\Gamma(z)} \Gamma_b(z) \qquad \Gamma(x+1) = x\,\Gamma(x)$$

simple poles $\quad z = -mb - nb^{-1} \quad m, n \in \mathbb{Z}_{\geq 0} \qquad x = -m \quad m \in \mathbb{Z}_{>0}$

## 2. Three-point coefficients

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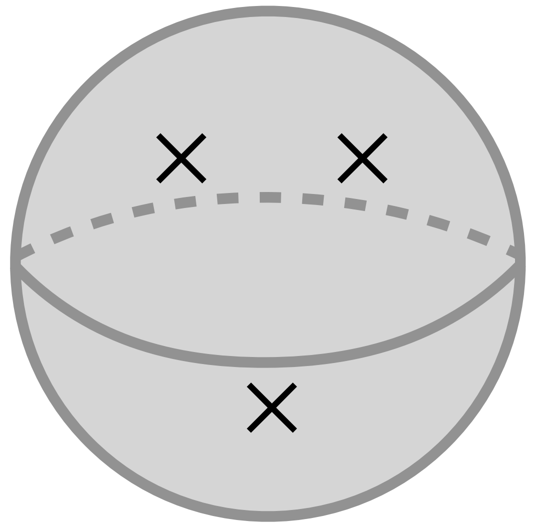

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★ A (double) infinite product formula you can play with!

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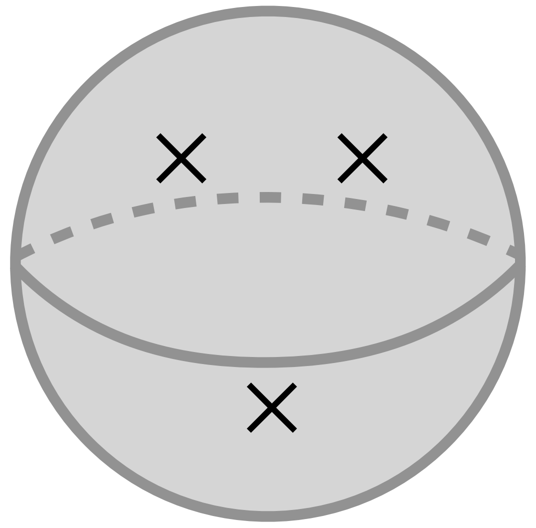

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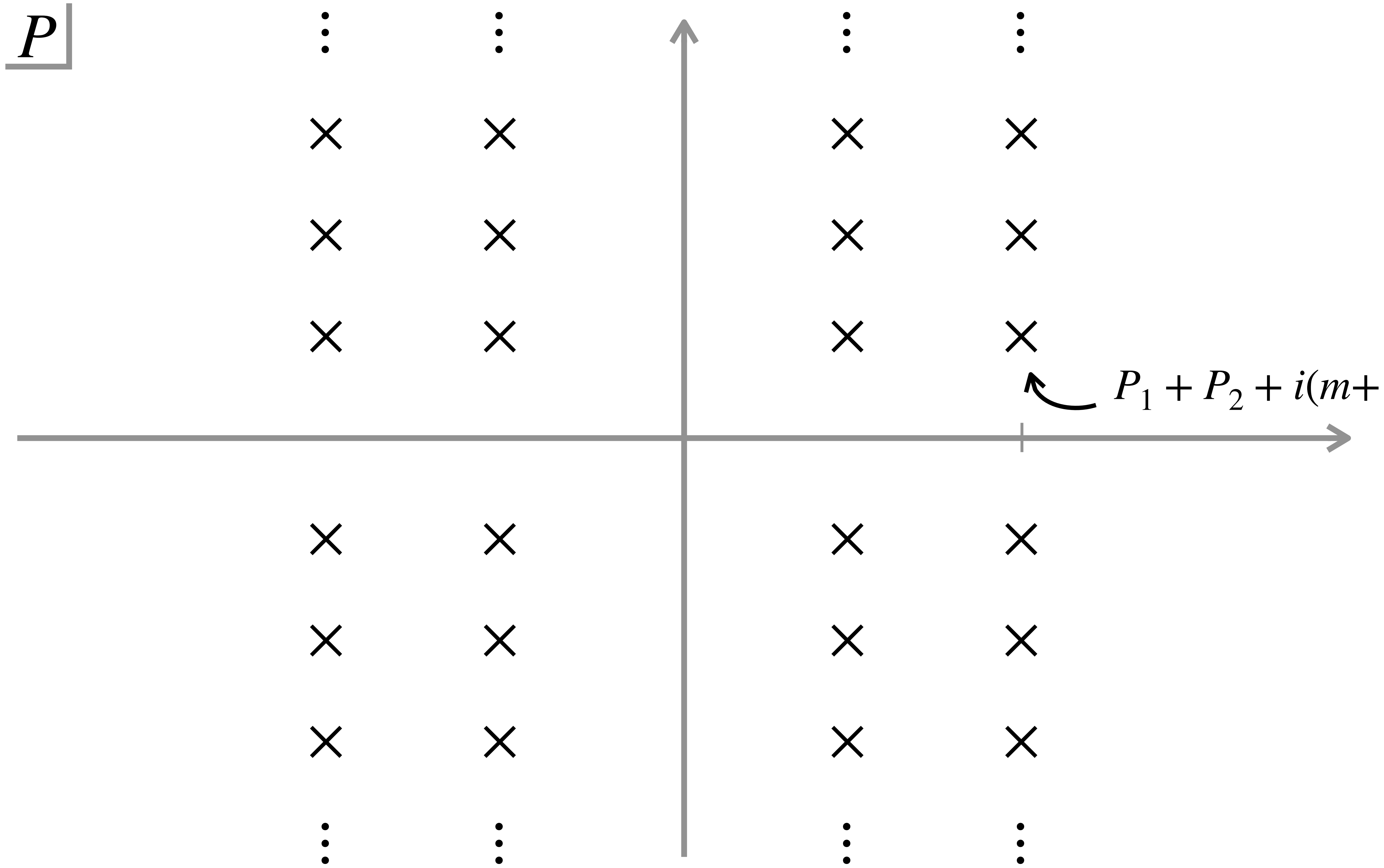
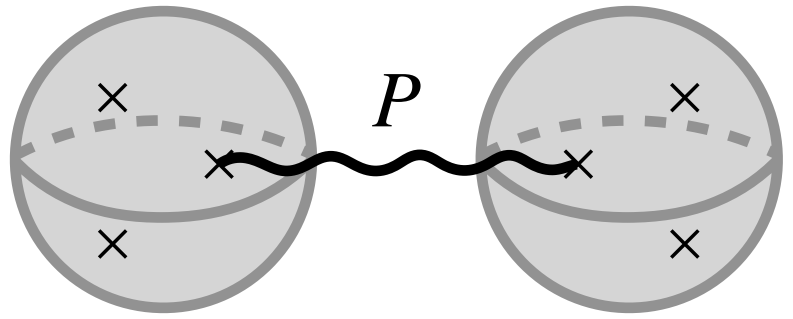
- Compute any correlation function on $\Sigma_{g,n}$ in Liouv. CFT via a conformal block decomposition

# 1. Liouville CFT

Analytic structure of DOZZ-1 $C(P_1, P_2, P)$




$$\langle V_{P_4}(\infty)V_{P_3}(1)V_{P_2}(z,\bar{z})V_{P_1}(0)\rangle_{c\geq 25\ \text{Liouv.}} = \int_0^\infty \mathrm{d}P\, C_b(P_1,P_2,P)C_b(P_3,P_4,P)F^{(b)}_{0,4}(h_4,h_3,h_2,h_1;h_P\,|\,z)F^{(b)}_{0,4}(h_4,h_3,h_2,h_1;h_P\,|\,\bar{z})$$

# 2. timelike Liouville CFT

Timelike Liouville CFT *bootstrap*

$$c = 1 + 6\,(b + b^{-1})^2 \geq 25, \quad b \in (0,1]$$

$$c = 1 + 6\,(b + b^{-1})^2 \leq 1 \qquad b \to i\hat{b}$$

The bootstrap solution is NOT simply the analytic continuation of Liouv CFT to $c \leq 1$ !

# 2. timelike Liouville CFT

Timelike Liouville CFT *bootstrap*

$$\hat{c} = 1 - 6\,(\hat{b}^{-1} - \hat{b})^2 \leq 1, \quad \hat{b} \in (0,1]$$

non-compact, non-unitary CFT (mild)

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$\hat{P} \in \mathbb{R}$ Liouv. momentum

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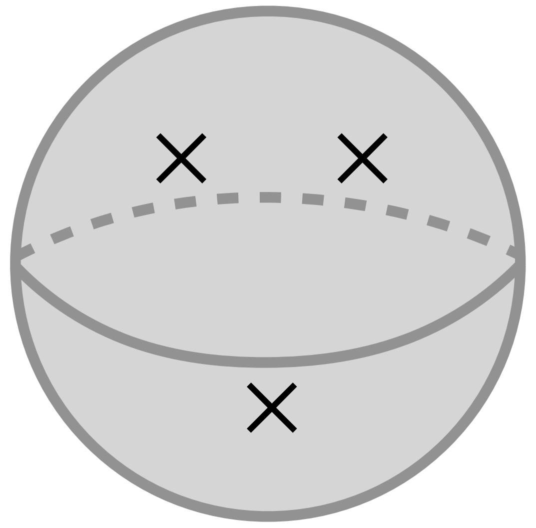
2. Three-point coefficients

[Zamolodchikov; Kostov Petkova ...]



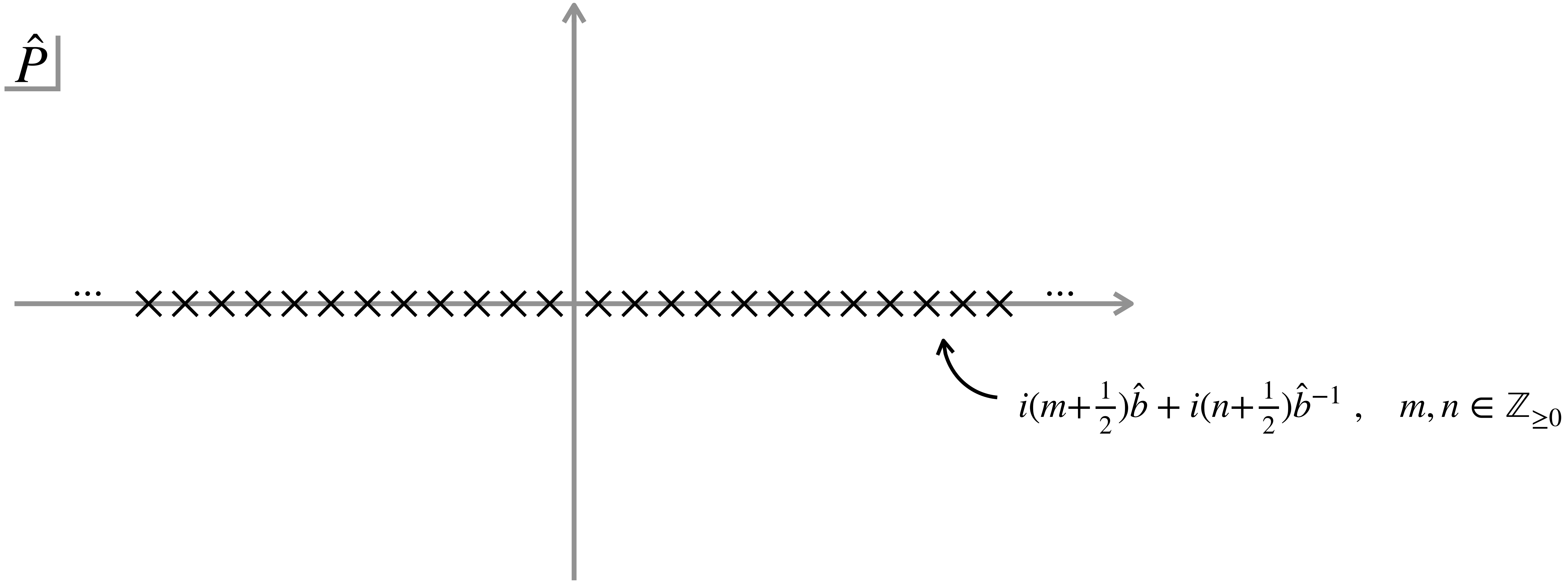
$$= \langle \hat{V}_{\hat{P}_1} \hat{V}_{\hat{P}_2} \hat{V}_{\hat{P}_3} \rangle^{(i\hat{b})} = \hat{C}_{\hat{b}}(\hat{P}_1, \hat{P}_2, \hat{P}_3)$$

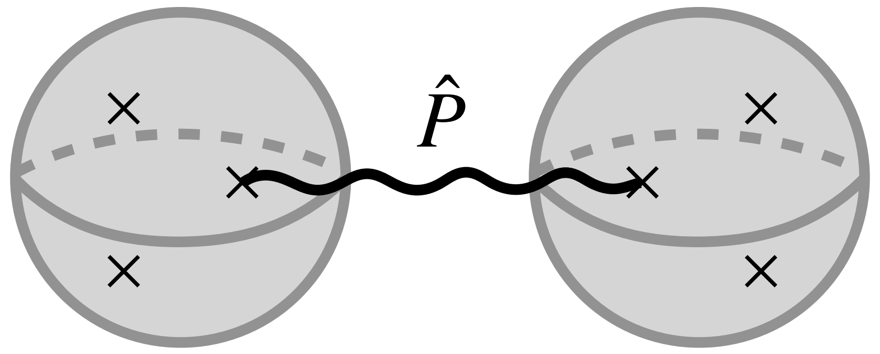
$$= \frac{1}{C_{\hat{b}}(i\hat{P}_1, i\hat{P}_2, i\hat{P}_3)}$$

$$= \frac{\sqrt{2}\,\Gamma_b(\hat{b} + \hat{b}^{-1})^3}{\Gamma_b(2\hat{b} + 2\hat{b}^{-1})} \frac{\prod_{k=1}^{3} \Gamma_b(\hat{b} + \hat{b}^{-1} \pm 2\hat{P}_k)}{\Gamma_b(\frac{\hat{b} + \hat{b}^{-1}}{2} \pm \hat{P}_1 \pm \hat{P}_2 \pm \hat{P}_3)}$$

# 2. timelike Liouville CFT

Analytic structure of DOZZ-2 $\hat{C}(\hat{P}, P_1, P_2)$




$$\langle \hat{V}_{P_4}(\infty)\hat{V}_{P_3}(1)\hat{V}_{P_2}(z,\bar{z})\hat{V}_{P_1}(0)\rangle_{\hat{c}\leq 1 \text{ Liouv.}} = \int_{\mathscr{C}} d\hat{P}\ \hat{C}(P_1,P_2,\hat{P})\hat{C}(P_3,P_4,\hat{P})F_{\hat{c}}(\hat{h}_4,\hat{h}_3,\hat{h}_2,\hat{h}_1;h_{\hat{P}}\,|\,z)F_{\hat{c}}(\hat{h}_4,\hat{h}_3,\hat{h}_2,\hat{h}_1;h_{\hat{P}}\,|\,\bar{z})$$

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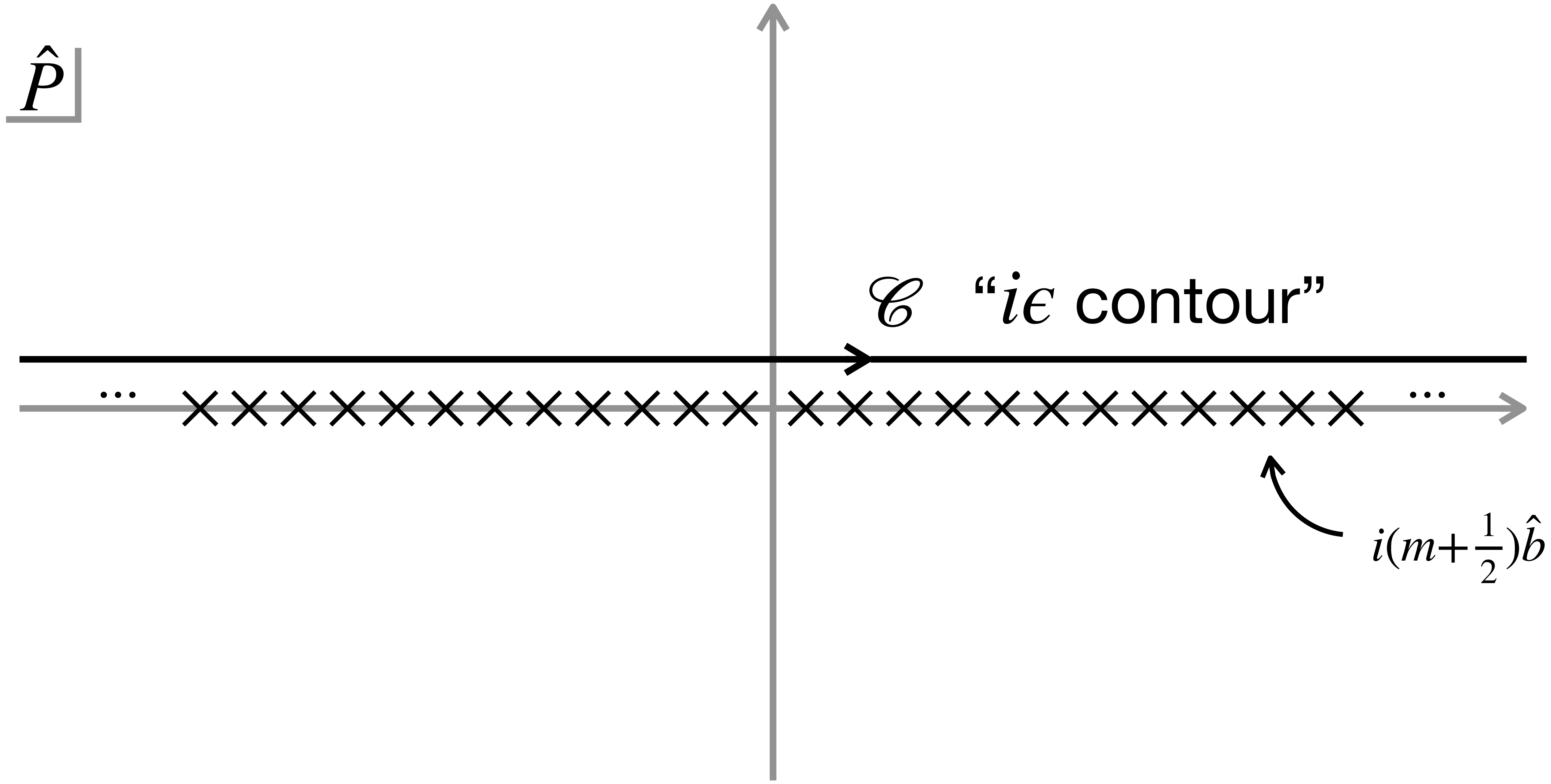
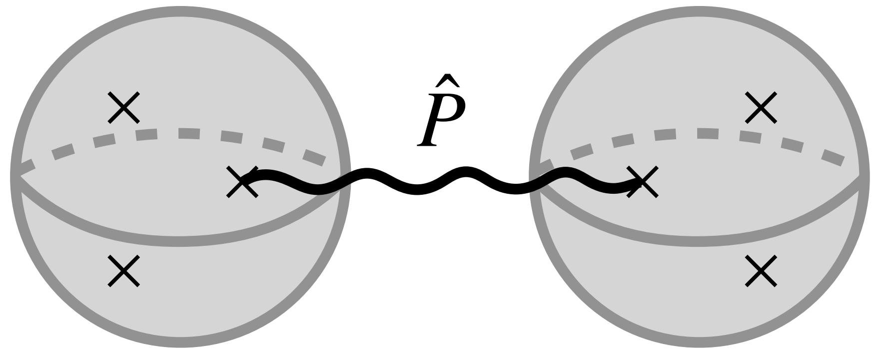


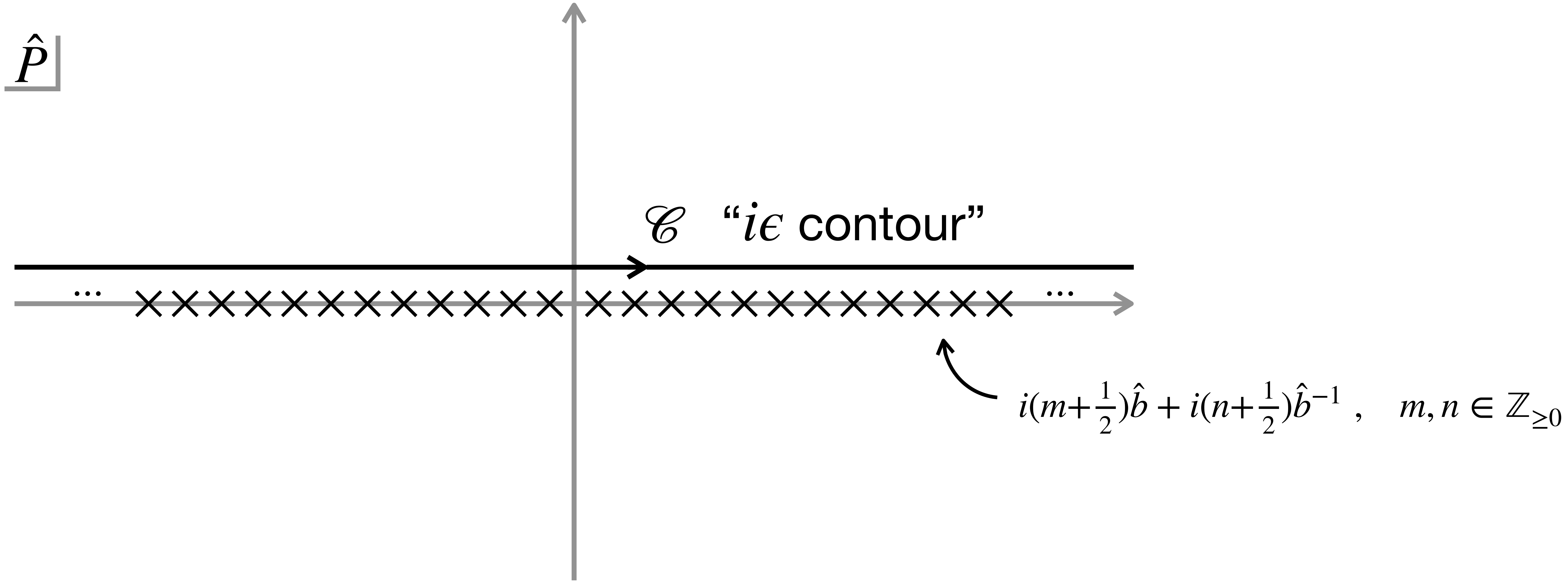



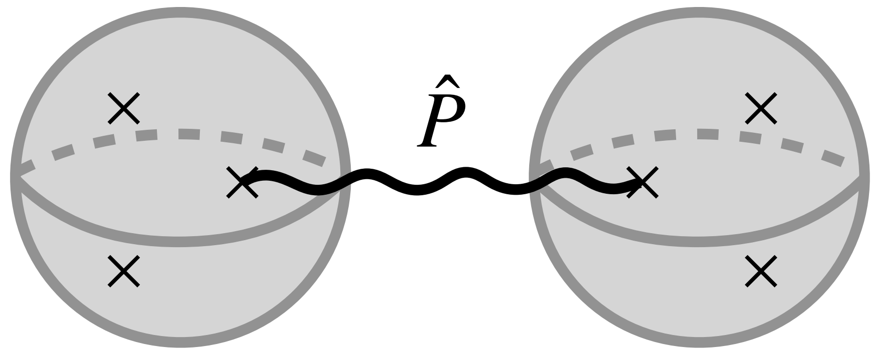

$$\langle \hat{V}_{P_4}(\infty)\hat{V}_{P_3}(1)\hat{V}_{P_2}(z,\bar{z})\hat{V}_{P_1}(0)\rangle_{\hat{c}\leq 1 \text{ Liouv.}} = \int_{\mathscr{C}} d\hat{P}\ \hat{C}(P_1, P_2, \hat{P})\hat{C}(P_3, P_4, \hat{P})F_{\hat{c}}(\hat{h}_4, \hat{h}_3, \hat{h}_2, \hat{h}_1; h_{\hat{P}}\,|\,z)F_{\hat{c}}(\hat{h}_4, \hat{h}_3, \hat{h}_2, \hat{h}_1; h_{\hat{P}}\,|\,\bar{z})$$

# 2. timelike Liouville CFT

Analytic structure of DOZZ-2 $\hat{C}(\hat{P}, P_1, P_2)$

[Ribault-Santachiara]




☑ satisfies crossing symmetry

$$\langle \hat{V}_{P_4}(\infty)\hat{V}_{P_3}(1)\hat{V}_{P_2}(z,\bar{z})\hat{V}_{P_1}(0)\rangle_{\hat{c}\leq 1\ \text{Liouv.}} = \int_{\mathscr{C}} d\hat{P}\ \hat{C}(P_1,P_2,\hat{P})\hat{C}(P_3,P_4,\hat{P})F_{\hat{c}}(\hat{h}_4,\hat{h}_3,\hat{h}_2,\hat{h}_1;h_{\hat{P}}\,|\,z)F_{\hat{c}}(\hat{h}_4,\hat{h}_3,\hat{h}_2,\hat{h}_1;h_{\hat{P}}\,|\,\bar{z})$$

# 3. a) A-series min. model CFT

$c = 1 - 6\,(\beta - \beta^{-1})^2 \leq 1, \quad \beta = \frac{p}{p'}$ similar parametrization

1. Operator spectrum: **discrete** $\hat{V}_{P_{r,s}}$ $P_{r,s} = \frac{1}{2}(r\beta^{-1} - s\beta)$

Liouv. momentum discrete

# 3. a) A-series min. model CFT

$c = 1 - 6\,(\beta - \beta^{-1})^2 \leq 1, \quad \beta = \frac{p}{p'}$  similar parametrization

1. Operator spectrum: **discrete** $\hat{V}_{P_{r,s}}$  $P_{r,s} = \frac{1}{2}(r\beta^{-1} - s\beta)$

Liouv. momentum discrete

$$\mathcal{S}^{\text{A-series}}_{p,p'} = \frac{1}{2} \bigoplus_{r=1}^{p-1} \bigoplus_{s=1}^{p'-1} \mathcal{R}_{r,s} \otimes \widetilde{\mathcal{R}}_{r,s}$$

$$h_{P_{r,s}} = \bar{h}_{P_{r,s}} = \frac{c-1}{24} + {P_{r,s}}^2$$

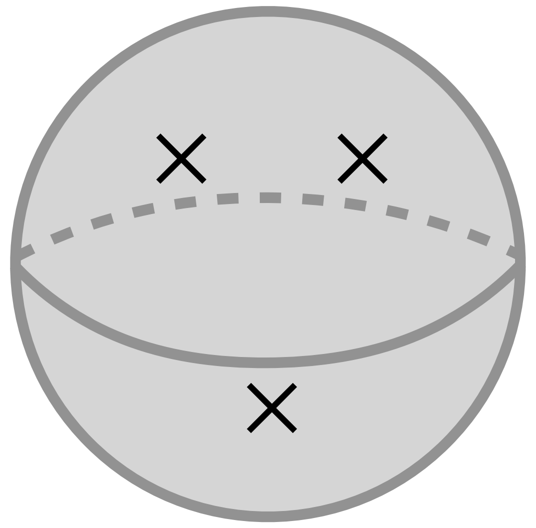
# 3. a) A-series min. model CFT

$c = 1 - 6\,(\beta - \beta^{-1})^2 \leq 1, \quad \beta = \frac{p}{p'}$ similar parametrization

1. Operator spectrum: **discrete** $\hat{V}_{P_{r,s}}$ $\quad P_{r,s} = \frac{1}{2}(r\beta^{-1} - s\beta)$

Liouv. momentum discrete

2. Three-point coefficients



$$= \hat{C}_\beta(P_{r_1,s_1}, P_{r_2,s_2}, P_{r_3,s_3}) \times f_{r_1,s_1;r_2,s_2;r_3,s_3}$$

enforces **fusion rule**

same timelike DOZZ

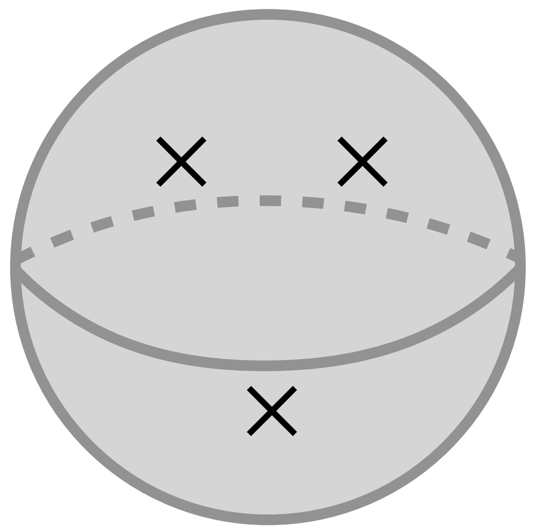
# 3. a) A-series min. model CFT

$c = 1 - 6\,(\beta - \beta^{-1})^2 \leq 1, \quad \beta = \frac{p}{p'}$   similar parametrization

1. Operator spectrum: **discrete** $\hat{V}_{P_{r,s}}$   $P_{r,s} = \frac{1}{2}(r\beta^{-1} - s\beta)$

Liouv. momentum discrete

2. Three-point coefficients



$$= \hat{C}_\beta(P_{r_1,s_1}, P_{r_2,s_2}, P_{r_3,s_3}) \times f_{r_1,s_1;r_2,s_2;r_3,s_3}$$

same timelike DOZZ

enforces **fusion rule**

$$\widehat{V}_{r_1,s_1} \times \widehat{V}_{r_2,s_2} = \sum_{r_3 \overset{2}{=} |r_1-r_2|+1}^{\min(r_1+r_2,2p-r_1-r_2)-1} \; \sum_{s_3 \overset{2}{=} |s_1-s_2|+1}^{\min(s_1+s_2,2p'-s_1-s_2)-1} \widehat{V}_{r_3,s_3}$$

# 3. b) D-series min. model CFT

$c = 1 - 6(\beta - \beta^{-1})^2 \leq 1, \quad \beta = \frac{p}{p'}$ similar parametrization

1. Operator spectrum: **discrete** $\hat{V}_{P_{r,s}}$ $P_{r,s} = \frac{1}{2}(r\beta^{-1} - s\beta)$

$$\mathcal{S}^{\text{D-series}}_{p,p'} \underset{p=2 \bmod 4}{=} \frac{1}{2} \bigoplus_{r \overset{2}{=} 1}^{p-1} \bigoplus_{s=1}^{p'-1} \underbrace{\mathcal{R}_{r,s} \otimes \widetilde{\mathcal{R}}_{r,s}}_{\text{“diagonal” primaries}} \oplus \frac{1}{2} \bigoplus_{r \overset{2}{=} 1}^{p-1} \bigoplus_{s=1}^{p'-1} \underbrace{\mathcal{R}_{r,s} \otimes \widetilde{\mathcal{R}}_{p-r,s}}_{\text{“non-diagonal” primaries}}$$

$P_{r,s}$ $P_{r,s}\,,\ \widetilde{P}_{r,s}$

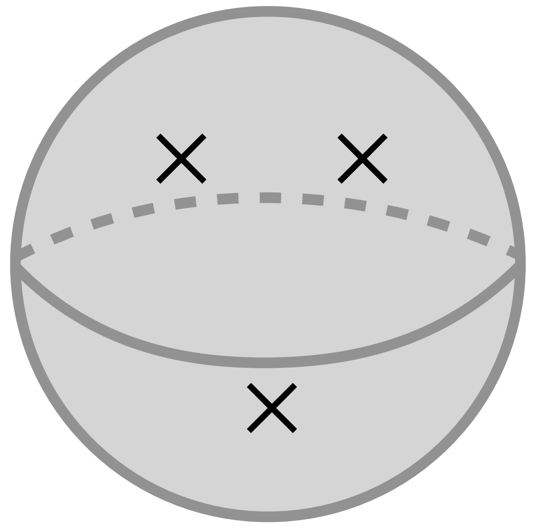
# 3. b) D-series min. model CFT

$c = 1 - 6\,(\beta - \beta^{-1})^2 \leq 1, \quad \beta = \frac{p}{p'}$ similar parametrization

1. Operator spectrum: **discrete** $\hat{V}_{P_{r,s}}$ $P_{r,s} = \frac{1}{2}(r\beta^{-1} - s\beta)$

$$P_{r,s}\ ,\ \widetilde{P}_{r,s}$$

2. Three-point coefficients $C_\beta^{DDD}\ ,\ C_\beta^{DNN}$



$$= C_\beta^{X_1X_2X_3}(P_{r_1,s_1}, P_{r_2,s_2}, P_{r_3,s_3}) \times f_{r_1,s_1;r_2,s_2;r_3,s_3}$$

enforces **fusion rule**

similar to timelike DOZZ

$$D \times D = D$$
$$D \times N = N$$
$$N \times N = D$$

similar fusion
+ conservation of
diagonality

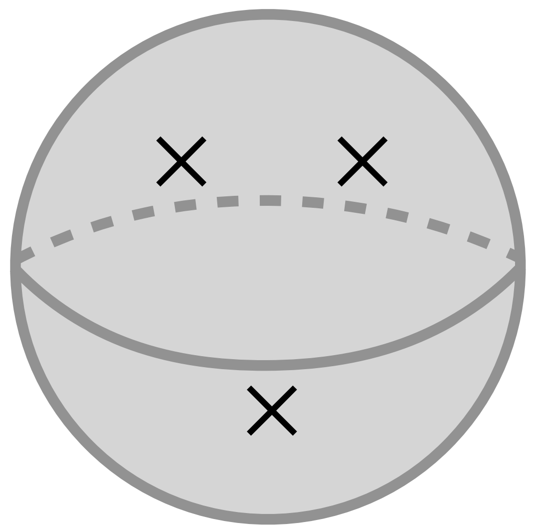
# 3. b) (12, p') E-series min. model CFT

$c = 1 - 6\,(\beta - \beta^{-1})^2 \leq 1, \quad \beta = \frac{p}{p'}$ similar parametrization

1. Operator spectrum: **discrete** $\hat{V}_{P_{r,s}}$ $P_{r,s} = \frac{1}{2}(r\beta^{-1} - s\beta)$

$P_{r,s}\ ,\ \widetilde{P}_{r,s}$

2. Three-point coefficients



$= C_\beta^{X_1X_2X_3}(P_{r_1,s_1}, P_{r_2,s_2}, P_{r_3,s_3}) \times f_{r_1,s_1;r_2,s_2;r_3,s_3}$

enforces **fusion rule**

similar to timelike DOZZ

nastier

[Nivesvivat-Ribault '25]

$D \times D = D$

$D \times N = N$

$N \times N = D$

similar fusion

+ conservation of

diagonality

this was all CFT
now the string theories

# Virasoro minimal string

VMS: Timelike Liouville CFT + Liouville CFT + b, c ghosts

| Timelike Liouville CFT | Liouville CFT | b, c ghosts |
|---|---|---|
| $\hat{c} < 1$ | $c > 25$ | $c_{\text{gh}} = -26$ |

- Central charge: $c = 1 + 6(b + b^{-1})^2$ $\hat{c} = 1 - 6\,(\hat{b}^{-1} - \hat{b})^2$

$\hat{c} + c = 26 \;\rightarrow\; \hat{b} = -ib$

Single parameter: $b$

# Virasoro minimal string

| | | | | | |
|---|---|---|---|---|---|
| VMS: | Timelike Liouville CFT | + | Liouville CFT | + | b, c ghosts |
| | $\hat{c} < 1$ | | $c > 25$ | | $c_{\text{gh}} = -26$ |

- Central charge: $c = 1 + 6(b + b^{-1})^2$ $\hat{c} = 1 - 6\,(\hat{b}^{-1} - \hat{b})^2$

$$\hat{c} + c = 26 \;\rightarrow\; \hat{b} = -ib$$

Single parameter: $b$

- "On-shell" vertex operators:

$$\mathscr{V}_P = c\bar{c}\; V_P^{(b)}\, \hat{V}_{iP}^{(-ib)}$$

Liouv. momenta: $P$

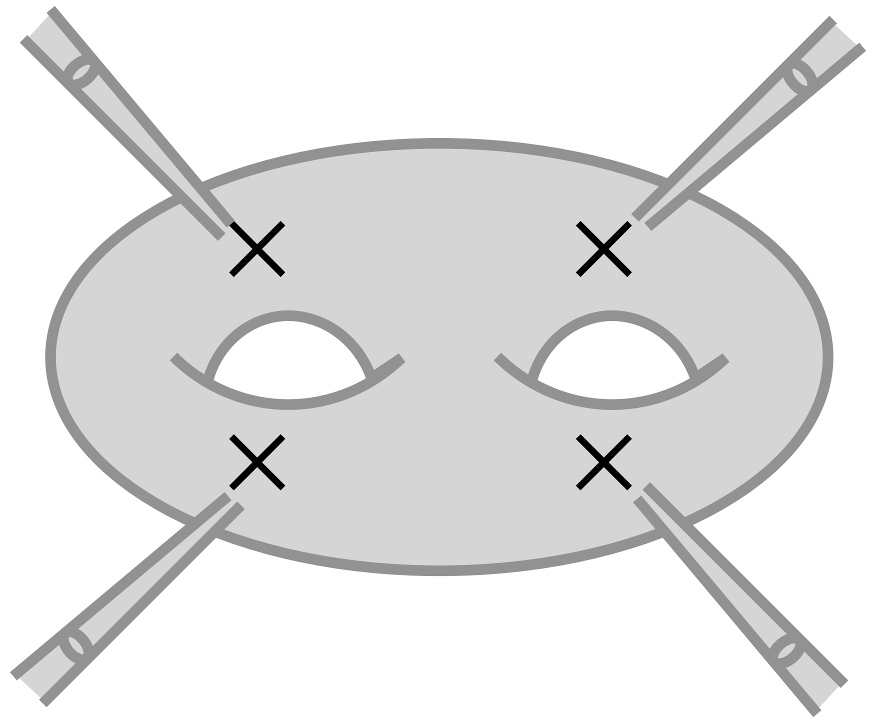
$$h_P + h_{\hat{P}} = 1 \;\rightarrow\; \hat{P} = iP$$

easy: poles in correlators dont move

# Virasoro minimal string

**VMS:** Timelike Liouville CFT $\hat{c} < 1$ + Liouville CFT $c > 25$ + b, c ghosts $c_{\text{gh}} = -26$

$$\mathsf{V}^{(b)}_{g,n}(P_1, \ldots, P_n) = \int_{\mathcal{M}(\Sigma_{g,n})} \left\langle \prod_{j=1}^{n} V_{P_j} \right\rangle^{(b)}_{\Sigma_g} \left\langle \prod_{j=1}^{n} \hat{V}_{iP_j} \right\rangle^{(-ib)}_{\Sigma_g} \times \left(b, c \text{ ghosts}\right)$$



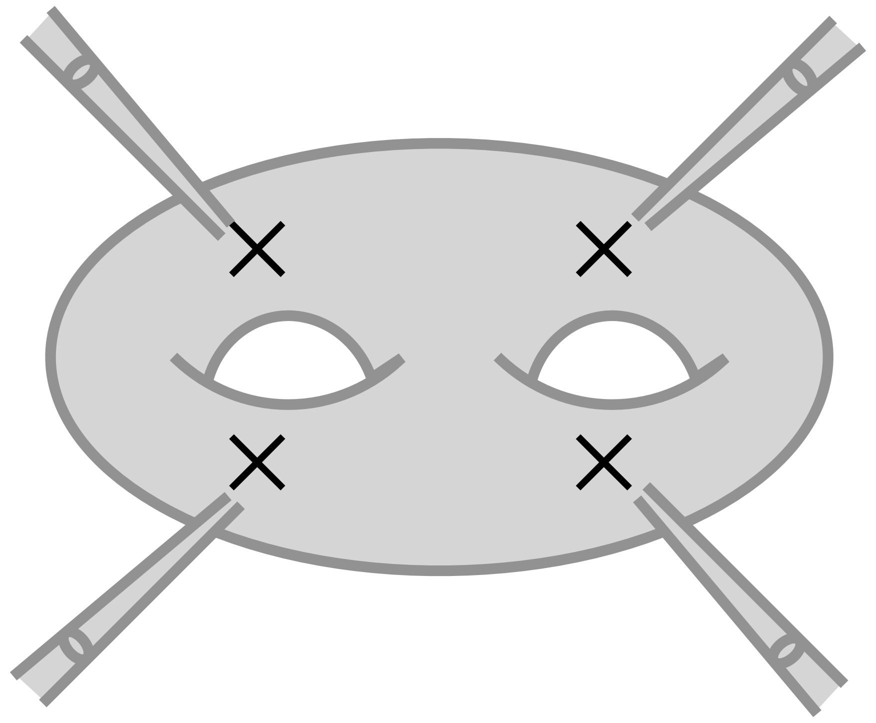
- Absolutely convergent moduli integrals
- NOT like S-matrix elements (discontinuities, etc.)

# Virasoro minimal string

VMS: Timelike Liouville CFT + Liouville CFT + b, c ghosts

$\hat{c} < 1$ | $c > 25$ | $c_{\text{gh}} = -26$

$$\mathsf{V}^{(b)}_{g,n}(P_1, \ldots, P_n) = \int_{\mathscr{M}(\Sigma_{g,n})} \left\langle \prod_{j=1}^{n} V_{P_j} \right\rangle^{(b)}_{\Sigma_g} \left\langle \prod_{j=1}^{n} \hat{V}_{iP_j} \right\rangle^{(-ib)}_{\Sigma_g} \times (b, c \text{ ghosts})$$



= polynomials !

$$\mathsf{V}^{(b)}_{0,4}(P_1, P_2, P_3, P_4) = \frac{c-13}{24} + P_1^2 + P_2^2 + P_3^2 + P_4^2$$

$$\mathsf{V}^{(b)}_{1,1}(P_1) = \frac{c-13}{576} + \frac{1}{24} P_1^2$$

# complex Liouville string

$\mathbb{C}$LS:

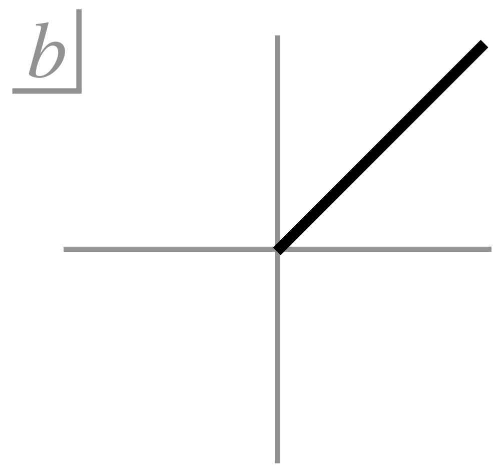
| | Liouville CFT | + | (Liouville CFT)* | + | b, c ghosts |
|---|---|---|---|---|---|
| | $c_+ = 13 + i\lambda$ | | $c_- = 13 - i\lambda$ | | $c_{\text{gh}} = -26$ |

- Central charge: $c = 1 + 6(b + b^{-1})^2 \in 13 + i\mathbb{R}$

$$c^+ + c^- = 26 \;\; \rightarrow \;\; b^- = -ib^+$$

$$(c^-)^* = c^+ \;\; \rightarrow \;\; (b^+)^2 \in i\mathbb{R}$$

Single parameter: $b \equiv b_+$



# complex Liouville string

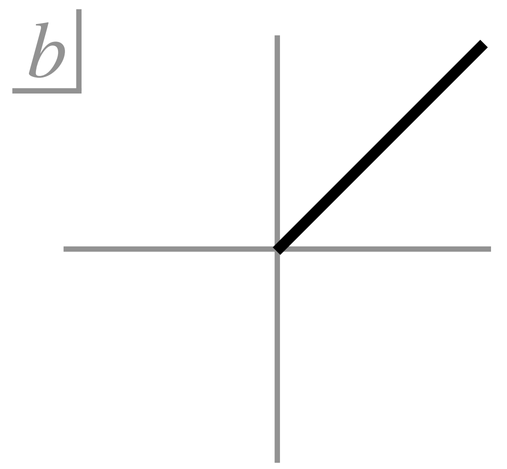
$\mathbb{C}$LS: Liouville CFT + (Liouville CFT)* + b, c ghosts

$c_+ = 13 + i\lambda$ $\quad c_- = 13 - i\lambda$ $\quad c_{\text{gh}} = -26$

- Central charge: $c = 1 + 6(b + b^{-1})^2 \in 13 + i\mathbb{R}$

$$c^+ + c^- = 26 \;\rightarrow\; b^- = -ib^+$$
$$(c^-)^* = c^+ \;\rightarrow\; (b^+)^2 \in i\mathbb{R}$$

Single parameter: $b \equiv b_+$

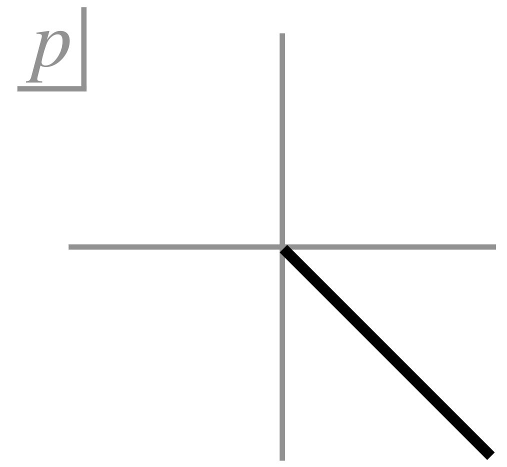



- "On-shell" vertex operators:

$$\mathscr{V}_p = c\bar{c}\; V_p^{(b)}\, V_{ip}^{(-ib)}$$

$$h_{p^+} + h_{p^-} = 1 \;\rightarrow\; p^- = ip^+$$
$$h_{p^-} = (h_{p^+})^* \;\rightarrow\; p^+ \in e^{-\frac{i\pi}{4}}\mathbb{R}_+$$

Liouv. momenta: $p \equiv p_+$



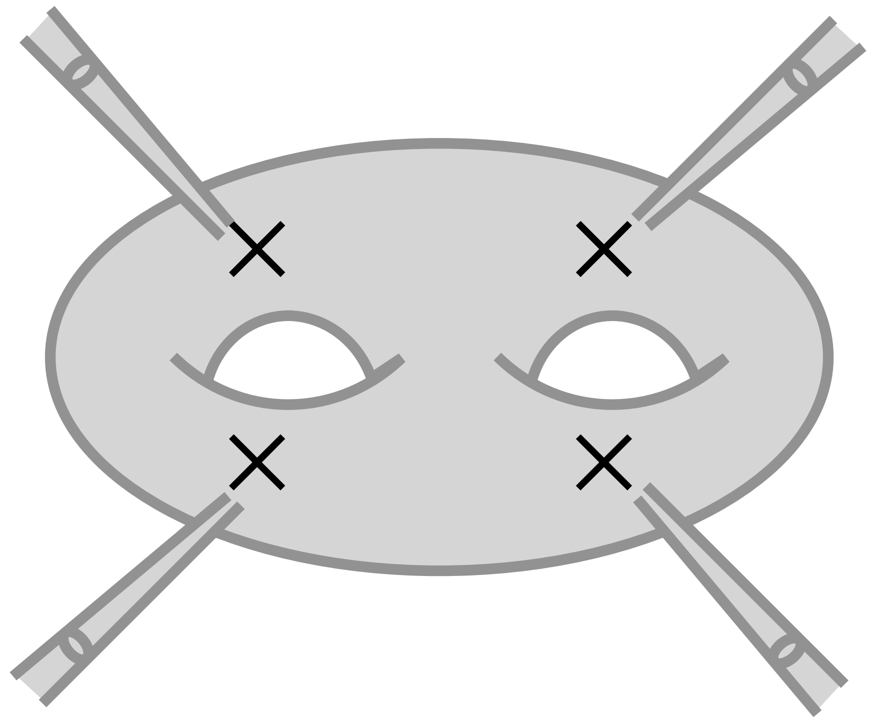

$$bp \in \mathbb{R}$$

# complex Liouville string

$\mathbb{C}$LS: Liouville CFT + (Liouville CFT)* + b, c ghosts

$c_+ = 13 + i\lambda$ $\quad c_- = 13 - i\lambda$ $\quad c_{\text{gh}} = -26$

$$\mathsf{A}^{(b)}_{g,n}(p_1,\ldots,p_n) = \left(\prod_{j=1}^{n} \mathcal{N}_b(p_j)\right) \int_{\mathcal{M}(\Sigma_{g,n})} \left| \left\langle \prod_{j=1}^{n} V_{p_j} \right\rangle^{(b)}_{\Sigma_g} \right|^2 \times (b, c\ \text{ghosts})$$



$$\mathsf{A}^{(b)}_{0,3}(p_1,p_2,p_3) = \sum_{m=1}^{\infty} \frac{2b(-1)^m \sin(2\pi m b p_1)\sin(2\pi m b p_2)\sin(2\pi m b p_3)}{\sin(\pi m b^2)}$$

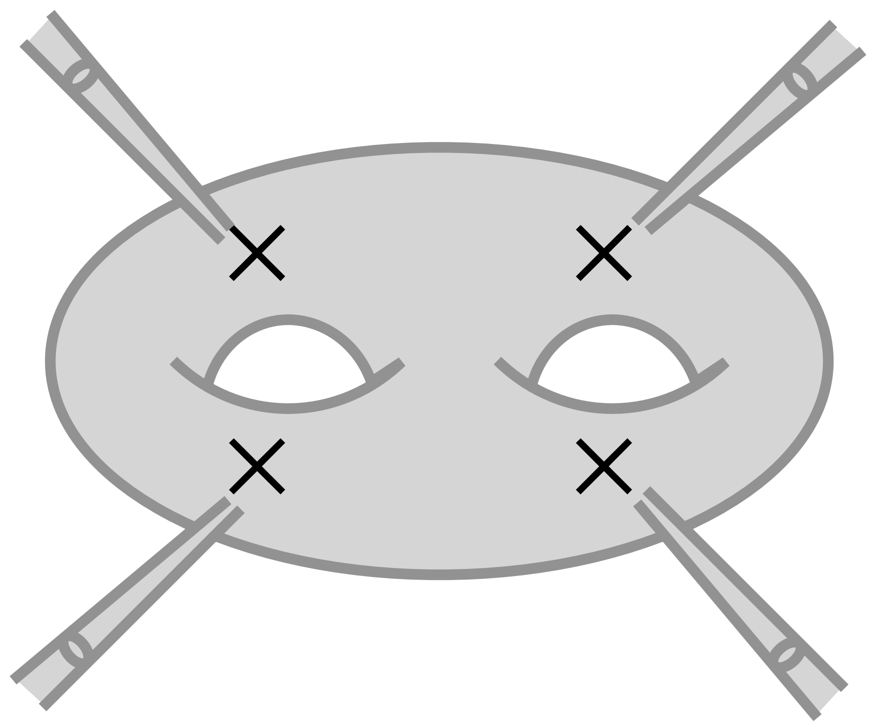
$$\mathsf{A}^{(b)}_{1,1}(p_1) = \sum_{m=1}^{\infty} \frac{(-1)^m b \sin(2\pi m b p_1)}{\sin(\pi m b^2)} \left(\mathsf{V}^{(b)}_{1,1}(ip_1) - \frac{1}{16\pi^2 b^2 m^2}\right)$$

# complex Liouville string

$\mathbb{C}$LS: Liouville CFT + (Liouville CFT)* + b, c ghosts

$c_+ = 13 + i\lambda$ $\quad c_- = 13 - i\lambda$ $\quad c_{\text{gh}} = -26$

$$\mathsf{A}^{(b)}_{g,n}(p_1, \ldots, p_n) = \left(\prod_{j=1}^{n} \mathcal{N}_b(p_j)\right) \int_{\mathcal{M}(\Sigma_{g,n})} \left| \left\langle \prod_{j=1}^{n} V_{p_j} \right\rangle^{(b)}_{\Sigma_g} \right|^2 \times \left(b, c \text{ ghosts}\right)$$



- Both VMS, CLS pert. amplitudes have been solved (e.g. dual topological recursion)

[Collier-Eberhardt-Muhlmann-**VAR** 23' 24']

# ADE minimal string

| | | | | | |
|---|---|---|---|---|---|
| ADE MS: | ADE minimal model CFT<br>$\hat{c} < 1$ | + | Liouville CFT<br>$c > 25$ | + | b, c ghosts<br>$c_{\text{gh}} = -26$ |

- Central charge: $c = 1 + 6(b + b^{-1})^2$ $\quad \hat{c} = 1 - 6\,(\beta^{-1} - \beta)^2$ $\quad (\beta^2 = p/p')$

$\hat{c} + c = 26 \;\; \rightarrow \;\; \beta = b$

Single parameter: $b$

# ADE minimal string

| | | | | | |
|---|---|---|---|---|---|
| ADE MS: | ADE minimal model CFT $\hat{c} < 1$ | + | Liouville CFT $c > 25$ | + | b, c ghosts $c_{\text{gh}} = -26$ |

- Central charge: $c = 1 + 6(b + b^{-1})^2$ $\hat{c} = 1 - 6\,(\beta^{-1} - \beta)^2$

$$\hat{c} + c = 26 \;\rightarrow\; \beta = b$$

Single parameter: $b$

- “On-shell” vertex operators:

$$\mathscr{V}_{P_{r,s}} = c\bar{c}\;\hat{V}^{(b)}_{P_{r,s}}\;V^{(b)}_{P\rightarrow iP_{r,s}}$$

Liouv. momenta: $P_{r,s}$

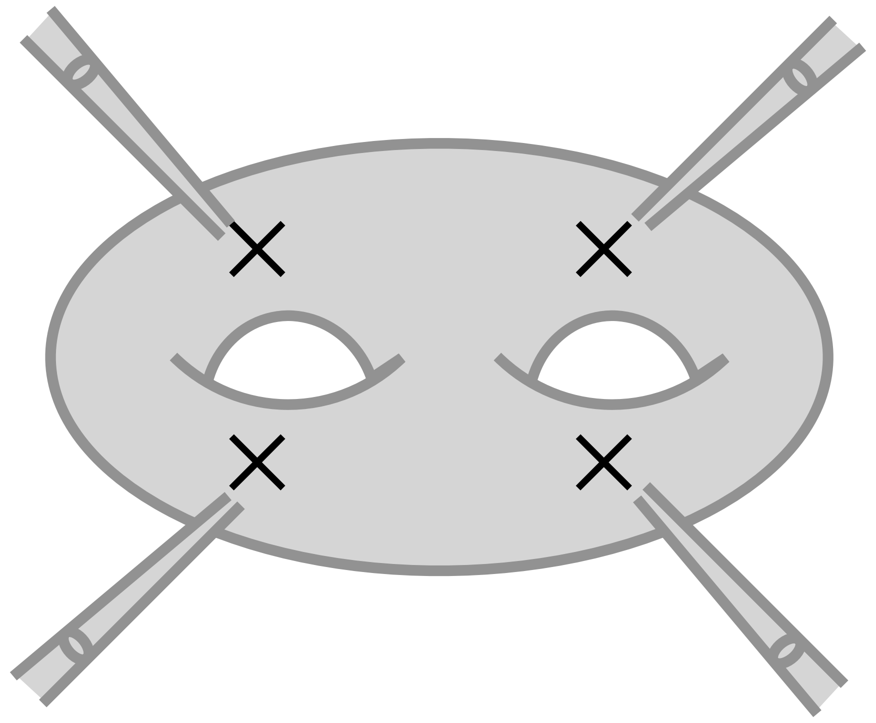
$$h_P + h_{P_{r,s}} = 1 \;\rightarrow\; P = iP_{r,s}$$

need to be careful!

# ADE minimal string

| | | | | | |
|---|---|---|---|---|---|
| ADE MS: | ADE minimal model CFT<br>$\hat{c} < 1$ | + | Liouville CFT<br>$c > 25$ | + | b, c ghosts<br>$c_{\text{gh}} = -26$ |

$$\mathrm{N}^{(b)}_{g,n}(P_{r_1,s_1},\ldots,P_{r_n,s_n}) = \int_{\mathscr{M}(\Sigma_{g,n})} \left\langle \prod_{j=1}^{n} V_{iP_{r_j,s_j}} \right\rangle^{(b)}_{\Sigma_g} \left\langle \prod_{j=1}^{n} \hat{V}_{P_{r_j,s_j}} \right\rangle^{(\beta=b)}_{\Sigma_g} \times \left(b, c\text{ ghosts}\right)$$

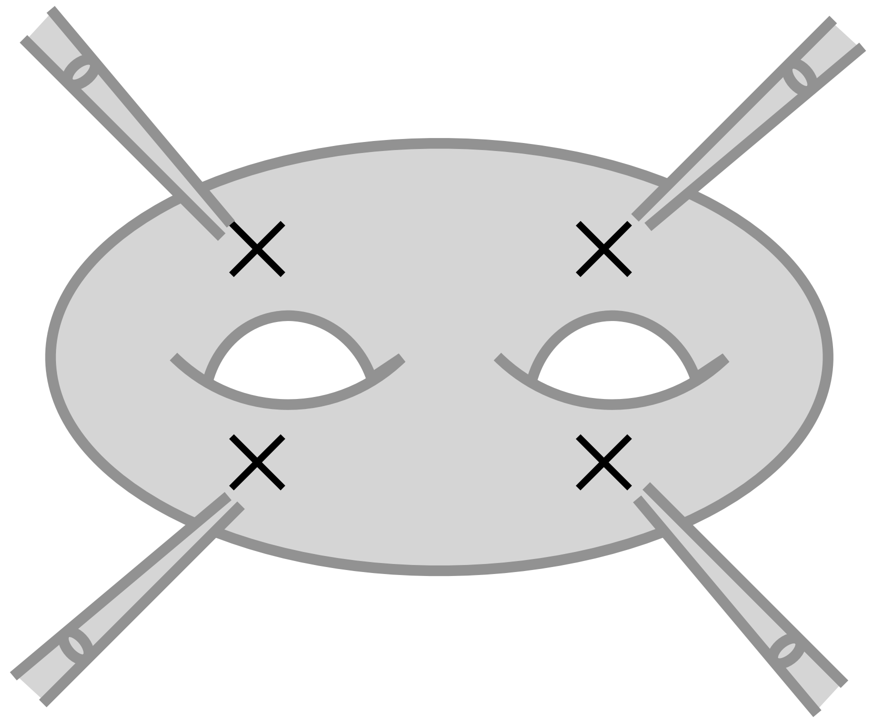


- NOT convergent moduli integrals, in general
- NOT like S-matrix elements (discontinuities, etc.)

# ADE minimal string

ADE MS: ADE minimal model CFT $\hat{c} < 1$ + Liouville CFT $c > 25$ + b, c ghosts $c_{\text{gh}} = -26$

$$\mathsf{N}^{(b)}_{g,n}(P_{r_1,s_1}, \ldots, P_{r_n,s_n}) = \int_{\mathcal{M}(\Sigma_{g,n})} \left\langle \prod_{j=1}^{n} V_{iP_{r_j,s_j}} \right\rangle^{(b)}_{\Sigma_g} \left\langle \prod_{j=1}^{n} \hat{V}_{P_{r_j,s_j}} \right\rangle^{(\beta=b)}_{\Sigma_g} \times \big(b, c\text{ ghosts}\big)$$



= integers !

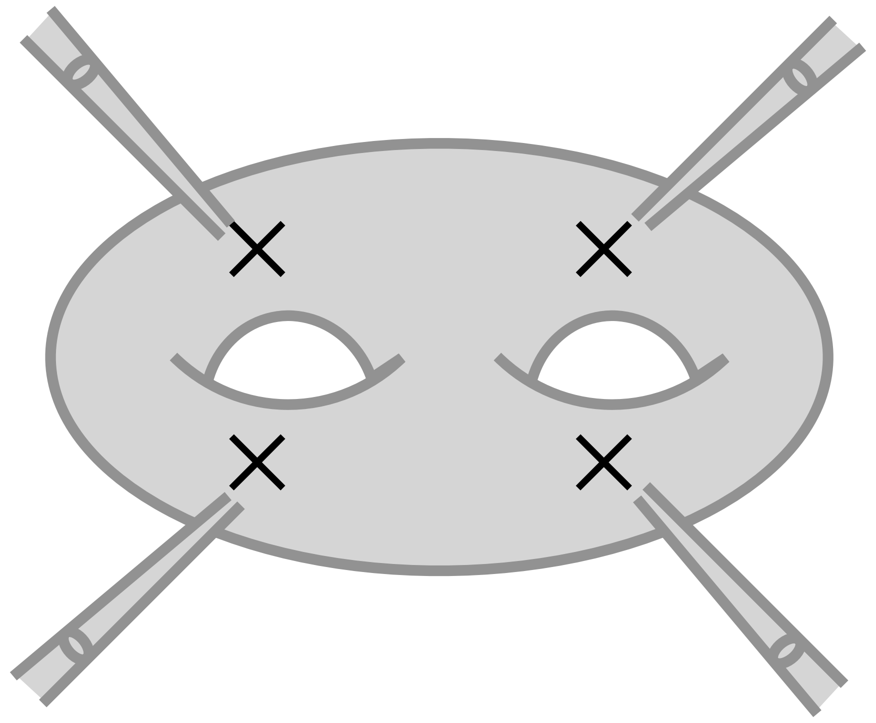
$$\mathsf{N}^{(b),\,\mathrm{BZ}}_{0,4}(P_{r_1,s_1}, P_{r_2,s_2}, P_{r_3,s_3}, P_{r_4,s_4}) = f(b)\left(r_1 s_1(r_1 p' + s_1 p) - \sum_{i=2}^{4} \sum_{r \overset{2}{=} 1-r_1}^{r_1-1} \sum_{s \overset{2}{=} 1-s_1}^{s_1-1} |(r_i - r)p' - (s_i - s)p|\right)$$

**AMS** — [BelavinZamolodchikov ’03]

# ADE minimal string

ADE MS: ADE minimal model CFT ($\hat{c} < 1$) + Liouville CFT ($c > 25$) + b, c ghosts ($c_{\text{gh}} = -26$)

$$\mathsf{N}^{(b)}_{g,n}(P_{r_1,s_1}, \ldots, P_{r_n,s_n}) = \int_{\mathscr{M}(\Sigma_{g,n})} \left\langle \prod_{j=1}^{n} V_{iP_{r_j,s_j}} \right\rangle^{(b)}_{\Sigma_g} \left\langle \prod_{j=1}^{n} \hat{V}_{P_{r_j,s_j}} \right\rangle^{(\beta=b)}_{\Sigma_g} \times \left(b, c\,\text{ghosts}\right)$$



= integers !

- Similar results for D- and E-series MS !

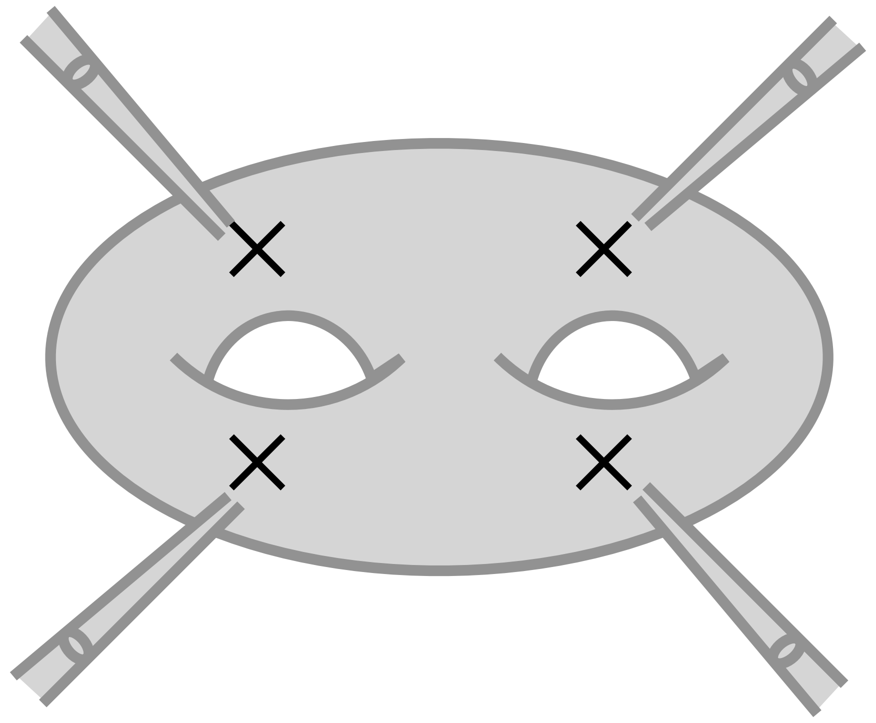
[**VAR**-Usatyuk-Wang ’25]

# ADE minimal string

ADE MS: ADE minimal model CFT +

$\hat{c} < 1$

$$\mathrm{N}^{(b)}_{g,n}(P_{r_1,s_1}, \ldots, P_{r_n,s_n}) = \int_{\mathscr{M}(\Sigma_{g,n})} \Big\langle$$



= integers !

| $(p,p')$ | DMS tachyon one-point string amplitude |
|---|---|
| 8,3 | $T_{1,1} = 10,\ T_{3,1} = 14,\ T_{5,1} = 6,\ T_{7,1} = 10$ |
| 10,3 | $T_{1,1} = 12,\ T_{3,1} = 20,\ T_{5,1} = 16,\ T_{7,1} = 0,\ T_{9,1} = -28$ |
| 14,3 | $T_{1,1} = 16,\ T_{3,1} = 32,\ T_{5,1} = 36,\ T_{7,1} = 28,\ T_{9,1} = 8,\ T_{11,1} = 0,\ T_{13,1} = -44$ |
| 16,3 | $T_{1,1} = 18,\ T_{3,1} = 38,\ T_{5,1} = 46,\ T_{7,1} = 42,\ T_{9,1} = 26,\ T_{11,1} = -2$<br>$T_{13,1} = -42,\ T_{15,1} = 26$ |
| 20,3 | $T_{1,1} = 22,\ T_{3,1} = 50,\ T_{5,1} = 66,\ T_{7,1} = 70,\ T_{9,1} = 62,\ T_{11,1} = 42,\ T_{13,1} = 10$<br>$T_{15,1} = -10,\ T_{17,1} = -66,\ T_{19,1} = 34$ |
| 22,3 | $T_{1,1} = 24,\ T_{3,1} = 56,\ T_{5,1} = 76,\ T_{7,1} = 84,\ T_{9,1} = 80,\ T_{11,1} = 64,\ T_{13,1} = 36$<br>$T_{15,1} = -4,\ T_{17,1} = -56,\ T_{19,1} = 0,\ T_{21,1} = -76$ |
| 6,5 | $T_{1,1} = 16,\ T_{3,1} = 0,\ T_{5,1} = -56$<br>$T_{1,3} = -16,\ T_{3,3} = 0,\ T_{5,3} = -4$ |
| 8,5 | $T_{1,1} = 20,\ T_{3,1} = 12,\ T_{5,1} = -36,\ T_{7,1} = 44$<br>$T_{1,3} = -18,\ T_{3,3} = 10,\ T_{5,3} = 18,\ T_{7,3} = 6$ |
| 12,5 | $T_{1,1} = 28,\ T_{3,1} = 36,\ T_{5,1} = 4,\ T_{7,1} = 4,\ T_{9,1} = -108,\ T_{11,1} = 76$<br>$T_{1,3} = -70,\ T_{3,3} = -18,\ T_{5,3} = 14,\ T_{7,3} = 26,\ T_{9,3} = 18,\ T_{11,3} = 14$ |
| 14,5 | $T_{1,1} = 32,\ T_{3,1} = 48,\ T_{5,1} = 24,\ T_{7,1} = -16,\ T_{9,1} = -120,\ T_{11,1} = 0$<br>$T_{13,1} = -184,\ T_{1,3} = -120,\ T_{3,3} = -56,\ T_{5,3} = -12,\ T_{7,3} = 12,\ T_{9,3} = 16$<br>$T_{11,3} = 0,\ T_{13,3} = -36$ |
| 16,5 | $T_{1,1} = 36,\ T_{3,1} = 60,\ T_{5,1} = 44,\ T_{7,1} = -12,\ T_{9,1} = -108,\ T_{11,1} = -28$<br>$T_{13,1} = -204,\ T_{15,1} = 108$<br>$T_{1,3} = -162,\ T_{3,3} = -86,\ T_{5,3} = -30,\ T_{7,3} = 6,\ T_{9,3} = 22,\ T_{11,3} = 18$<br>$T_{13,3} = -6,\ T_{15,3} = 22$ |

# Matrix Models
# (brief)

# Duals

| | | | |
|---|---|---|---|
| VMS: | 1-matrix integral | (special case of 2-matrix) | |
| $\mathbb{C}$LS: | 2-matrix integral | | |
| ADE MS: | 2-matrix integral | 4+ matrix integral | 4+ matrix integral |
| | A-series | D-series | E-series |

# 2-matrix integrals

- A class of two-matrix models with minimal coupling [review Eynard-Ribault ;…]

$$\int_{\mathbb{R}^{N\times N}} \mathrm{d}M_1\,\mathrm{d}M_2\ \mathrm{e}^{-N\,\mathrm{tr}(V_1(M_1)+V_2(M_2)-M_1M_2)}$$

- Main observable are resolvents, $R^{(j)}(x) \equiv \mathrm{tr}\,\dfrac{1}{x-M_j}$

$$R(x_1,\ldots,x_n) = \prod_{j=1}^{n} R(x_j)$$

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$R(x_1,\ldots,x_n) = \prod_{j=1}^{n} R(x_j)$

- Admit a 'genus' expansion

$$\langle R(x_1,\ldots,x_n)\rangle_c = \sum_{g=0}^{\infty} R_{g,n}(x_1,\ldots,x_n)\,N^{2-2g-n}$$

multi-valued functions in $x_j$

- Cuts along the real axis, e.g. 'eigenvalue density'

$$\rho_0(x) = -\frac{1}{2\pi i}(R_{0,1}(x+i\varepsilon) - R_{0,1}(x-i\varepsilon))$$

$\rho_0(x) = \sum_i \delta(x-\lambda_i)$

# Spectral curves

💪

[review Eynard-Ribault; Eynard-Orantin,…]

- Nice fact: all resolvents $R_{g,n}(\boldsymbol{x})$ are *recursively* determined by spectral curve (Riemann surface on which resolvents are single-valued)

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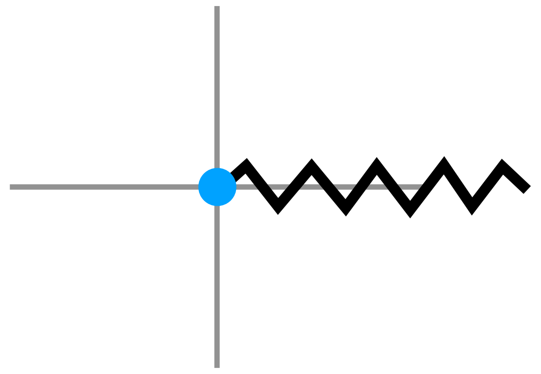
JT gravity

$$\mathrm{x}(z) = -z^2\,,$$
$$\mathrm{y}(z) = \frac{\sin(2\pi z)}{z}$$

  - 1 branch pt $\mathrm{dx}(z_m^*) = 0$

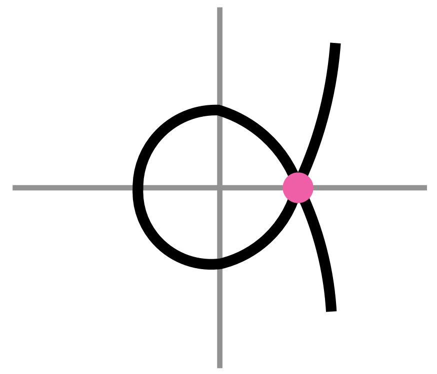
$$z^* = 0$$



  - ∞ many nodal singularities

$$\mathrm{x}(z^+_{(r,s)}) = \mathrm{x}(z^-_{(r,s)})$$
$$\mathrm{y}(z^+_{(r,s)}) = \mathrm{y}(z^-_{(r,s)})$$

$$z^\pm_{(r,s)} = (rb \pm sb^{-1})^2\,, \qquad r,s \in \mathbb{Z}_{\geq 1}$$



associated with instantons

# Spectral curves

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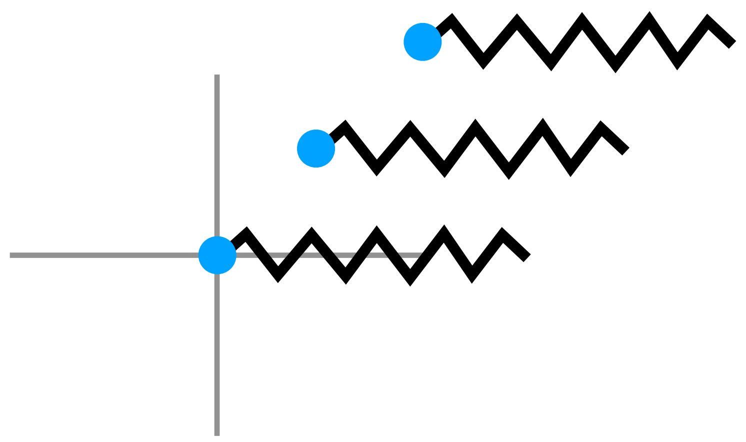
[review Eynard-Ribault; Eynard-Orantin,…]

- $\mathbb{C}$LS:

$$\mathrm{x}(z) = -2\cos(\pi b^{-1}\sqrt{z})\ , \quad \mathrm{y}(z) = 2\cos(\pi b\sqrt{z})$$

  - $\infty$ many branch pts $\quad \mathrm{dx}(z_m^*) = 0$

$$z_m^* = (mb)^2\ , \qquad m \in \mathbb{Z}_{\geq 1}$$



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$$\mathrm{x}(z_{(r,s)}^+) = \mathrm{x}(z_{(r,s)}^-)$$
$$\mathrm{y}(z_{(r,s)}^+) = \mathrm{y}(z_{(r,s)}^-)$$

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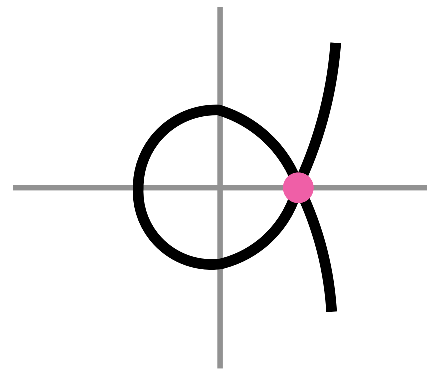


associated with instantons

# Spectral curves

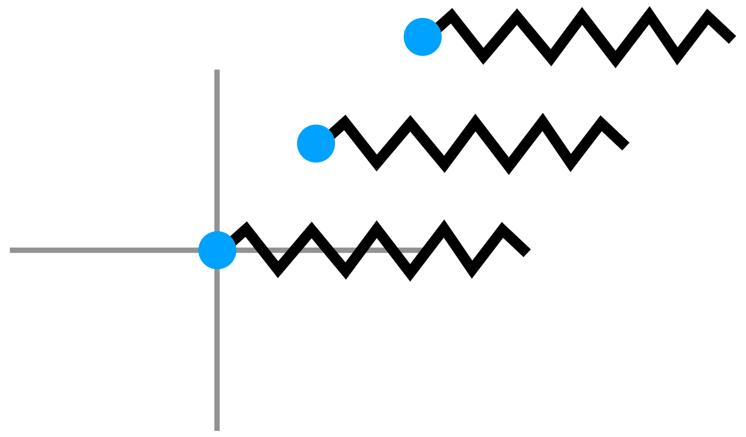
- Nice fact: all resolvents $R_{g,n}(x)$ are *recursively* determined by spectral curve (Riemann surface on which resolvents are single-valued) [review Eynard-Ribault; Eynard-Orantin,…]

- ADE MS: $\mathrm{x}(z) = -2T_p(z)\,, \quad \mathrm{y}(z) = 2T_{p'}(z)$ $\quad (\beta^2 = p/p')$

  Chebyshev polys

  - finitely many branch pts $\mathrm{dx}(z_m^*) = 0$

    $$z_m^* = (mb)^2\,, \qquad m \in \mathbb{Z}_{\geq 1}$$



  - finitely many nodal singularities $\mathrm{x}(z_{(r,s)}^+) = \mathrm{x}(z_{(r,s)}^-)$, $\mathrm{y}(z_{(r,s)}^+) = \mathrm{y}(z_{(r,s)}^-)$

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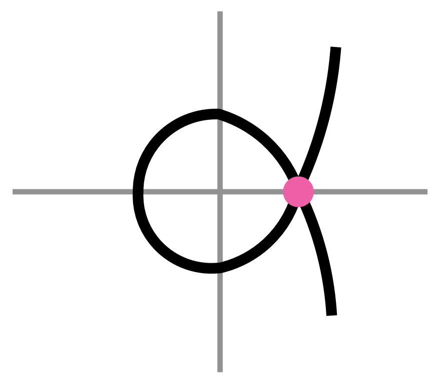

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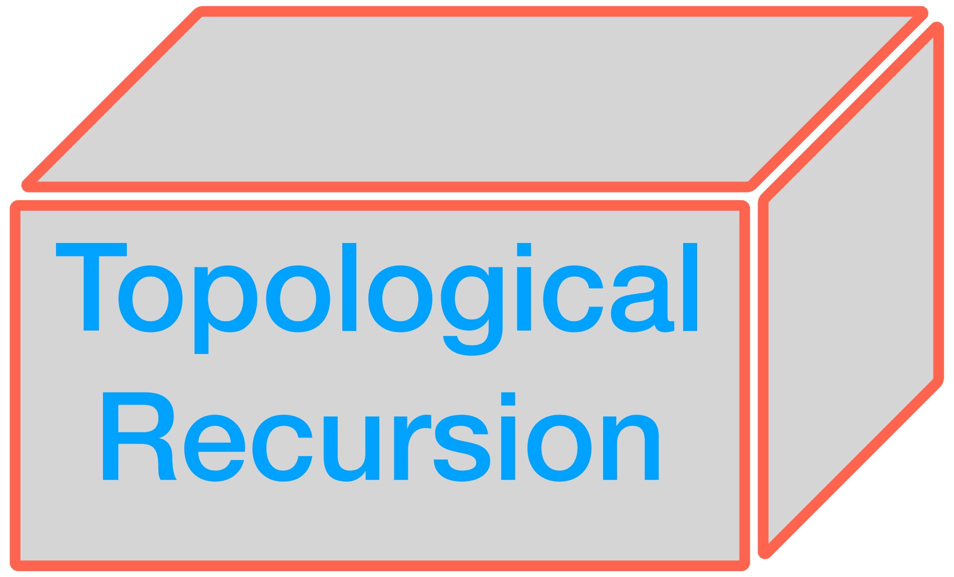
[review Eynard-Ribault; Eynard-Orantin,…]

spectral curve
$\mathsf{x}(z)\,,\,\mathsf{y}(z)$ → 




→ all resolvents
$R_{g,n}(\mathsf{x}(z_1),\ldots,\mathsf{x}(z_n))$

[Eynard; Chekhov-Eynard-Orantin;…]

- Dictionary: integral transform back to string amplitudes $R_{g,n}(\boldsymbol{x}) \leftrightarrow \mathsf{A}_{g,n}(\boldsymbol{P})$

VMS , ℂLS , AMS
✓ ✓ ?

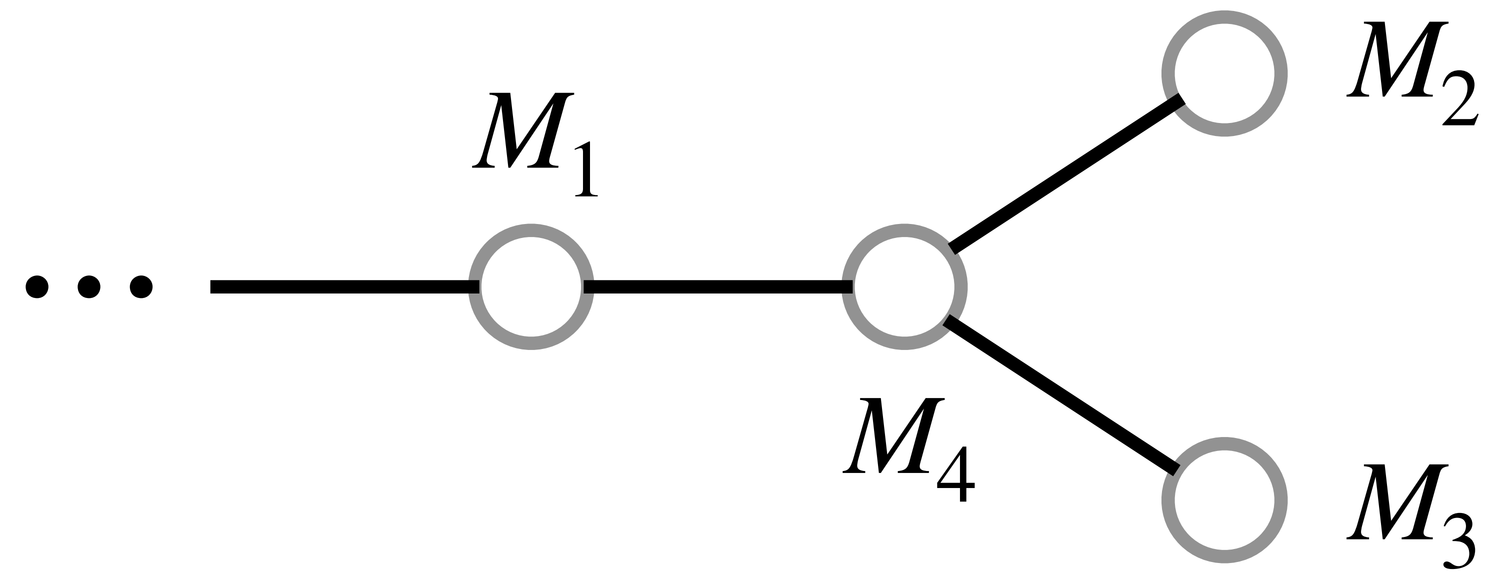
# Beyond standard TR?

- D-series and E-series minimal string ?

$$\int_{\mathbb{R}^{N\times N}} \prod_{j=1}^{4} \mathrm{d}M_j \quad \mathrm{e}^{-N\,\mathrm{tr}\left(\sum_j V_j(M_j) - M_4(M_1+M_2+M_3)\right)}$$

[VAR-Usatyuk-Wang]

evidence for 4+ matrix structure from conformal boundaries on the string ws




- No known topological recursion structure!

# Positivity bootstrap

[Lin, Kazakov-Zheng,…]

- Bootstrap

$$\int_{\mathbb{R}^{N\times N}} \prod_{j=1}^{4} \mathrm{d}M_j \;\; \mathrm{e}^{-N\,\mathrm{tr}(\sum_j V_j(M_j) - h\, M_4(M_1+M_2+M_3))}, \quad V_j(M) = \frac{1}{2}M^2 + g\frac{1}{4}M^4$$

see Zechuan's talk!

# Positivity bootstrap

[Lin, Kazakov-Zheng,…]

- Bootstrap

$$\int_{\mathbb{R}^{N\times N}} \prod_{j=1}^{4} \mathrm{d}M_j \;\; \mathrm{e}^{-N\,\mathrm{tr}(\sum_j V_j(M_j) - h\, M_4(M_1+M_2+M_3))}, \qquad V_j(M) = \frac{1}{2}M^2 + g\frac{1}{4}M^4$$

see Zechuan's talk!

1. Loop equations

$$\int_{\mathbb{R}^{N\times N}} \prod_{j=1}^{4} \mathrm{d}M_j \frac{\partial}{\partial M_i}\left(\mathcal{O}\,\mathrm{e}^{-N\,\mathrm{tr} V_{\mathrm{tot}}}\right) = 0$$

$\mathcal{O}$ = arbitrary word in $M_j$

e.g. $\mathcal{O} = M_1^7 M_2^3 M_3 M_4^2$

2. Positivity

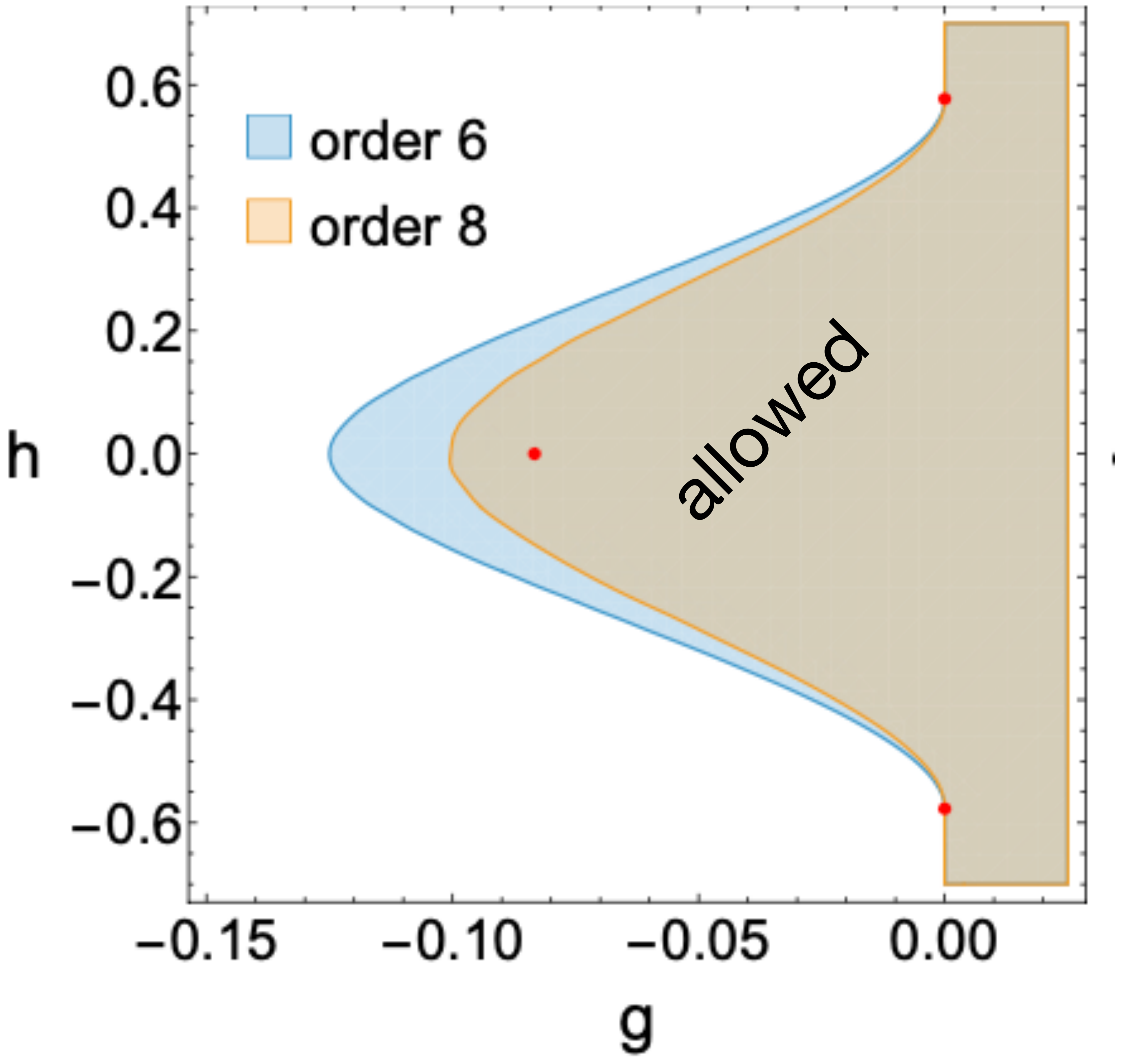
$$\langle \mathrm{tr}\, O^\dagger O \rangle \geq 0 \quad \leftrightarrow \quad \boldsymbol{M} \geq 0 \text{ with } \boldsymbol{M}_{ij} = \langle \mathrm{tr}\, O_i^\dagger O_j \rangle$$

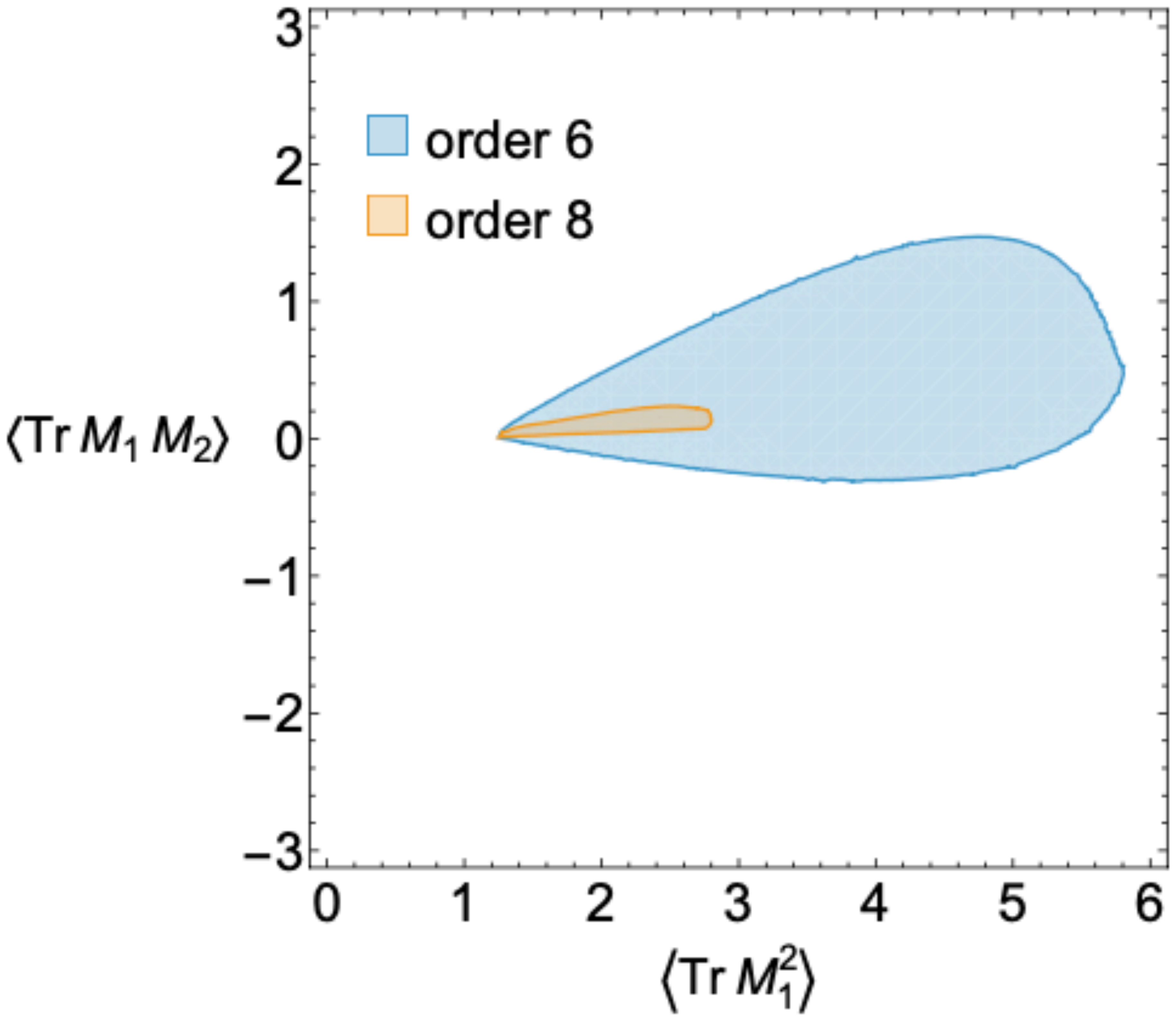
topological recursion ~ analyticity instead of positivity

# Positivity bootstrap

[Lin, Kazakov-Zheng,…]

- Bootstrap $$\int_{\mathbb{R}^{N\times N}} \prod_{j=1}^{4} \mathrm{d}M_j \;\; \mathrm{e}^{-N\,\mathrm{tr}(\sum_j V_j(M_j) - h\, M_4(M_1+M_2+M_3))}, \quad V_j(M) = \frac{1}{2}M^2 + g\frac{1}{4}M^4$$




[**VAR**-Usatyuk-Wang ’25]

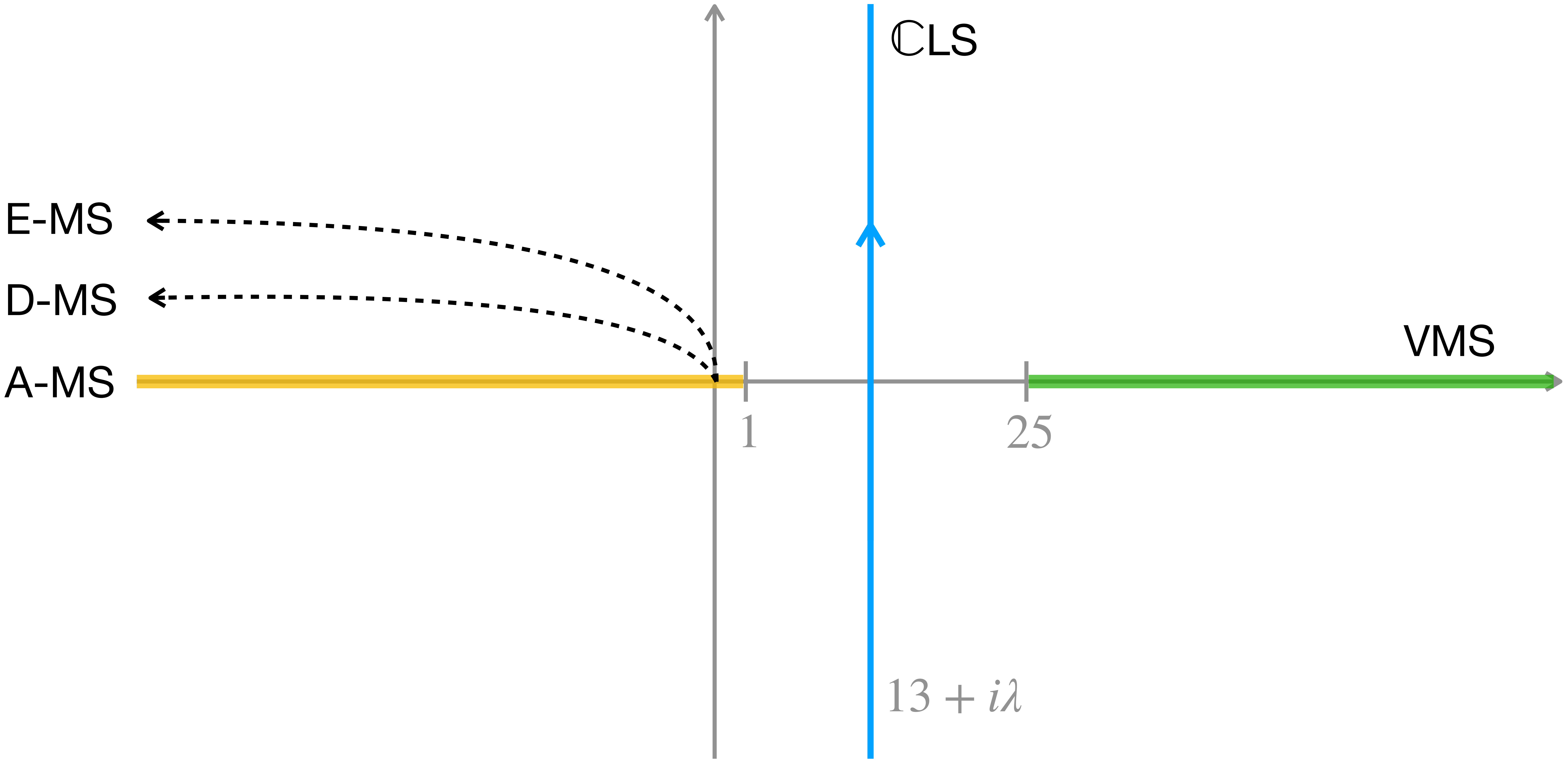
# Outlook

- DMS and EMS — not solved, but amplitudes are very simple (integers)
  there should be some TR structure

- Unknown: - open strings on D-branes - GMM - non-analytic Liouv. - …

- $\mathbb{C}$LS as grandparent of all others?
  is there a string theory like this in higher-d?

- $c = 1$ string theory ↔ Matrix *Quantum Mechanics* 0+1 dim
  S-matrix of $c = 1$ [WIP Collier, Eberhardt, **VAR**]


ℂLS
E-MS
D-MS
A-MS
VMS
1
25
$13 + i\lambda$