BOOTSTRAP METHOD IN THE BFSS MODEL

2410.14647 && 2507.21007 &&2511.08560

Zechuan Zheng

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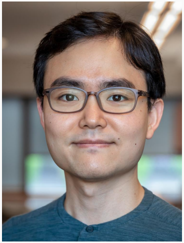
Perimeter Institute

Co-authors

2410.14647 w/ H. Lin
2507.21007 w/ H. Lin
2511.08560



Minjae

Barak

Henry

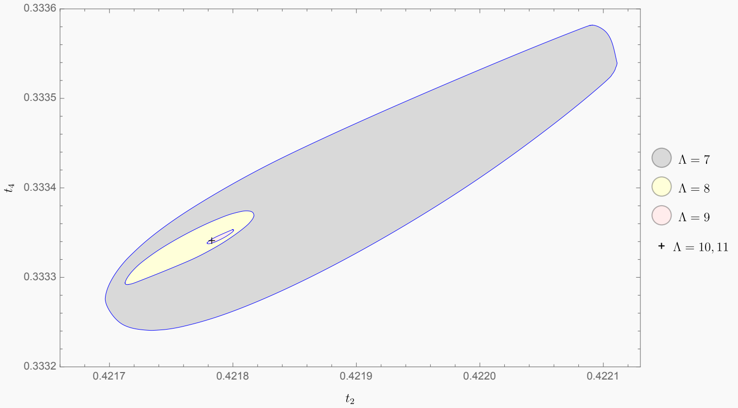
Jessica

Background

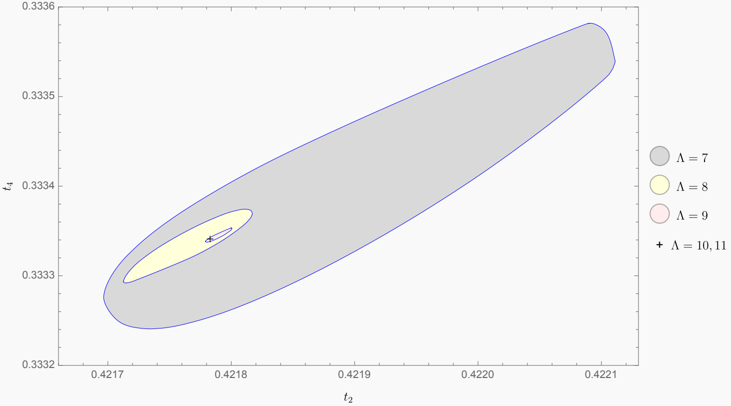
- Pioneering work on matrix quantum mechanics bootstrap[Han et al., 2020].
- Pioneering BFSS bootstrap [Lin, 2023]
- See also
  [Anderson and Kruczenski, 2017][Lin, 2020][Kazakov and Zheng, 2022]
  [Kazakov and Zheng, 2023][Kazakov and Zheng, 2024]
  [Cho et al., 2024]

Multi-matrix bootstrap[Kazakov and Zheng, 2022]

$$Z = \lim_{N\to\infty} \int d^{N^2}A\, d^{N^2}B\, \mathrm{e}^{-N\mathrm{tr}\left(-h[A,B]^2/2 + A^2/2 + gA^4/4 + B^2/2 + gB^4/4\right)} \quad (1)$$



Multi-matrix bootstrap[Kazakov and Zheng, 2022]



$$\Lambda = 11,\ g = h = 1: \begin{cases} 0.421783612 \leq \langle \mathrm{Tr} A^2 \rangle \leq 0.421784687 \\ 0.333341358 \leq \langle \mathrm{Tr} A^4 \rangle \leq 0.333342131 \end{cases} \quad (3)$$

Small pieces of Optimization theory

Basically bootstrap method is solving problems in theoretical physics by optimization theory.

- Quadratic programming:

$$\begin{aligned} \min \quad & y \\ \text{s.t.} \quad & y = x^2 + 3x + 1 \end{aligned} \tag{4}$$

- Linear programming:

$$\begin{aligned} \max \quad & 300x + 100y \\ \text{s.t.} \quad & 6x + 3y \leq 40 \\ & x - 3y \leq 0 \\ & x + \tfrac{1}{4}y \leq 4 \end{aligned} \tag{5}$$

Semi-definite Programming

- Semi-definite Programming:

$$\begin{aligned} \min \quad & 2x + 3y \\ \text{s.t.} \quad & \begin{pmatrix} x & 1 \\ 1 & y \end{pmatrix} \succeq 0 \end{aligned} \tag{6}$$

- Linear programming and Quadratic programming are special situations of Semi-definite Programming(SDP).
- They all fall into the class of Convex Optimization.
- Generally we cannot solve large-scale non-convex optimization problems.

Why bootstrap

MC:

- physics simplifies at large N but the computation gets harder
- sign problem, finite volume truncation, finite N truncation

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Bootstrap:

- works in large N directly; gives rigorous bounds
- no sign problem, no finite volume or finite N truncation

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- works in large N directly; gives rigorous bounds
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The Hamiltonian is chosen to be [Banks, Fischler, Shenker, Susskind 97]:

$$H = \frac{1}{2}\,\mathrm{Tr}\left(g^2 P_I^2 - \frac{1}{2g^2}\left[X_I, X_J\right]^2 - \psi_\alpha \gamma^I_{\alpha\beta}\left[X_I, \psi_\beta\right]\right) \tag{7}$$

Here:

$$[X_{ij}, P_{kl}] = i\delta_{il}\delta_{jk},\ \{\psi_{\alpha,ij}, \psi_{\beta,kl}\} = \delta_{\alpha\beta}\delta_{il}\delta_{kj} \tag{8}$$

Dual to the dynamics of the D0-brane.

The matrices are in multiples of the $SO(9)$ symmetry, with the supercharge:

$$Q_\alpha = g\,\mathrm{tr} P_I \gamma^I_{\alpha\beta}\psi_\beta - \frac{i}{2g}\mathrm{tr}[X^I, X^J]\gamma^{IJ}_{\alpha\beta}\psi_\beta \tag{9}$$

The goal of this work is to bootstrap the ground state of this model.

Toy model

The Hamiltonian is chosen to be($g = 1$):

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}gx^2y^2 \tag{10}$$

This model is not trivially solved by numerical method/analytical method[Hoppe, 1980] [Simon, 1983][Komatsu et al., 2024]. It keeps certain key feature of the BFSS Hamiltonian:

$$H = \frac{1}{2}\,\mathrm{Tr}\left(g^2P_I^2 - \frac{1}{2g^2}\left[X_I, X_J\right]^2 - \psi_\alpha\gamma^I_{\alpha\beta}\left[X_I, \psi_\beta\right]\right) \tag{11}$$

Equations of motion

Our goal is to solve all the eigenvalues and all the expectations of the operators under different eigenstates.

For an eigenstate with eigenvalue E, the corresponding loop equations are:

$$\langle [H, \mathcal{O}] \rangle = 0, \ \forall \mathcal{O} \tag{12}$$

$$\langle H\mathcal{O} \rangle = E \langle \mathcal{O} \rangle, \ \forall \mathcal{O} \tag{13}$$

together with the Ward identities:

$$\langle \mathcal{O}_g \rangle = \langle \mathcal{O} \rangle, \ \forall \mathcal{O} \tag{14}$$

These are all linear equations, we can expand any operators as:

$$\mathcal{O} = \sum \alpha_{mnts} p_x^m p_y^n x^t y^s \tag{15}$$

Equations of motion

Here $\pi_x = -ip_x$ and $\pi_y = -ip_y$:

$$
\begin{aligned}
&\tfrac{1}{2}\left\langle x^2y^4\right\rangle - \tfrac{1}{2}\left\langle \pi_x^2 y^2\right\rangle - E\left\langle y^2\right\rangle - \tfrac{1}{2}\left\langle \pi_y^2 y^2\right\rangle - 2\left\langle \pi_y y\right\rangle - 1 = 0\\
&\tfrac{1}{2}\left\langle x^4y^2\right\rangle - \tfrac{1}{2}\left\langle \pi_y^2 x^2\right\rangle - E\left\langle x^2\right\rangle - \tfrac{1}{2}\left\langle \pi_x^2 x^2\right\rangle - 2\left\langle \pi_x x\right\rangle - 1 = 0\\
&\tfrac{1}{2}\left\langle \pi_y x^2y^3\right\rangle - \tfrac{1}{2}\left\langle \pi_y\pi_x^2 y\right\rangle - \left\langle \pi_y^2\right\rangle - E\left\langle \pi_y y\right\rangle - \tfrac{1}{2}\left\langle \pi_y^3 y\right\rangle = 0\\
&\tfrac{1}{2}\left\langle \pi_y x^3y^2\right\rangle - E\left\langle \pi_y x\right\rangle - \left\langle \pi_y\pi_x\right\rangle - \tfrac{1}{2}\left\langle \pi_y^3 x\right\rangle - \tfrac{1}{2}\left\langle \pi_y\pi_x^2 x\right\rangle = 0\\
&\tfrac{1}{2}\left\langle \pi_x x^2y^3\right\rangle - E\left\langle \pi_x y\right\rangle - \tfrac{1}{2}\left\langle \pi_x^3 y\right\rangle - \left\langle \pi_y\pi_x\right\rangle - \tfrac{1}{2}\left\langle \pi_y^2\pi_x y\right\rangle = 0\\
&\tfrac{1}{2}\left\langle \pi_x x^3y^2\right\rangle - \tfrac{1}{2}\left\langle \pi_y^2\pi_x x\right\rangle - \left\langle \pi_x^2\right\rangle - E\left\langle \pi_x x\right\rangle - \tfrac{1}{2}\left\langle \pi_x^3 x\right\rangle = 0\\
&-\left\langle \pi_x y\right\rangle - \left\langle \pi_y x\right\rangle - \tfrac{1}{2}\left\langle \pi_x^2 xy\right\rangle - \tfrac{1}{2}\left\langle \pi_y^2 xy\right\rangle = 0
\end{aligned}
$$

Positivity by inner product

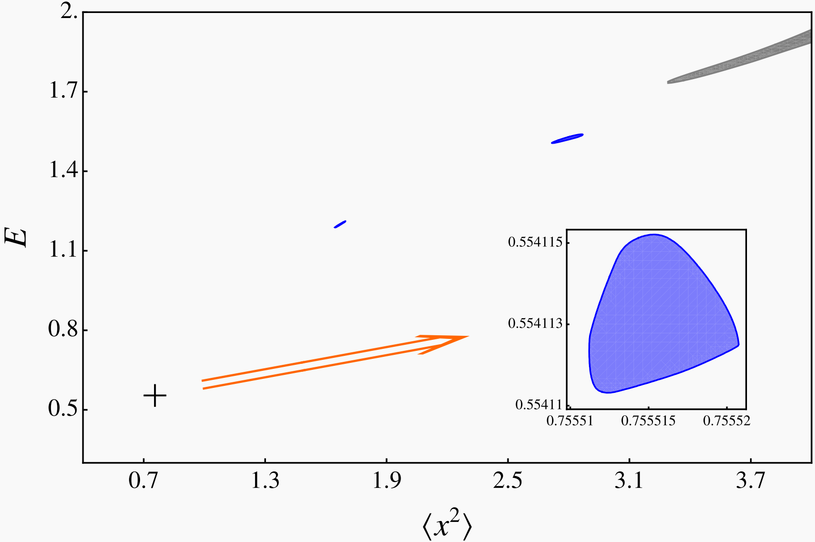
Generalization: Any inner products defined on the vector space of operators or its subspace could leads to positivity condition:

$$\langle \mathcal{O}|\mathcal{O}\rangle = \langle \mathcal{O}^\dagger \mathcal{O}\rangle = \alpha^{*\mathrm{T}} \mathcal{M} \alpha \geq 0,\ \forall \alpha \Leftrightarrow \mathcal{M} \succeq 0. \tag{16}$$

Here we do the expansion $\mathcal{O} = \sum \alpha_i \mathcal{O}_i$, $\mathcal{M}_{ij} = \langle \mathcal{O}_i^\dagger \mathcal{O}_j \rangle$.

$$\begin{array}{c} \\ 1 \\ x^2 \\ p_x^2 \\ p_x x \\ \vdots \end{array}
\begin{array}{c} \begin{array}{ccccc} 1 & x^2 & p_x^2 & x p_x & \cdots \end{array} \\
\begin{pmatrix}
1 & \langle x^2 \rangle & \langle p_x^2 \rangle & \langle x p_x \rangle & \cdots \\
\langle x^2 \rangle & \langle x^4 \rangle & \langle x^2 p_x^2 \rangle & \langle x^3 p_x \rangle & \cdots \\
\langle p_x^2 \rangle & \langle p_x^2 x^2 \rangle & \langle p_x^4 \rangle & \langle p_x^2 x p_x \rangle & \cdots \\
\langle p_x x \rangle & \langle p_x x^3 \rangle & \langle p_x x p_x^2 \rangle & \langle p_x x^2 p_x \rangle & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} \end{array} \succeq 0.$$

Toy model($\Lambda = 12$)



Ground State Positivity

For the ground state, or more generally, any stationary state, the corresponding positivities are:

$$\langle \mathcal{O}^\dagger \mathcal{O} \rangle \geq 0,\ \forall \mathcal{O} \tag{17}$$

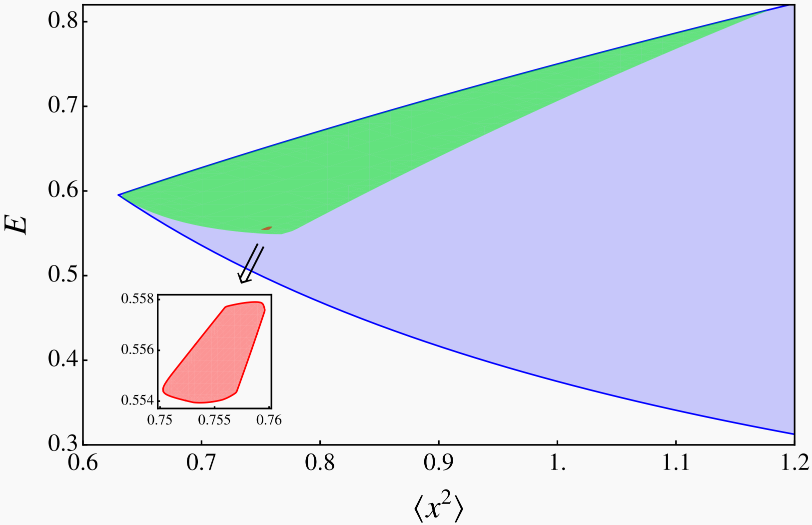
$$\langle \mathcal{O}^\dagger [H, \mathcal{O}] \rangle \geq 0,\ \forall \mathcal{O} \tag{18}$$

The later positivity is specialized for the ground state. For more general thermal state with inverse temperature β [Fawzi, Fawzi and Scalet 23][Cho et al., 2024],

$$\langle \mathcal{O}^\dagger \mathcal{O} \rangle \log \frac{\langle \mathcal{O}^\dagger \mathcal{O} \rangle}{\langle \mathcal{O} \mathcal{O}^\dagger \rangle} \leq \beta \langle \mathcal{O}^\dagger [H, \mathcal{O}] \rangle,\ \forall \mathcal{O} \tag{19}$$

Mathematically, these positivities together with the loop equations are necessary and sufficient.

Ground state($\Lambda = 4, 6, 8$)



Ground state($\Lambda = 10$)

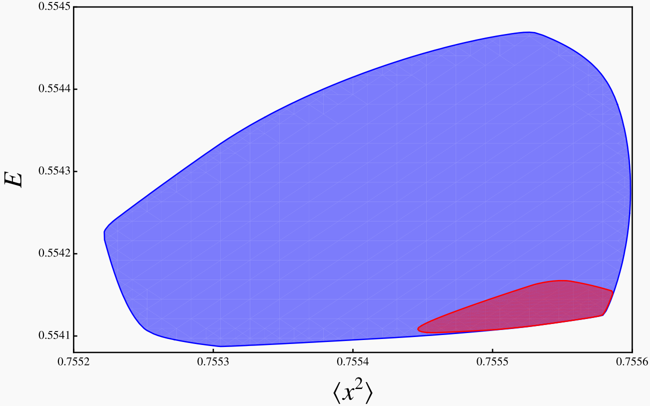


Figure 1: The bootstrap allowed region at $\Lambda = 10$ with/without ground state positivity.

One-matrix quantum mechanics

The Hamiltonian is chosen to be:

$$H = \mathrm{tr}(P^2 + X^2 + gX^4) \tag{20}$$

Here X is a large N Hermitian matrix:

$$[X_{ij}, P_{kl}] = i\delta_{il}\delta_{jk} \tag{21}$$

The ground state is known to be solvable.

Loop equations

The corresponding loop equations are:

$$\langle [H, \mathcal{O}] \rangle = 0, \ \forall \mathcal{O} \tag{22}$$

$$\langle \mathrm{tr}(G\mathcal{O}) \rangle = 0, \ \forall \mathcal{O} \tag{23}$$

together with the cyclicity of $\mathrm{tr}\mathcal{O}$. $G = i[X, P] + I$ is the generator of the $SU(N)$ gauge symmetry.

Result: general words in P and X can be reduced to polynomials of $\mathrm{tr}X^m$.

$$\langle \mathrm{tr}P^2X^2P^2X^4 \rangle = \frac{12}{77}g^2\langle \mathrm{tr}X^{14} \rangle - \frac{2}{3}g\langle \mathrm{tr}X^2 \rangle\langle \mathrm{tr}X^6 \rangle - \frac{1}{5}g\langle \mathrm{tr}X^8 \rangle + \frac{40}{231}g\langle \mathrm{tr}X^{12} \rangle + \frac{\langle \mathrm{tr}X^2 \rangle}{24} - \frac{1}{3}\langle \mathrm{tr}X^2 \rangle\langle \mathrm{tr}X^4 \rangle - \frac{\langle \mathrm{tr}X^6 \rangle}{10} + \frac{\langle \mathrm{tr}X^{10} \rangle}{21} \tag{24}$$

Positivity

For the ground state, or more generally, any stationary state, the corresponding loop equations are:

$$\langle \mathcal{O}^\dagger \mathcal{O} \rangle \geq 0, \ \forall \mathcal{O} \tag{25}$$

$$\langle \mathcal{O}^\dagger [H, \mathcal{O}] \rangle \geq 0, \ \forall \mathcal{O} \tag{26}$$

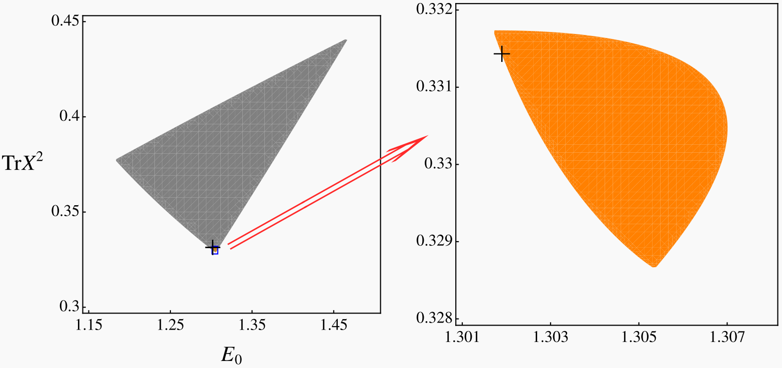
The later positivity is specialized for the ground state. For more general thermal state with inverse temperature β [Fawzi, Fawzi and Scalet 23] [Cho et al., 2024],

$$\langle \mathcal{O}^\dagger \mathcal{O} \rangle \log \frac{\langle \mathcal{O}^\dagger \mathcal{O} \rangle}{\langle \mathcal{O} \mathcal{O}^\dagger \rangle} \leq \beta \langle \mathcal{O}^\dagger [H, \mathcal{O}] \rangle, \ \forall \mathcal{O} \tag{27}$$

Mathematically, these positivities together with the loop equations is necessary and sufficient.

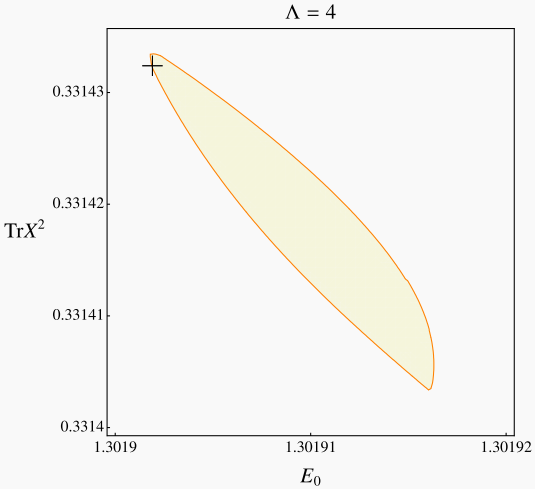
Convergence

The illustration of convergence, the left one is $\Lambda = 2$, whereas the right one corresponds to $\Lambda = 3$. The size of the SDP matrix is $2, 2, 2$ and $3, 3, 2, 3$, respectively,



Convergence

The size of the SDP matrices are $5, 4, 4, 4$.



BFSS model

The Hamiltonian is chosen to be [Banks, Fischler, Shenker, Susskind 97]:

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Here:

$$[X_{ij}, P_{kl}] = i\delta_{il}\delta_{jk},\ \{\psi_{\alpha,ij}, \psi_{\beta,kl}\} = \delta_{\alpha\beta}\delta_{il}\delta_{kj} \tag{29}$$

Dual to the dynamics of the D0-brane.

The matrices are in multiples of the $SO(9)$ symmetry, with the supercharge:

$$Q_\alpha = g\,\mathrm{tr} P_I \gamma^I_{\alpha\beta}\psi_\beta - \frac{i}{2g}\mathrm{tr}[X^I, X^J]\gamma^{IJ}_{\alpha\beta}\psi_\beta \tag{30}$$

BFSS model bootstrap

Equalities:

$$\langle [H, \mathcal{O}] \rangle = 0 \tag{31}$$

$$\langle \{Q_\alpha, \mathcal{O}_\alpha\} \rangle = 0 \tag{32}$$

$$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle = \langle \mathcal{O}_2 \mathcal{O}_1 \rangle - \langle [\mathcal{O}_1, \mathcal{O}_2] \rangle \tag{33}$$

$$\langle \mathrm{tr}(C_{ij} \mathcal{O}_{ji}) \rangle = 0,\ \forall \mathcal{O} \tag{34}$$

This is the gauge singlet condition:

$$C_{ij} = -\mathrm{i}[X^I, P^I]_{ij} - \psi^\alpha_{ik} \psi^\alpha_{kj} - \mathbf{1}_{ij} \tag{35}$$

Positivities:

$$\langle \mathcal{O}^\dagger \mathcal{O} \rangle \geq 0,\ \forall \mathcal{O} \tag{36}$$

$$\langle \mathcal{O}^\dagger [H, \mathcal{O}] \rangle \geq 0,\ \forall \mathcal{O} \tag{37}$$

BFSS MODEL BOOTSTRAP

Equalities:

$$\cancel{\langle [H, \mathcal{O}] \rangle = 0} \tag{38}$$

$$\langle \{ Q_\alpha, \mathcal{O}_\alpha \} \rangle = 0 \tag{39}$$

$$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle = \langle \mathcal{O}_2 \mathcal{O}_1 \rangle - \langle [\mathcal{O}_1, \mathcal{O}_2] \rangle \tag{40}$$

$$\langle \mathrm{tr}(C_{ij} \mathcal{O}_{ji}) \rangle = 0, \ \forall \mathcal{O} \tag{41}$$

This is the gauge singlet condition:

$$C_{ij} = -\mathrm{i}[X^I, P^I]_{ij} - \psi^\alpha_{ik} \psi^\alpha_{kj} - \mathbf{1}_{ij} \tag{42}$$

Positivities:

$$\langle \mathcal{O}^\dagger \mathcal{O} \rangle \geq 0, \ \forall \mathcal{O} \tag{43}$$

$$\cancel{\langle \mathcal{O}^\dagger [H, \mathcal{O}] \rangle \geq 0}, \ \forall \mathcal{O} \tag{44}$$

Variables:

$$\langle \mathrm{tr}\mathcal{O} \rangle, \quad \mathcal{O} = \mathcal{O}_{ind}\mathcal{I}^{ind}, \tag{45}$$

$$\text{e.g.} \quad \langle \mathrm{tr}\psi_{\alpha_1}\psi_{\alpha_2}\psi_{\alpha_3}\psi_{\alpha_4}\rangle \gamma^I_{\alpha_1\alpha_2}\gamma^I_{\alpha_3\alpha_4}, \ \langle \mathrm{tr}X^{I_1}\cdots X^{I_9}\rangle \epsilon^{I_1 I_2 \cdots I_9}$$

The algebra of SO(9) invariant tensors can be generated from:

$$\delta^{IJ}, \quad \epsilon^{I_1 I_2 \ldots I_9}, \quad \gamma^I_{\alpha\beta}. \tag{46}$$

A linear basis of invariant tensors is a non-trivial task.

$$\delta_{\alpha\eta}\delta_{\beta\epsilon} = \frac{1}{16}\gamma^I_{\alpha\beta}\gamma^I_{\eta\epsilon} + \frac{1}{16}\delta_{\alpha\beta}\delta_{\eta\epsilon} + \frac{1}{32}\gamma^{IJ}_{\alpha\beta}\gamma^{IJ}_{\eta\epsilon} + \frac{1}{96}\gamma^{IJK}_{\alpha\beta}\gamma^{IJK}_{\eta\epsilon} + \frac{1}{384}\gamma^{IJKL}_{\alpha\beta}\gamma^{IJKL}_{\eta\epsilon} \tag{47}$$

BFSS MODEL BOOTSTRAP

Equations:

$$0 = \gamma^{I}_{\beta\alpha}\left\langle \mathrm{tr} P^{I}\psi_\beta\psi_\alpha\right\rangle + \gamma^{I}_{\beta\alpha}\left\langle \mathrm{tr}\psi_\beta P^{I}\psi_\alpha\right\rangle$$
$$0 = -\gamma^{I}_{\beta\alpha}\left\langle \mathrm{tr}\psi_\beta X^{I}\psi_\alpha\right\rangle - \gamma^{I}_{\beta\alpha}\left\langle \mathrm{tr}\psi_\beta\psi_\alpha X^{I}\right\rangle$$
$$0 = -16\left\langle \mathrm{tr} P^{I}P^{I}\right\rangle - 8\gamma^{I}_{\beta\alpha}\left\langle \mathrm{tr}\psi_\beta X^{I}\psi_\alpha\right\rangle + 8\gamma^{I}_{\beta\alpha}\left\langle \mathrm{tr}\psi_\beta\psi_\alpha X^{I}\right\rangle$$
$$0 = -16\left\langle \mathrm{tr} X^{I}X^{J}X^{I}X^{J}\right\rangle + 16\left\langle \mathrm{tr} X^{I}X^{J}X^{J}X^{I}\right\rangle - 8\gamma^{I}_{\beta\alpha}\left\langle \mathrm{tr}\psi_\beta X^{I}\psi_\alpha\right\rangle + 8\gamma^{I}_{\beta\alpha}\left\langle \mathrm{tr}\psi_\beta\psi_\alpha X^{I}\right\rangle$$
$$0 = \gamma^{IJ}_{\beta\alpha}\left\langle \mathrm{tr} X^{I}X^{J}\psi_\beta\psi_\alpha\right\rangle - \gamma^{IJ}_{\beta\alpha}\left\langle \mathrm{tr}\psi_\beta X^{I}X^{J}\psi_\alpha\right\rangle - \gamma^{I}_{\beta\alpha}\left\langle \mathrm{tr} P^{I}\psi_\beta\psi_\alpha\right\rangle + \gamma^{I}_{\beta\alpha}\left\langle \mathrm{tr}\psi_\beta P^{I}\psi_\alpha\right\rangle$$
$$0 = -\gamma^{IJ}_{\beta\alpha}\left\langle \mathrm{tr} X^{I}X^{J}\psi_\beta\psi_\alpha\right\rangle + \gamma^{IJ}_{\beta\alpha}\left\langle \mathrm{tr}\psi_\beta X^{I}X^{J}\psi_\alpha\right\rangle - \gamma^{I}_{\beta\alpha}\left\langle \mathrm{tr} P^{I}\psi_\beta\psi_\alpha\right\rangle + \gamma^{I}_{\beta\alpha}\left\langle \mathrm{tr}\psi_\beta P^{I}\psi_\alpha\right\rangle$$
$$0 = -\gamma^{IJ}_{\beta\alpha}\left\langle \mathrm{tr}\psi_\beta X^{I}X^{J}\psi_\alpha\right\rangle + \gamma^{IJ}_{\beta\alpha}\left\langle \mathrm{tr}\psi_\beta X^{I}\psi_\alpha X^{J}\right\rangle - \gamma^{I}_{\beta\alpha}\left\langle \mathrm{tr}\psi_\beta\psi_\alpha P^{I}\right\rangle$$
$$0 = -\gamma^{IJ}_{\beta\alpha}\left\langle \mathrm{tr}\psi_\beta X^{I}X^{J}\psi_\alpha\right\rangle + \gamma^{IJ}_{\beta\alpha}\left\langle \mathrm{tr}\psi_\beta X^{I}\psi_\alpha X^{J}\right\rangle - \gamma^{IJ}_{\beta\alpha}\left\langle \mathrm{tr}\psi_\beta\psi_\alpha X^{I}X^{J}\right\rangle$$

BFSS model bootstrap

The hierarchy is defined by sorting operators into *levels*: we assign the basic fields $\ell(X) = 1$, $\ell(P) = 2$, and $\ell(\psi) = 3/2$. The level of an operator is the sum of the levels of all its fields: schematically, $\ell(X^{n_X} P^{n_P} \psi^{n_\psi}) = n_X + 3n_P + 3n_\psi/2$.

- $\ell(\{Q_\alpha, \mathcal{O}_\alpha\}) = \ell(\mathcal{O}_\alpha) + \frac{1}{2}$
- To obtain the positivity conditions involving variables up to level ℓ_{cutoff}, we need to select operators up to level $\frac{1}{2}\ell_{\text{cutoff}}$ and take their inner products.

level cutoff	total variables	free variables
4	11	3
5	38	4
6	140	11
7	569	18
8	2528	59
9	12077	149

For example, the level 5 free variables are:

$$\left\langle \mathrm{tr} P^I P^I \right\rangle, \left\langle \mathrm{tr} X^I X^I \right\rangle, \left\langle \mathrm{tr} X^I X^I X^J X^J \right\rangle, \left\langle \mathrm{tr} X^I X^I \psi_\alpha \psi_\alpha \right\rangle \tag{48}$$

Positivity

By Schur's lemma, operators in different irreducible representations are orthogonal to each other. We need to classify the operators with respect to their SO(9) quantum number. Here is an example from the singlet positivity:

$$
\begin{array}{c} \\ 1 \\ X^J X^J \\ iP^J X^J \\ O \end{array}
\begin{array}{c} \begin{array}{cccc} 1 & X^I X^I & -iX^I P^I & O \end{array} \\
\begin{pmatrix}
1 & \mathrm{tr}X^2 & 9/2 & 8 \\
\mathrm{tr}X^2 & \mathrm{tr}X^2\,X^2 & -i\mathrm{tr}X^2\,XP & \mathrm{tr}X^2\,O \\
9/2 & -i\mathrm{tr}X^2\,XP & \mathrm{tr}PX\,XP & i\mathrm{tr}PX\,O \\
8 & \mathrm{tr}X^2\,O & i\mathrm{tr}PX\,O & \mathrm{tr}OO
\end{pmatrix} \end{array} \succeq 0.
$$

Here $O = \psi_\alpha \psi_\alpha$

Numerical result

Numerical result

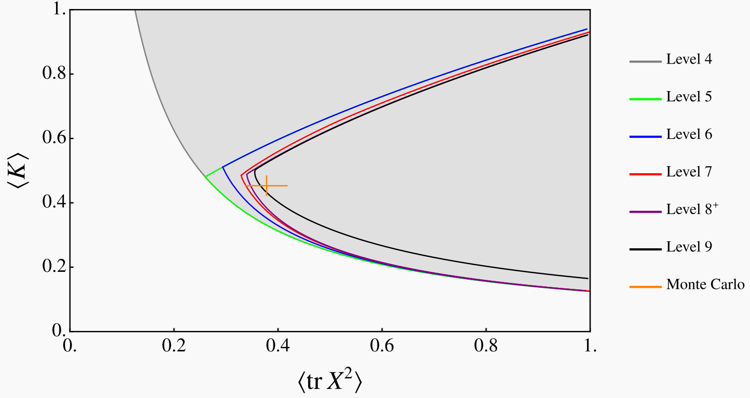
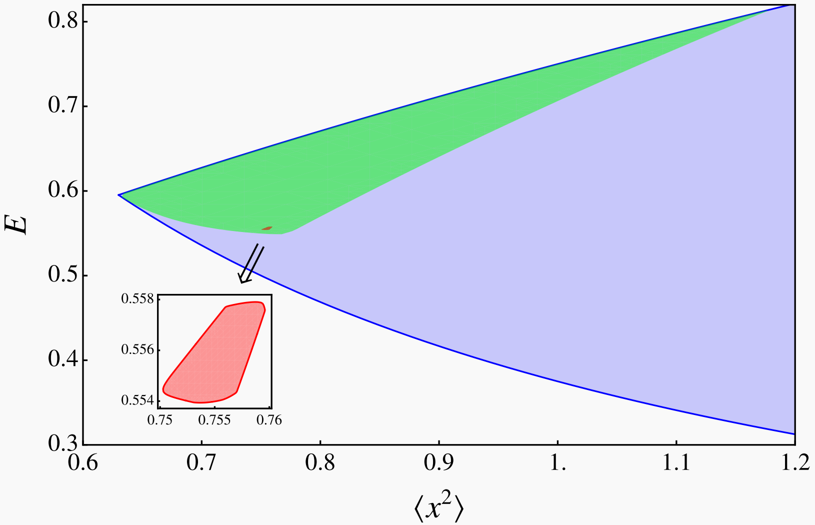


Figure 2: $\langle K \rangle \propto \langle \mathrm{tr} P^2 \rangle \propto \langle \mathrm{tr}[X^I, X^J]^2 \rangle$ We also show the extrapolation of the Monte Carlo [Berkowitz et al., 2018] results.

Convergence

method	$\langle \mathrm{tr} X^2 \rangle$
Monte Carlo [Berkowitz et al., 2018]	$\approx 0.378 \pm 0.04$
Monte Carlo [Pateloudis et al., 2023]	$[0.346, 0.430]$
primitive bootstrap [Lin, 2023]	≥ 0.1875
bootstrap level 5	≥ 0.260
bootstrap level 6	≥ 0.294
bootstrap level 7	≥ 0.329
bootstrap level 8^+	≥ 0.340
bootstrap level 9	≥ 0.355

Compare with toy model



The Hamiltonian:

$$H = \frac{1}{2}\sum_{I=1}^{D}\left(\mathrm{Tr}P^I P^I + M^2\mathrm{Tr}X^I X^I\right) - \sum_{I,J=1}^{D}\mathrm{Tr}[X^I, X^J]^2 \tag{49}$$

Especially for the D=2 case, we have:

$$Z = \frac{1}{\sqrt{2}}(X + \mathrm{i}Y), \quad \bar{Z} = \frac{1}{\sqrt{2}}(X - \mathrm{i}Y), \tag{50}$$

$$P = \frac{1}{\sqrt{2}}(P_X + \mathrm{i}P_y), \quad \bar{P} = \frac{1}{\sqrt{2}}(P_X - \mathrm{i}P_y), \tag{51}$$

$$H = -\mathrm{tr}\Pi\,\bar{\Pi} + M^2\mathrm{tr}Z\bar{Z} + \mathrm{tr}[Z, \bar{Z}]^2. \tag{52}$$

The two positivities of X^I and P^I read:

$$\begin{pmatrix} \frac{D}{2} & \mathrm{tr} P^I P^I \\ \mathrm{tr} P^I P^I & (D-1)\mathrm{tr} X^I X^I + M^2 \end{pmatrix} \succeq 0 \tag{53}$$

$$\begin{pmatrix} \mathrm{tr} X^I X^I & D/2 \\ D/2 & \mathrm{tr} P^I P^I \end{pmatrix} \succeq 0 \tag{54}$$

Which means:

$$\frac{3}{16\langle \mathrm{tr} X^2\rangle} + \frac{M^2\langle \mathrm{tr} X^2\rangle}{4} \leq \frac{\mathcal{E}}{D} \leq \frac{1}{8}\left[2M^2\langle \mathrm{tr} X^2\rangle + 3\left(2(D-1)\langle \mathrm{tr} X^2\rangle + 2\frac{M^2}{D}\right)^{1/2}\right], \tag{55}$$

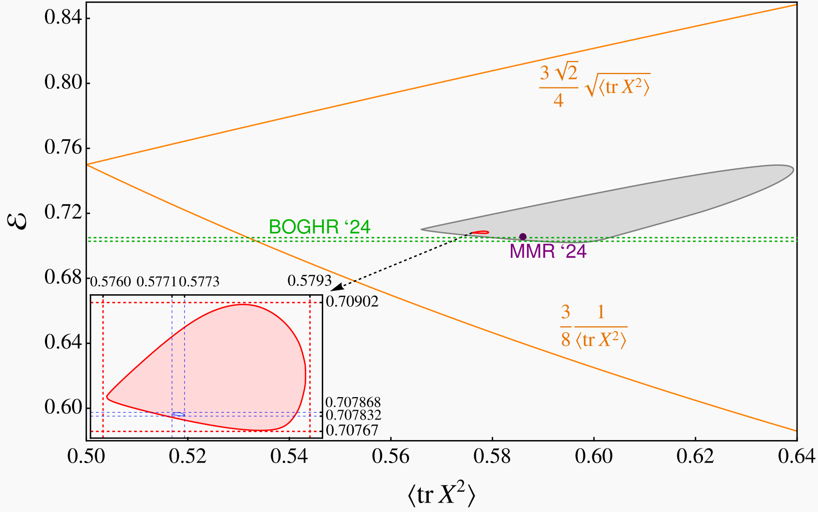
$$\mathcal{E} \equiv \frac{E_0}{N^2}(g_{\mathrm{YM}}^2 N)^{-1/3} = \frac{3}{4}\langle \mathrm{tr} P^I P^I\rangle + \frac{1}{4}M^2\langle \mathrm{tr} X^I X^I\rangle \tag{56}$$

Benchmark (2507.21007)

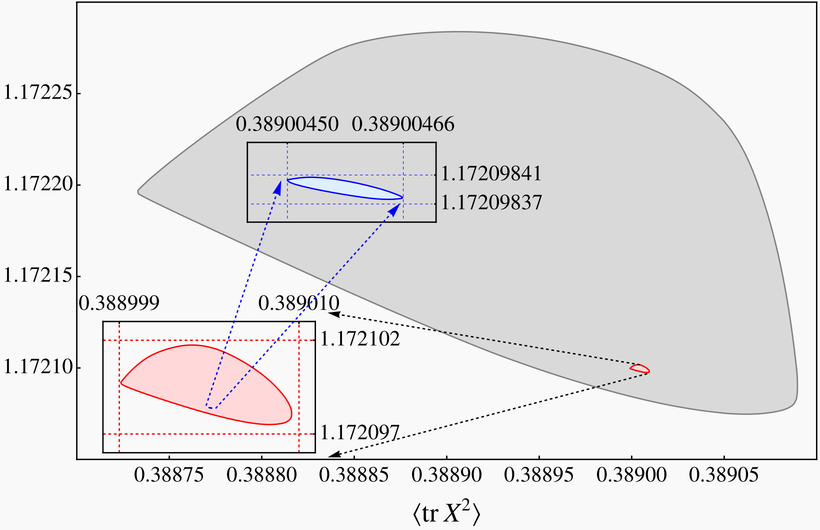
Model	$\mathcal{E}$	$\langle \mathrm{tr}\, X_I X_I \rangle$	level 4 operator
$M^2 = 0,\ D = 2$			$\langle \mathrm{tr}(Z^2 \bar{Z}^2) \rangle$
Bootstrap	[0.707832, 0.707868]	[1.15420, 1.15460]	[0.37055, 0.37085]
Monte Carlo	0.7039(11)	—	—
Loop truncation	0.7056(2)	1.172(1)	0.383(2)
Large D expansion	0.756	0.985	0.177
$M^2 = 1,\ D = 2$			$\langle \mathrm{tr}(Z^2 \bar{Z}^2) \rangle$
Bootstrap	[1.172098376, 1.172098408]	[0.77800898, 0.77800934]	[0.15850588, 0.15850607]
Monte Carlo	1.1654(11)	—	—
Loop truncation	1.17198	0.7784	0.1588
$M^2 = 0,\ D = 9$			$\langle \mathrm{tr} X_I X_I X_J X_J \rangle$
Bootstrap	[6.69946, 6.69968]	[2.29195, 2.29218]	[5.7787, 5.7804]
Monte Carlo	6.695(5)	2.291(1)	—
Large D expansion	6.713	2.279	5.646

Table 1: Ground state energy $\mathcal{E}$ bounds and expectation values ...

O(2) model(massless)

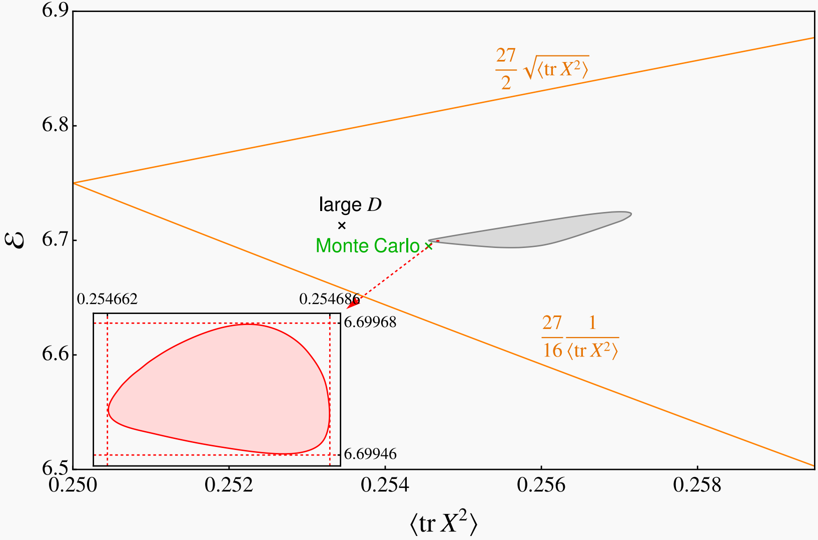


O(2) MODEL($M^2 = 1$)



1.17225
1.17220
1.17215
1.17210
0.38900450
0.38900466
1.17209841
1.17209837
0.388999
0.389010
1.172102
1.172097
0.38875
0.38880
0.38885
0.38890
0.38895
0.38900
0.38905
$\langle \mathrm{tr}\, X^2 \rangle$

O(9) model(massless)



Benchmark (2507.21007)

Model	$\mathcal{E}$	$\langle \mathrm{tr}\, X_I X_I \rangle$	level 4 operator
$M^2 = 0,\ D = 2$			$\langle \mathrm{tr}(Z^2 \bar{Z}^2) \rangle$
Bootstrap	[0.707832, 0.707868]	[1.15420, 1.15460]	[0.37055, 0.37085]
Monte Carlo	0.7039(11)	—	—
Loop truncation	0.7056(2)	1.172(1)	0.383(2)
Large D expansion	0.756	0.985	0.177
$M^2 = 1,\ D = 2$			$\langle \mathrm{tr}(Z^2 \bar{Z}^2) \rangle$
Bootstrap	[1.172098376, 1.172098408]	[0.77800898, 0.77800934]	[0.15850588, 0.15850607]
Monte Carlo	1.1654(11)	—	—
Loop truncation	1.17198	0.7784	0.1588
$M^2 = 0,\ D = 9$			$\langle \mathrm{tr} X_I X_I X_J X_J \rangle$
Bootstrap	[6.69946, 6.69968]	[2.29195, 2.29218]	[5.7787, 5.7804]
Monte Carlo	6.695(5)	2.291(1)	—
Large D expansion	6.713	2.279	5.646

Table 2: Ground state energy $\mathcal{E}$ bounds and expectation values ...

Euclidean two-point correlator(2511.08560)

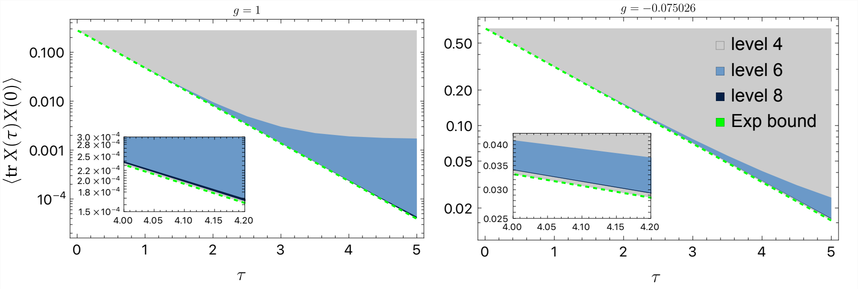


Figure 3: Euclidean two-point correlator $\langle \mathrm{tr} X(\tau) X(0) \rangle$ in the ground state of the 1-MQM with $V = \frac{1}{2}\mathrm{Tr}X^2 + g\mathrm{Tr}X^4$ at $g = 1$ (left) and $g = -0.075026 \approx g_c$ (right). The shaded region indicates the allowed bootstrap region at level-$\{4, 6, 8\}$.

Euclidean two-point correlator(2511.08560)

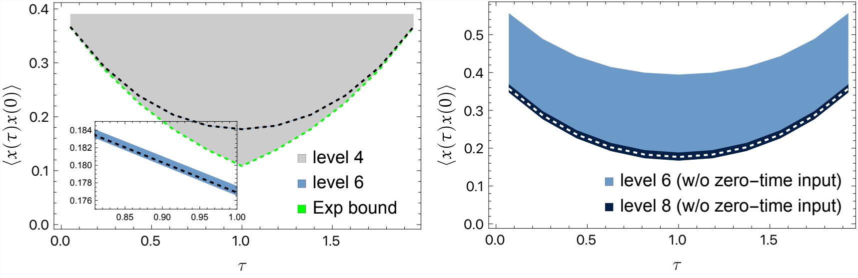


Figure 4: Bootstrap bounds on the Euclidean two-point correlator in the thermal state of anharmonic oscillator with $H = \frac{1}{2}x^2 + \frac{1}{2}p^2 + \frac{1}{4}x^4$ at $\beta = 2$.

Euclidean two-point correlator(2511.08560)

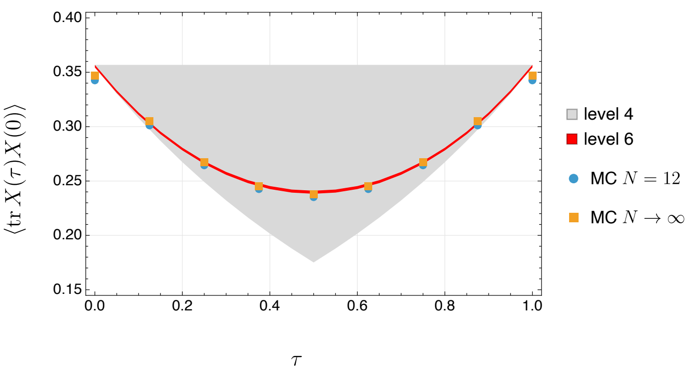


Figure 5: Bootstrap bounds on the thermal two-point correlator $\langle \mathrm{tr}\, X(\tau) X(0) \rangle$ for the ungauged 1-MQM with $\beta = 1$ and $g = 1$, at levels 4 and 6.

Euclidean two-point correlator(2511.08560)

This is simply a condition on the connected two-point correlator $G_c(\tau) = \langle \bar{\mathcal{O}}(\tau)\mathcal{O}(0)\rangle - \langle \bar{\mathcal{O}}\rangle\langle \mathcal{O}\rangle$. By the inner product of $\{\mathcal{O}, [H, \mathcal{O}]\}$ at $\frac{\tau}{2}$ and $-\frac{\tau}{2}$

$$\mathcal{M}_c(\tau) = \begin{pmatrix} G_c(\tau) & G_c'(\tau) \\ G_c'(\tau) & G_c''(\tau) \end{pmatrix} \succeq 0 \implies (\log G_c(\tau))'' \geq 0. \tag{57}$$

$$\log G_c(\tau) \geq \log G_c(0) + \tau \frac{G_c'(0^+)}{G_c(0)} \Rightarrow G_c(\tau) \geq G_c(0) \exp\left(-\mu\tau\right), \tag{58}$$

$$\mu = \frac{-G_c'(0^+)}{G_c(0)}. \tag{59}$$

Euclidean two-point correlator(2511.08560)

$$\log G_c(\tau) \geq \log G_c(0) + \tau \frac{G_c'(0^+)}{G_c(0)} \tag{60}$$

$$\log \frac{G(\beta)}{G(0)} \geq \beta \frac{G'(0)}{G(0)}. \tag{61}$$

$$G(0) = \langle \mathcal{O}^\dagger \mathcal{O} \rangle_\beta, \quad G'(0) = -\langle \mathcal{O}^\dagger [H, \mathcal{O}] \rangle_\beta \tag{62}$$

$$G(\beta) = \mathrm{tr}\{\rho e^{\beta H} \mathcal{O}^\dagger e^{-\beta H} \mathcal{O}\} = \mathrm{tr}\{\mathcal{O}\mathcal{O}^\dagger \rho\} = \langle \mathcal{O}\mathcal{O}^\dagger \rangle_\beta. \tag{63}$$

In the last line we used the KMS condition. Plugging into (61), we arrive at the Energy-Entropy balance (EEB) inequality [Araki and Sewell, 1977]:

$$\langle \mathcal{O}^\dagger \mathcal{O} \rangle \log \frac{\langle \mathcal{O}^\dagger \mathcal{O} \rangle}{\langle \mathcal{O}\mathcal{O}^\dagger \rangle} \leq \beta \langle \mathcal{O}^\dagger [H, \mathcal{O}] \rangle, \ \forall \mathcal{O} \tag{64}$$

Euclidean two-point correlator(2511.08560)

$$\begin{aligned} &\text{minimize} \quad \langle x(T)x(0)\rangle, \\ &\text{subject to } \mathcal{M}(\tau) = \begin{pmatrix} \langle x(\tau)x(0)\rangle & -i\langle x(\tau)p(0)\rangle \\ -i\langle x(\tau)p(0)\rangle & \langle p(\tau)p(0)\rangle \end{pmatrix} \succeq 0, \end{aligned} \tag{65}$$

with the prototype:

$$\begin{aligned} \text{minimize}\, \mathrm{Tr}\, \widehat{O}\mathcal{M}(T) \\ \text{subject to } \mathcal{M}(\tau) \succeq 0 \\ \mathrm{Tr}\left(D^{(k)} - C^{(k)}\frac{\mathrm{d}}{\mathrm{d}\tau}\right)\mathcal{M}(\tau) = 0. \end{aligned} \tag{66}$$

$$\begin{aligned} \text{minimize}\ \mathrm{Tr}\,\widehat{O}\mathcal{M}(T) \\ \text{subject to}\ \mathcal{M}(\tau) \succeq 0 \\ \mathrm{Tr}\left(D^{(k)} - C^{(k)}\frac{\mathrm{d}}{\mathrm{d}\tau}\right)\mathcal{M}(\tau) = 0. \end{aligned} \tag{67}$$

$$\mathcal{O} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{68}$$

$$C^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C^{(2)} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \tag{69}$$

$$D^{(1)} = \begin{pmatrix} 0 & -1/2 \\ -1/2 & 0 \end{pmatrix}, \quad D^{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \tag{70}$$

$$\mathcal{M}(0) = \begin{pmatrix} \langle x^2 \rangle & 1/2 \\ 1/2 & \langle p^2 \rangle \end{pmatrix}. \tag{71}$$

Euclidean two-point correlator(2511.08560)

$$\begin{aligned} \text{minimize}\ \mathrm{Tr}\, \widehat{O}\mathcal{M}(T) \\ \text{subject to}\ \mathcal{M}(\tau) \succeq 0 \\ \mathrm{Tr}\left(D^{(k)} - C^{(k)}\frac{\mathrm{d}}{\mathrm{d}\tau}\right)\mathcal{M}(\tau) = 0. \end{aligned} \tag{72}$$

$$\begin{aligned} I &= \mathrm{Tr}\Big\{\widehat{O}\mathcal{M}(T) + \int_0^T \mathrm{d}\tau\left[\lambda_D^{(k)}\left(D^{(k)} - C^{(k)}\partial_\tau\right) - \Lambda_{\mathcal{M}}\right]\mathcal{M}(\tau)\Big\} \\ &= \mathrm{Tr}\Big\{\int_0^T \mathrm{d}\tau\left[\left(D^{(k)} + C^{(k)}\frac{\mathrm{d}}{\mathrm{d}\tau}\right)\lambda_D^{(k)} - \Lambda_{\mathcal{M}}\right]\mathcal{M}(\tau) + \left[\widehat{O} - \lambda_D^{(k)}(T)C^{(k)}\right]\mathcal{M}(T)\Big\} \end{aligned}$$

$$\begin{aligned} \text{minimize}\ & \operatorname{Tr}\widehat{O}\mathcal{M}(T) \\ \text{subject to}\ & \mathcal{M}(\tau)\succeq 0 \\ & \operatorname{Tr}\left(D^{(k)}-C^{(k)}\frac{\mathrm{d}}{\mathrm{d}\tau}\right)\mathcal{M}(\tau)=0. \end{aligned} \tag{73}$$

$$\begin{aligned} I &= \mathrm{Tr}\left\{\widehat{O}\mathcal{M}(T)+\int_0^T \mathrm{d}\tau\left[\lambda_D^{(k)}\left(D^{(k)}-C^{(k)}\partial_\tau\right)-\Lambda_{\mathcal{M}}\right]\mathcal{M}(\tau)\right\} \\ &= \mathrm{Tr}\left\{\int_0^T \mathrm{d}\tau\left[\left(D^{(k)}+C^{(k)}\frac{\mathrm{d}}{\mathrm{d}\tau}\right)\lambda_D^{(k)}-\Lambda_{\mathcal{M}}\right]\mathcal{M}(\tau)+\left[\widehat{O}-\lambda_D^{(k)}(T)C^{(k)}\right]\mathcal{M}(T)\right\} \end{aligned}$$

$$\boxed{\begin{aligned} \text{maximize}\ & \lambda_D^{(k)}(0)\operatorname{Tr}C^{(k)}\mathcal{M}(0) \\ \text{subject to}\ & \widehat{O}-\lambda_D^{(k)}(T)C^{(k)}=0 \\ & \left(D^{(k)}+C^{(k)}\partial_\tau\right)\lambda_D^{(k)}(\tau)\succeq 0 \end{aligned}} \tag{74}$$

Euclidean two-point correlator(2511.08560)

$$\begin{aligned} &\text{minimize} \quad \langle x(T)x(0)\rangle, \\ &\text{subject to } \mathcal{M}(\tau) = \begin{pmatrix} \langle x(\tau)x(0)\rangle & -i\langle x(\tau)p(0)\rangle \\ -i\langle x(\tau)p(0)\rangle & \langle p(\tau)p(0)\rangle \end{pmatrix} \succeq 0, \end{aligned} \tag{75}$$

dual to:

$$\begin{aligned} \text{maximize} \quad & \lambda_D^{(1)}(0)\langle x^2\rangle + \frac{1}{2}\lambda_D^{(2)}(0), \\ \text{subject to} \quad & \begin{pmatrix} \lambda_D^{(1)}(T) & \lambda_D^{(2)}(T)/2 \\ \lambda_D^{(2)}(T)/2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\ & \begin{pmatrix} \lambda_D'^{(1)}(\tau) & \lambda_D'^{(2)}(\tau)/2 \\ \lambda_D'^{(2)}(\tau)/2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \lambda_D^{(1)}(\tau)/2 \\ \lambda_D^{(1)}(\tau)/2 & \lambda_D^{(2)}(\tau) \end{pmatrix} \succeq 0. \end{aligned} \tag{76}$$

Prospect

- Other systems with sign-problem: Yang-Mills with theta terms, IKKT model...
- Gauged matrix model, subleading in $\frac{1}{N}$
- Lorentzian-time correlator, complex-time correlator
- S-matrix and the BFSS conjecture

Questions?

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