New Computational Methods for Matrix Models

Enrico Rinaldi

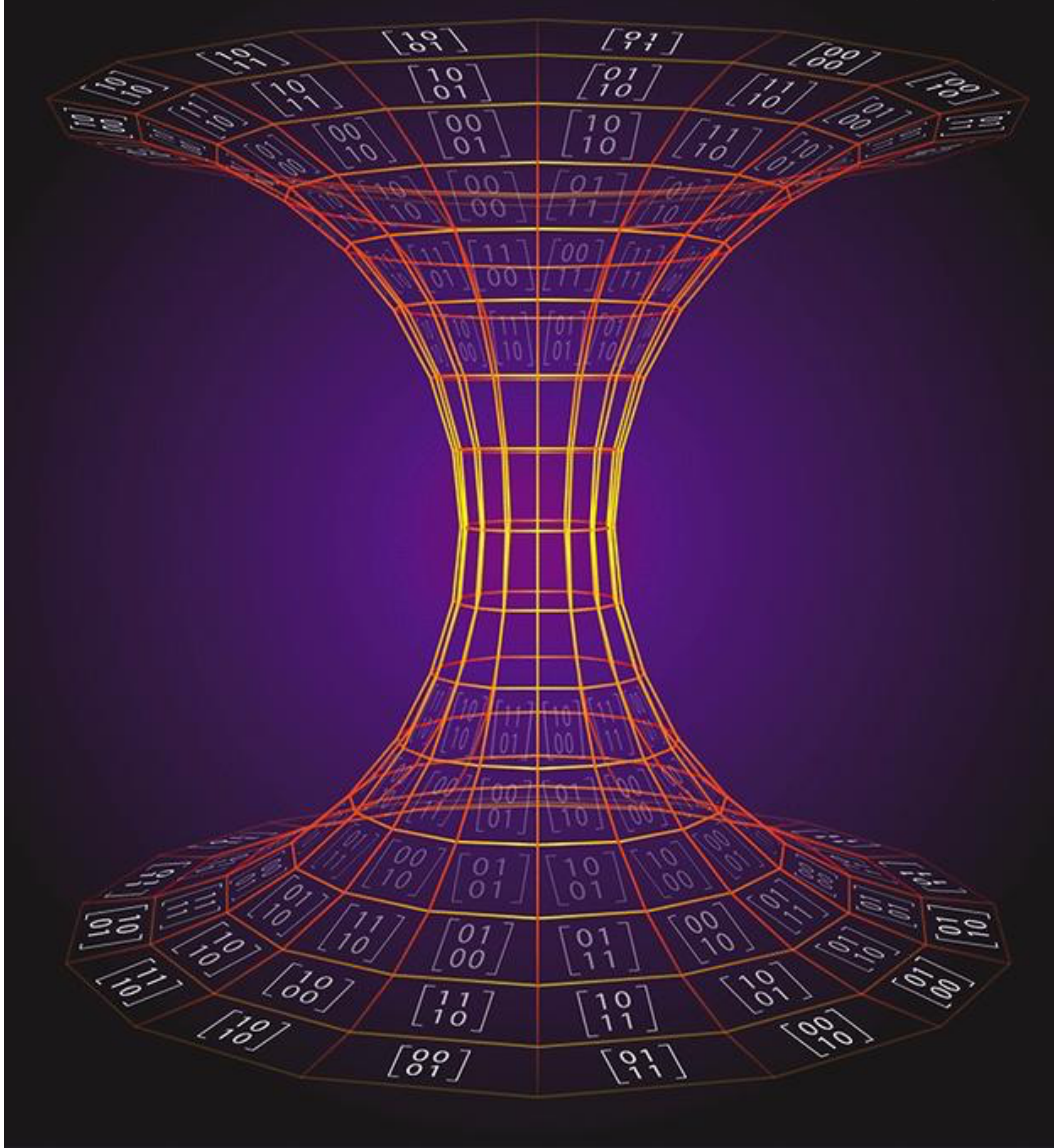
(Main) | Quantinuum (London) ⟩ + (Visitor) | RIKEN [RQC] ⟩

2025-12-03 Matrix Model for Superstring/M-theory (YITP) @ Kyoto

[Rinaldi et al., PRX Quantum 3 (2022) 010324] [Brehm et al., JHEP 09 (2025) 116]

Outline

Credits:APS/Alan Stonebraker. Physics Magazine



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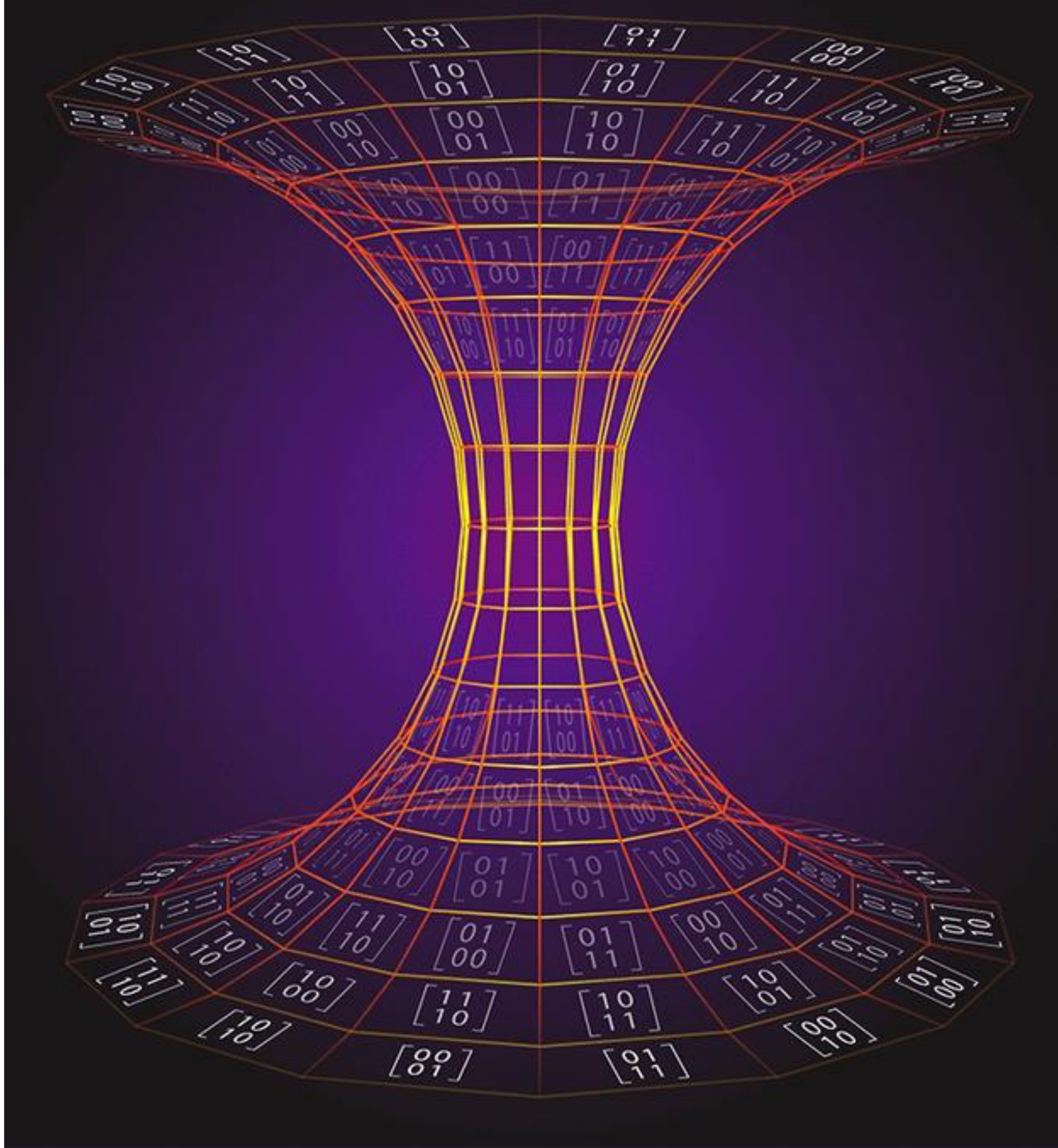

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- **Numerical techniques for matrix quantum mechanics**:
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  - Truncated Hamiltonian
  - Quantum Computing
  - Tensor Networks

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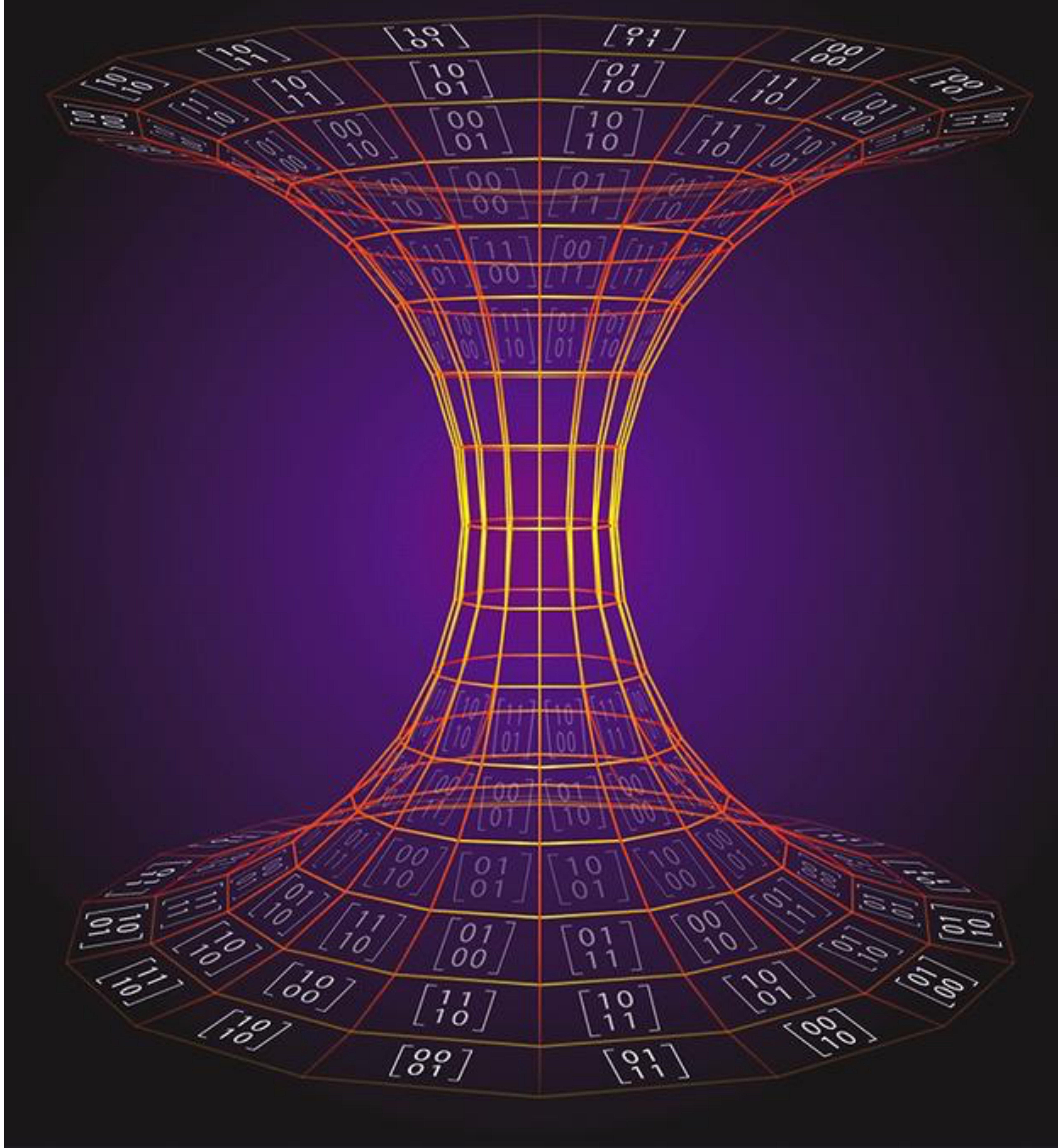

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- Conclusions and challenges

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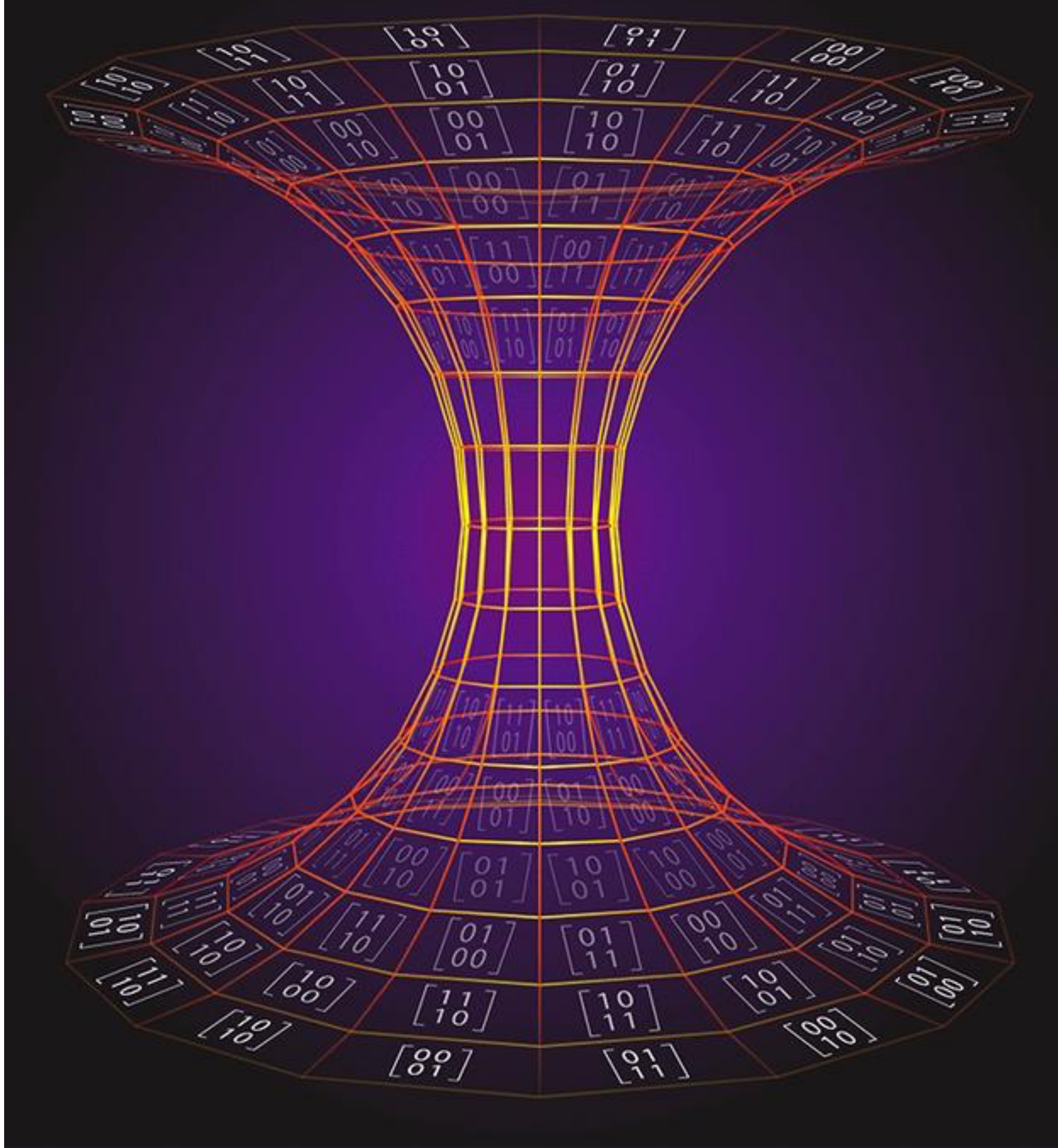


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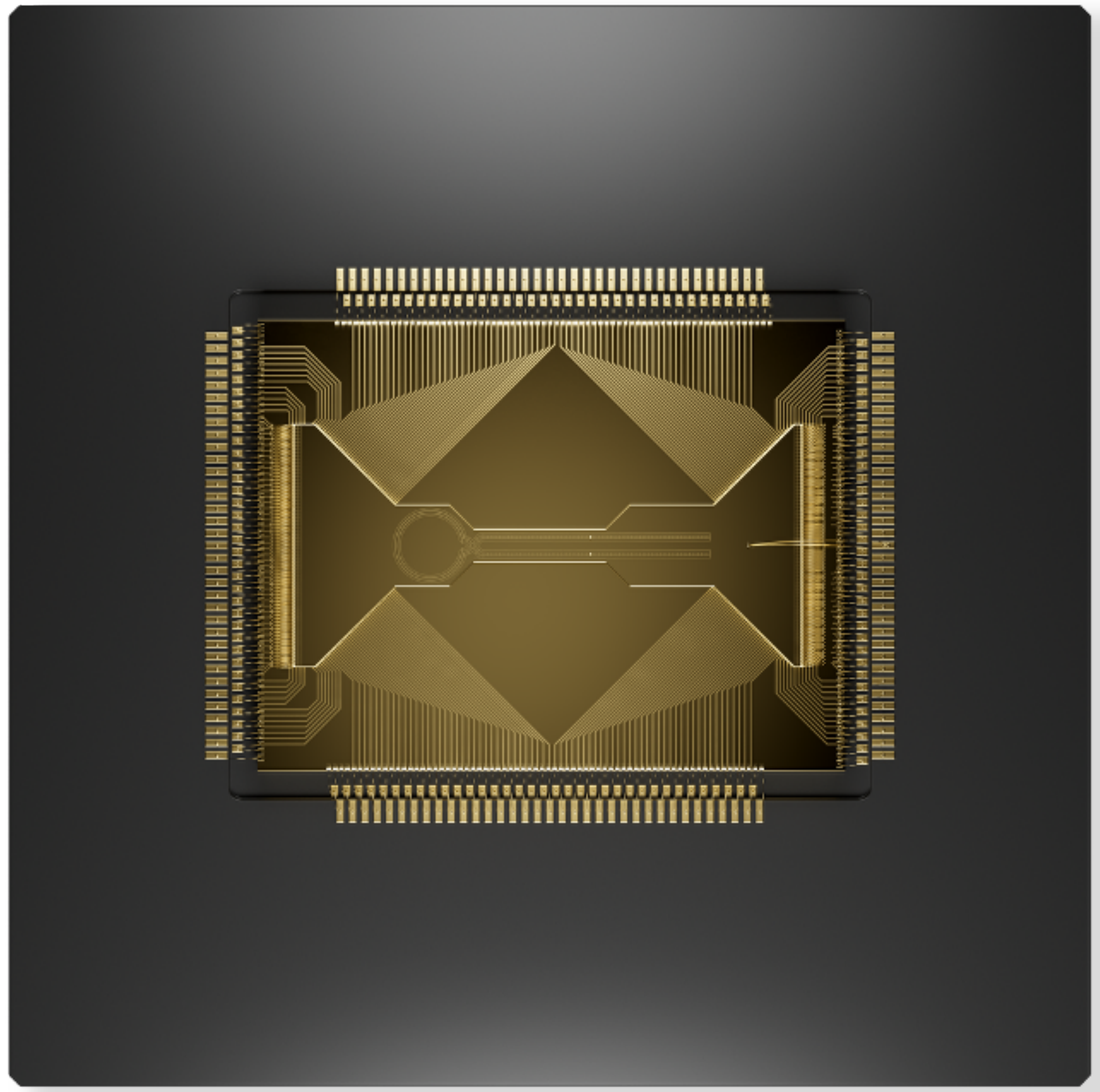

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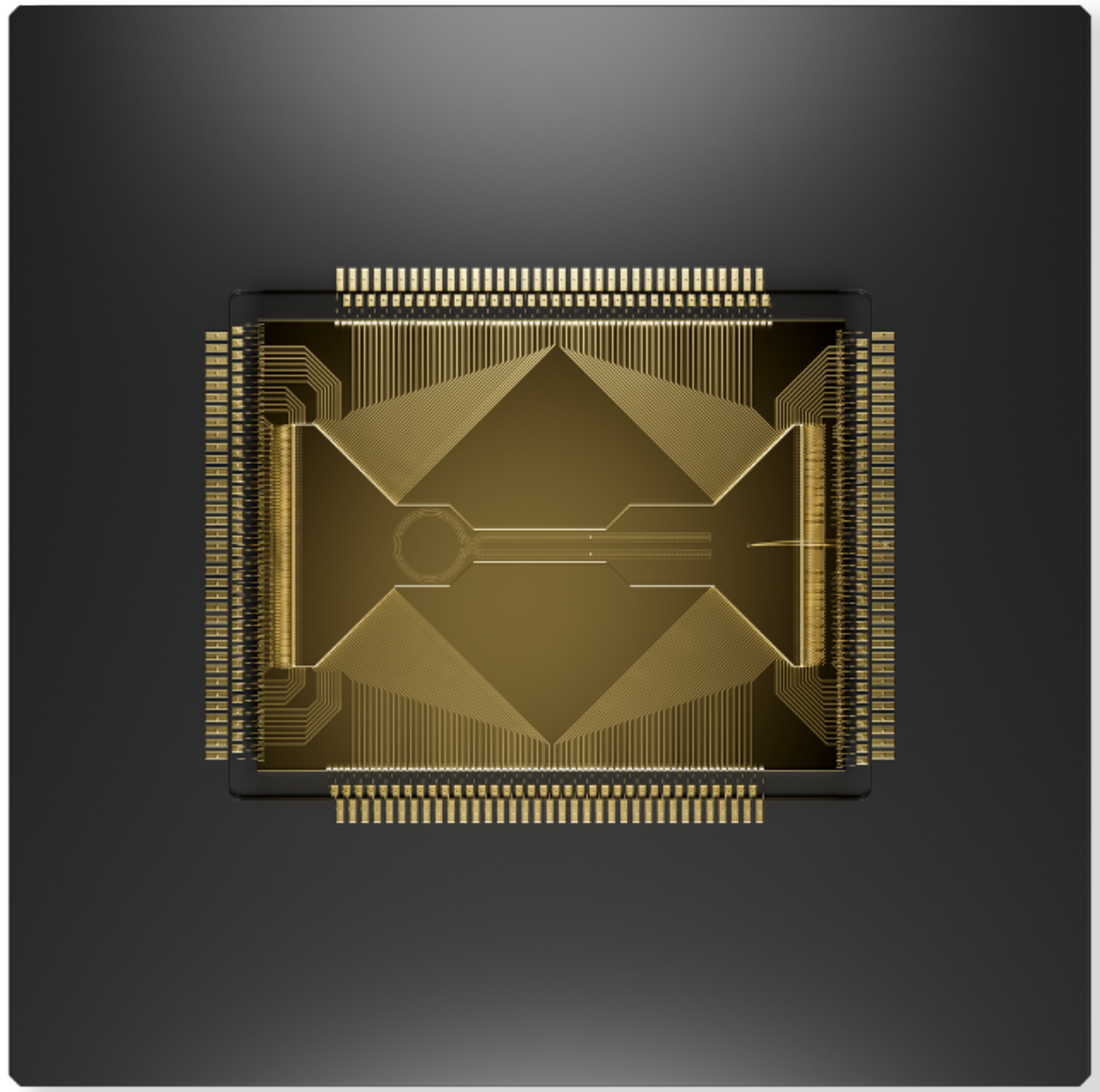
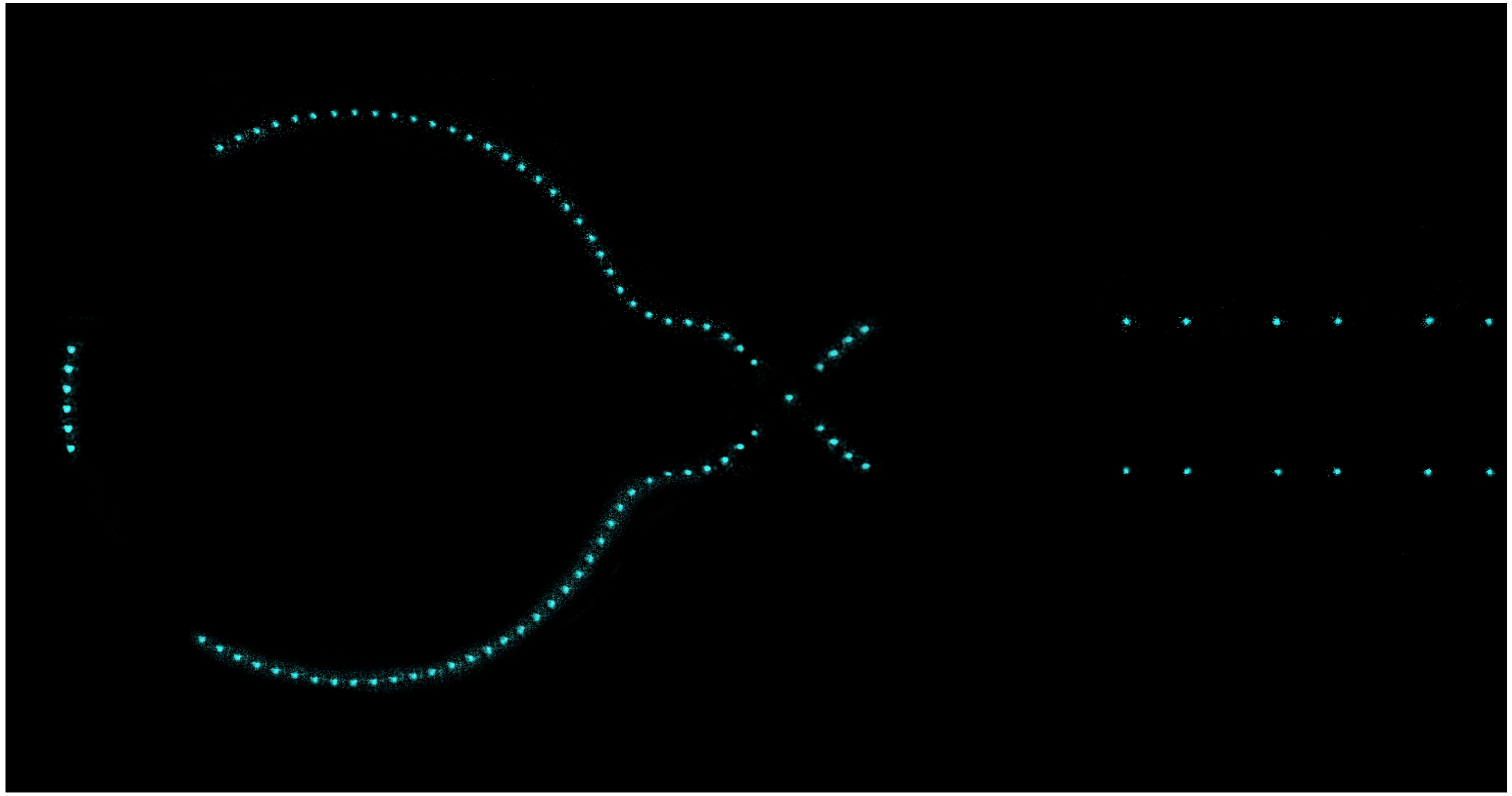
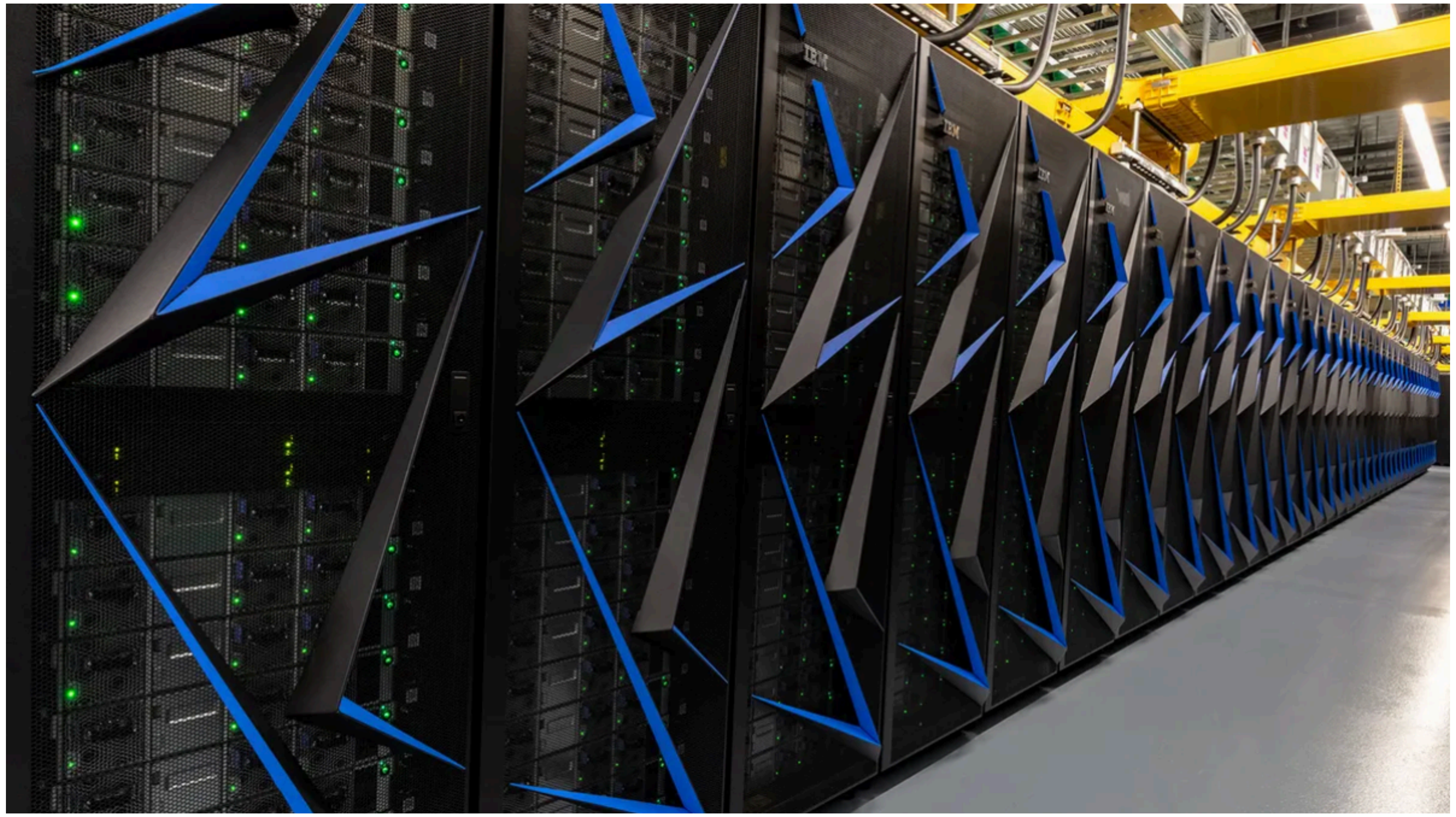
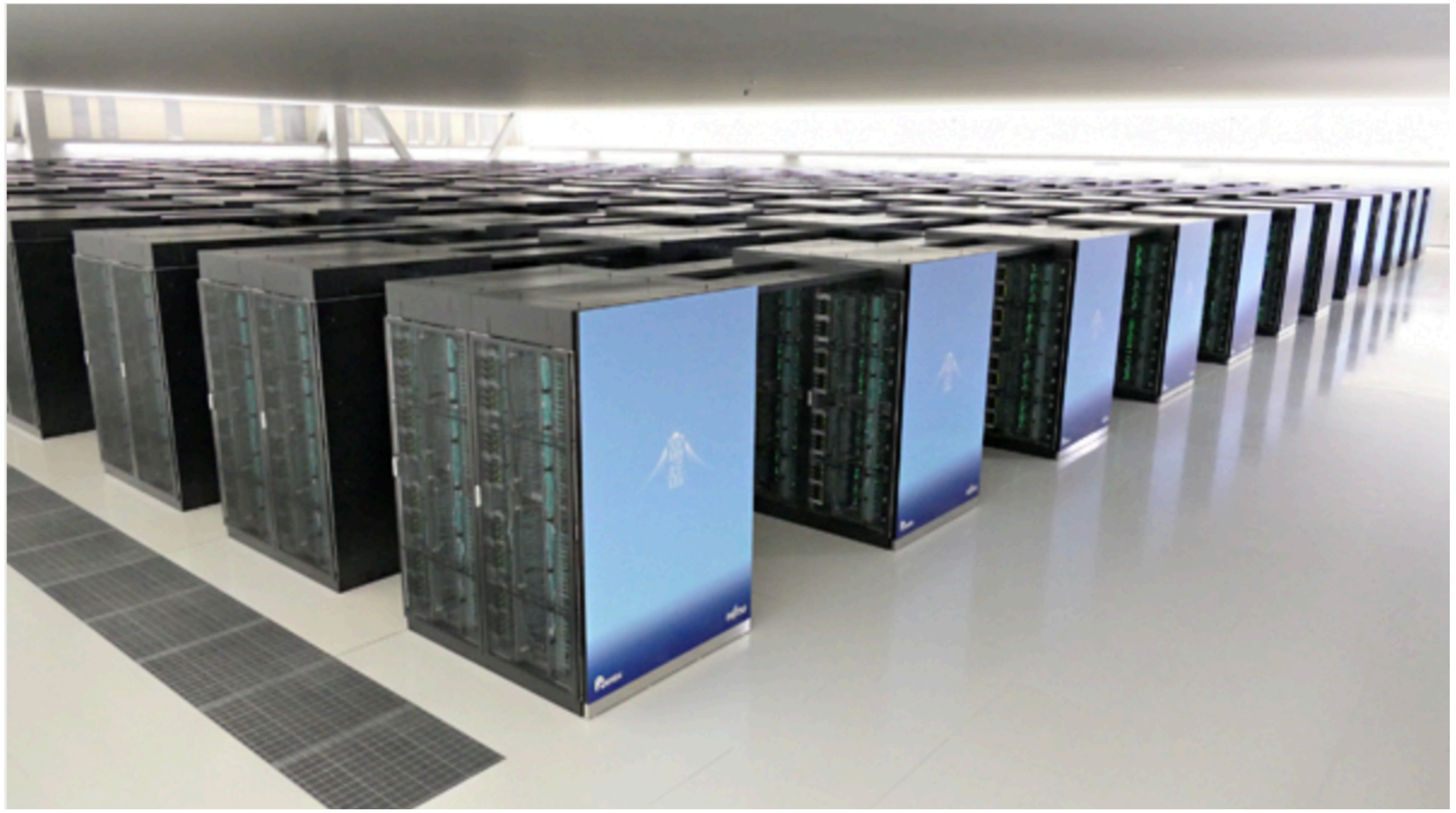




Summit @ ORNL

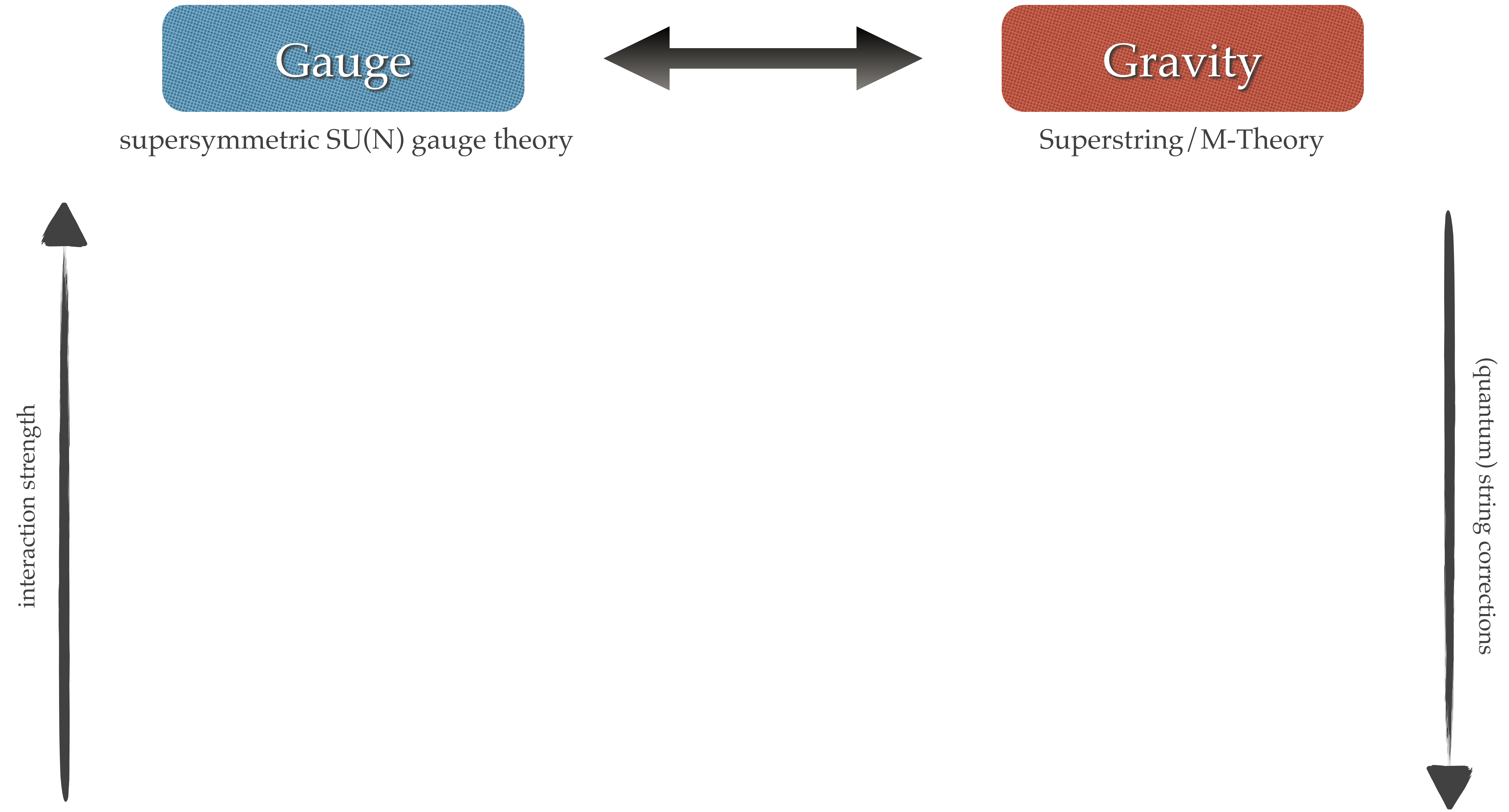


Fugaku @ R-CCS



[Maldacena ’97, Itzhaki,Maldacena,Sonnenschein,Yankielowic ’98] [’t Hooft, Susskind, Maldacena …]

GAUGE/GRAVITY DUALITY



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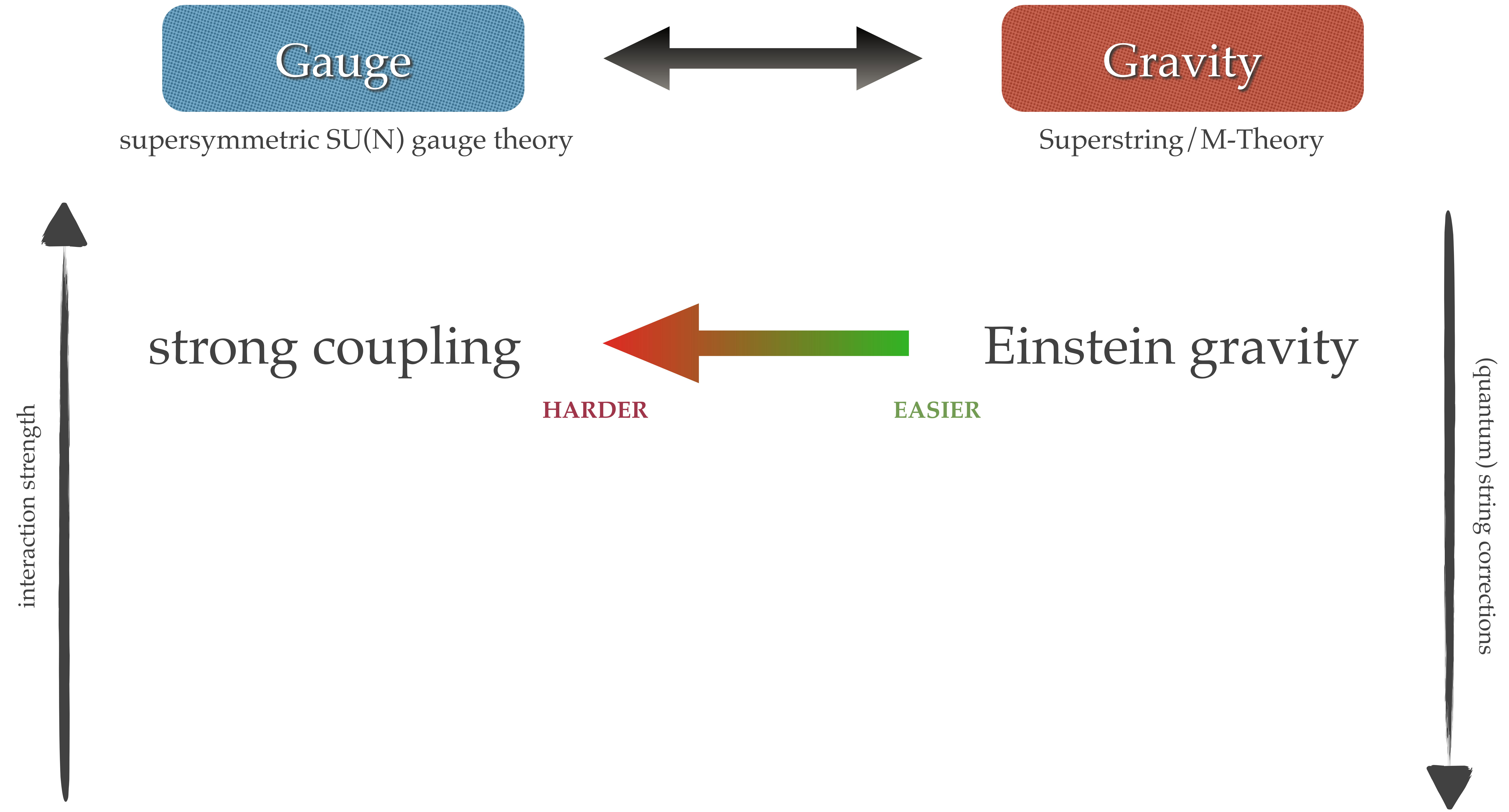

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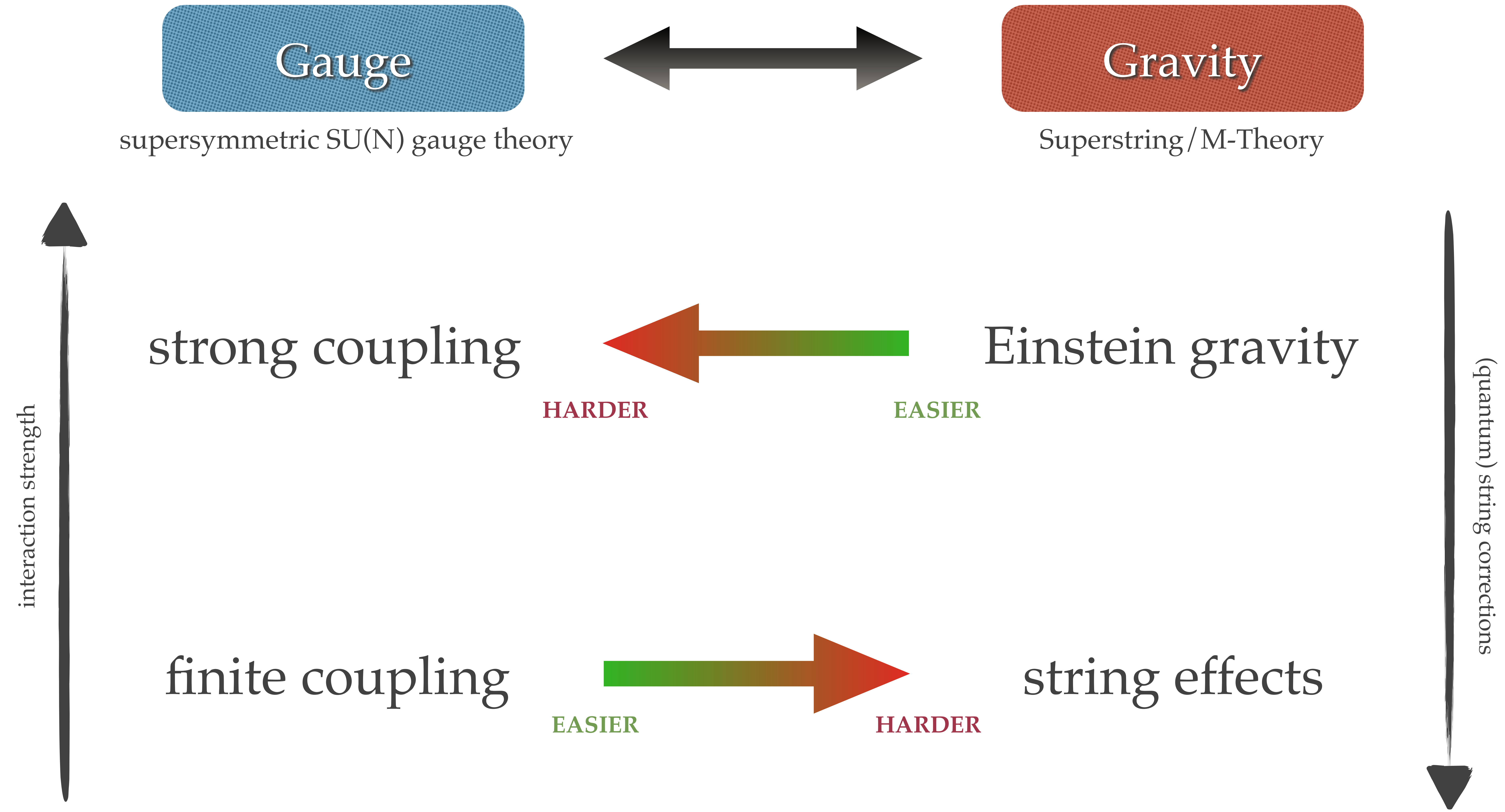

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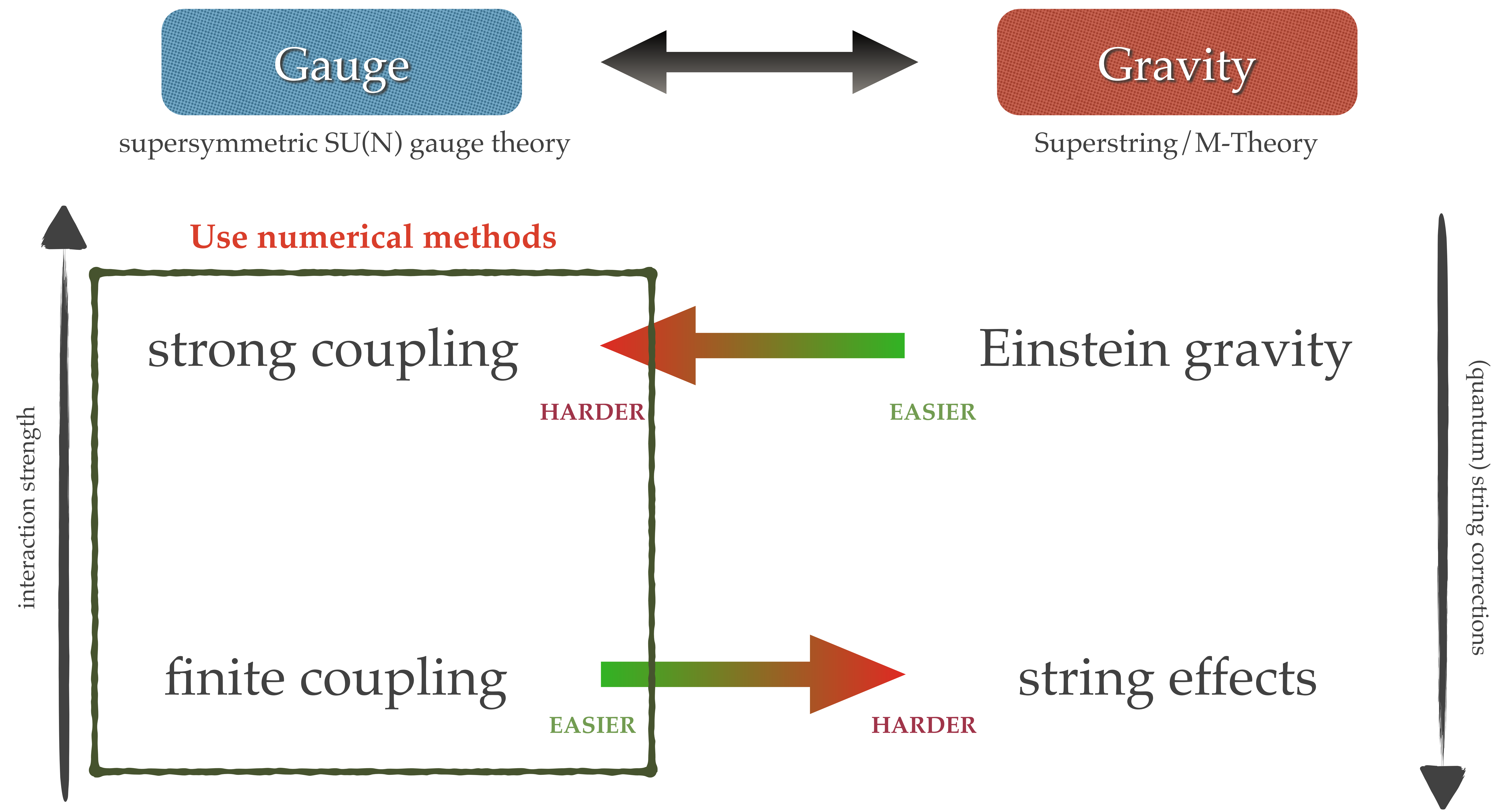

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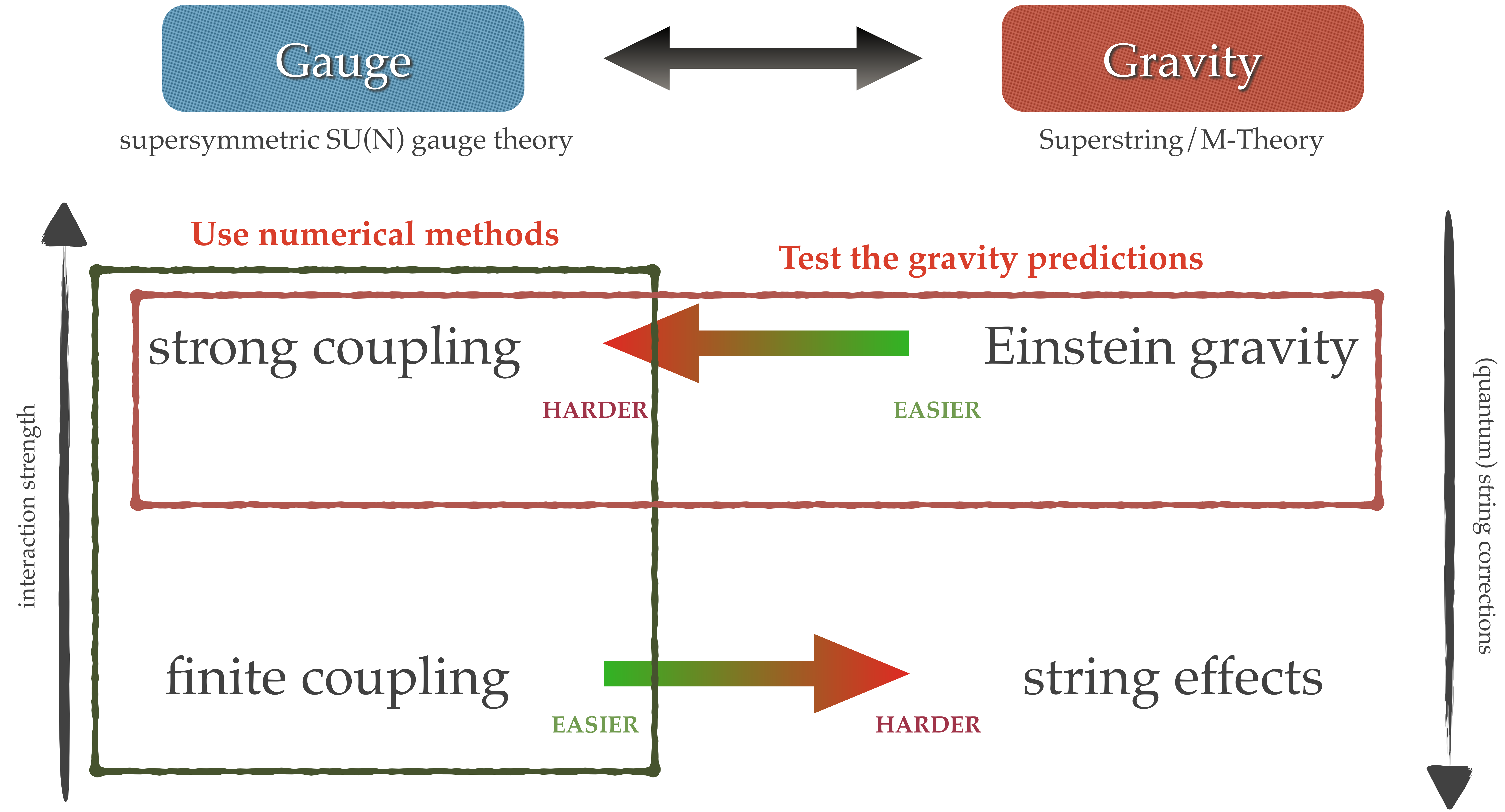

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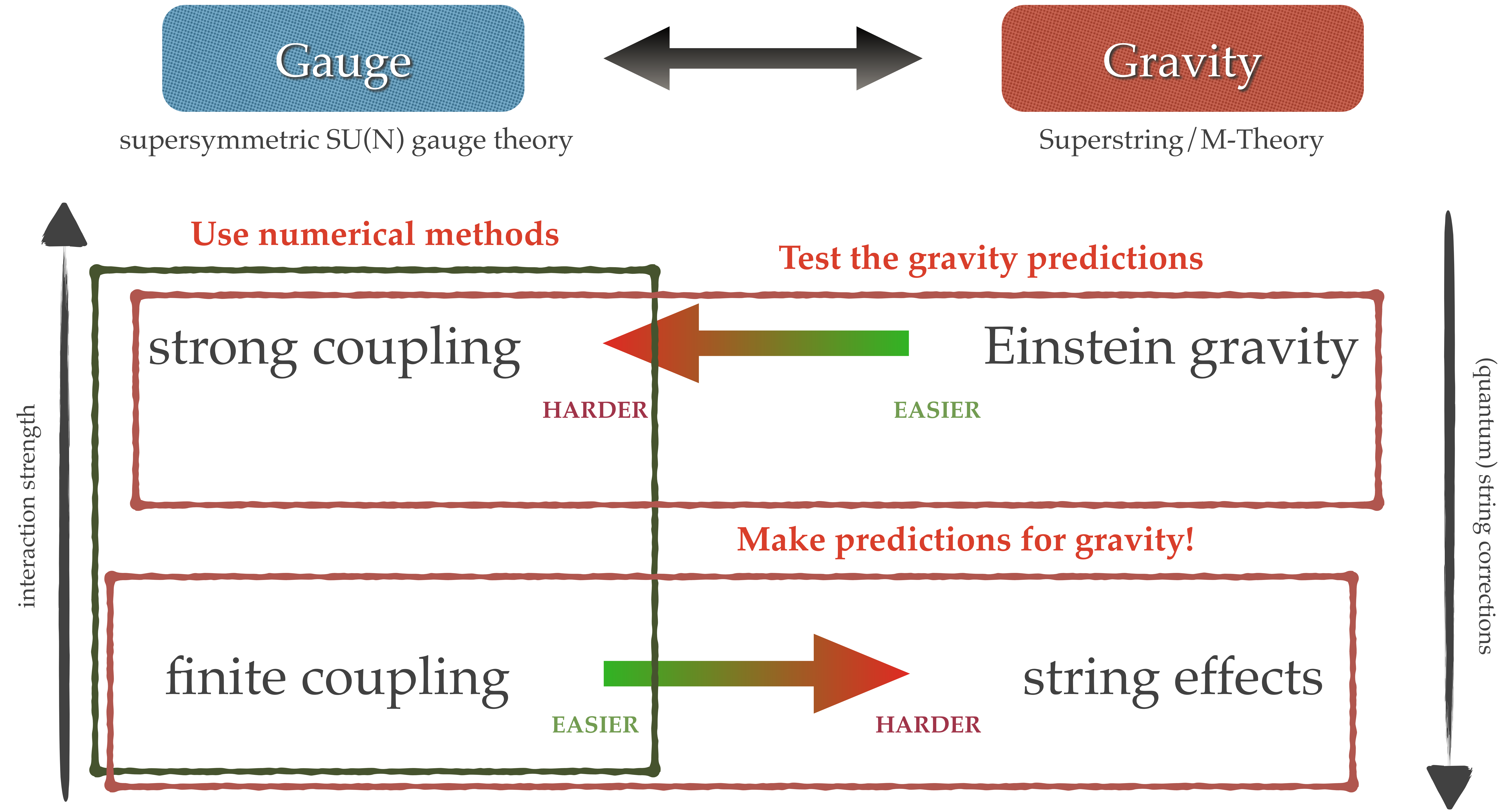

Matrix Quantum Mechanics

Interpretation

$$L = \frac{1}{2g_{YM}^2}\mathrm{Tr}\left\{(D_t X_M)^2 + [X_M, X_{M'}]^2 + i\bar{\psi}^\alpha D_t \psi^\beta + \bar{\psi}^\alpha \gamma^M_{\alpha\beta}[X_M, \psi^\beta]\right\}$$

obtained from $\mathcal{N}$=1 U(N) SYM in (9+1)d via dimensional reduction to (0+1)d

or equivalently from $\mathcal{N}$=4 U(N) SYM in (3+1)d: it is maximally supersymmetric

$$S = \int_0^{1/T} dt\, L$$

$$\lambda = g_{\mathrm{YM}}^2 N$$

't Hooft coupling

$X_M\,, M = 1, \ldots, 9\ (N \times N) \rightarrow$ hermitian scalars

$\psi^\alpha\,, \alpha = 1, \ldots, 16\ (N \times N) \rightarrow$ adjoint fermions

$D_t \cdot\ = \partial_t \cdot - i[A_t, \cdot] \rightarrow$ gauge covariant derivative

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[BFSS, hep-th/9610043]

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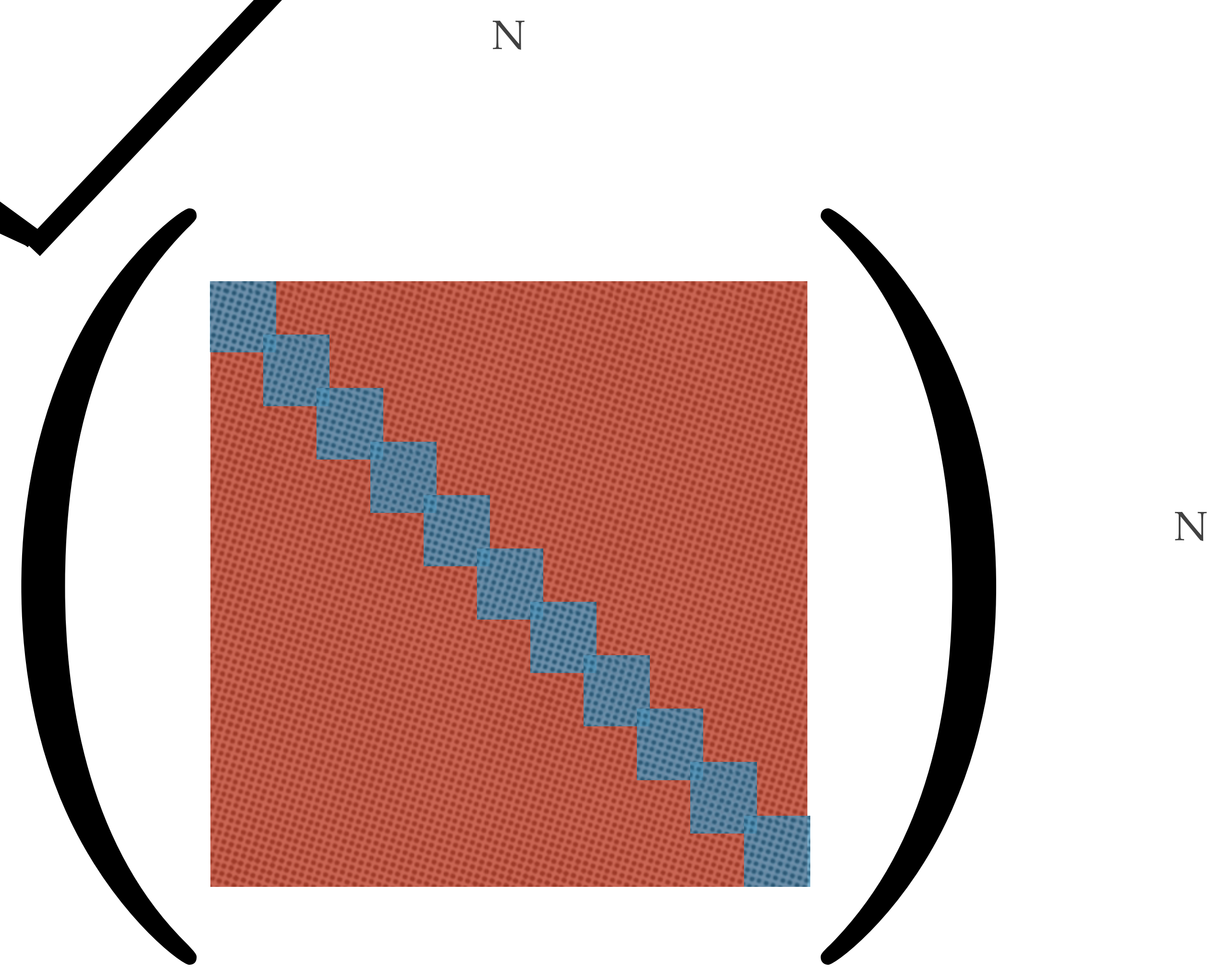

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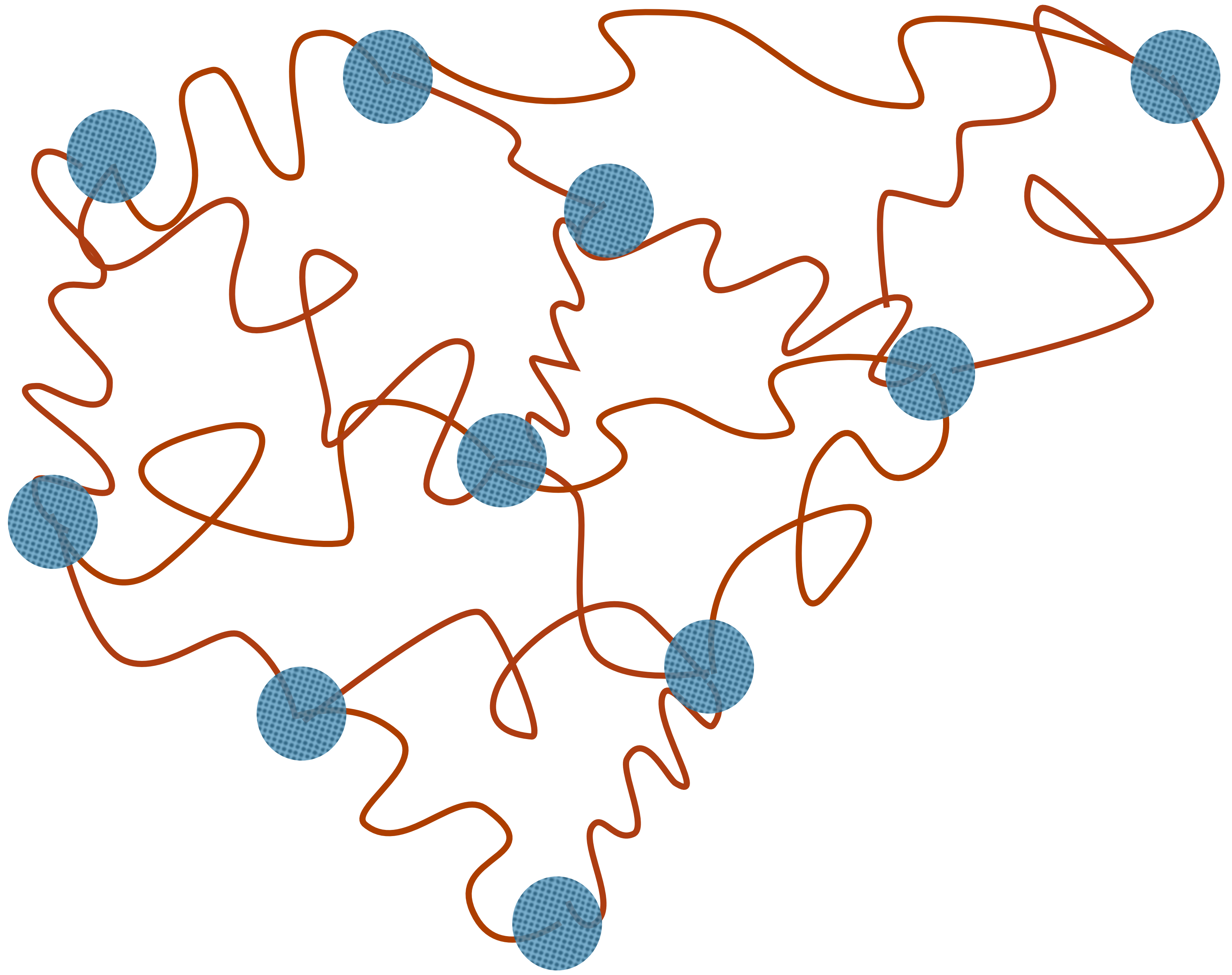
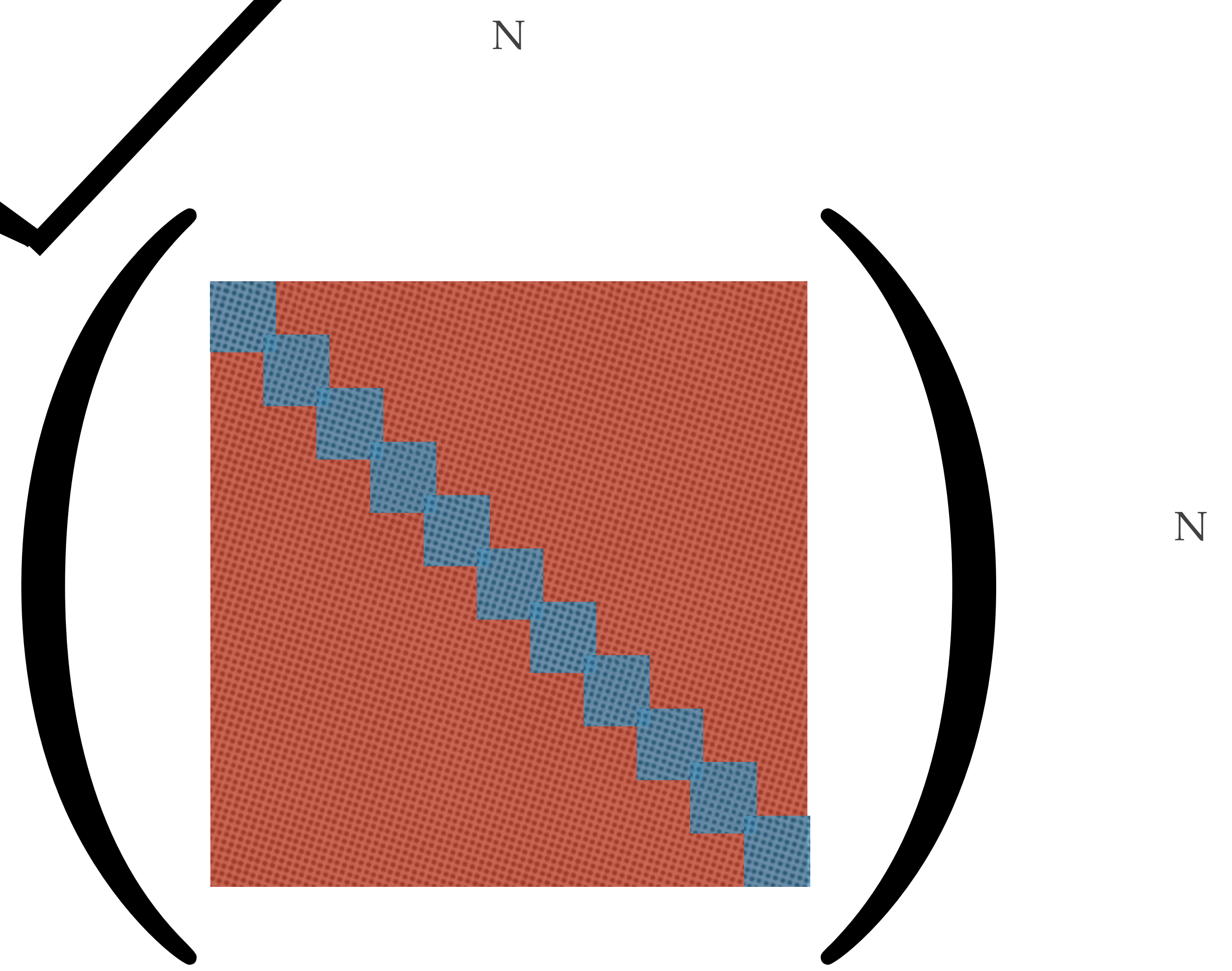


D0-branes

coordinates

couplings



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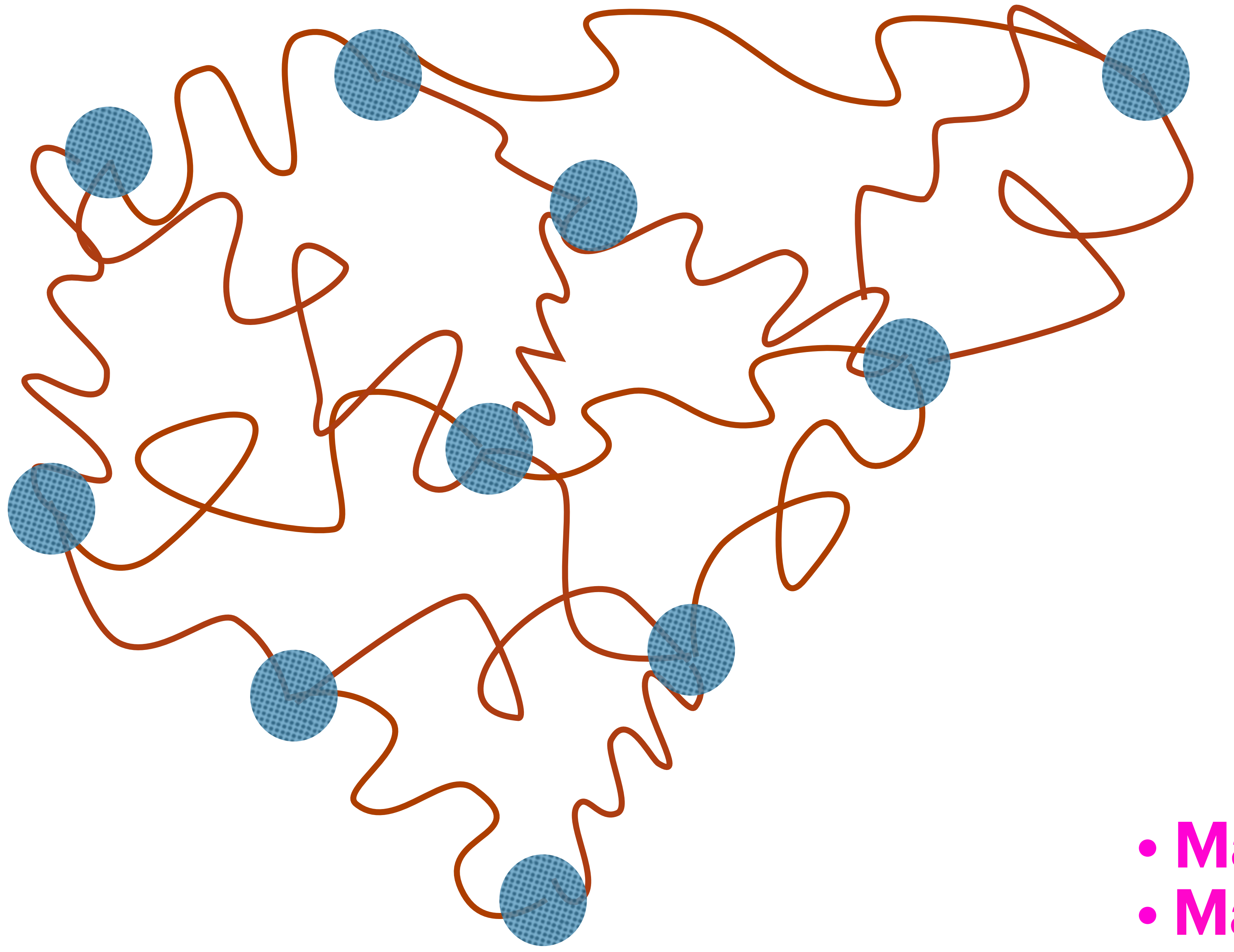
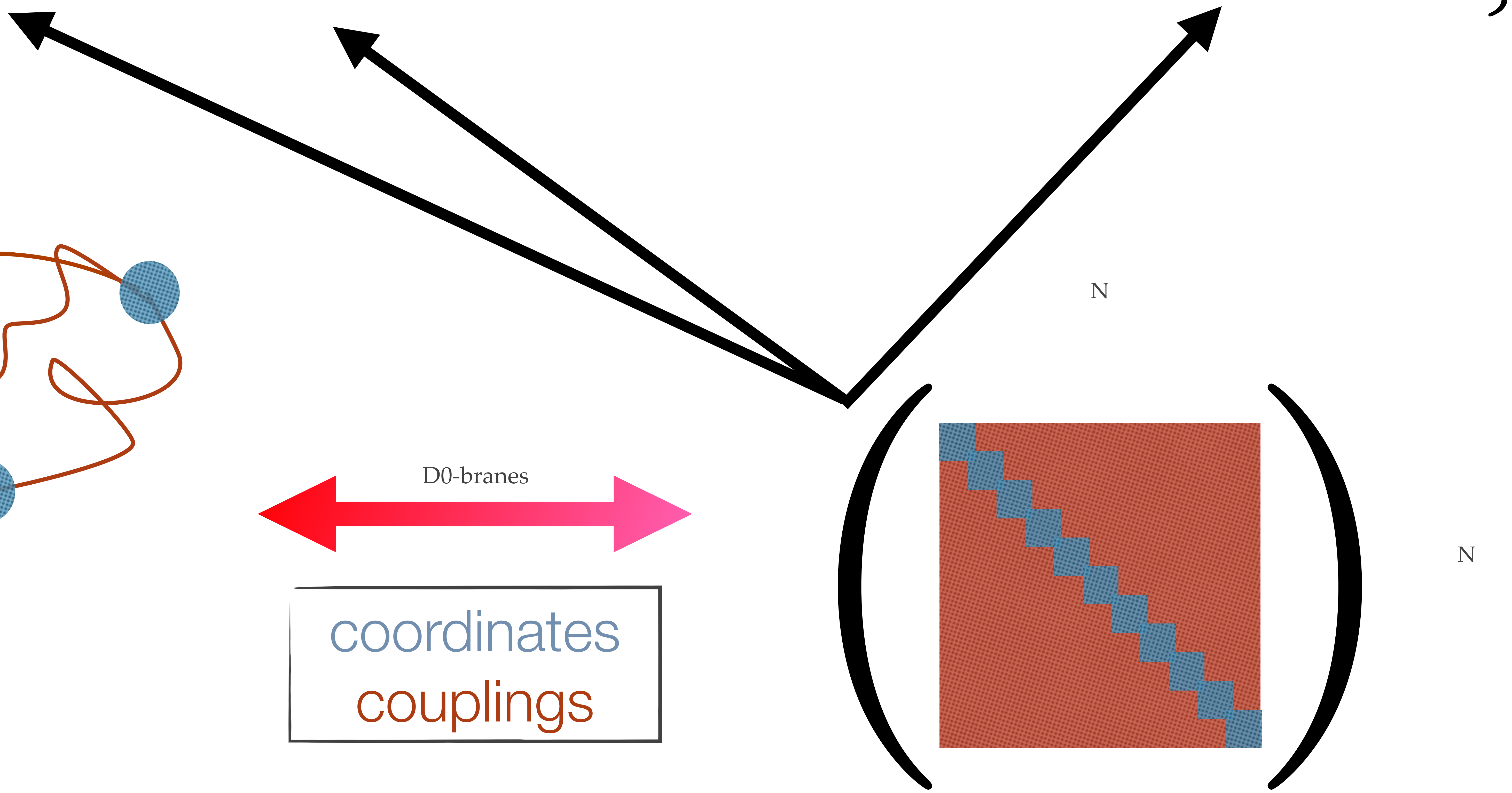




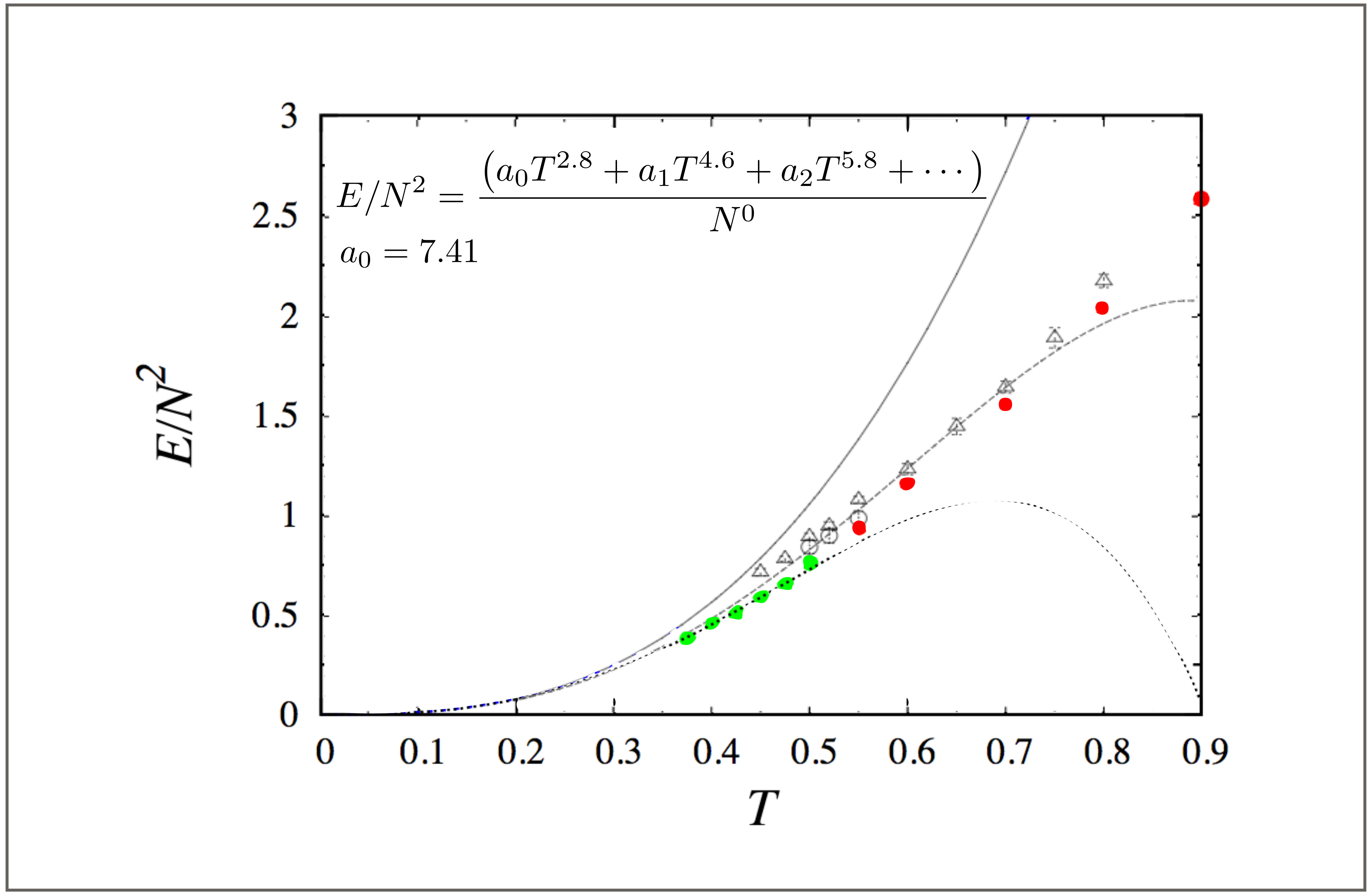
- Matrix regularization of 11D supermembrane [De Wit, Hope, Nicolai, 1988]
- Matrix model of M-theory [BFSS, 1996]
- Dual to type IIA black 0-brane near 't Hooft limit [IMSY, 1998]

Previous results

A long history

[Agnastopoulos et al. arxiv:0707:4454]
[Catteral, Wiseman arxiv:0803.4273]
[Hanada et al. arxiv:0811.3102]
[Hanada et al. arxiv:1311.5603]
[Kadoh, Kamata arxiv:1503.08499]
[Filev, O'Connor arxiv:1506.01366]
…

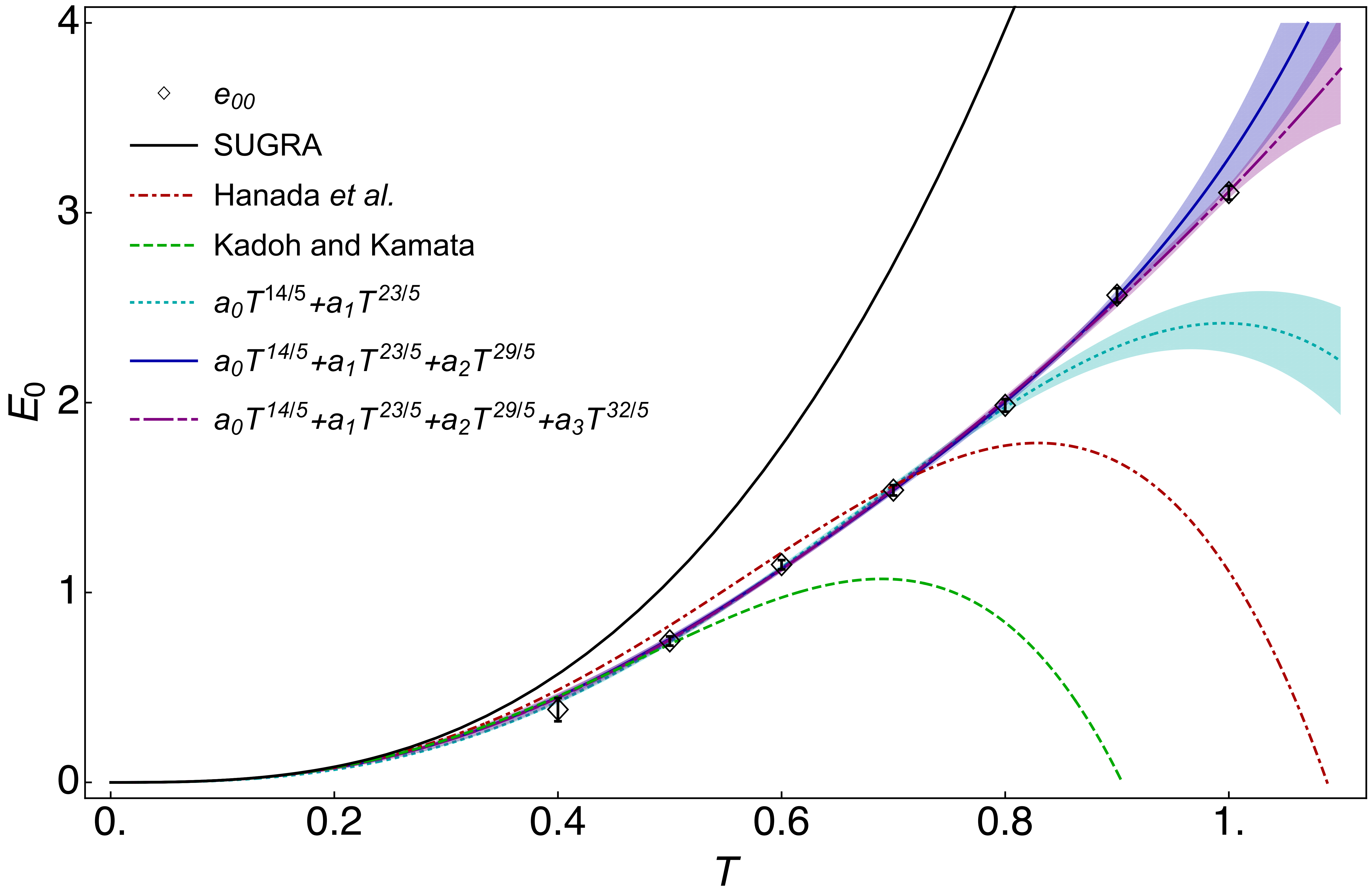
- Different cutoff regulator
- Different discretizations
- Finite N
- Finite cutoff
- Qualitative agreement
- Not enough precision for quantitative predictions



[Hanada et al. arxiv:0811.3102, Kadoh and Kamata arxiv:1503.08499]

TEST OF DUALITY IN BFSS

$$\frac{E}{N^2} = a_0 T^{2.8} + a_1 T^{4.6} + a_2 T^{5.8}$$



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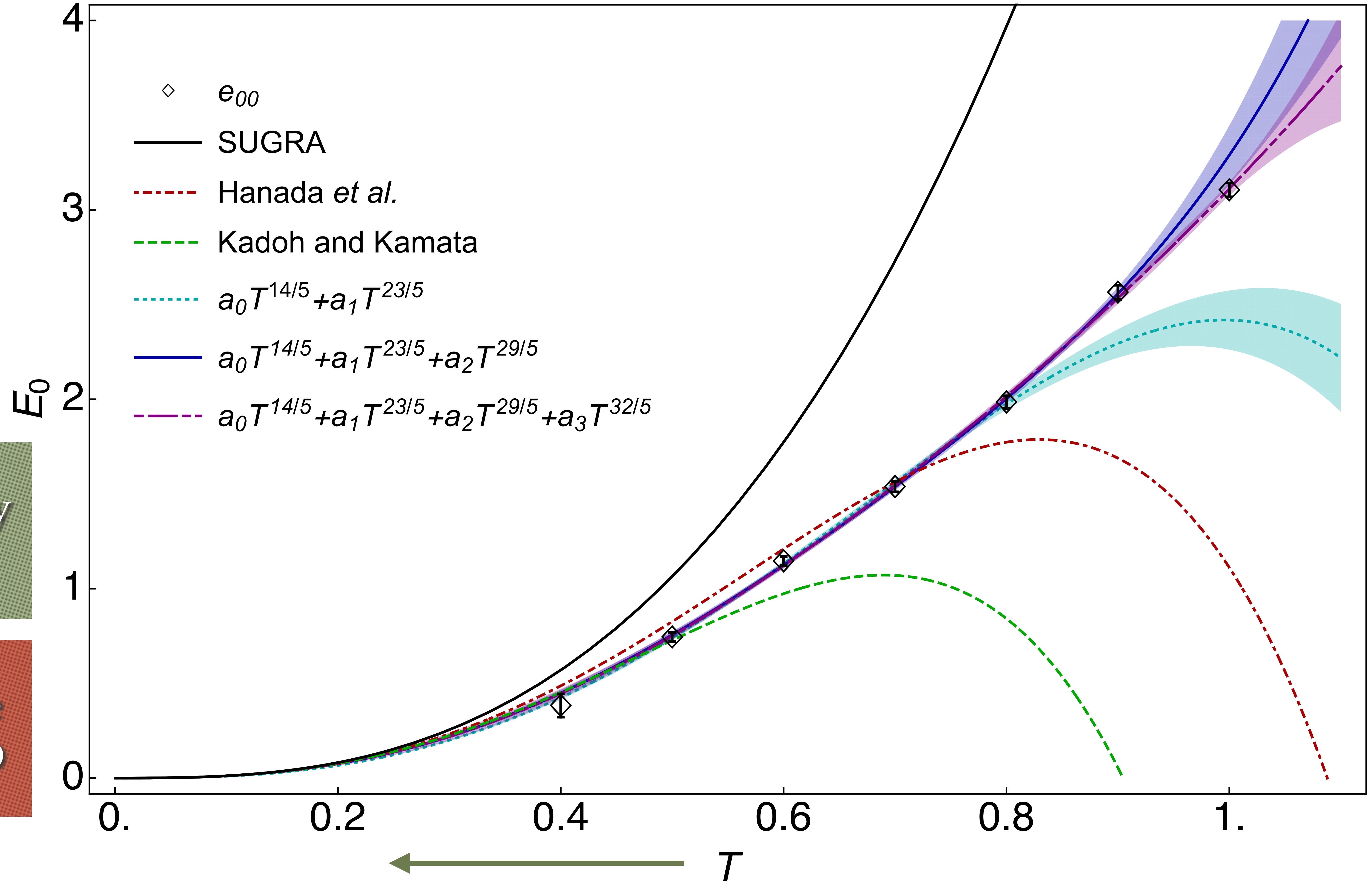
$a_0 = 7.41$ Supergravity

Gravity EASY

Gauge HARD



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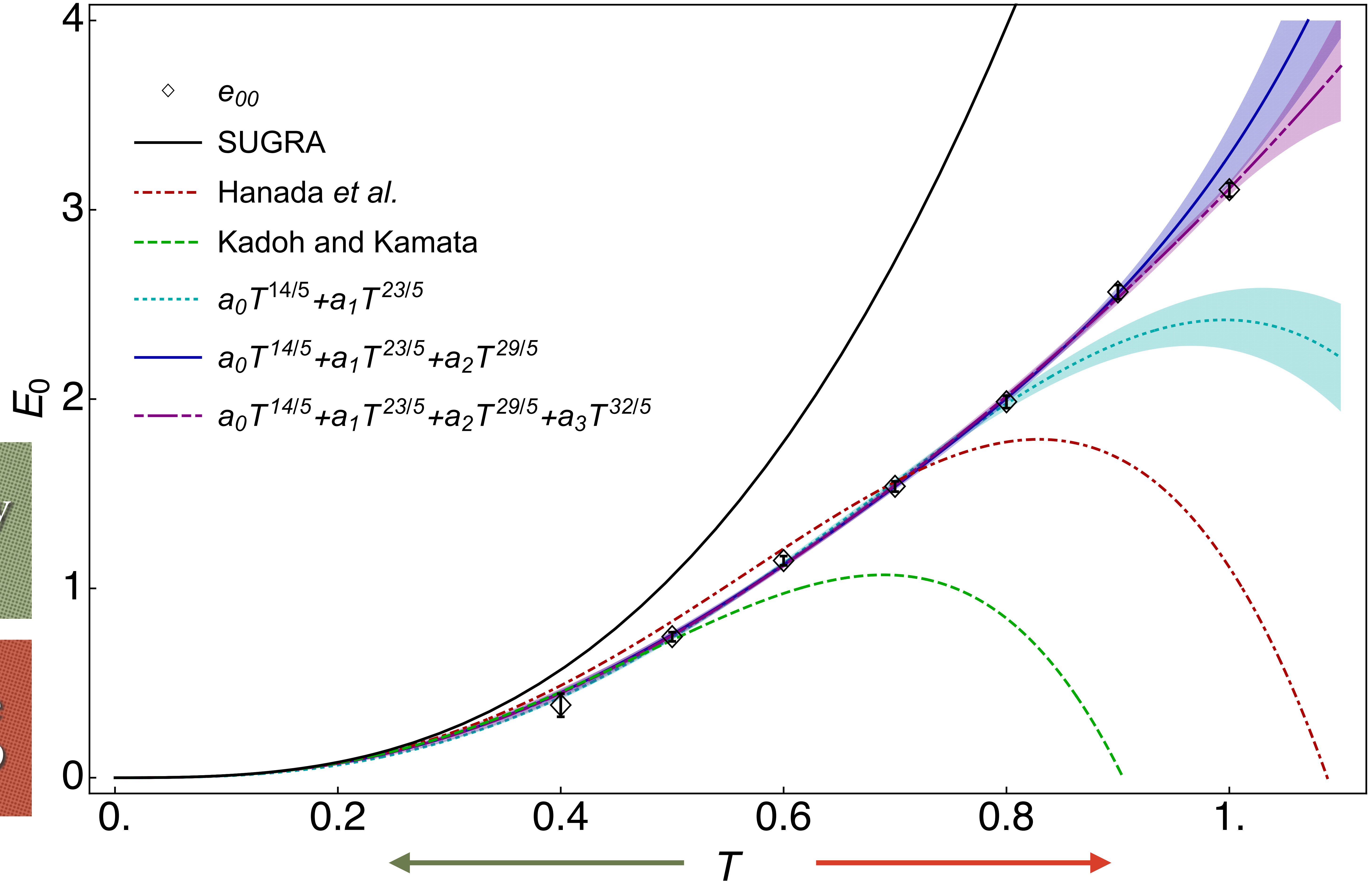

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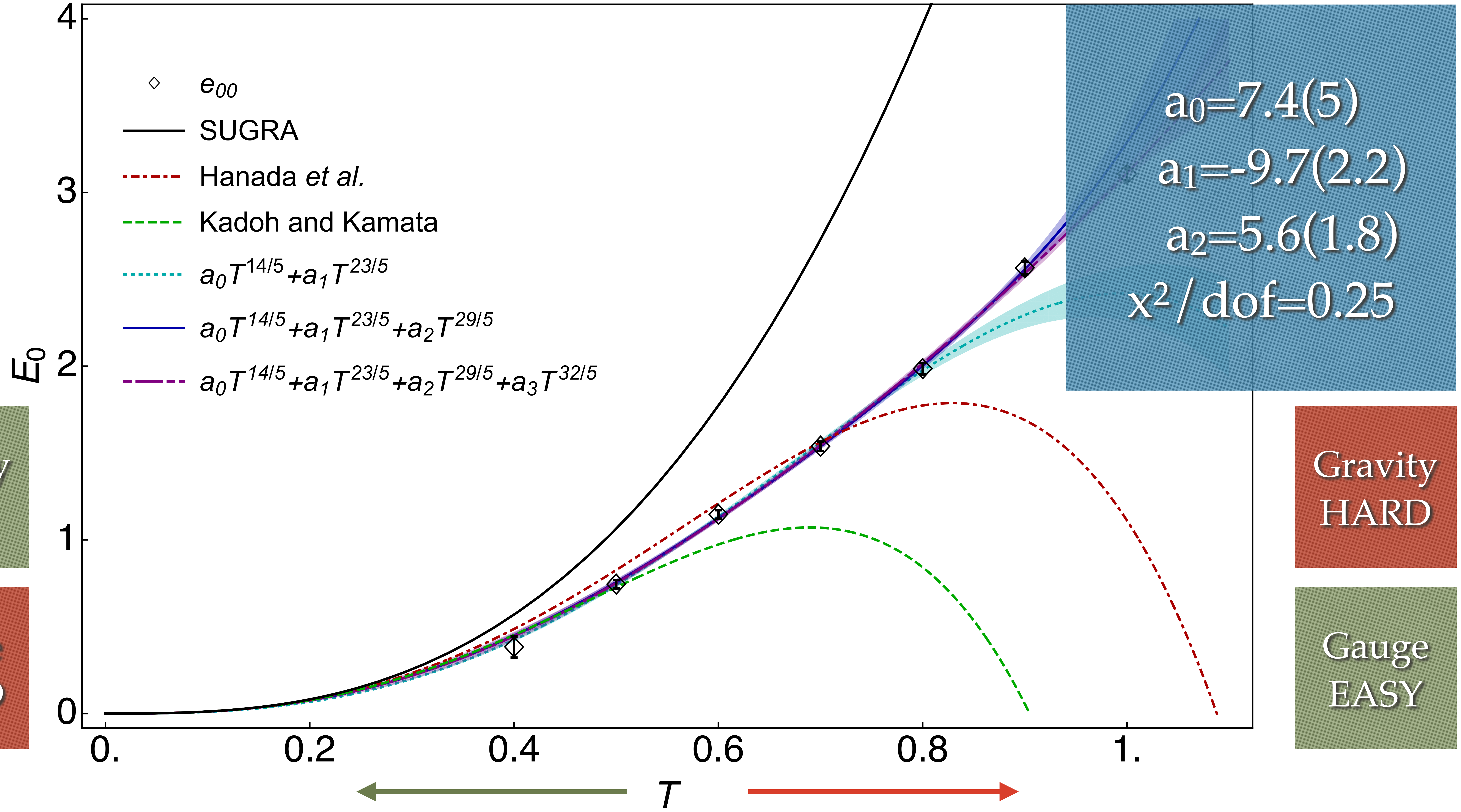

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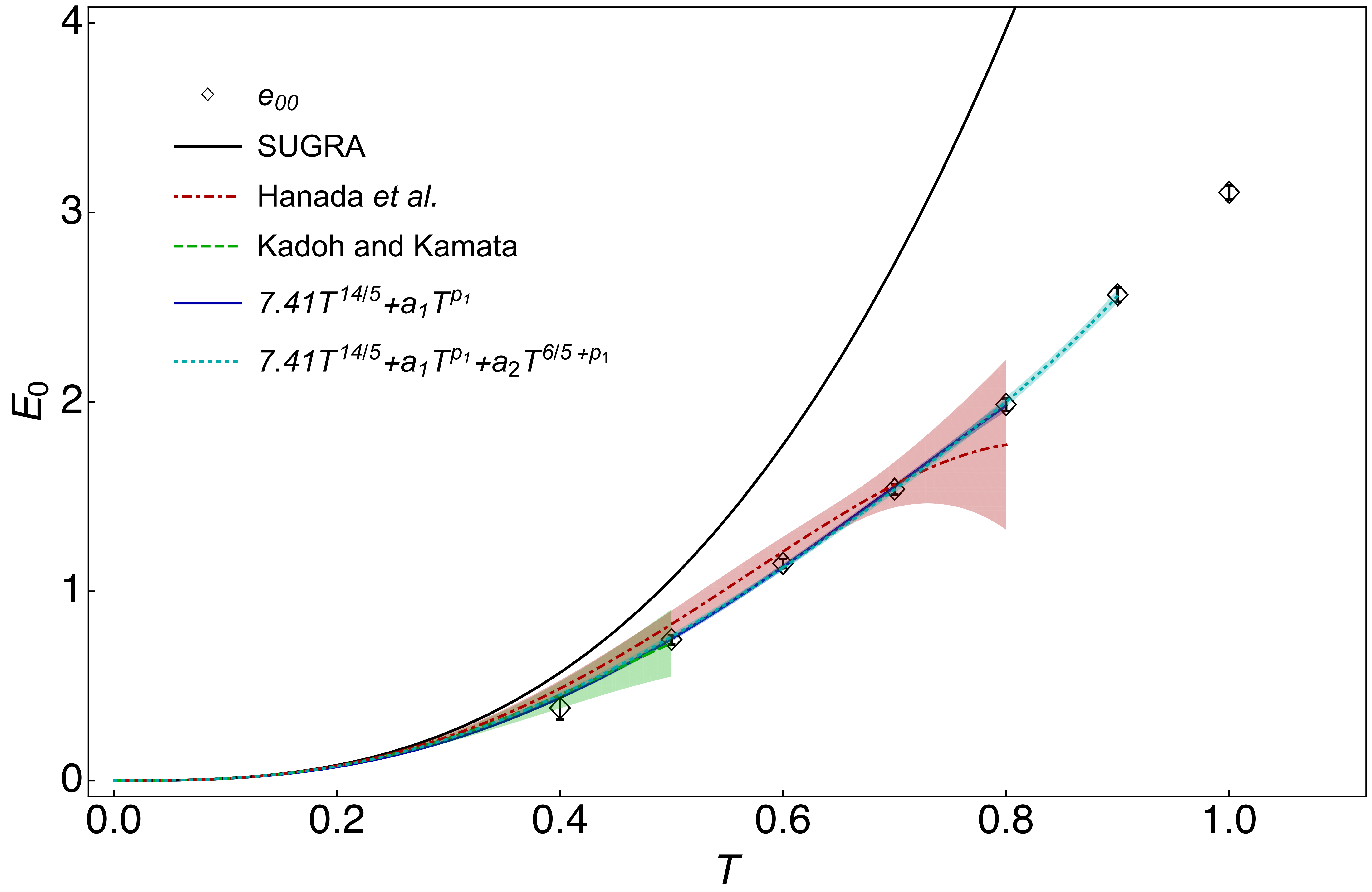
TEST STRING EFFECTS

$$\frac{E}{N^2} = a_0 T^{2.8} + a_1 T^{p_1} + a_2 T^{p_1 + 6/5}$$



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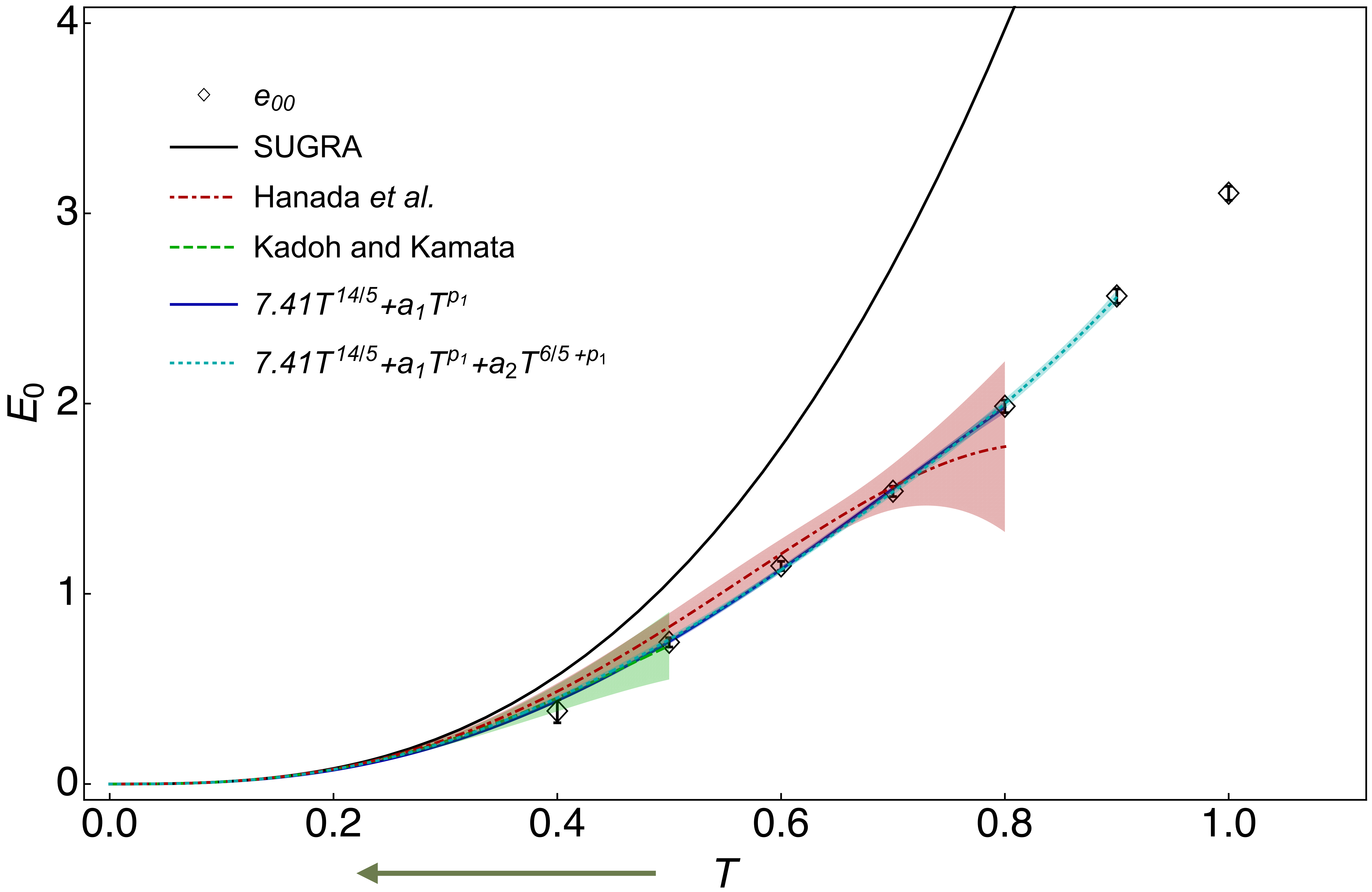

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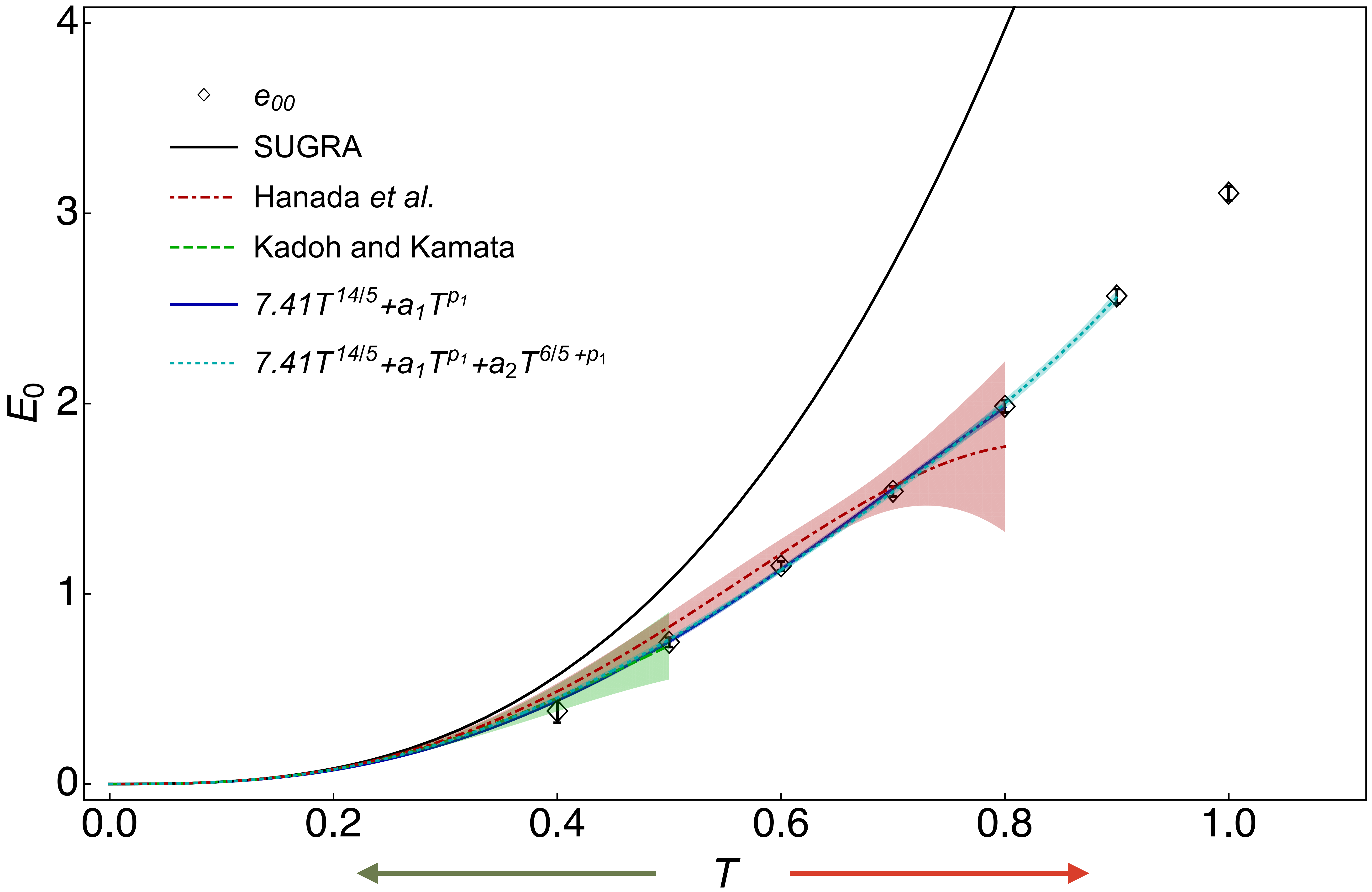

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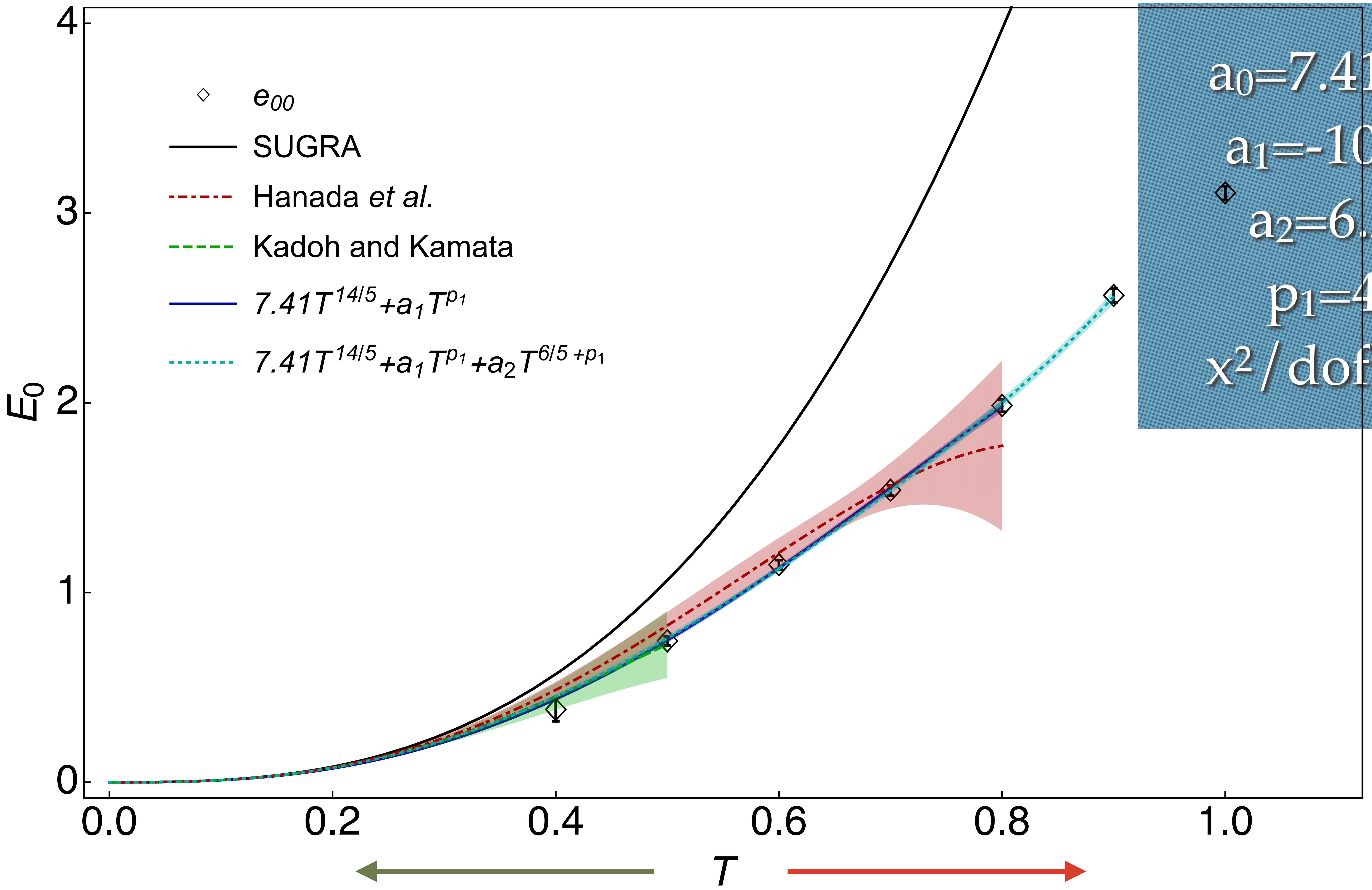

a_0=7.41 (fix)
a_1=-10.2(2.4)
a_2=6.2(2.6)
p_1=4.6(3)
x^2/dof=0.87

Gravity EASY

Gauge HARD

Gravity HARD

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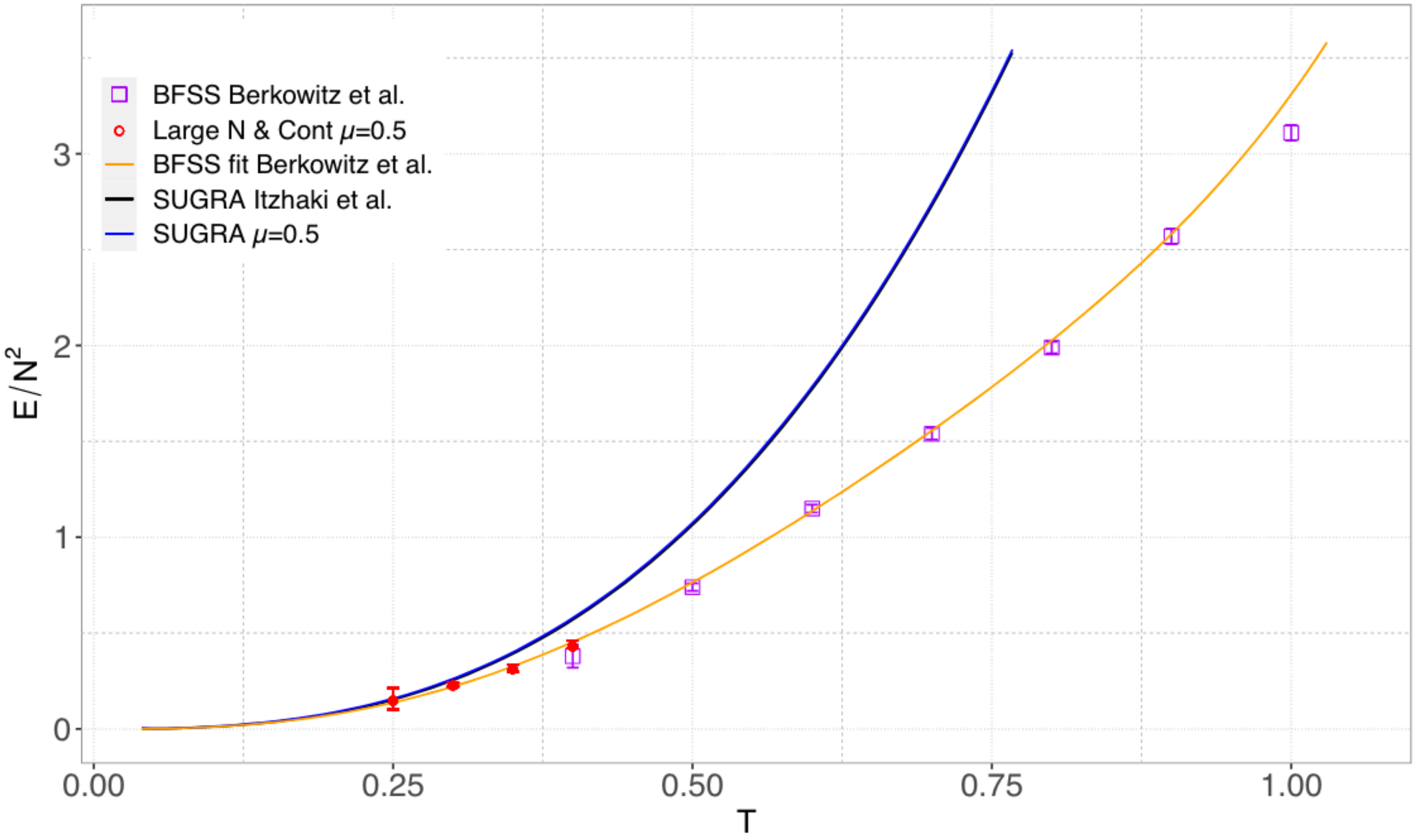
[Pateloudis et al., JHEP03(2023)071]

State-of-the-art results

6 years later...



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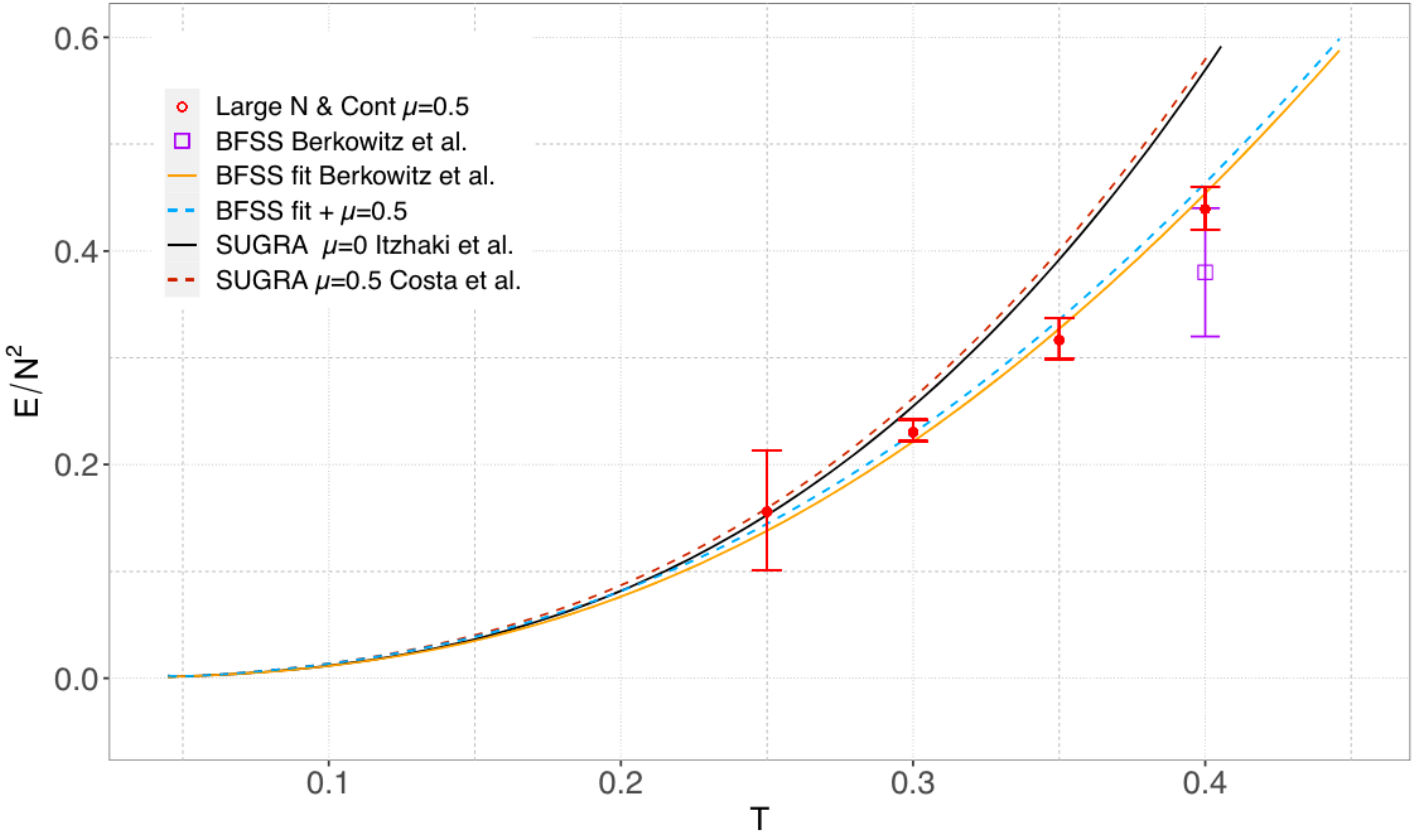

New Directions

★HPC simulations using Path Integral-based methods on discrete grids: Monte Carlo sampling of quantum mechanical paths.

➡**Challenges**:

- Sign problem → paths are not weighted with a standard probability distribution (*i.e.* chem. pot., **time evolution**)
- Wave function → physics applications require knowledge of entanglement (*i.e.* **information paradox, geometry**)

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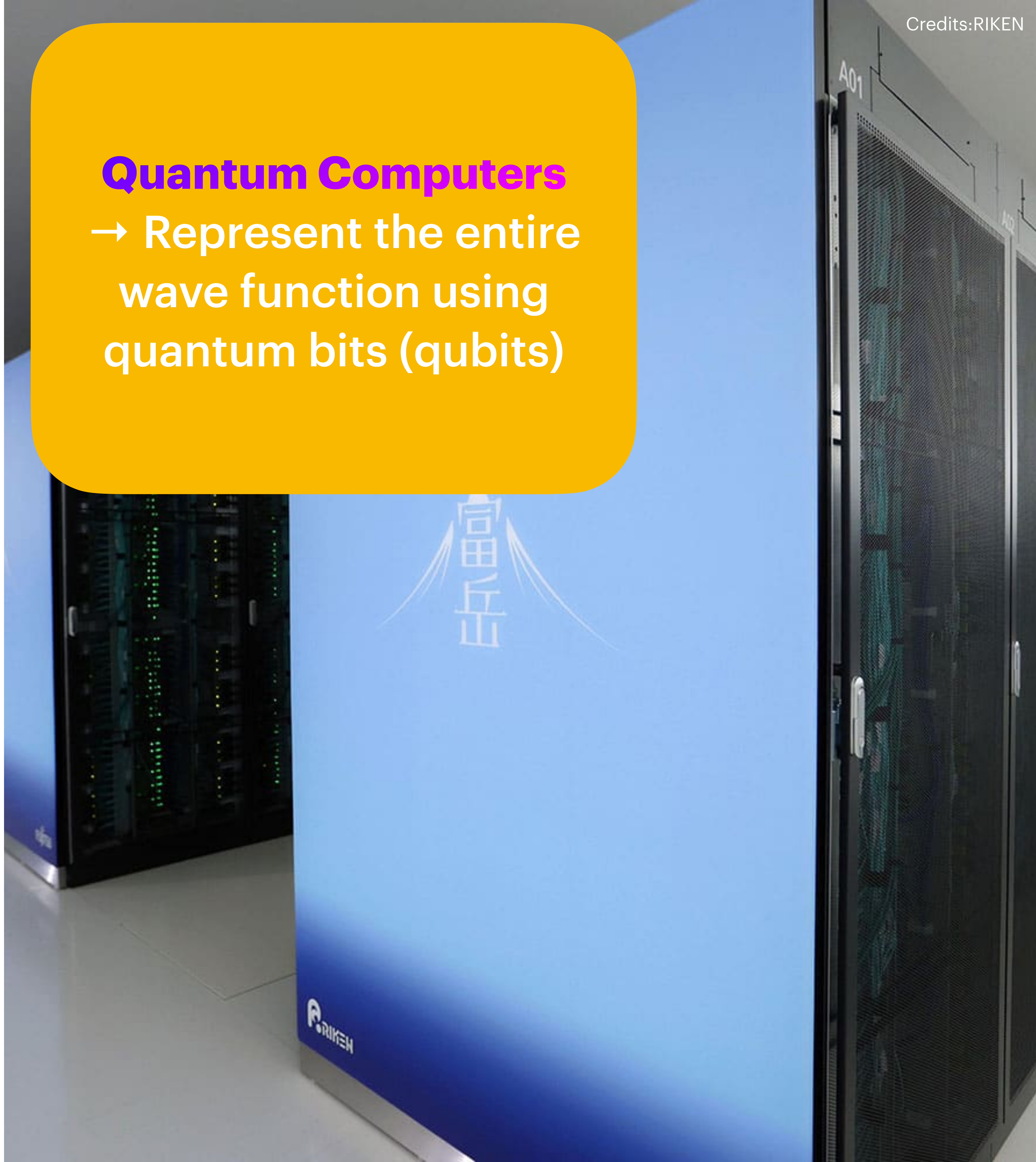

Credits:RIKEN

Quantum Computers
→ Represent the entire wave function using quantum bits (qubits)

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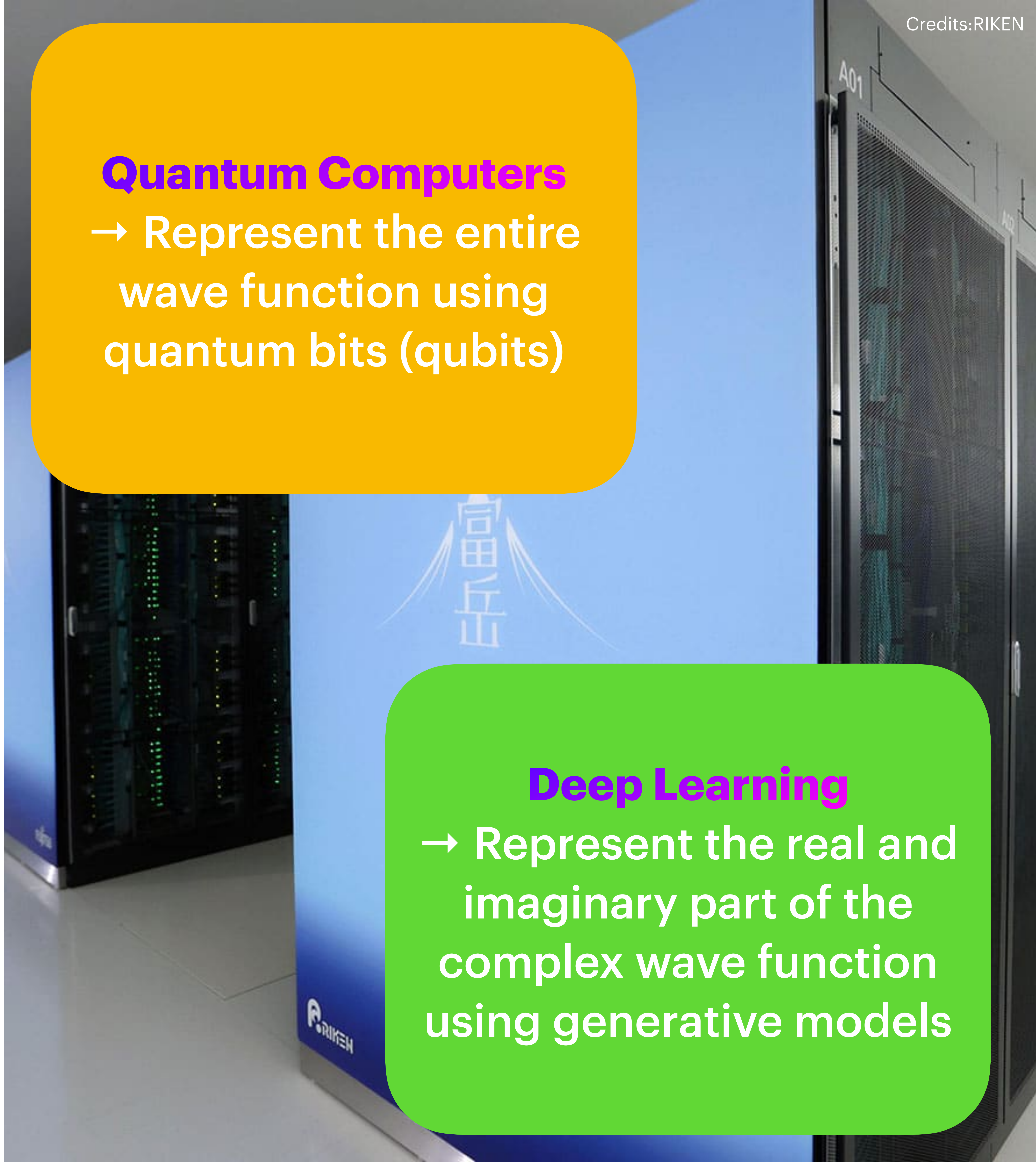

Credits:RIKEN

Quantum Computers
→ Represent the entire wave function using quantum bits (qubits)

Deep Learning
→ Represent the real and imaginary part of the complex wave function using generative models

Quantum Computing

arxiv:2303.11534

- Juan Maldacena is not immune to the quantum “hype" 😁
- He estimates the number of qubits and gates to describe the evolution of a black hole:

$$n_q \sim N^2 \left[8 + 9 \log_2 \left(\frac{\lambda^{1/3}}{\omega} \right) \right] \sim 7,000, \quad \text{for} \quad N = 16, \quad \frac{\lambda^{1/3}}{\omega} \sim 4$$

A simple quantum system that describes a black hole

Juan Maldacena

Institute for Advanced Study, Princeton, NJ 08540, USA

Abstract

During the past decades, theorists have been studying quantum mechanical systems that are believed to describe black holes. We review one of the simplest examples. It involves a collection of interacting oscillators and Majorana fermions. It is conjectured to describe a black hole in an emergent universe governed by Einstein equations. Based on previous numerical computations, we make an estimate of the necessary number of qubits necessary to see some black hole features.

[Rinaldi et al., PRX Quantum 3 (2022) 010324]

Numerical Methods for MQM

Prototypes

Bosonic Model

$$\hat{H}_{B2} = \mathrm{Tr}\left(\frac{1}{2}\hat{P}_I^2 + \frac{m^2}{2}\hat{X}_I^2 - \frac{g^2}{4}\left[\hat{X}_I, \hat{X}_J\right]^2\right)$$

Physical states are invariant under SU(N) Gauge Symmetry

Supersymmetric Model

$$\hat{H} = \hat{H}_{B2} + \mathrm{Tr}\left(\frac{g}{2}\hat{\xi}\left[-\hat{X}_1 - i\hat{X}_2, \hat{\xi}\right] + \frac{g}{2}\hat{\xi}^\dagger\left[-\hat{X}_1 + i\hat{X}_2, \hat{\xi}^\dagger\right] + \frac{3\mu}{2}\hat{\xi}^\dagger\hat{\xi}\right) - \left(N^2 - 1\right)\mu$$

[Rinaldi et al., PRX Quantum 3 (2022) 010324]

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$$\hat{X}_I = \sum_{\alpha=1}^{N^2-1} \hat{X}_I^\alpha \tau_\alpha \qquad I = 1,\ldots,D$$

$\hat{X}_I^\alpha$ → bosonic degrees of freedom

τ_α → generators of SU(N) group

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Numerical Methods for MQM

Prototypes

Bosonic Model

FREE

$$\hat{H}_{B2} = \mathrm{Tr}\left(\frac{1}{2}\hat{P}_I^2 + \frac{m^2}{2}\hat{X}_I^2 - \frac{g^2}{4}\left[\hat{X}_I, \hat{X}_J\right]^2\right)$$

Physical states are invariant under SU(N) Gauge Symmetry

$$\hat{X}_I = \sum_{\alpha=1}^{N^2-1} \hat{X}_I^{\alpha}\tau_{\alpha} \qquad I = 1,\ldots,D$$

$\hat{X}_I^{\alpha}$ → bosonic degrees of freedom

τ_{α} → generators of SU(N) group

Supersymmetric Model

$$\hat{H} = \hat{H}_{B2} + \mathrm{Tr}\left(\frac{g}{2}\hat{\xi}\left[-\hat{X}_1 - i\hat{X}_2, \hat{\xi}\right] + \frac{g}{2}\hat{\xi}^{\dagger}\left[-\hat{X}_1 + i\hat{X}_2, \hat{\xi}^{\dagger}\right] + \frac{3\mu}{2}\hat{\xi}^{\dagger}\hat{\xi}\right) - \left(N^2 - 1\right)\mu$$

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[Rinaldi et al., PRX Quantum 3 (2022) 010324]

Numerical Methods for MQM

Prototypes

Bosonic Model

FREE | BOS. INTERACTION

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FREE · BOS. INTERACTION

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FERM.-BOS. INTERACTION

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FERM.-BOS. INTERACTION — ZERO EN.

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Numerical Methods for MQM

Prototypes

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FERM.-BOS. INTERACTION

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Challenge: numerical methods on quantum computers have a limited number of qubits!

[Rinaldi et al., PRX Quantum 3 (2022) 010324]

Numerical Methods for MQM

Prototype: small-scale system

Bosonic Model

FREE | BOS. INTERACTION

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Example: N=2, D=2

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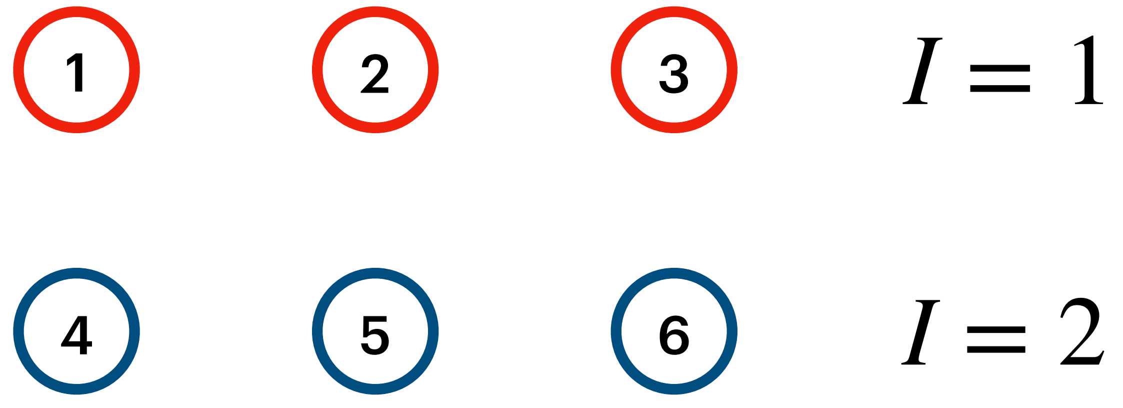

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FREE

BOS. INTERACTION

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SYMMETRIES

1 2 3 $I = 1$

4 5 6 $I = 2$

$g^2 = 0$

$g^2 > 0$

[Rinaldi et al., PRX Quantum 3 (2022) 010324]

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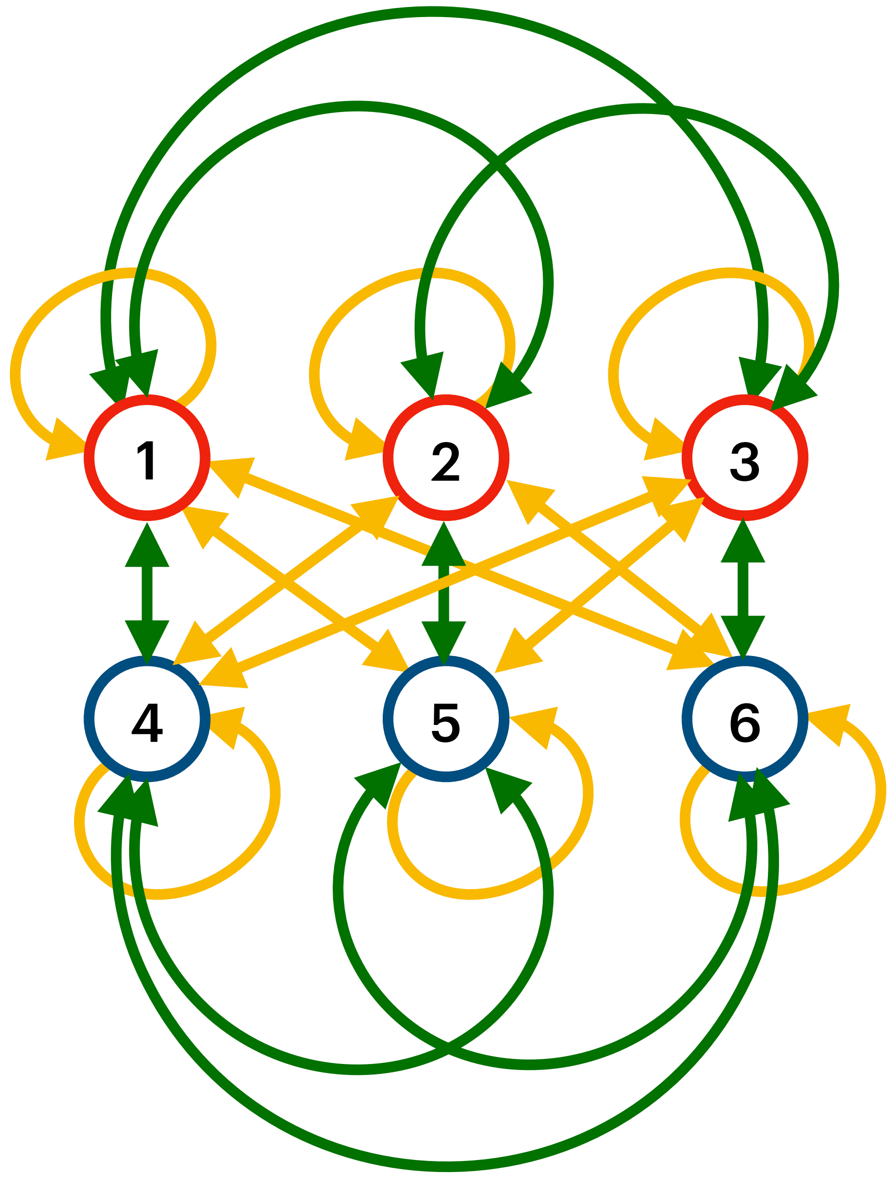

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$$|\mathrm{VACUUM}\rangle = \left(\otimes_{I,\alpha}|0\rangle_{I\alpha}\right) \rightarrow_{g^2>0} |\mathrm{Ground\ State}\rangle = (???)$$

[Gharibyan et al., JHEP07(2021)140]

Hilbert space regularization

Truncation in Fock space

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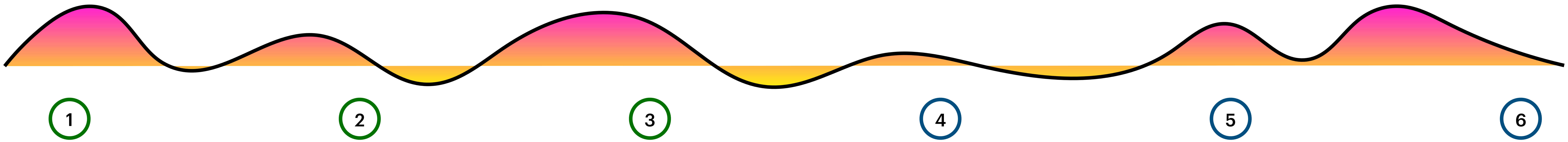

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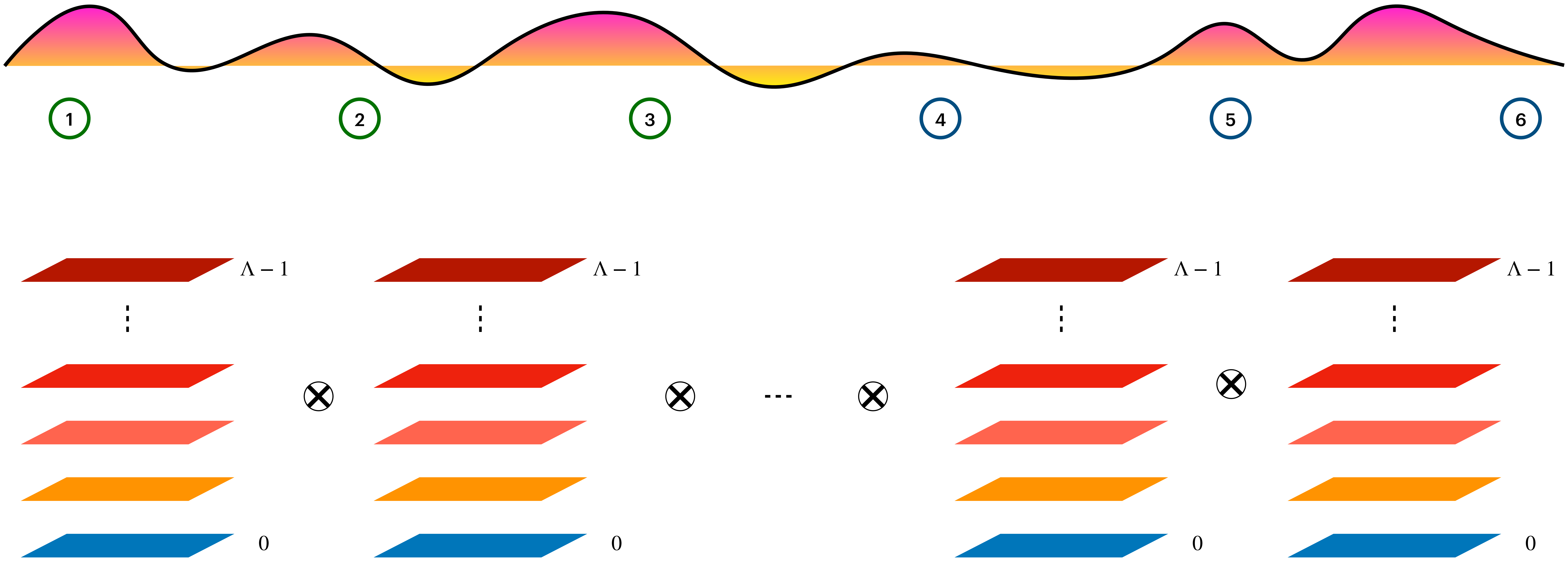

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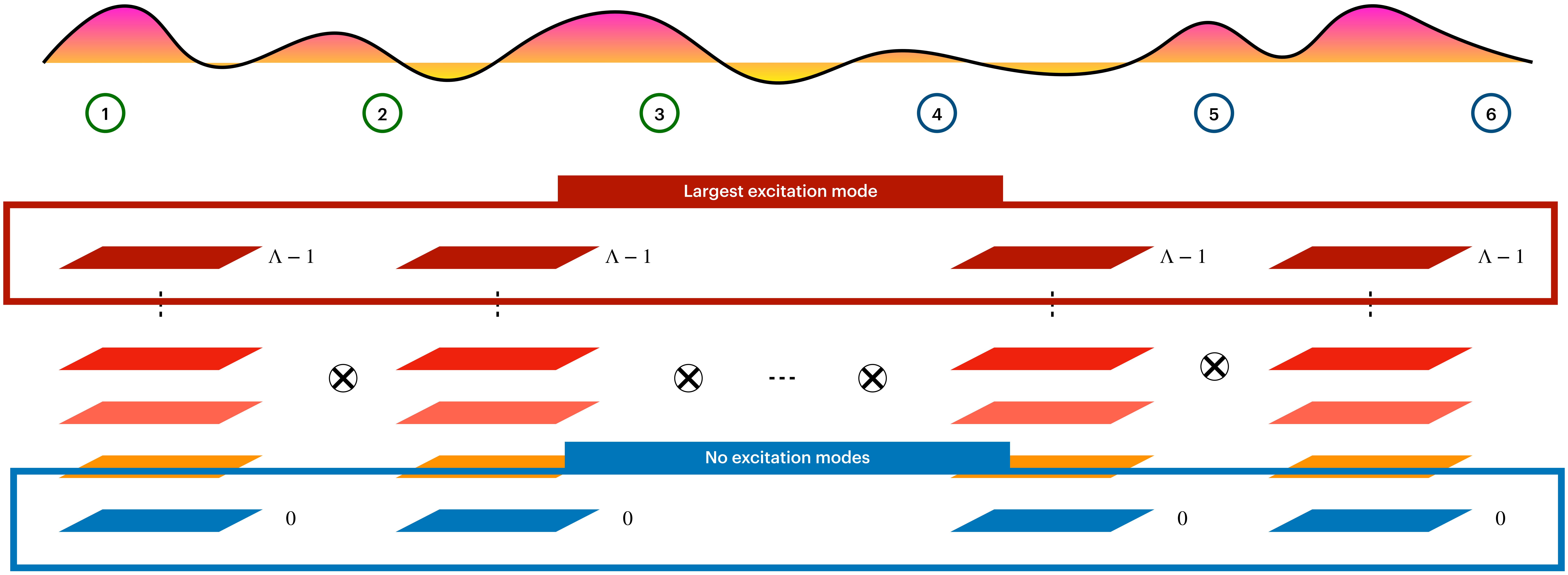

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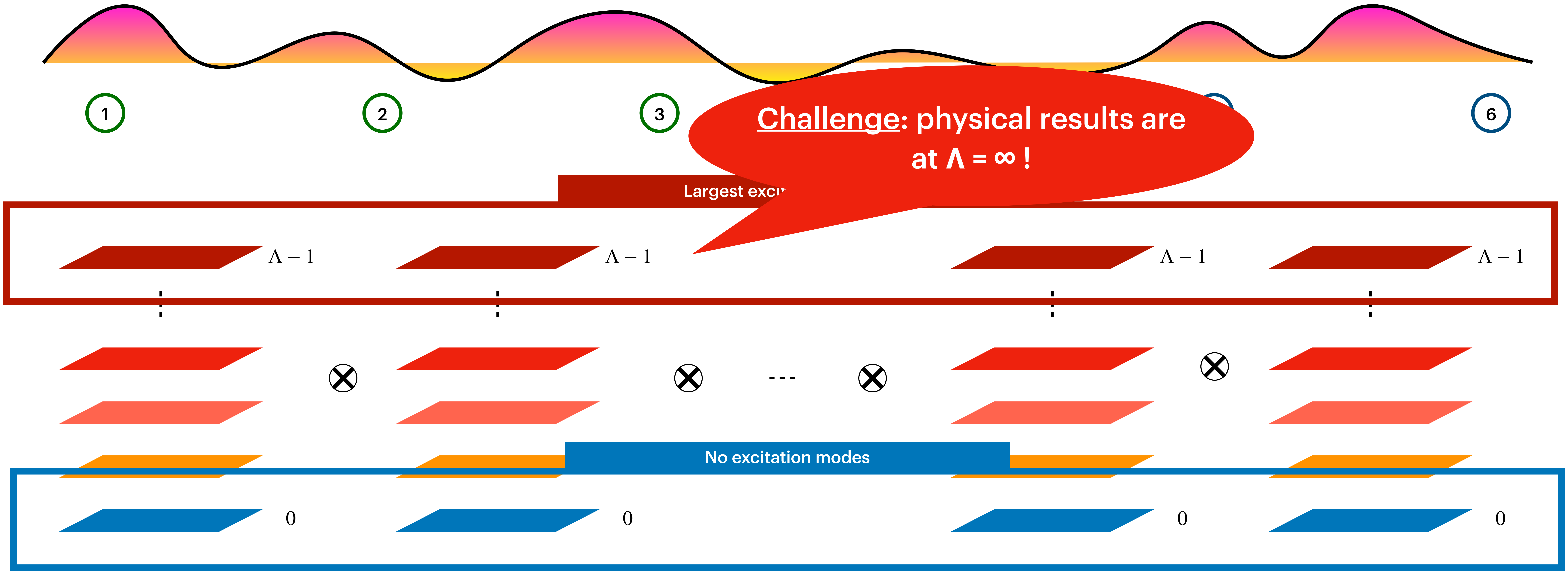
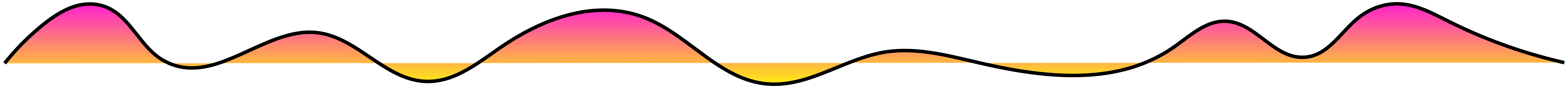


Hilbert space regularization

Truncation in the coordinate basis

$$\hat{x}\,|x\rangle = x\,|x\rangle$$



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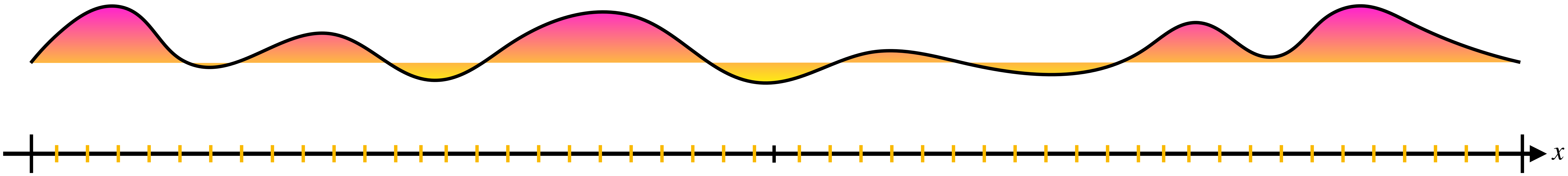

[Hanada, Liu, Rinaldi, Tezuka, MLST 4, 045021]

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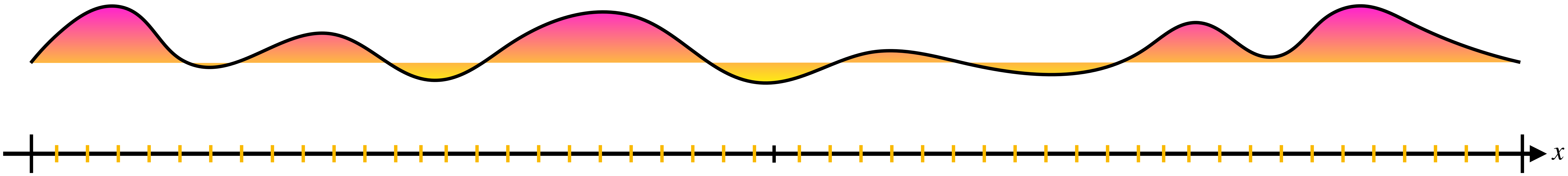

$$x(n) = -R + n a_{\mathrm{dig}}, \quad a_{\mathrm{dig}} = \frac{2R}{\Lambda - 1}, \quad n = 0, 1, \cdots, \Lambda - 1$$

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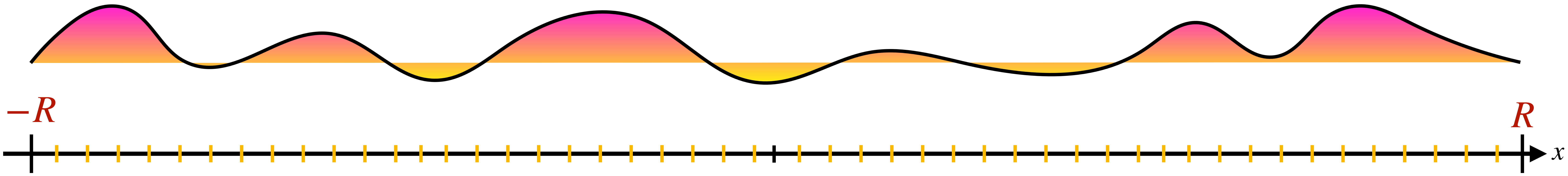

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- R is the "infrared" cutoff $\rightarrow \infty$

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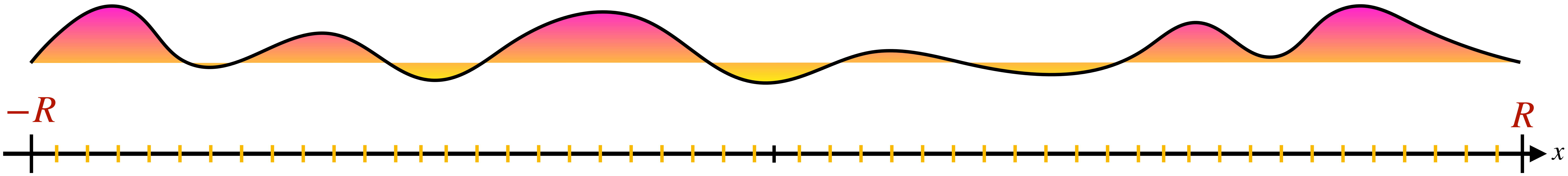

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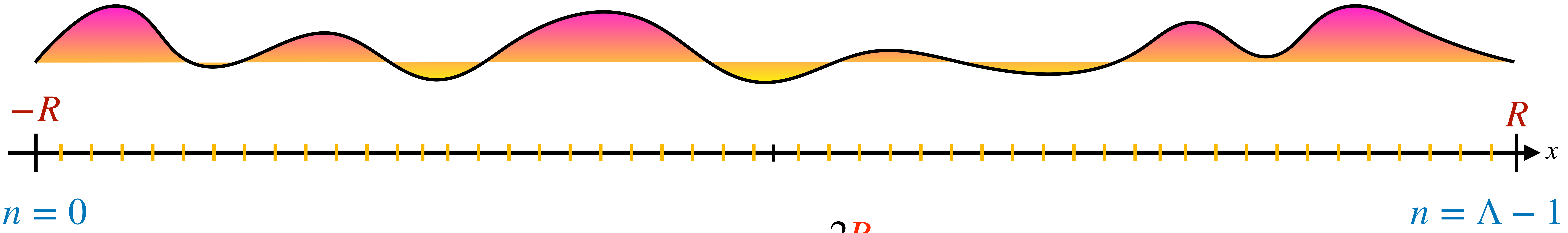

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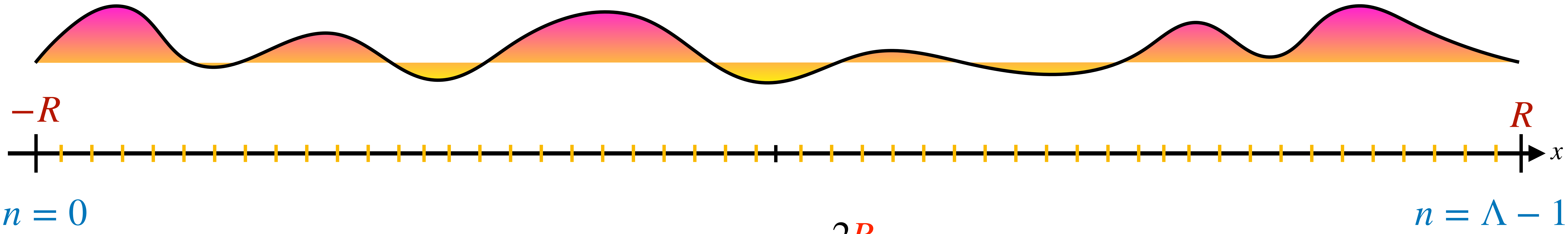

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Note: can extend this to many bosons $N_{bos} > 1$!

Hilbert space regularization

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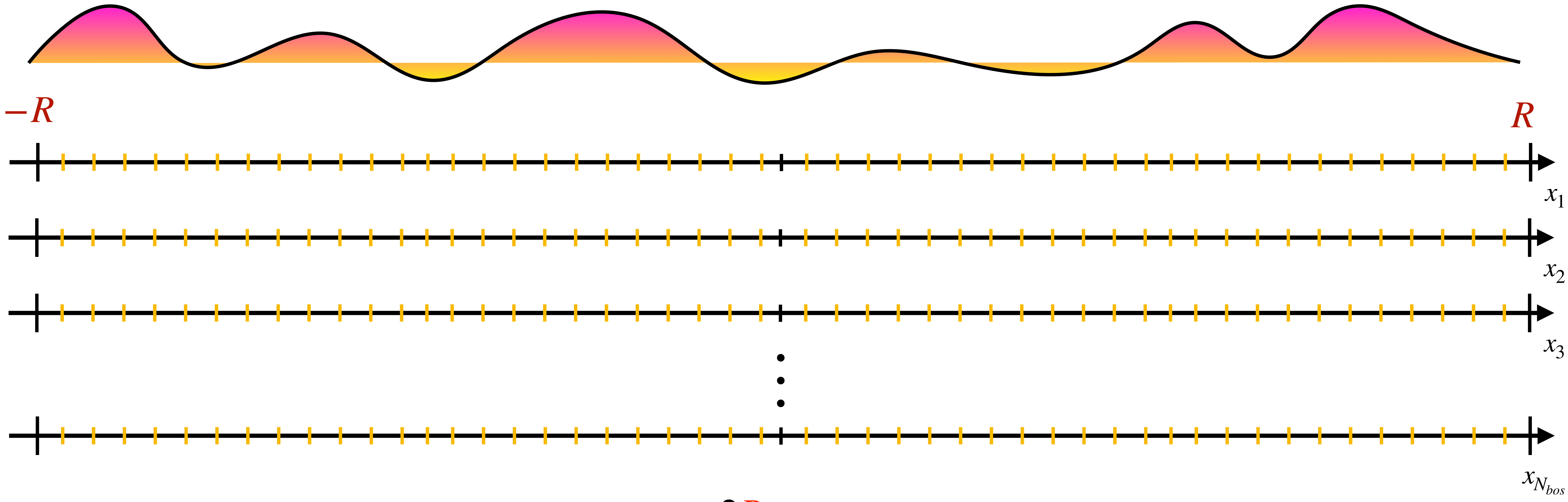

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[Halimeh, et al., arxiv:2411.13161]

Universal Framework

General paradigm across MQM and LGT

- Digitizing matrix elements or fields on lattices is a general strategy that makes these systems amenable to quantum computing algorithms
- Any Hamiltonian $\hat{H} = \frac{1}{2}\sum_a \hat{p}_a^2 + V(\hat{x})$ can be studied in this framework
- "**Space**" and "**Time**" resources can be easily computed for time evolution algorithms

	Number of qubits	# T gates in $V(\hat{x})$ term	# T gates in $\hat{p}^2$ term
Scalar QFT	$V_{\text{lattice}}Q$	$V_{\text{lattice}}\binom{Q}{4}$	$V_{\text{lattice}}Q(Q-1)$
Matrix Model	dN^2Q	$d(d-1)N^4Q^4$	$dN^2Q(Q-1)$
Orbifold YM	$2dN^2V_{\text{lattice}}Q$	$d^2V_{\text{lattice}}N^4Q^4$	$N^2dV_{\text{lattice}}Q(Q-1)$

Space

Time

https://github.com/erinaldi/bmn2-qutip

[Rinaldi et al., PRX Quantum 3 (2022) 010324]

QuTiP

Quantum Toolbox in Python

Results

Small-scale: N=2, D=2, $\Lambda \to \infty$

https://github.com/erinaldi/bmn2-qutip

QuTiP

Quantum Toolbox in Python

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Results

Small-scale: N=2, D=2, $\Lambda \to \infty$

- **Benchmark**: compute the lowest states via exact diagonalization

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QuTiP
Quantum Toolbox in Python

[Rinaldi et al., PRX Quantum 3 (2022) 010324]

Results

Small-scale: N=2, D=2, $\Lambda \to \infty$

- **Benchmark**: compute the lowest states via exact diagonalization
- Study the convergence to $\Lambda \to \infty$

QuTiP
Quantum Toolbox in Python

Results

Small-scale: N=2, D=2, Λ → ∞

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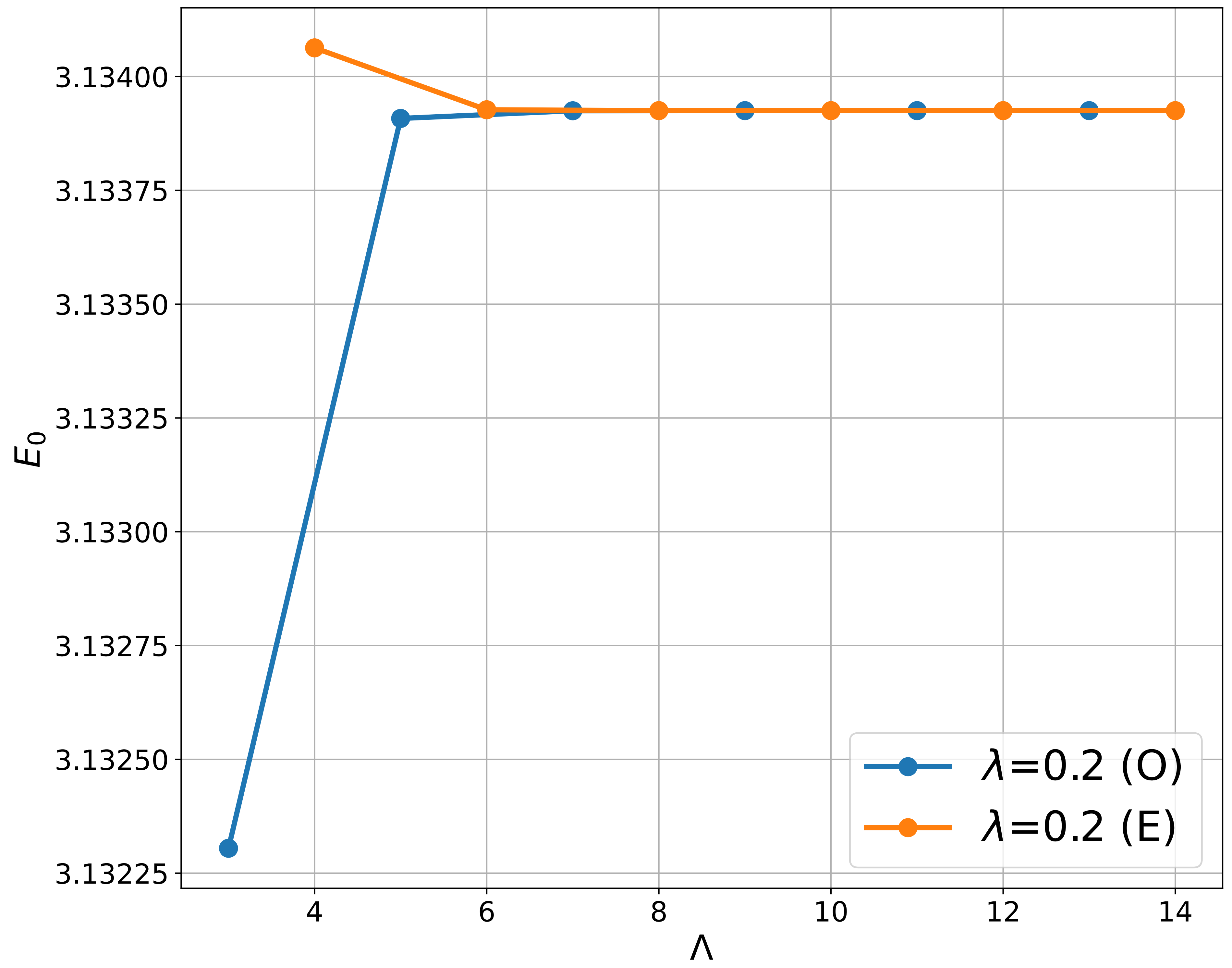
$$E_0 = \left\langle E_0 \,|\, \hat{H} \,|\, E_0 \right\rangle$$



QuTiP
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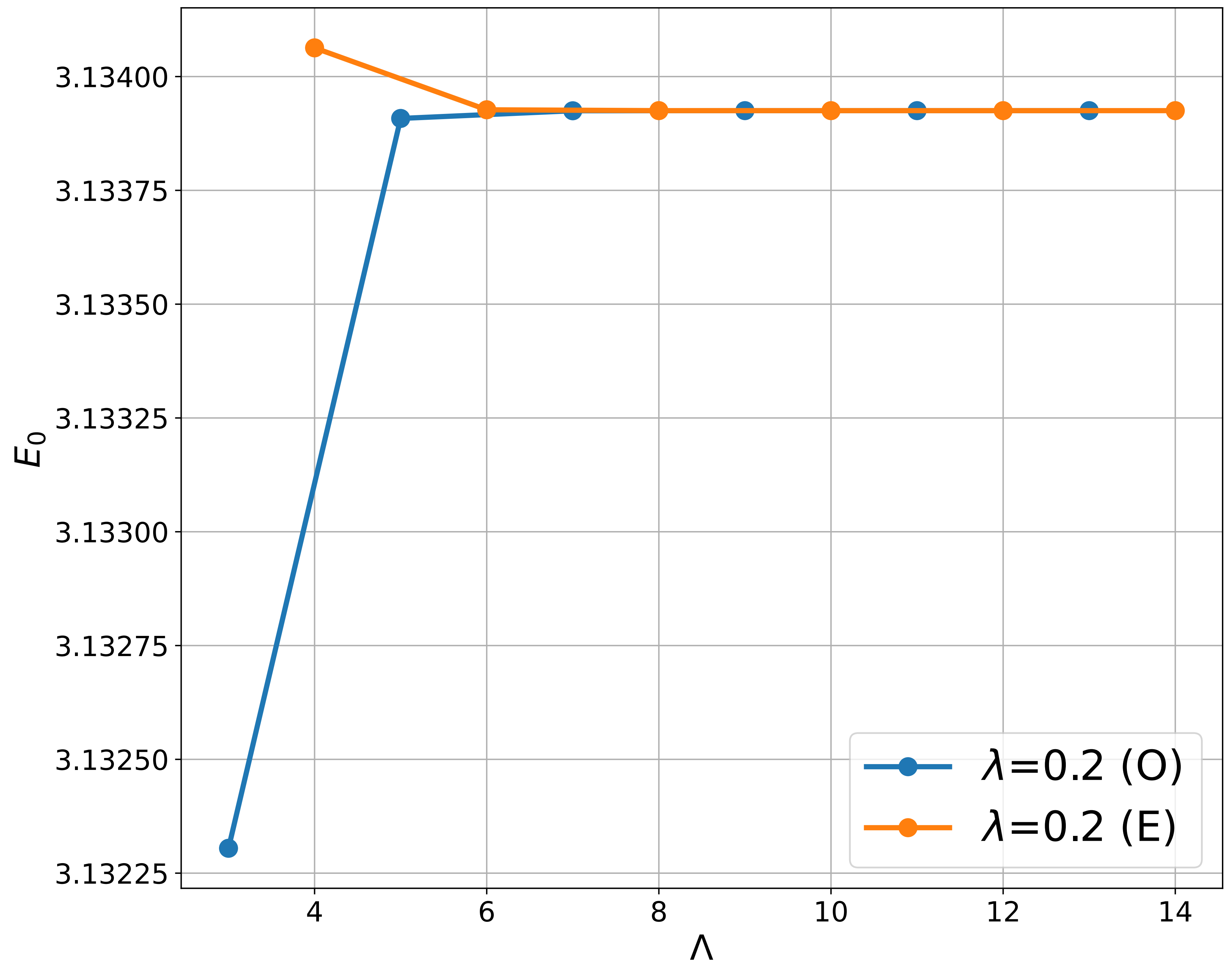

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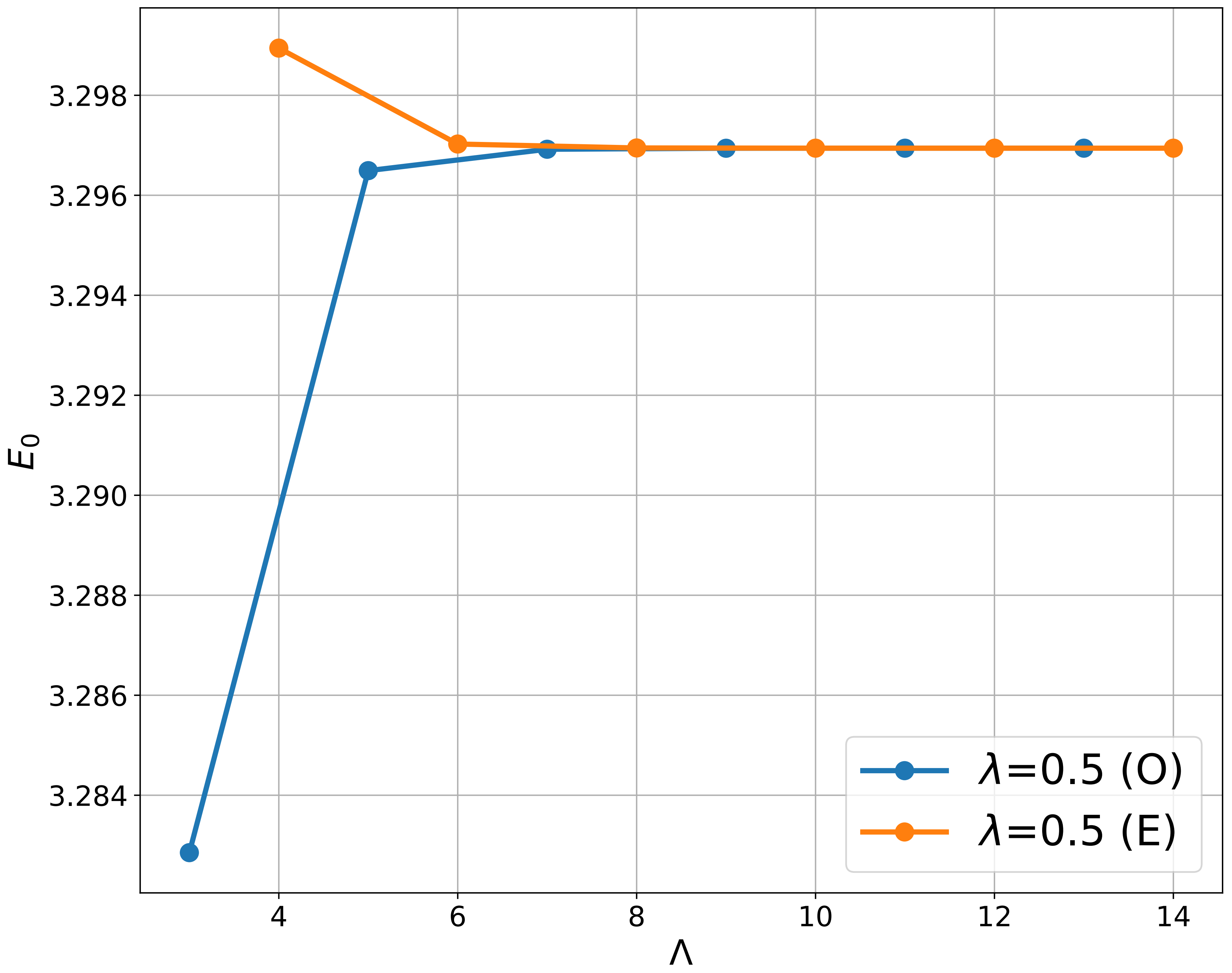

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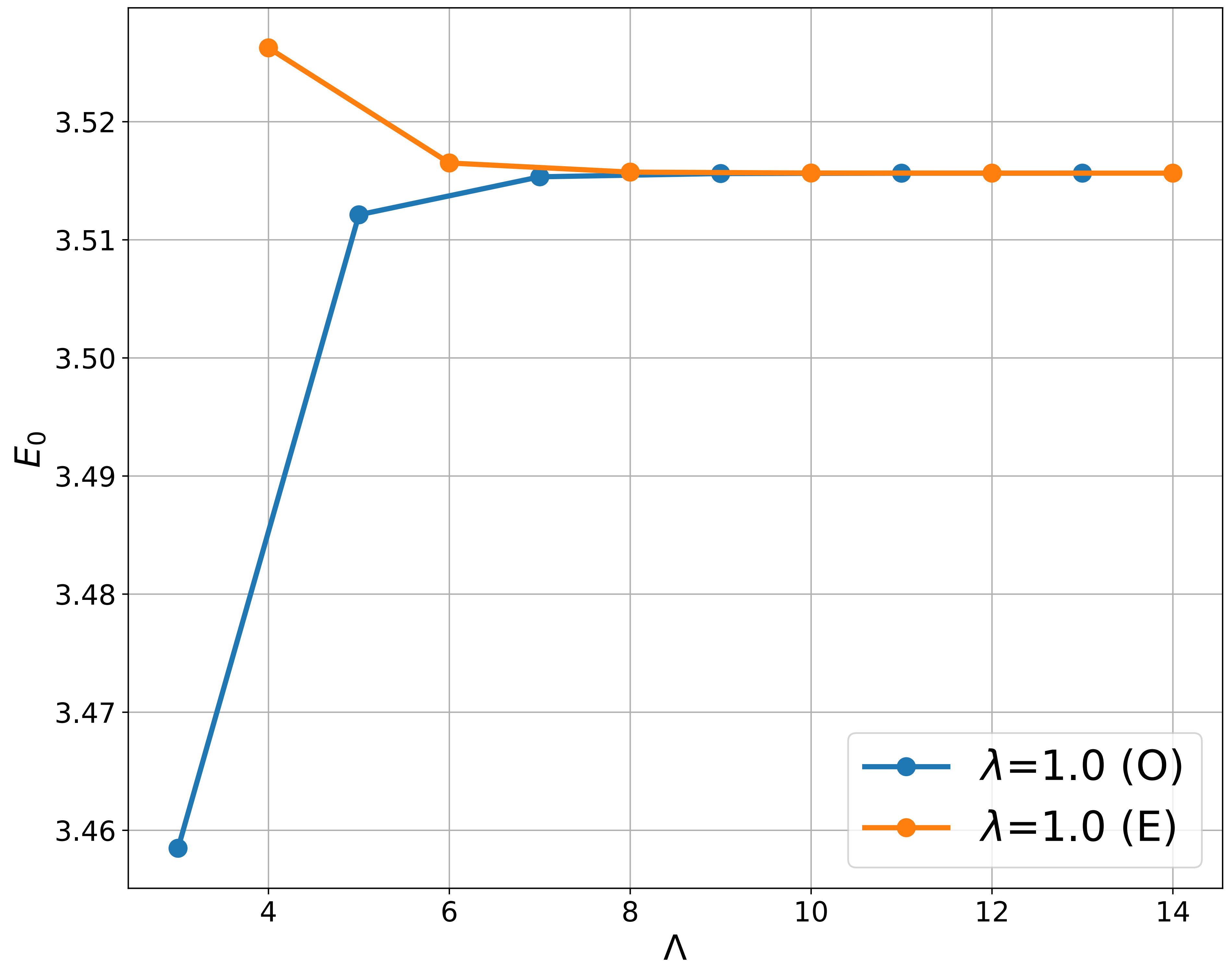

https://github.com/erinaldi/bmn2-qutip

QuTiP

Quantum Toolbox in Python

[Rinaldi et al., PRX Quantum 3 (2022) 010324]

Results

Small-scale: N=2, D=2, Λ → ∞

- **Benchmark**: compute the lowest states via exact diagonalization
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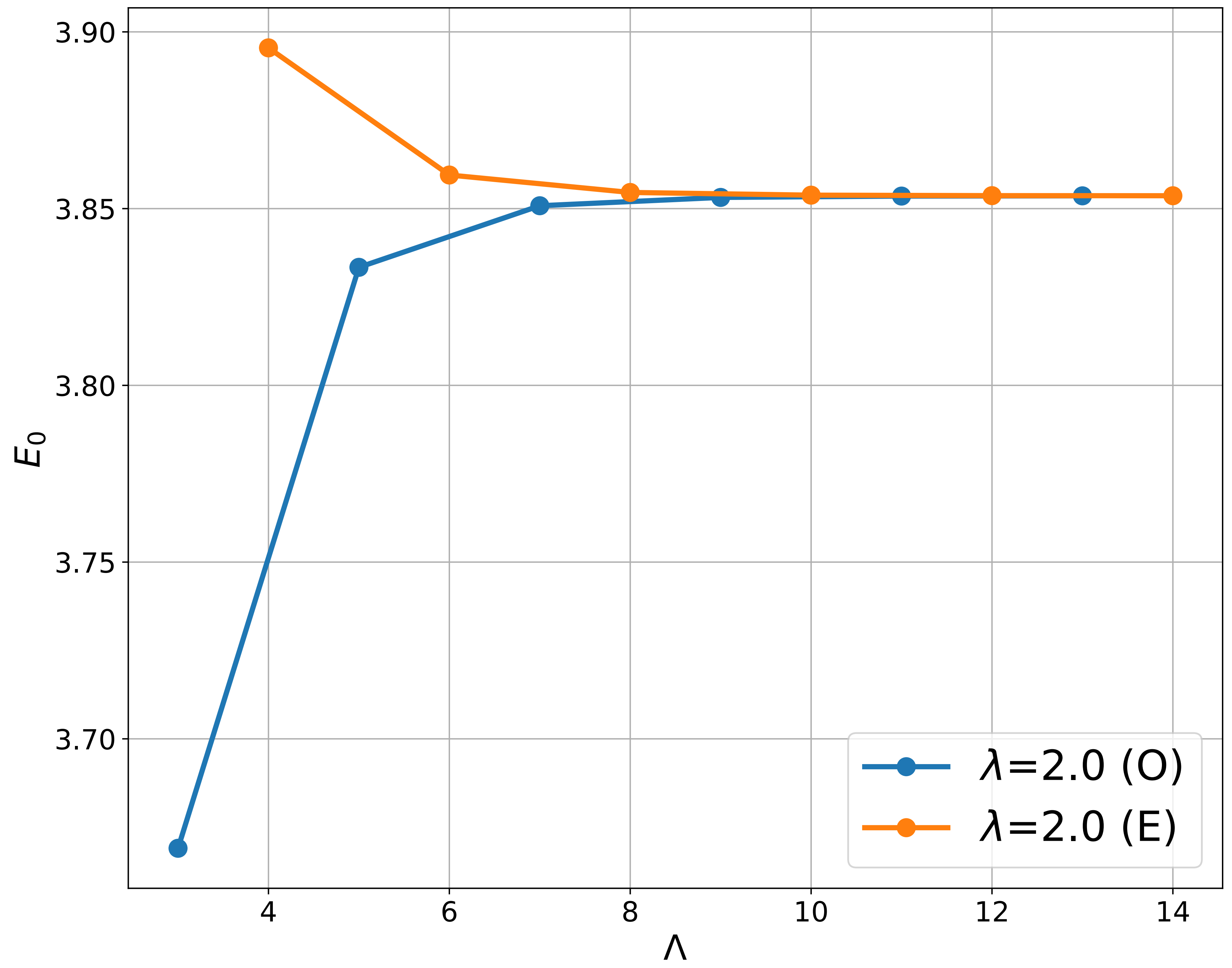

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- Study the gauge-singlet constraint

$$\hat{G}_\alpha = i \sum_{\beta,\gamma,I} f_{\alpha\beta\gamma} \hat{a}^\dagger_{I\beta} \hat{a}_{I\gamma} \longrightarrow \hat{G}_\alpha \left(\otimes_{I,\beta} |0\rangle_{I\beta} \right) = 0$$

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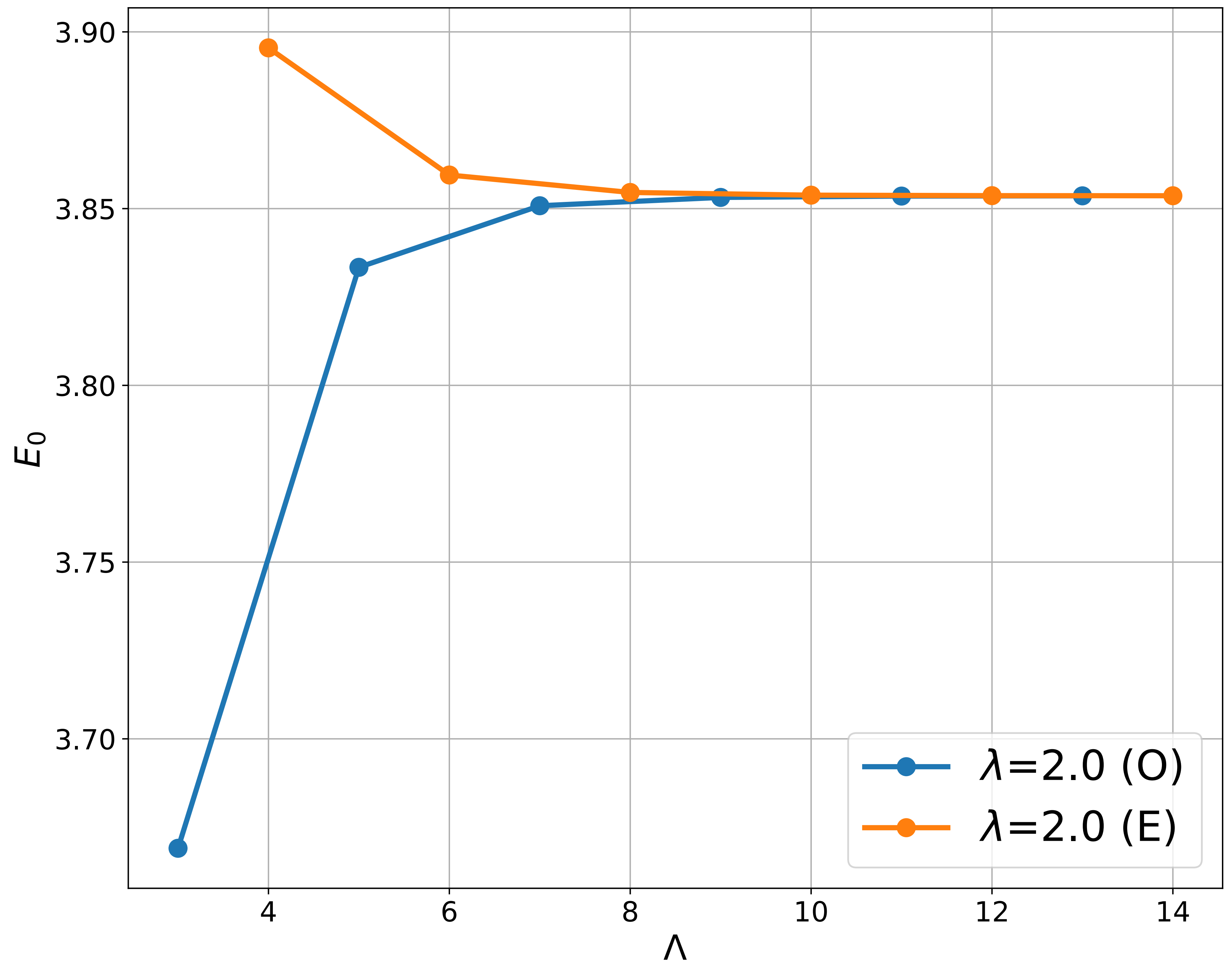

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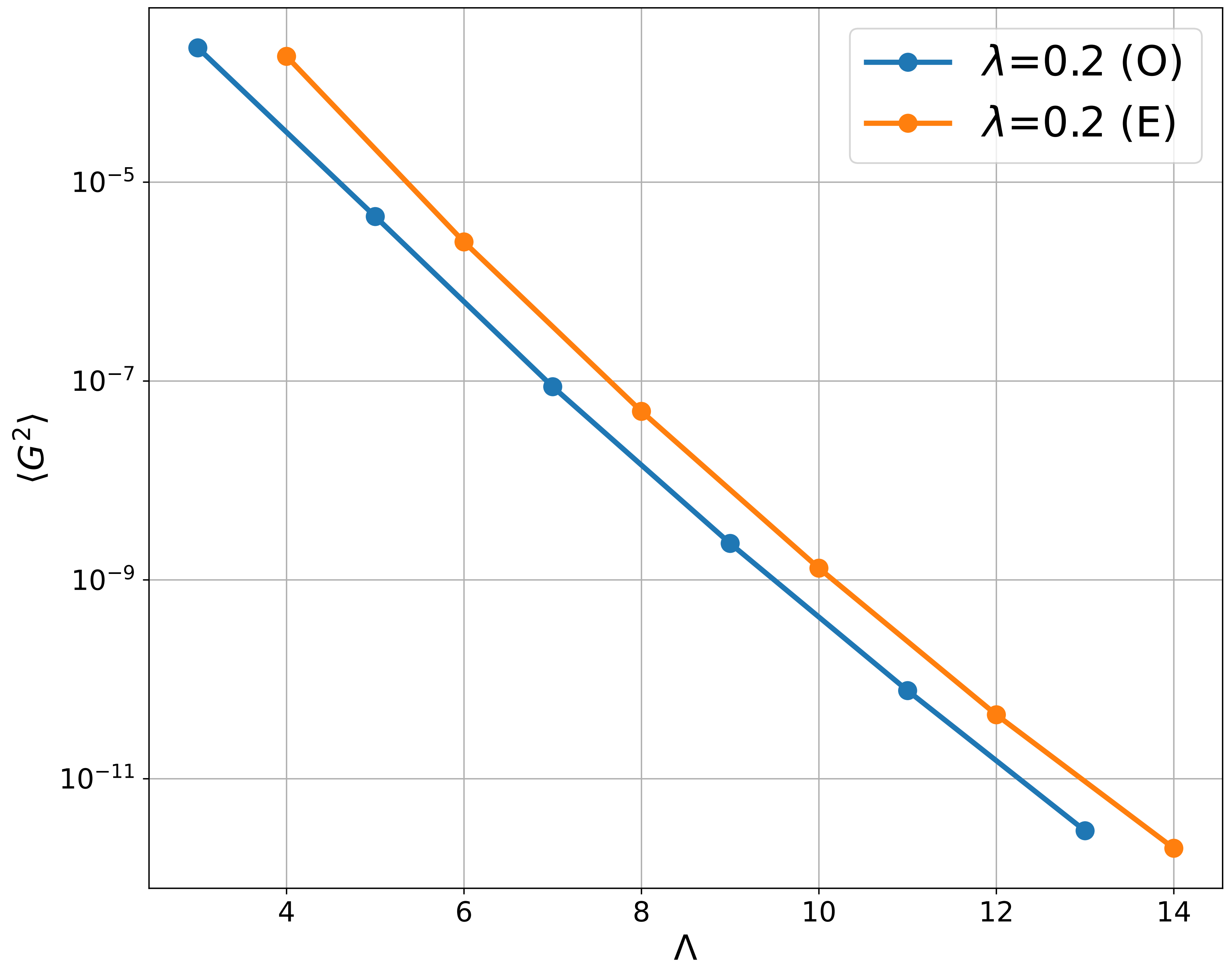

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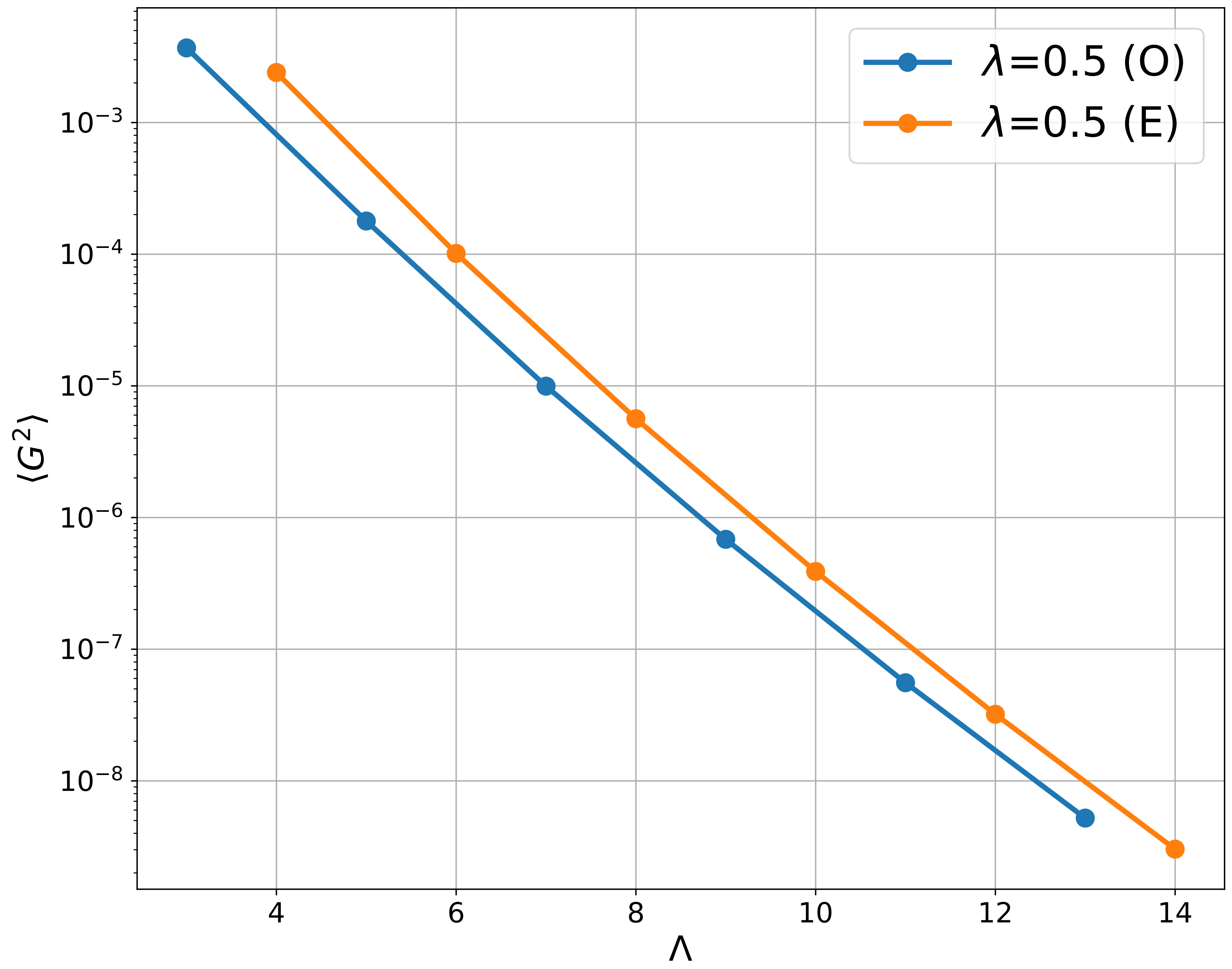

QuTiP

Quantum Toolbox in Python

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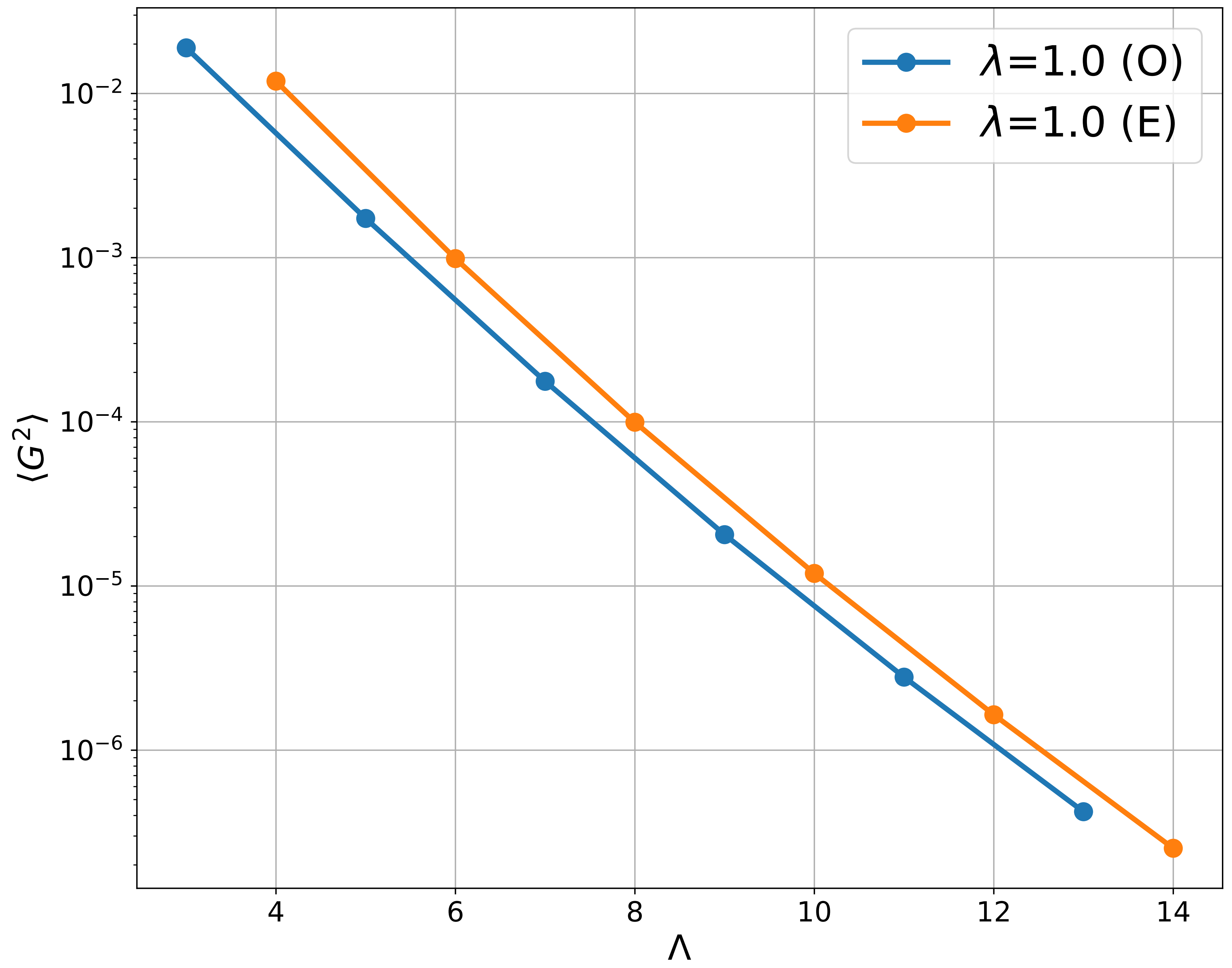

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Quantum Toolbox in Python

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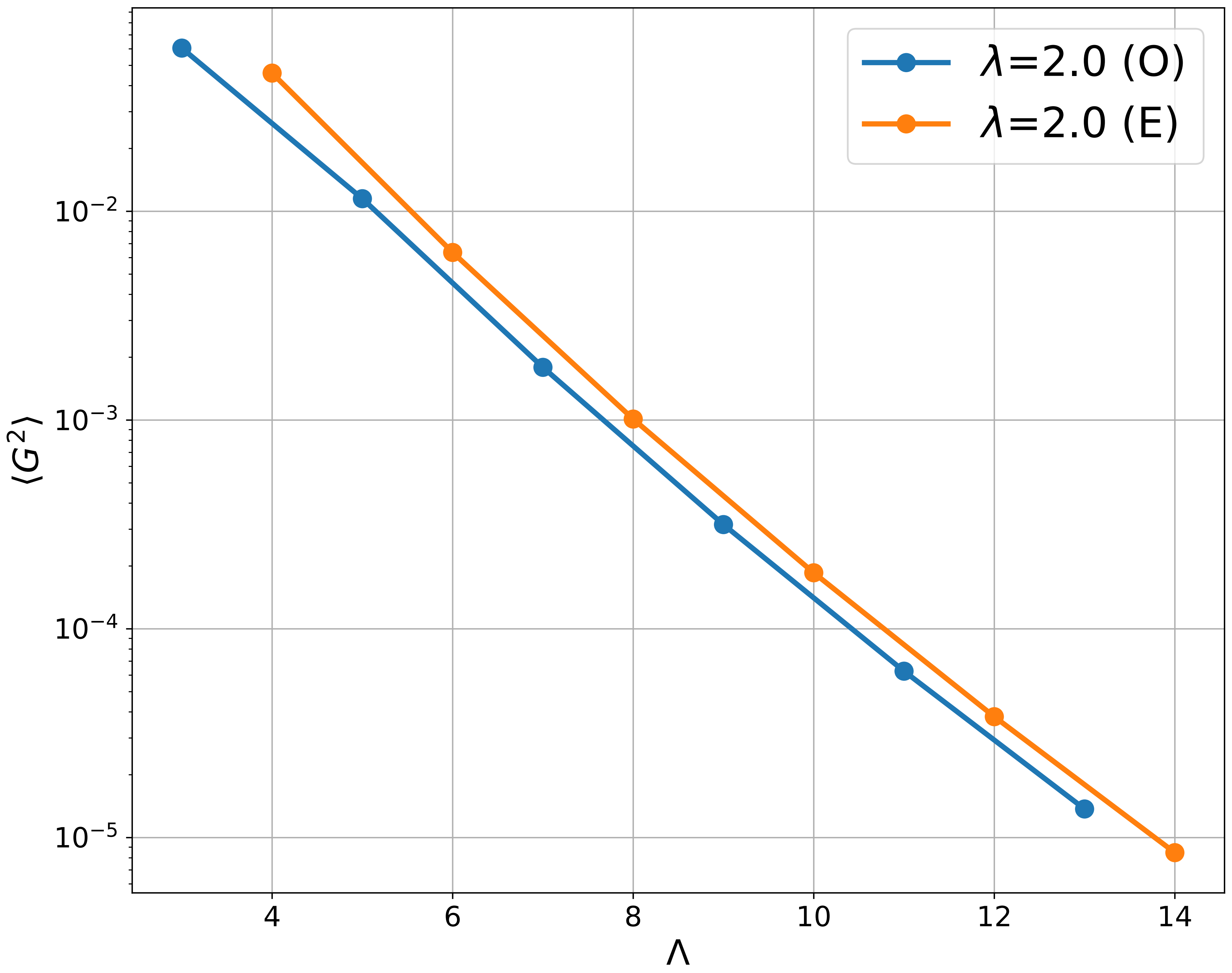

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Quantum Toolbox in Python

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[Rinaldi et al., PRX Quantum 3 (2022) 010324]

QuTiP
Quantum Toolbox in Python

Results

Small-scale: N=2, D=2, $\Lambda \to \infty$

- **Benchmark**: compute the lowest states via exact diagonalization

$$\hat{H}' = \hat{H} + c \sum \hat{G}_\alpha^2$$

- Study the convergence to $\Lambda \to \infty$
- Study the effects of different couplings
- Study the gauge-singlet constraint

$$\hat{G}_\alpha = i \sum_{\beta,\gamma,I} f_{\alpha\beta\gamma} \hat{a}^\dagger_{I\beta} \hat{a}_{I\gamma} \longrightarrow \hat{G}_\alpha \left(\otimes_{I,\beta} |0\rangle_{I\beta} \right) = 0$$

- Study a perturbed Hamiltonian with gauge penalty: increase energy iff not singlet

QuTiP
Quantum Toolbox in Python

Results

Small-scale: N=2, D=2, $\Lambda \to \infty$

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$$\hat{G}_\alpha = i \sum_{\beta,\gamma,I} f_{\alpha\beta\gamma} \hat{a}^\dagger_{I\beta} \hat{a}_{I\gamma} \longrightarrow \hat{G}_\alpha \left(\otimes_{I,\beta} |0\rangle_{I\beta} \right) = 0$$

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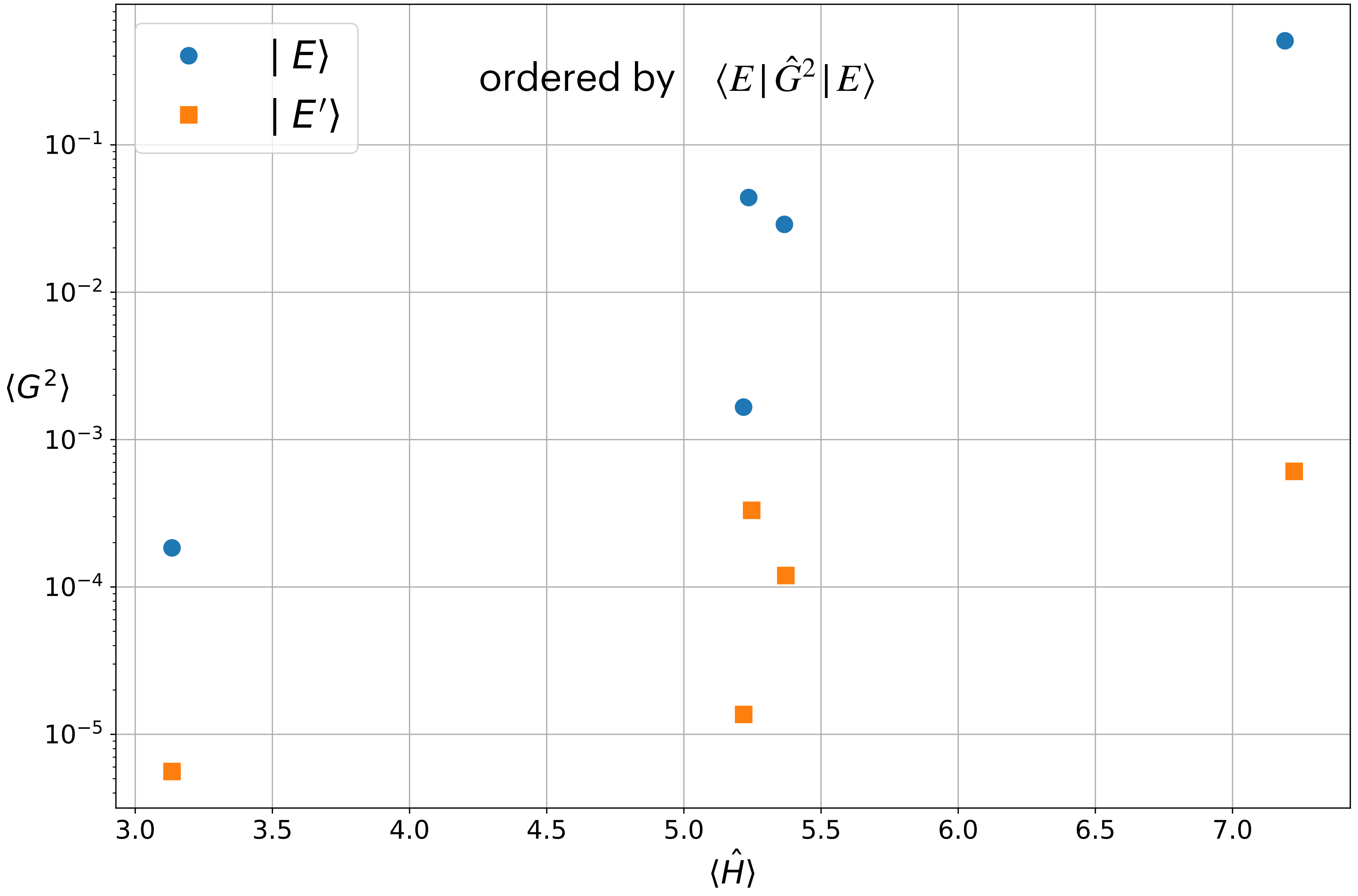
$$\hat{H}' = \hat{H} + c \sum \hat{G}^2_\alpha$$

$c = \Lambda = 4 \qquad \lambda = 0.2$



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QuTiP
Quantum Toolbox in Python

[Rinaldi et al., PRX Quantum 3 (2022) 010324]

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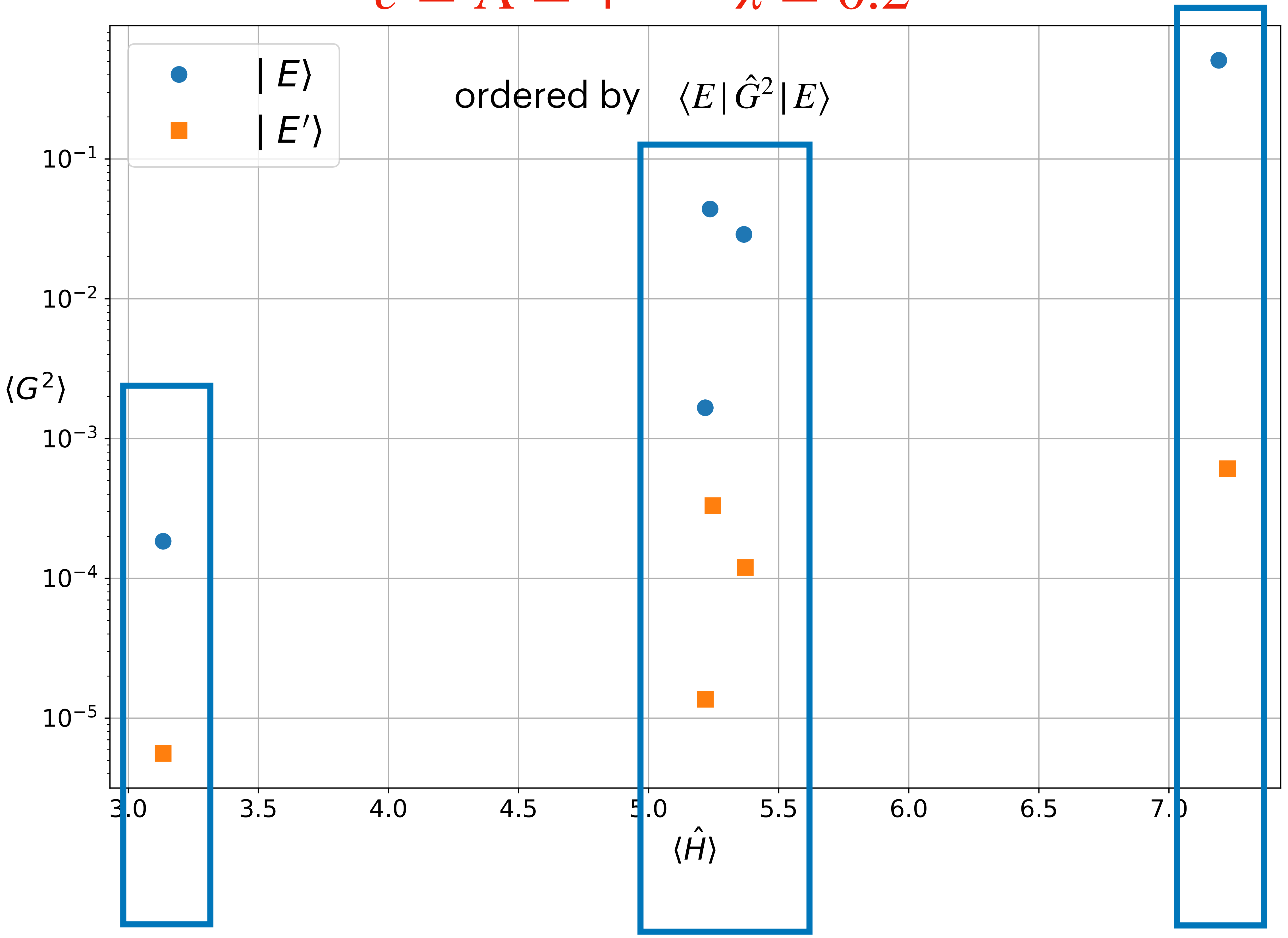
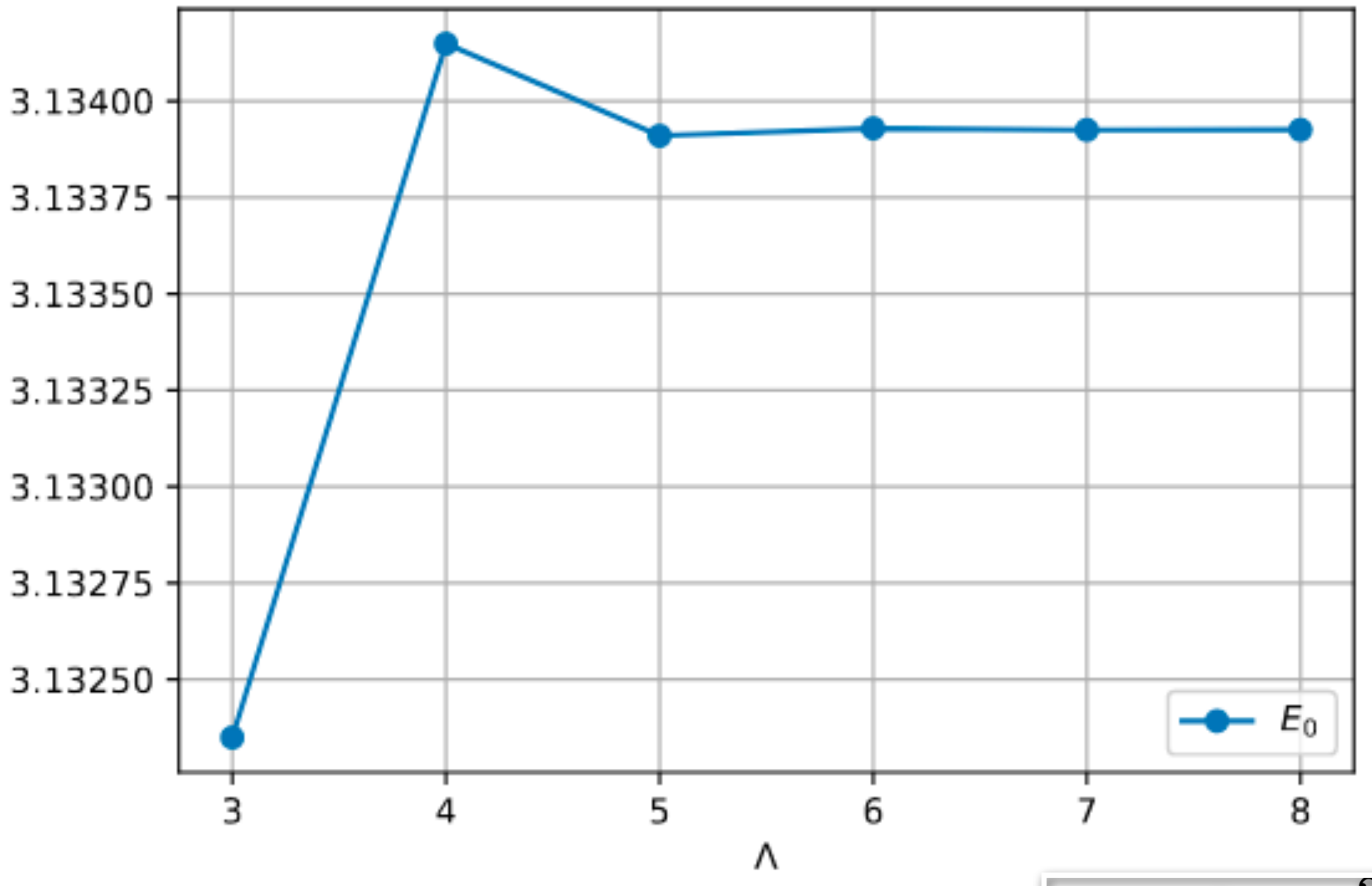


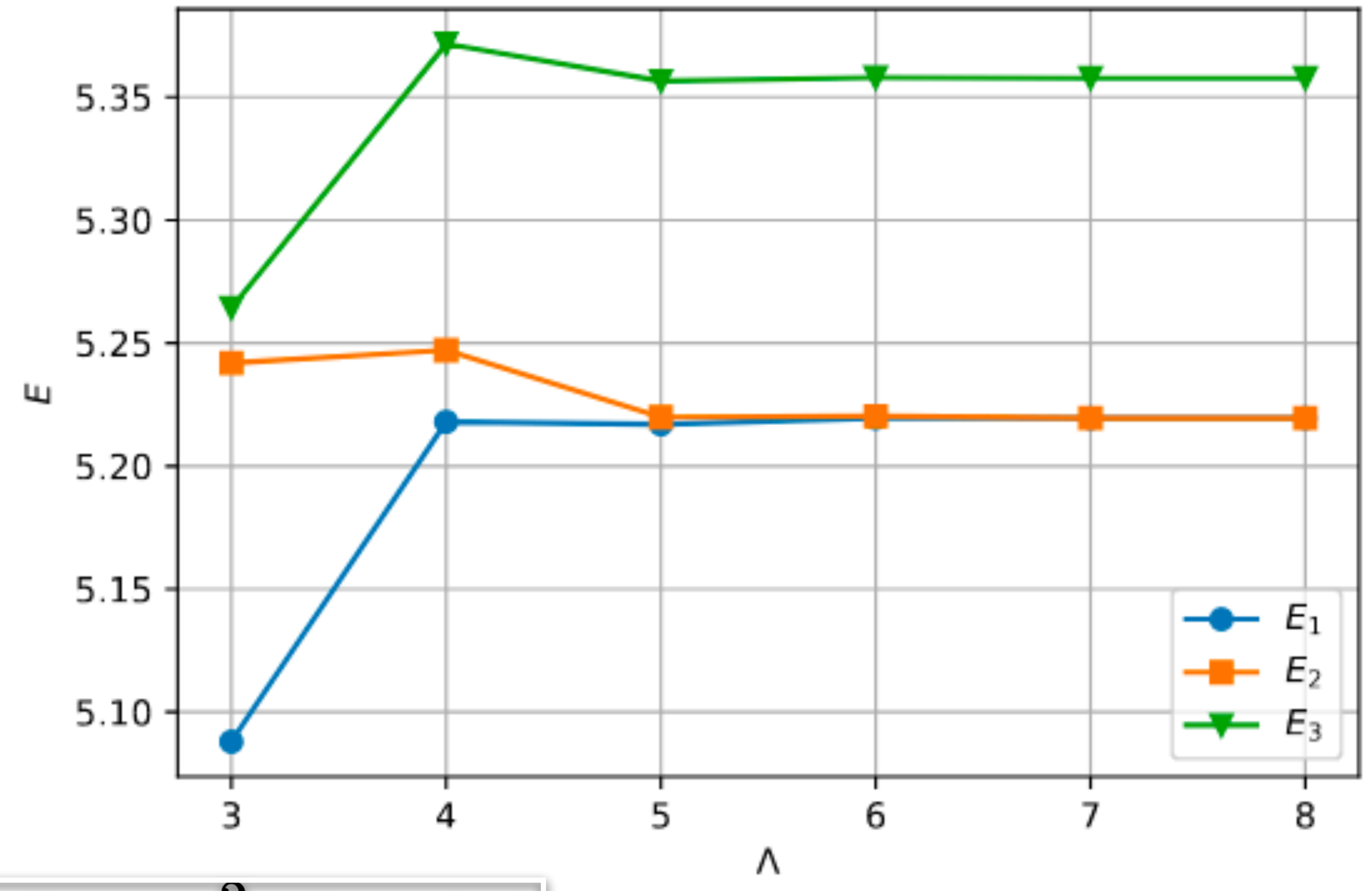
[Rinaldi et al., PRX Quantum 3 (2022) 010324]

QuTiP
Quantum Toolbox in Python

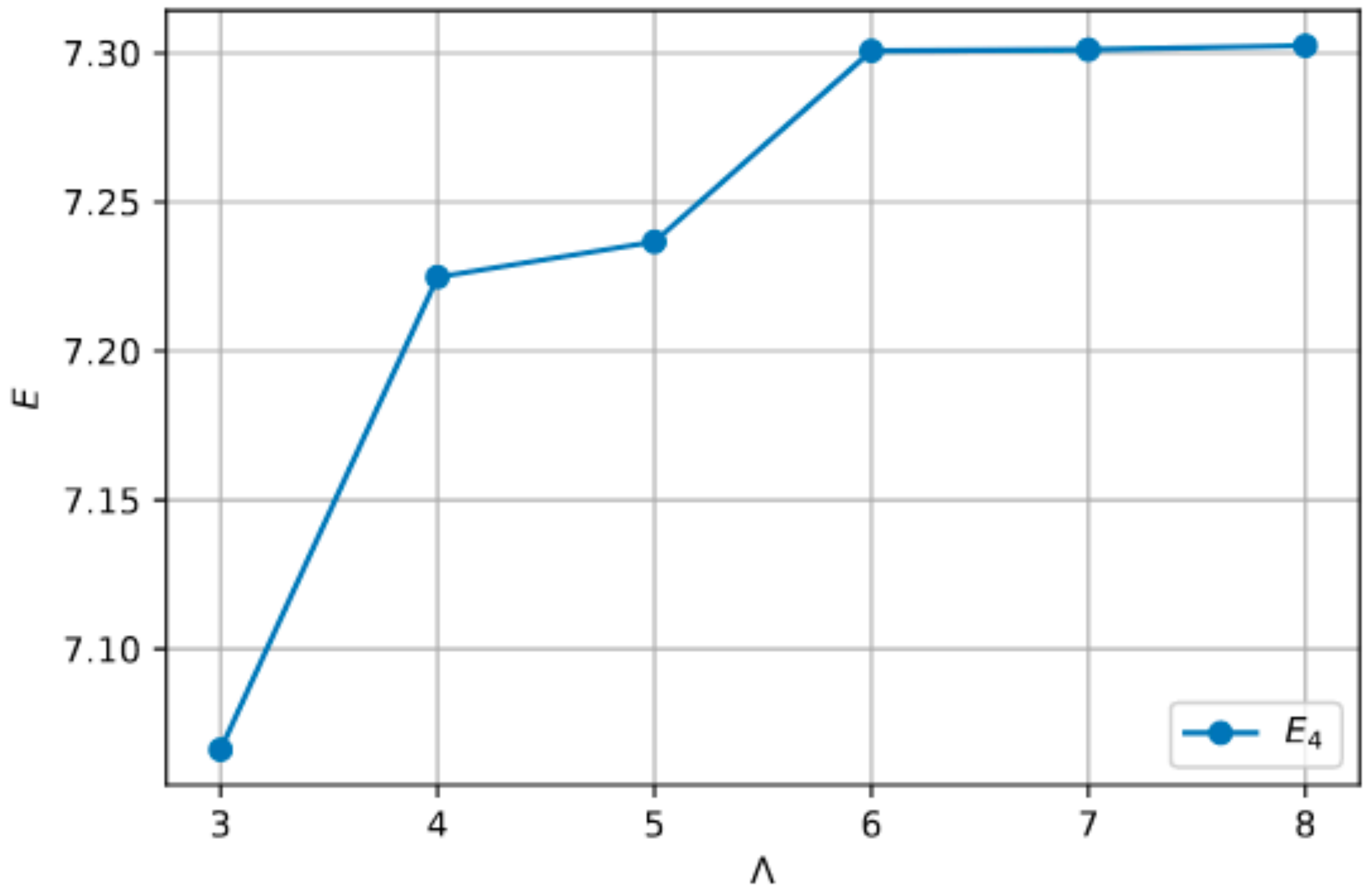
Excited states

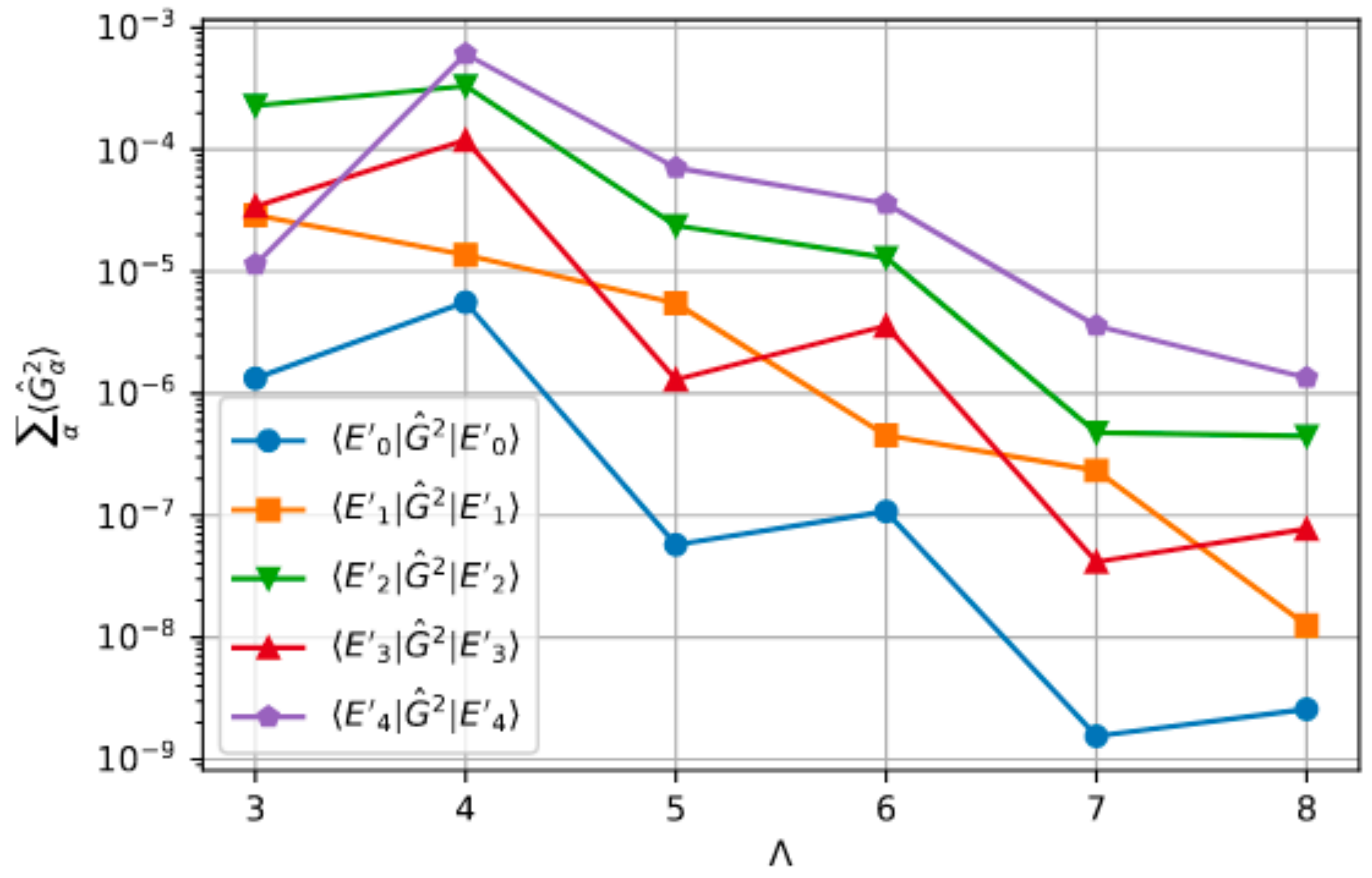
Small-scale: N=2, D=2, Λ → ∞





$$c = \Lambda, m^2 = 1, \lambda = g^2 N = 0.2$$

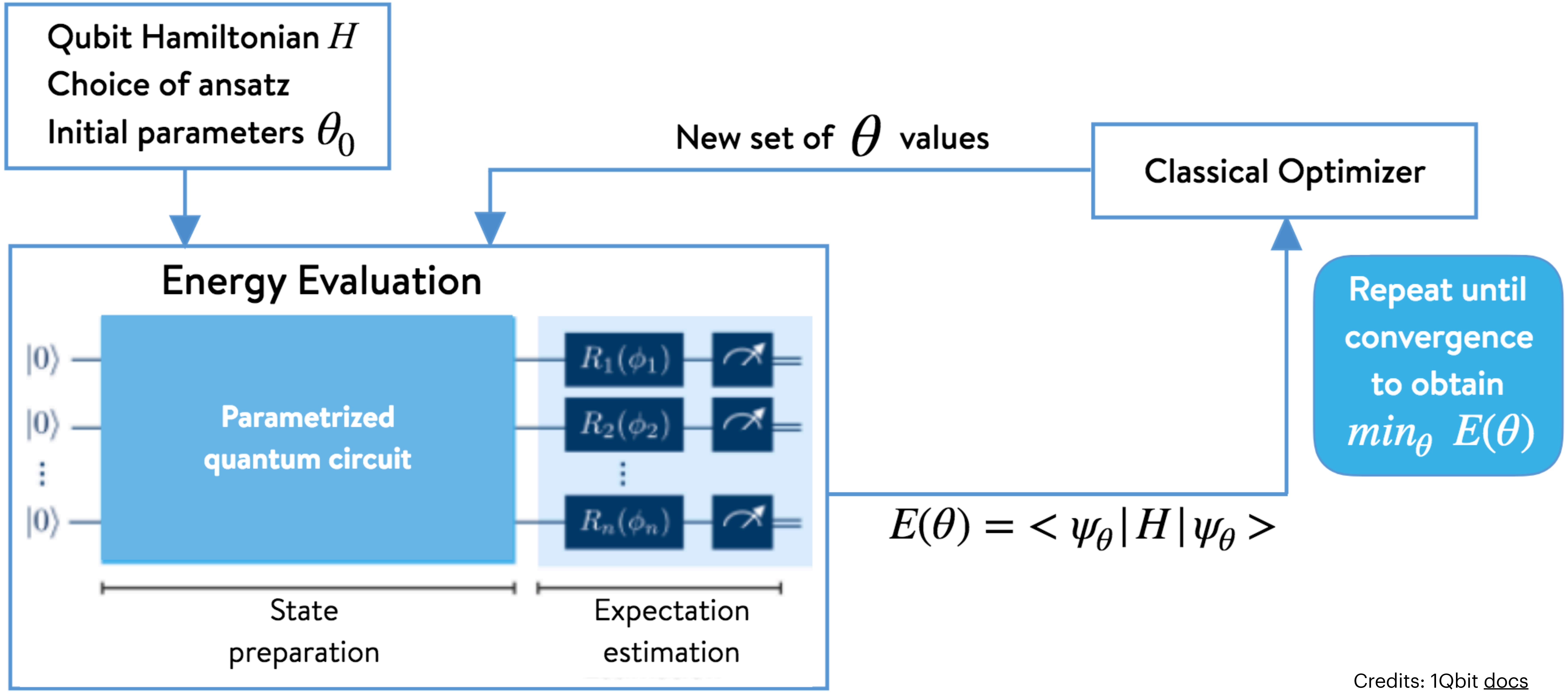




[Peruzzo et al., arxiv:1304.3061]

Quantum Computing

Variational Quantum Eigensolver - VQE

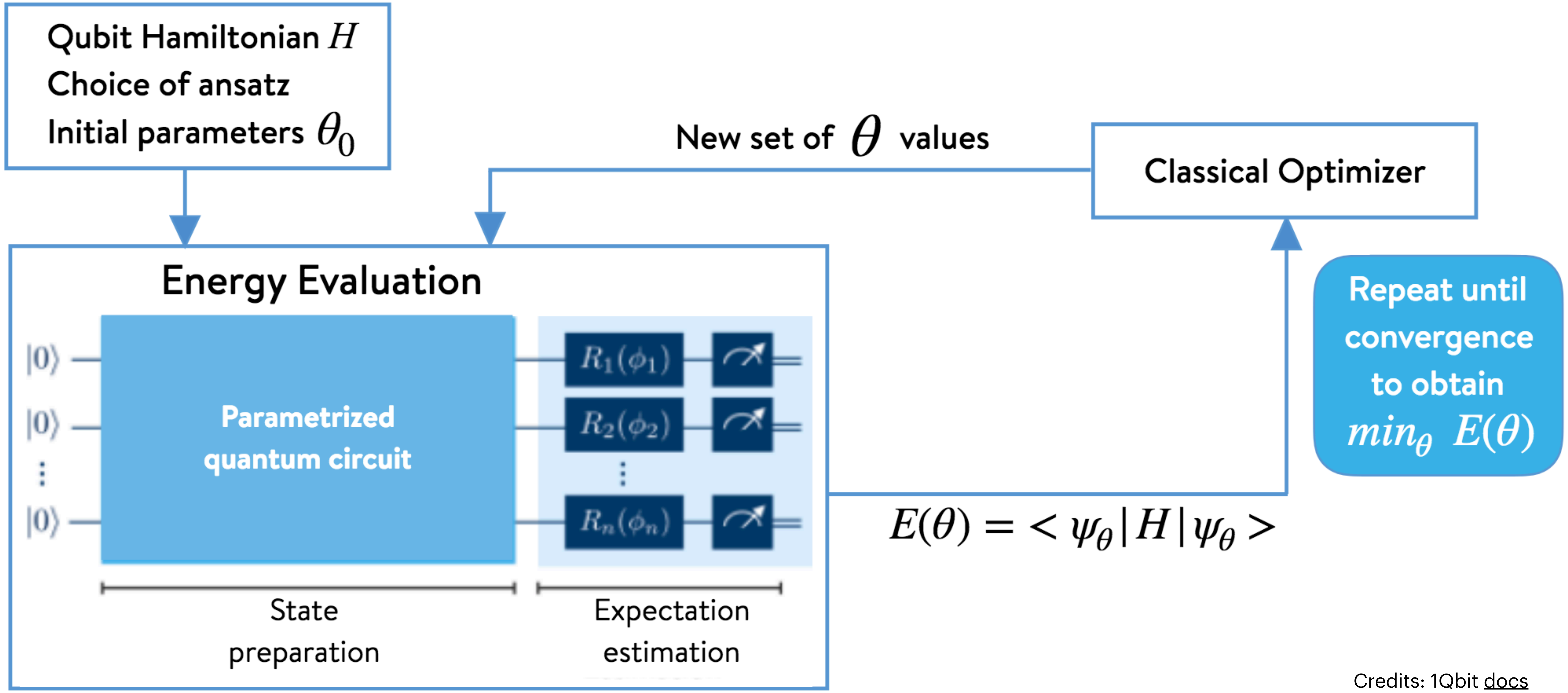


Credits: 1Qbit docs

[Peruzzo et al., arxiv:1304.3061]

Quantum Computing

Variational Quantum Eigensolver - VQE



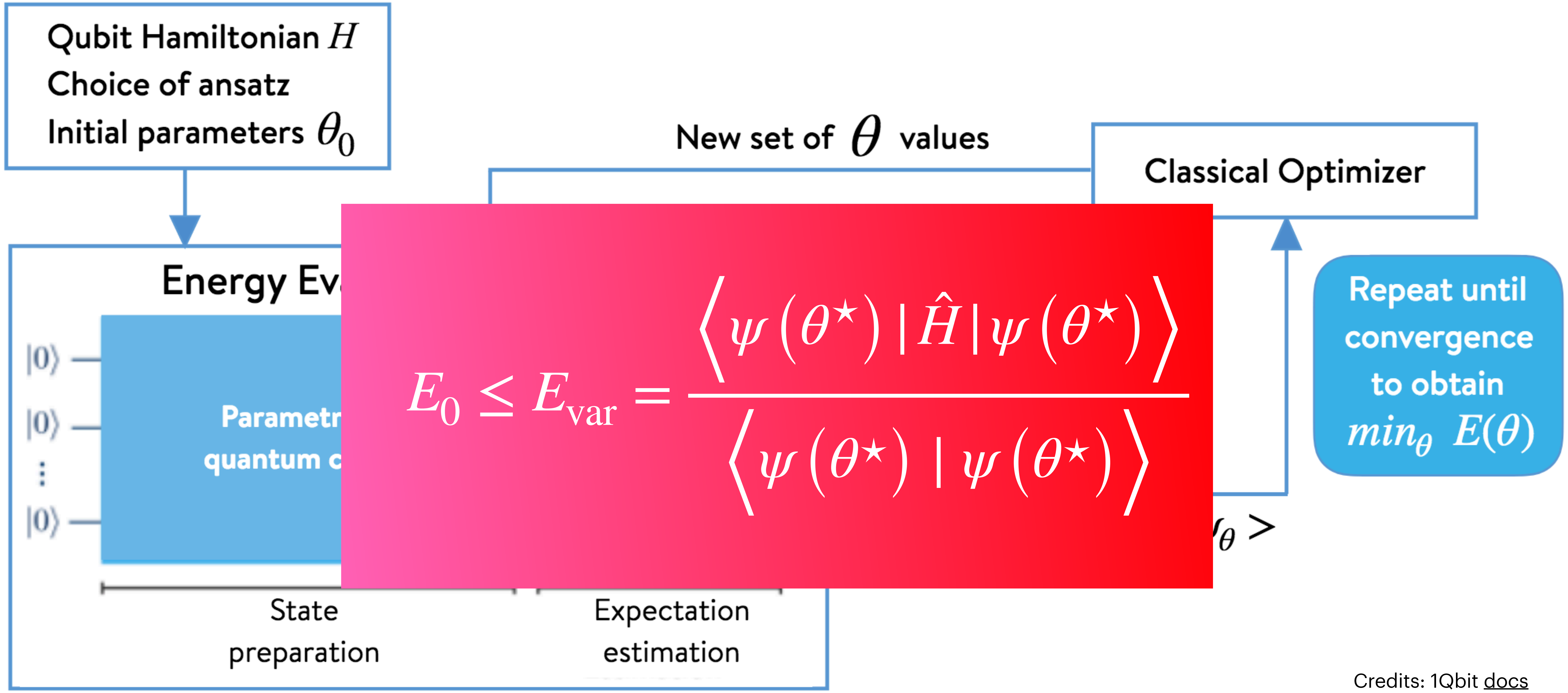
Credits: 1Qbit docs

PQC → Variational Ansatz for |Φ 〉 → Evaluation of cost function → E(θ) → Optimize parameters → θ*

[Peruzzo et al., arxiv:1304.3061]

Quantum Computing

Variational Quantum Eigensolver - VQE



$$E_0 \leq E_{var} = \frac{\left\langle \psi\left(\theta^{\star}\right) \middle| \hat{H} \middle| \psi\left(\theta^{\star}\right) \right\rangle}{\left\langle \psi\left(\theta^{\star}\right) \middle| \psi\left(\theta^{\star}\right) \right\rangle}$$

Credits: 1Qbit docs

PQC → Variational Ansatz for |Φ⟩ → Evaluation of cost function → E(θ) → Optimize parameters → θ*

https://github.com/erinaldi/bmn2-qiskit

[Rinaldi et al., PRX Quantum 3 (2022) 010324]

Qiskit

VQE details

Small-scale: N=2, D=2

$\Lambda = 2$

$\log_2 \Lambda^6 = 6$ qubits

Qiskit

VQE details

Small-scale: N=2, D=2

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→

Choose Variational Ansatz

Qiskit

VQE details

Small-scale: N=2, D=2

$\Lambda = 2$

$\log_2 \Lambda^6 = 6$ qubits

→

Choose Variational Ansatz

PQC → Variational Ansatz for $|\Phi\rangle$

Qiskit

VQE details

Small-scale: N=2, D=2

Λ = 2

$\log_2 \Lambda^6$ = 6 qubits

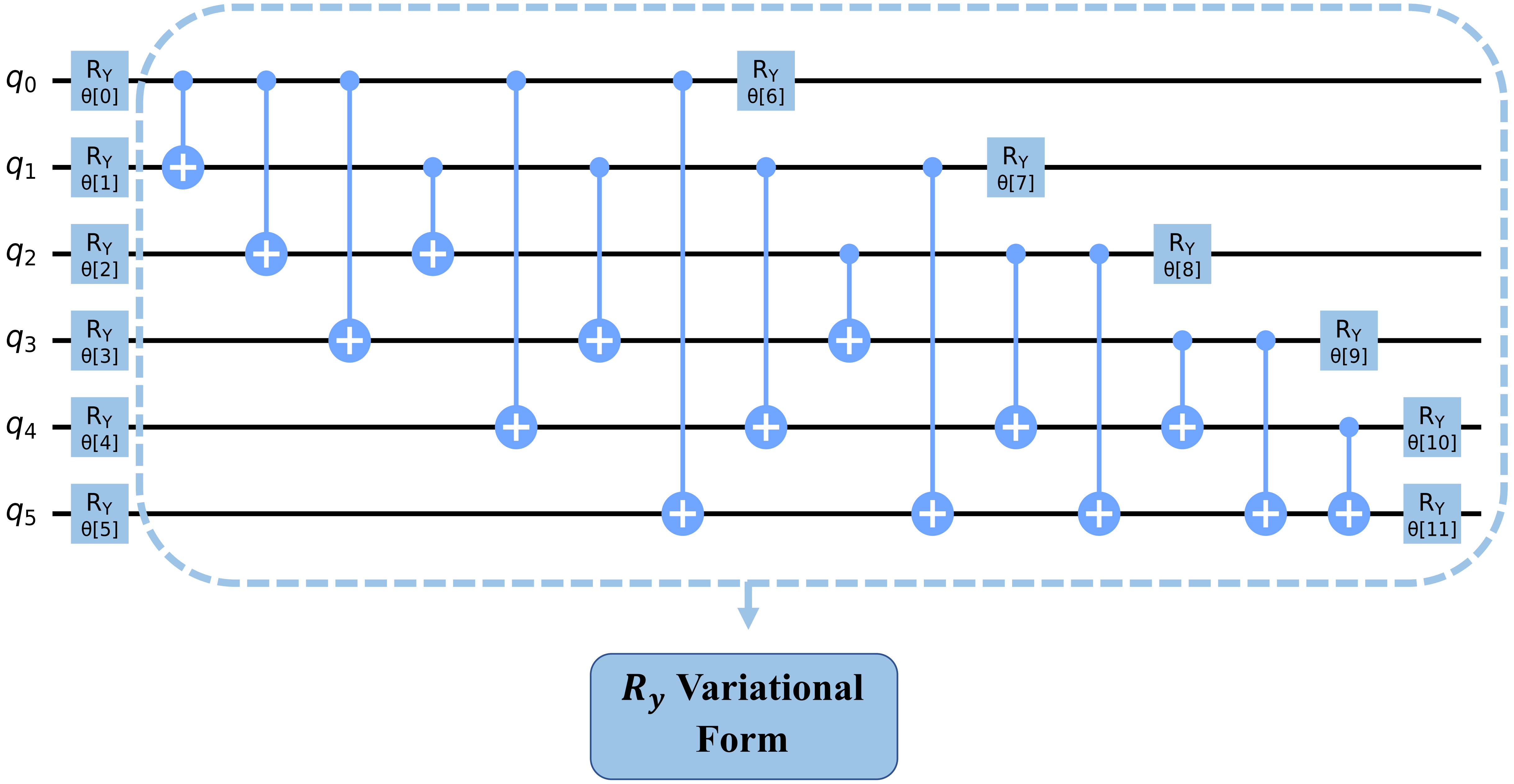
→

Choose Variational Ansatz

PQC → Variational Ansatz for |Φ⟩

depth = 1

parameters = 12



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[Rinaldi et al., PRX Quantum 3 (2022) 010324]

Qiskit

VQE details

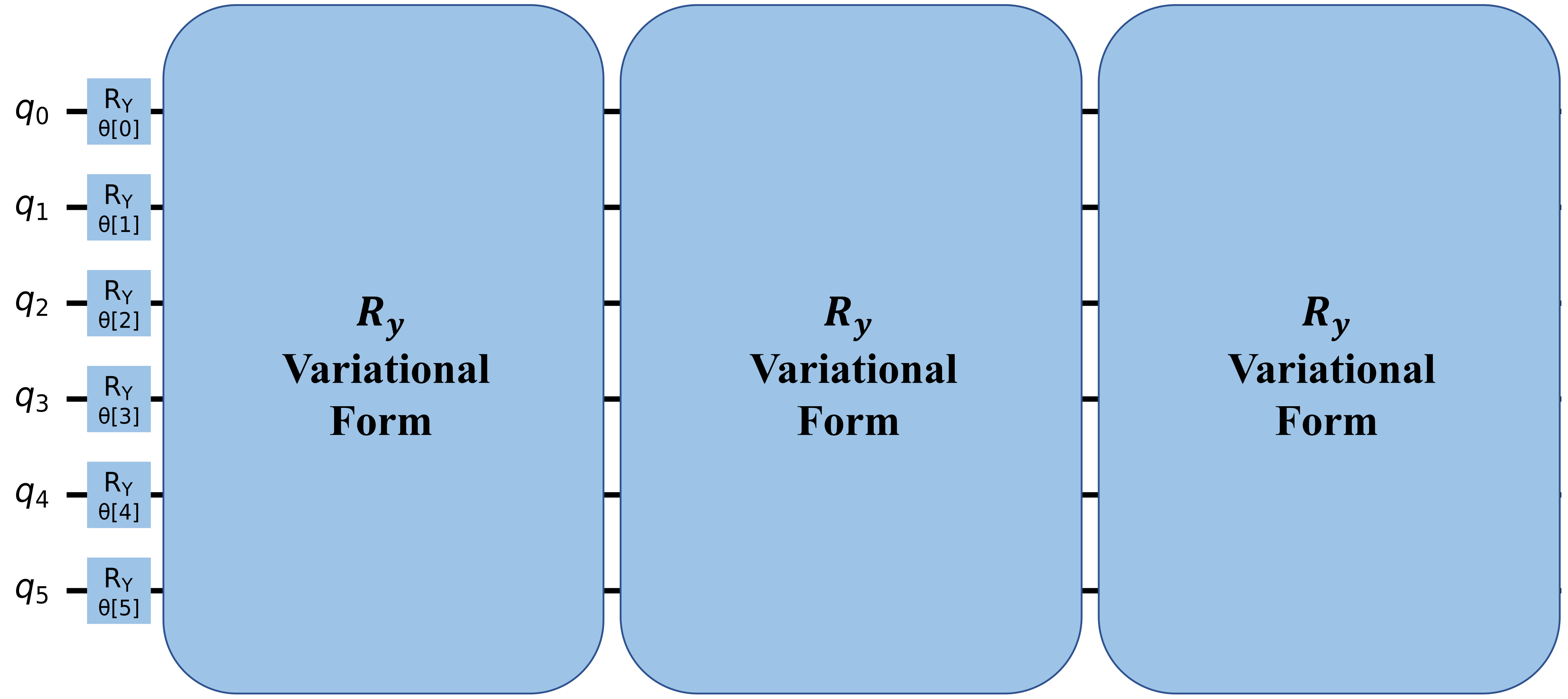
Small-scale: N=2, D=2

$\Lambda = 2$ | $\log_2 \Lambda^6 = 6$ qubits → Choose Variational Ansatz

PQC → Variational Ansatz for $|\Phi\rangle$

depth = 3

parameters = 24



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[Rinaldi et al., PRX Quantum 3 (2022) 010324]

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VQE details

Small-scale: N=2, D=2

Λ = 2

$\log_2 \Lambda^6$ = 6 qubits

→

Choose Variational Ansatz

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depth = 3

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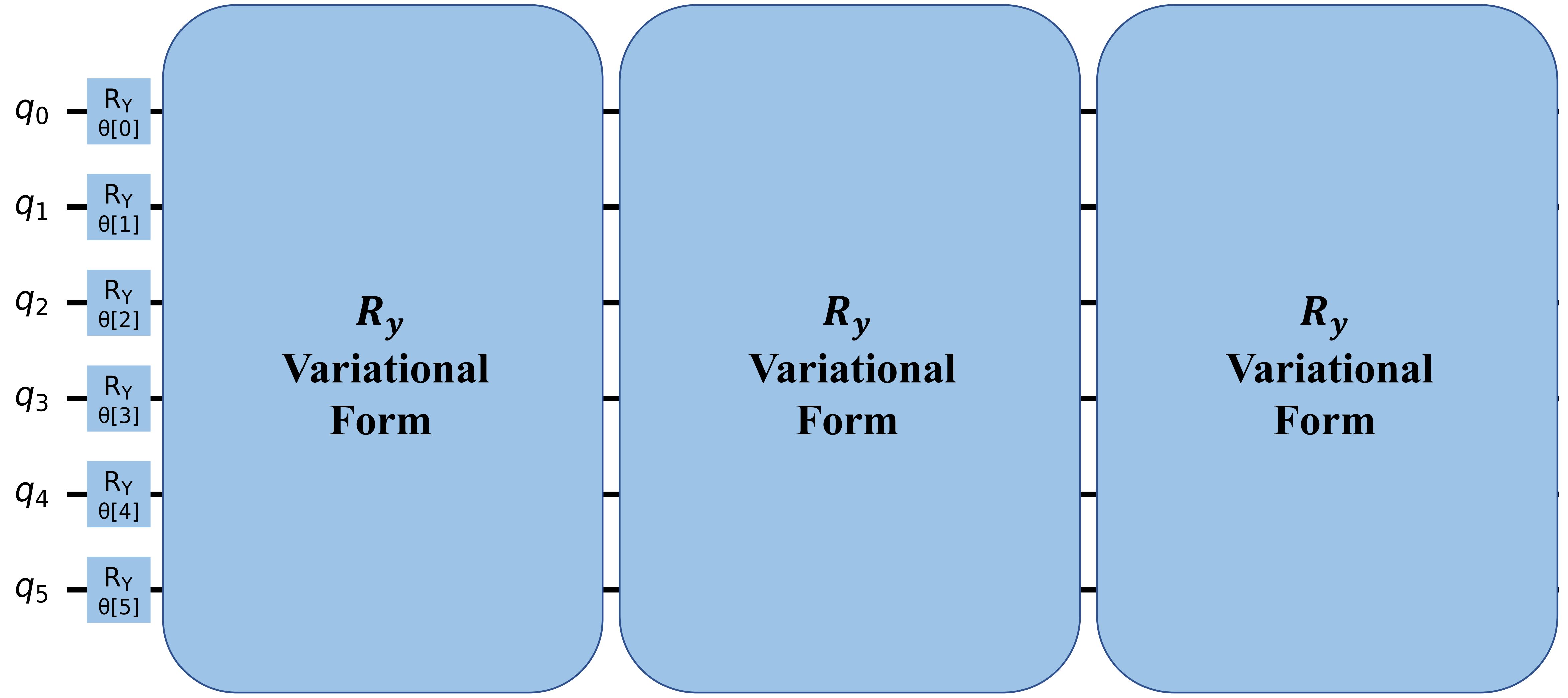


Run each multiple instances of PQC from different initial points

Qiskit

VQE details

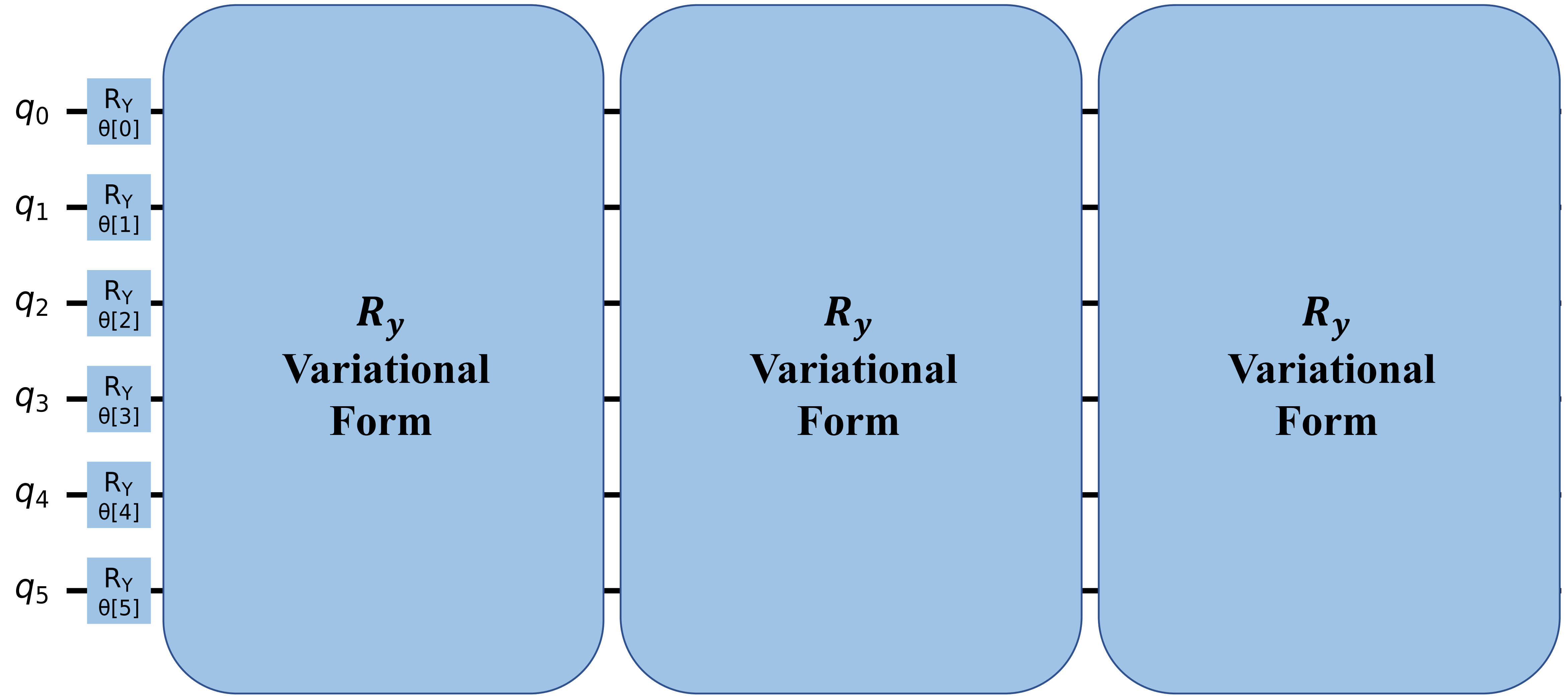
Small-scale: N=2, D=2

$\Lambda = 2$ | $\log_2 \Lambda^6$ = 6 qubits → Choose Variational Ansatz → Choose Quantum Simulator

PQC → Variational Ansatz for |Φ⟩

depth = 3

parameters = 24



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[Rinaldi et al., PRX Quantum 3 (2022) 010324]

Qiskit

VQE details

Small-scale: N=2, D=2

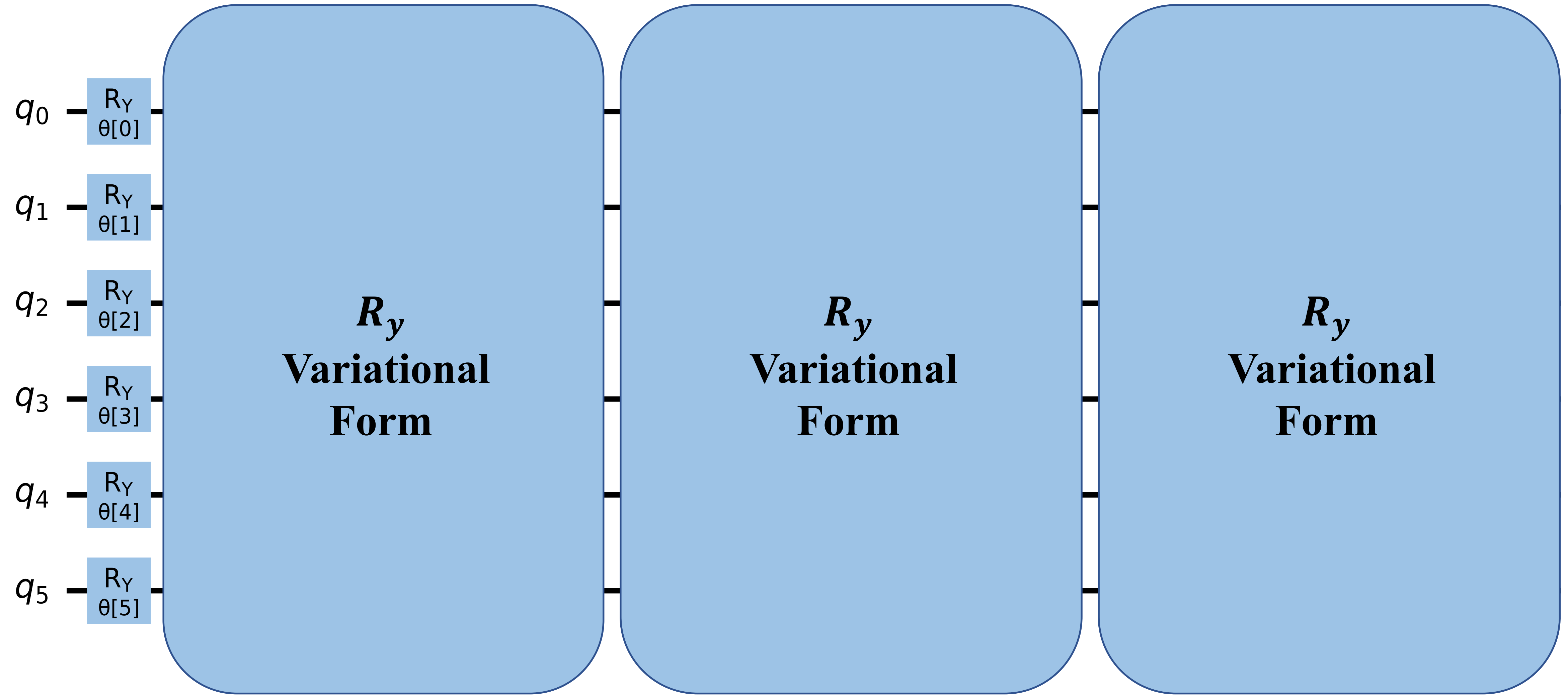
$\Lambda = 2$ | $\log_2 \Lambda^6 = 6$ qubits → Choose Variational Ansatz → Choose Quantum Simulator

PQC → Variational Ansatz for $|\Phi\rangle$

depth = 3

parameters = 24

Evaluation of cost function → $E(\theta)$



- Statevector simulator

Run each multiple instances of PQC from different initial points

https://github.com/erinaldi/bmn2-qiskit

[Rinaldi et al., PRX Quantum 3 (2022) 010324]

Qiskit

VQE details

Small-scale: N=2, D=2

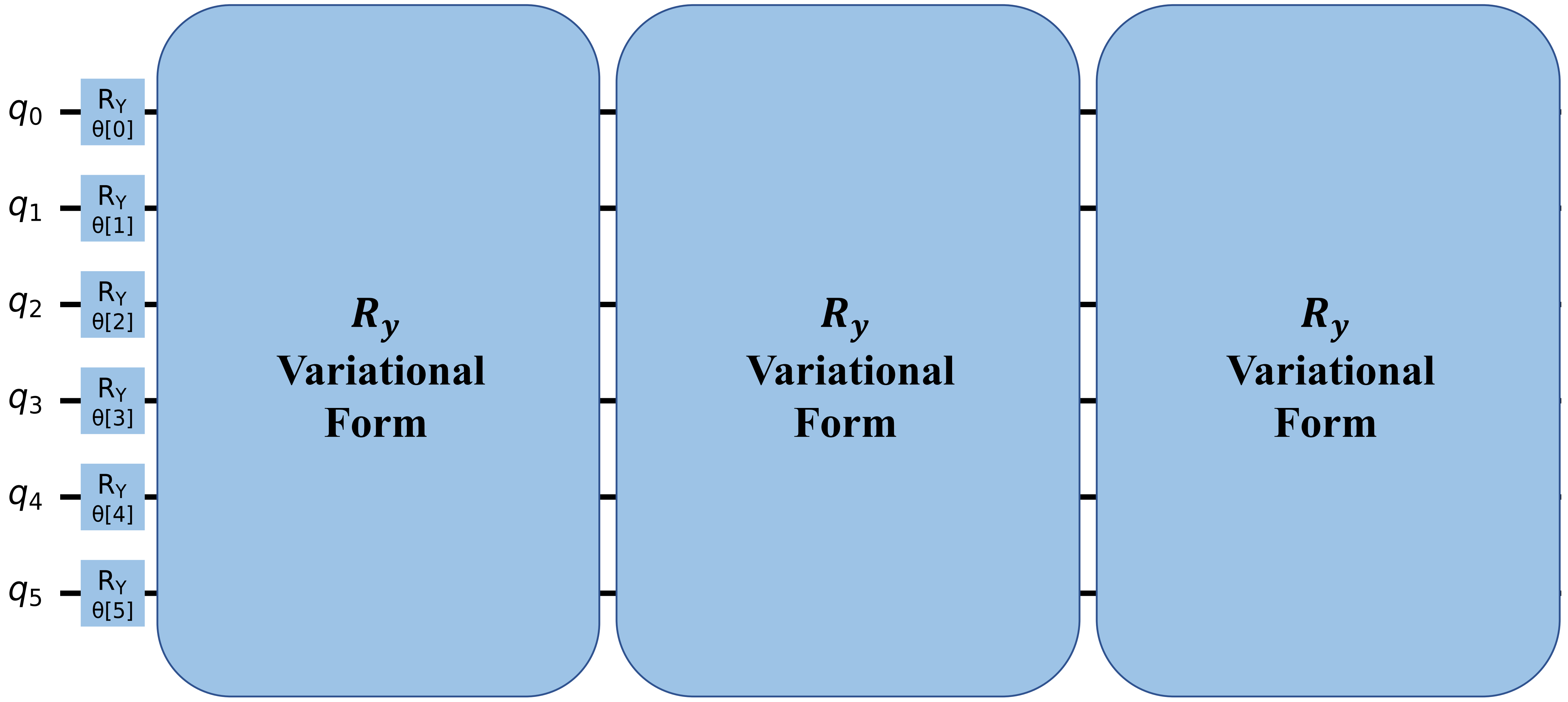
$\Lambda = 2$ | $\log_2 \Lambda^6 = 6$ qubits → Choose Variational Ansatz → Choose Quantum Simulator → Choose Classical Optimizer

PQC → Variational Ansatz for $|\Phi\rangle$

depth = 3

parameters = 24

Evaluation of cost function → $E(\theta)$



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[Rinaldi et al., PRX Quantum 3 (2022) 010324]

Qiskit

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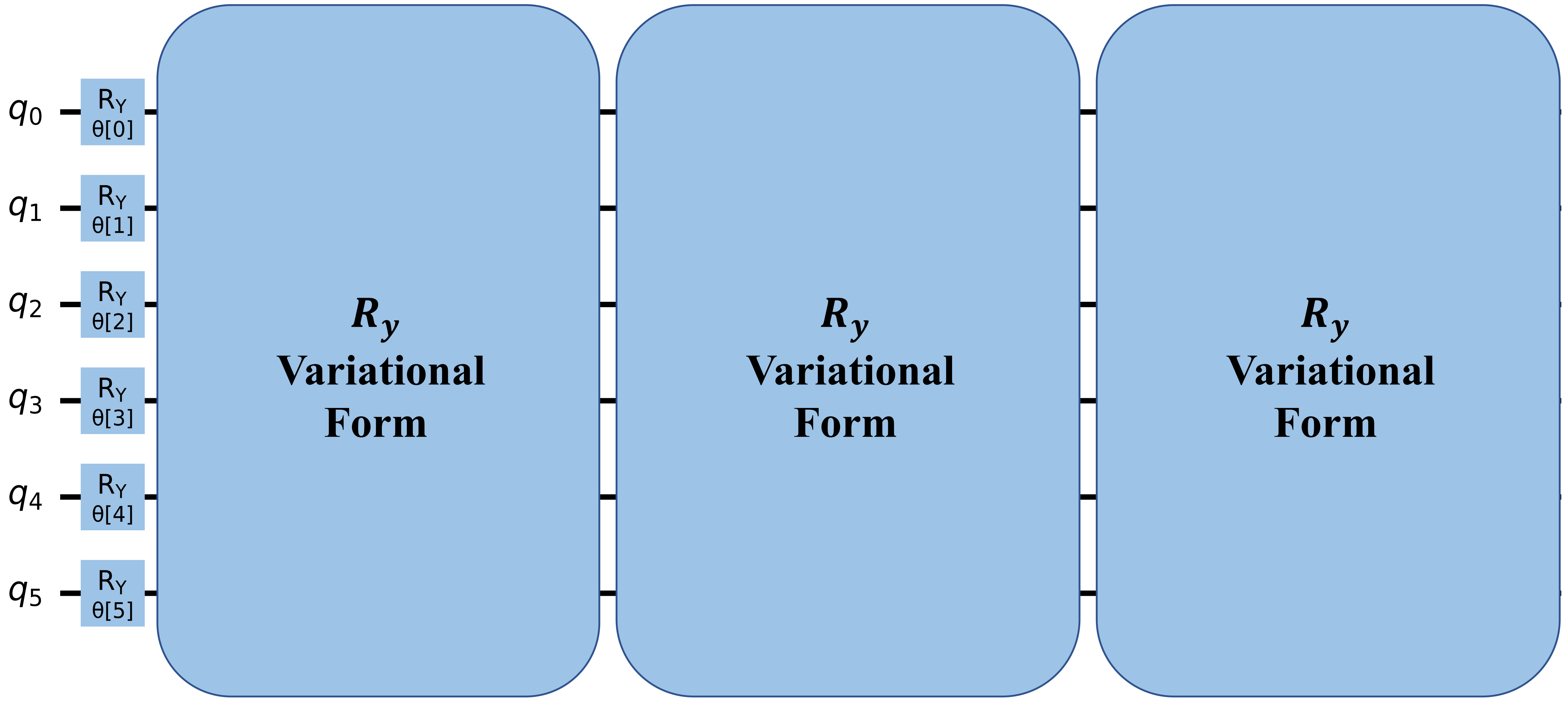
Evaluation of cost function → $E(\theta)$

Optimize parameters → θ^*



- Statevector simulator

- Least SQuares Programming optimizer (SLSQP)
- Constrained Optimization By Linear Approximation optimizer (COBYLA)
- Limited-memory BFGS Bound optimizer (L-BFGS-B)
- Nelder-Mead

Run each optimizer with a max. number of iterations

Run each multiple instances of PQC from different initial points

Qiskit

Qubitization of MQM

Small-scale: N=2, D=2, $\Lambda \to \infty$

Qiskit

Qubitization of MQM

Small-scale: N=2, D=2, $\Lambda \to \infty$

Truncation Level

$\Lambda = 2$

$\Lambda = 4$

$\Lambda = 8$

Qiskit

Qubitization of MQM

Small-scale: N=2, D=2, $\Lambda \to \infty$

Truncation Level

$\Lambda = 2$

Each boson is 1 qubit

$\Lambda = 4$

$\Lambda = 8$

https://github.com/erinaldi/bmn2-qiskit

[Rinaldi et al., PRX Quantum 3 (2022) 010324] [Gharibyan et al., JHEP07(2021)140]

Qiskit

Qubitization of MQM

Small-scale: N=2, D=2, Λ → ∞

Truncation Level

Λ = 2

Each boson is 1 qubit

$\log_2 \Lambda^6$ = 6 qubits

Λ = 4

Λ = 8

Qiskit

Qubitization of MQM

Small-scale: N=2, D=2, $\Lambda \to \infty$

Truncation Level

$\Lambda = 2$	Each boson is 1 qubit	$\log_2 \Lambda^6 = 6$ qubits
$\Lambda = 4$	Each boson is 2 qubits	$\log_2 \Lambda^6 = 12$ qubits
$\Lambda = 8$		

Qiskit

Qubitization of MQM

Small-scale: N=2, D=2, Λ → ∞

Truncation Level

Λ = 2	Each boson is 1 qubit	$\log_2 \Lambda^6$ = 6 qubits
Λ = 4	Each boson is 2 qubits	$\log_2 \Lambda^6$ = 12 qubits
Λ = 8	Each boson is 3 qubits	$\log_2 \Lambda^6$ = 18 qubits

https://github.com/erinaldi/bmn2-qiskit

[Rinaldi et al., PRX Quantum 3 (2022) 010324] [Gharibyan et al., JHEP07(2021)140]

Qiskit

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$\Lambda = 8$	Each b[illegible]on i[illegible] qubits	$\log_2 \Lambda^6$ = 18 qubits

limited resources

[Rinaldi et al., PRX Quantum 3 (2022) 010324] [Gharibyan et al., JHEP07(2021)140]

Qiskit

Qubitization of MQM

Small-scale: N=2, D=2, $\Lambda \to \infty$

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$\Lambda = 8$	Each b on i qubits	$\log_2 \Lambda^6 = 18$ qubits

limited resources

$$\hat{H}_B = \sum_{\alpha,I}\left(\frac{1}{2}\hat{P}^2_{I\alpha} + \frac{m^2}{2}\hat{X}^2_{I\alpha}\right) + \frac{g^2}{4}\sum_{\gamma,I,J}\left(\sum_{\alpha,\beta} f_{\alpha\beta\gamma}\hat{X}^\alpha_I \hat{X}^\beta_J\right)^2 \qquad I = 1,2 \qquad \alpha = 1,2,3$$

Build matrix Hamiltonian which gets mapped to qubits

[Rinaldi et al., PRX Quantum 3 (2022) 010324] [Gharibyan et al., JHEP07(2021)140]

Qiskit

Qubitization of MQM

Small-scale: N=2, D=2, Λ → ∞

Truncation Level

Λ = 2	Each boson is 1 qubit	$\log_2 \Lambda^6$ = 6 qubits
Λ = 4	Each boson is 2 qubits	$\log_2 \Lambda^6$ = 12 qubits
Λ = 8	Each b…on … qubits	$\log_2 \Lambda^6$ = 18 qubits

limited resources

Rewrite $X_i \to a_i$
Annihilation operator for site "i"

$$\hat{a}_i = \hat{I}_1 \otimes \ldots \otimes \hat{I}_{i-1} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \hat{I}_{i+1} \otimes \ldots \otimes \hat{I}_6$$

$$\hat{a}_i = \hat{I}_1 \otimes \ldots \otimes \hat{I}_{i-1} \otimes \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \hat{I}_{i+1} \otimes \ldots \otimes \hat{I}_6$$

$$\hat{H}_B = \sum_{\alpha,I} \left(\frac{1}{2} \hat{P}^2_{I\alpha} + \frac{m^2}{2} \hat{X}^2_{I\alpha} \right) + \frac{g^2}{4} \sum_{\gamma,I,J} \left(\sum_{\alpha,\beta} f_{\alpha\beta\gamma} \hat{X}^\alpha_I \hat{X}^\beta_J \right)^2 \qquad I = 1,2 \qquad \alpha = 1,2,3$$

Build matrix Hamiltonian which gets mapped to qubits

https://github.com/erinaldi/bmn2-qiskit

[Rinaldi et al., PRX Quantum 3 (2022) 010324]

Qiskit

Results

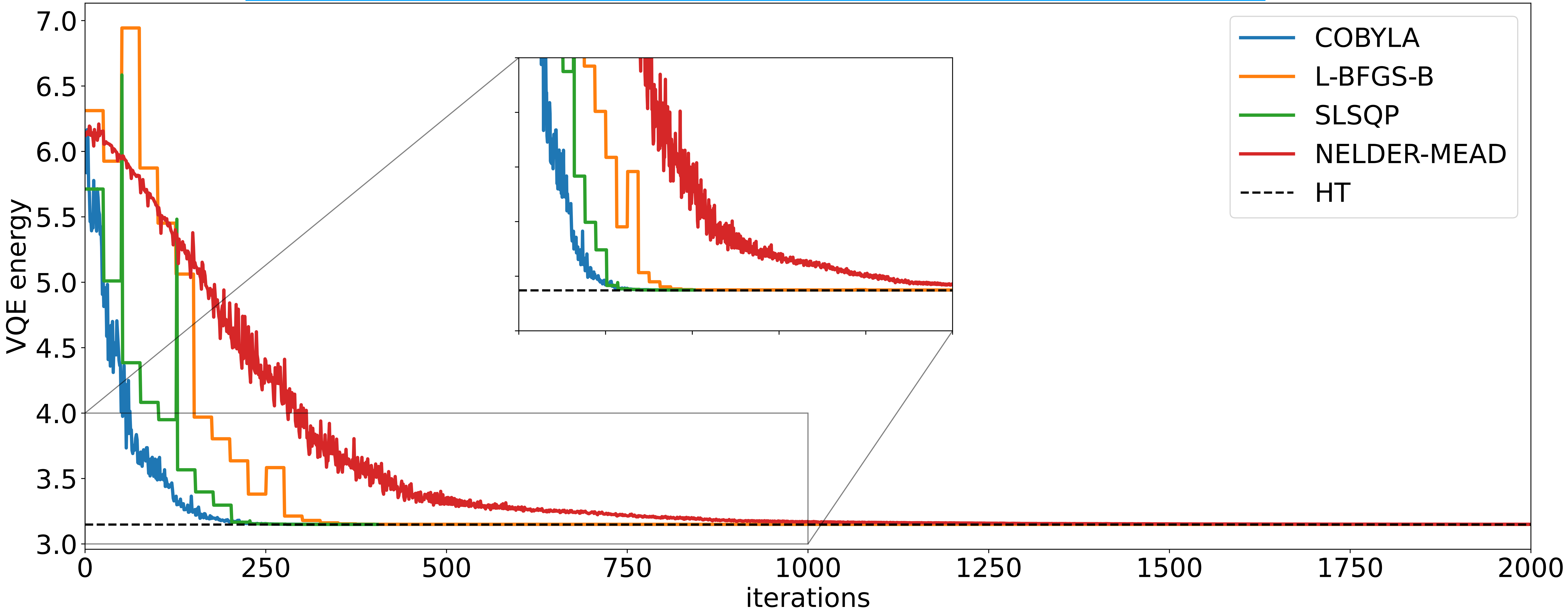
Small-scale: N=2, D=2, Λ → ∞

Λ = 2

$\log_2 \Lambda^6$ = 6 qubits

Optimizer	Var. form: R_y				Var. form: $R_y R_z$			
	Min.	Max.	Mean	Std.	Min.	Max.	Mean	Std.
COBYLA	3.149370	4.147156	3.159740	0.099739	**3.149157**	3.150034	3.149862	0.000202
L-BFGS-B	3.149268	4.150000	3.159886	0.100012	3.149375	4.148751	3.159925	0.099882
SLSQP	3.149397	4.150000	3.164968	0.111340	3.149377	4.149946	3.164980	0.111349
NELDER-MEAD	**3.148972**	3.195922	3.150774	0.005065	3.149516	4.149891	3.171468	0.140469

PQC with y rotation gates: depth = 3 → 24 parameters | Best out of 100 runs



Best VQE (100 runs): $E_0 = 3.148972$

Exact Diagonalization: $E_0 = 3.14808$

https://github.com/erinaldi/bmn2-qiskit

[Rinaldi et al., PRX Quantum 3 (2022) 010324]

Qiskit

Results

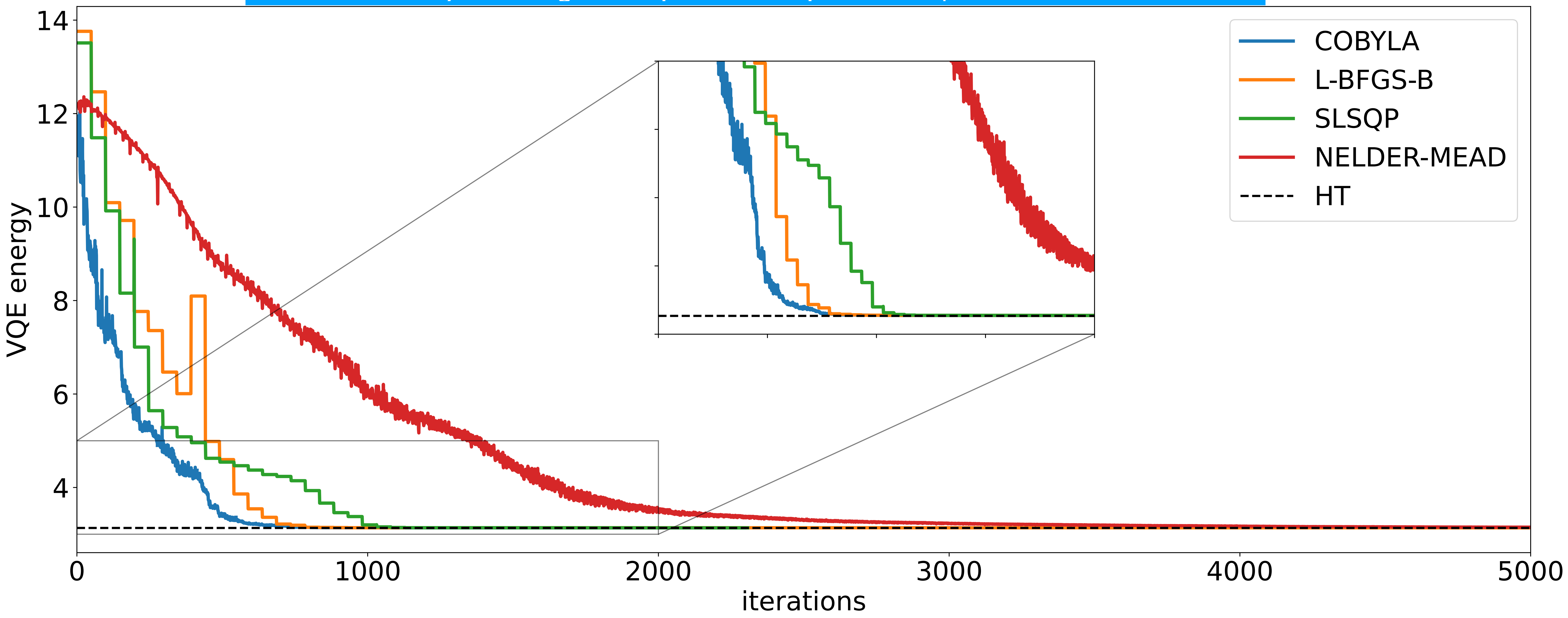
Small-scale: N=2, D=2, Λ → ∞

Λ = 4

$\log_2 \Lambda^6$ = 12 qubits

Optimizer	Var. form: R_y				Var. form: R_yR_z			
	Min.	Max.	Mean	Std.	Min.	Max.	Mean	Std.
COBYLA	**3.137059**	4.769101	3.251414	0.347646	3.137237	4.782013	3.378628	0.472015
L-BFGS-B	**3.137059**	5.769553	3.283462	0.434162	**3.137050**	4.286367	3.243110	0.307549
SLSQP	3.137060	5.769554	3.327706	0.471957	3.137059	4.232419	3.236925	0.290855
NELDER-MEAD	3.137471	5.713976	3.492673	0.478810	3.273614	6.443055	4.428032	0.758732

PQC with y rotation gates: depth = 3 → 36 parameters | Best out of 100 runs



Best VQE (100 runs): $E_0 = 3.137$

Exact Diagonalization: $E_0 = 3.13406$

[Rinaldi et al., PRX Quantum 3 (2022) 010324]

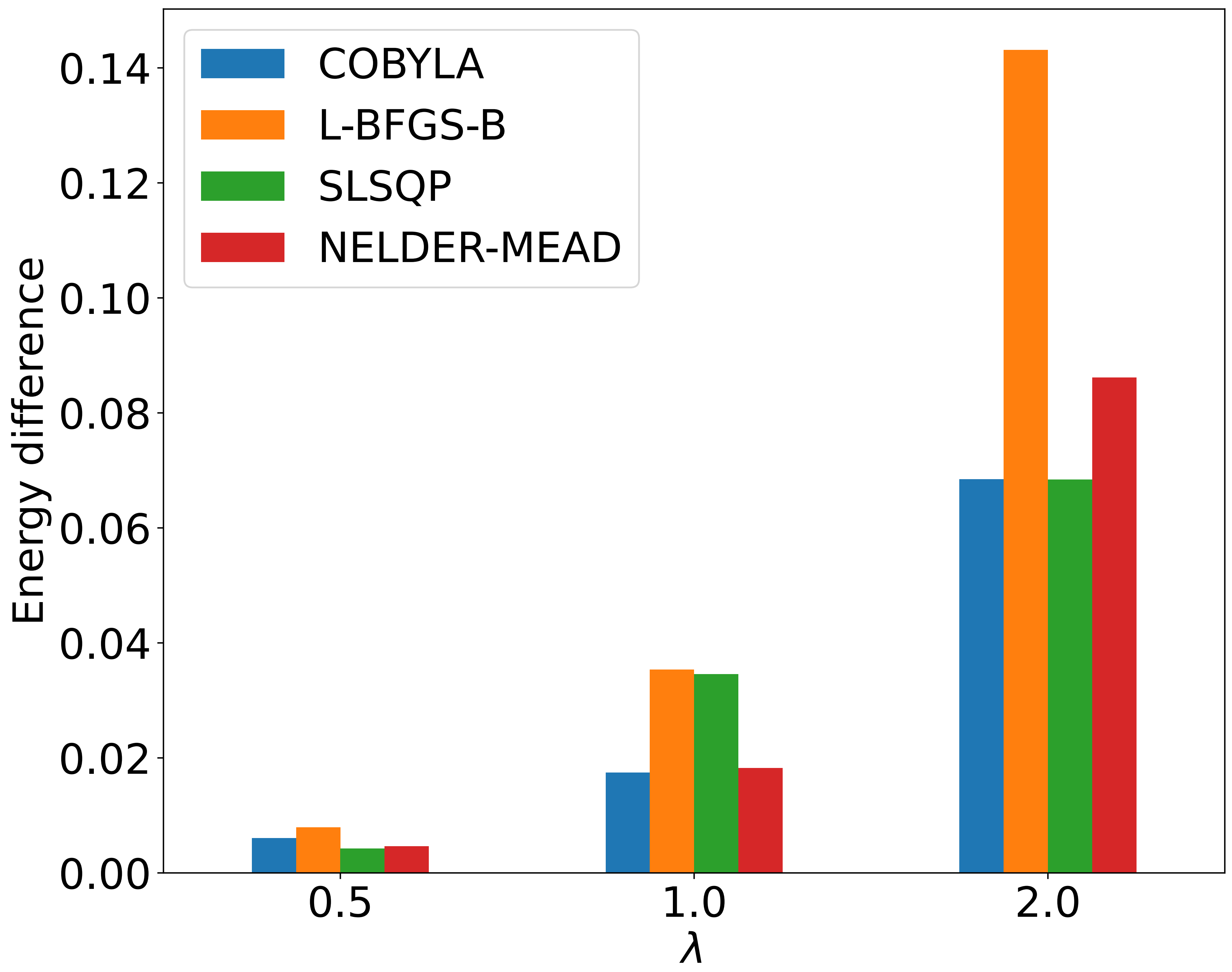
Qiskit

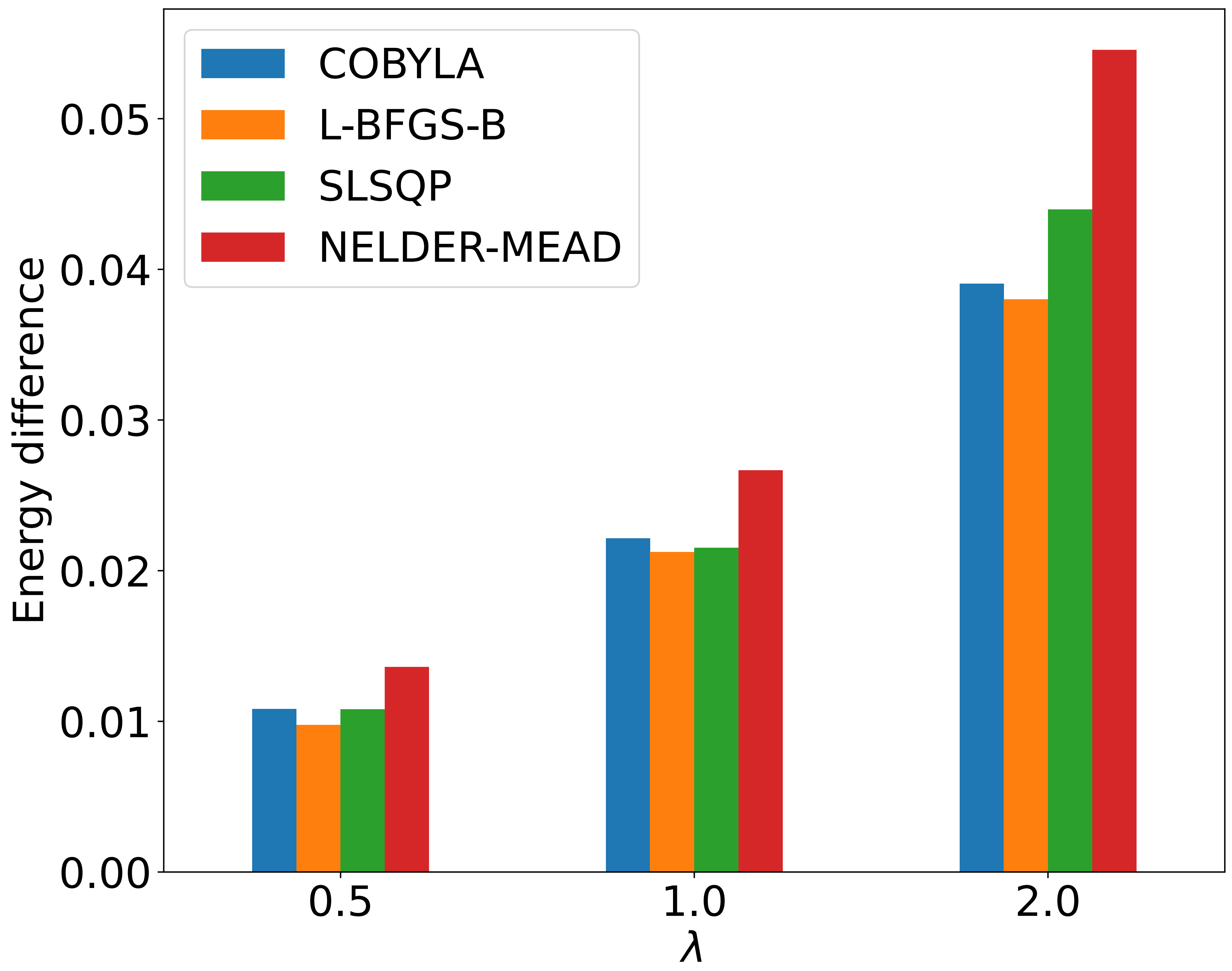
Results

Small-scale: N=2, D=2, Λ → ∞

Λ = 2 | $\log_2 \Lambda^6$ = 6 qubits

Λ = 4 | $\log_2 \Lambda^6$ = 12 qubits





PQC with y rotation gates: depth = 3 | Best out of 100 runs

https://github.com/erinaldi/bmn2-qiskit

[Rinaldi et al., PRX Quantum 3 (2022) 010324]

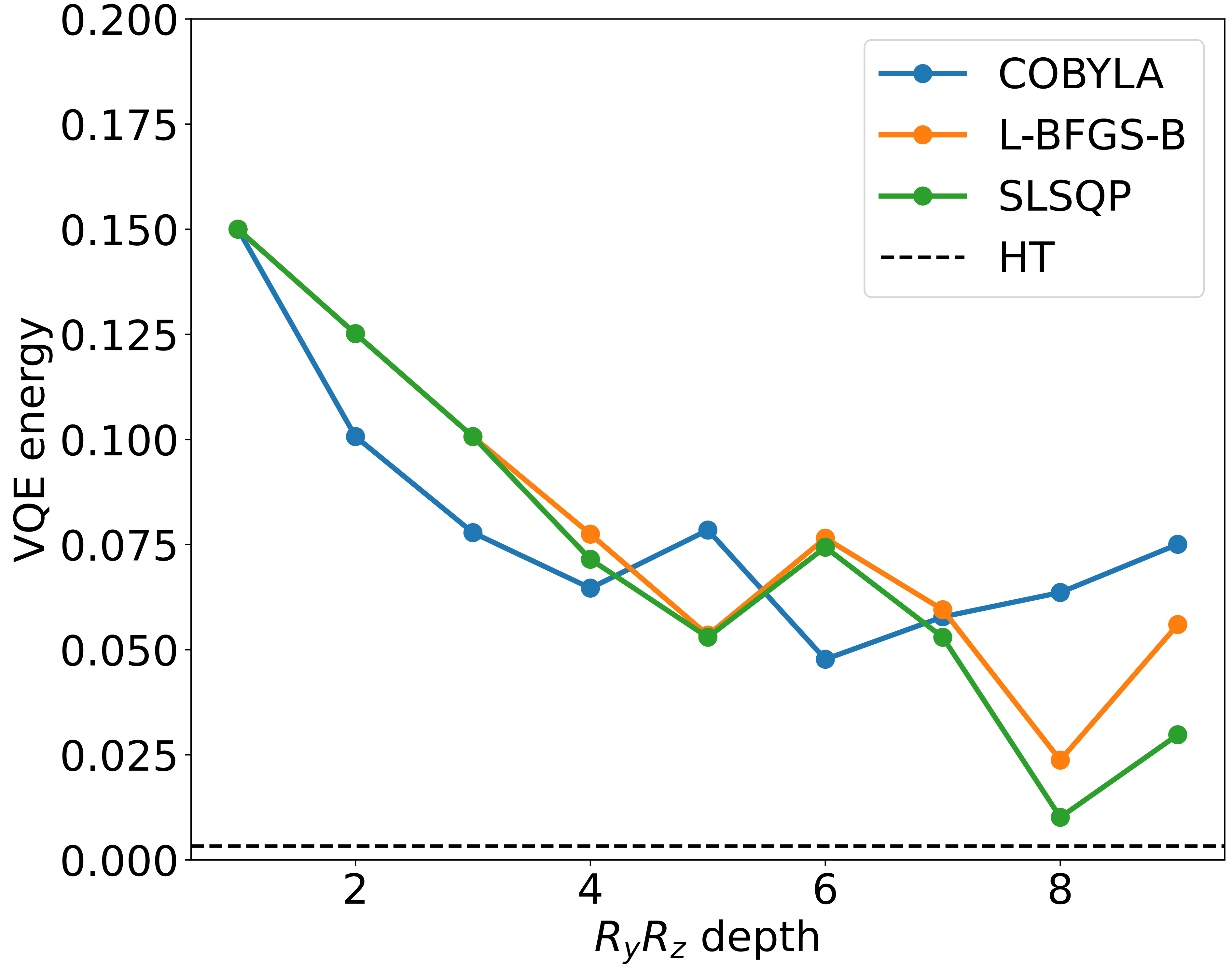
Qiskit

Results

Supersymmetric N=2 D=2 at large coupling

$\Lambda = 2$

$\log_2 \Lambda^6 = 9$ qubits



	depth = 5				depth = 9	
λ	COBYLA	L-BFGS-B	SLSQP	NELDER-MEAD	Best	HT (exact)
0.5	0.088492	0.139702	0.134517	0.406003	0.02744	0.01690
1.0	0.135800	0.219268	0.308781	0.752459	0.07900	0.04829
2.0	0.387977	0.622704	0.522396	1.271939	0.17688	0.08385

[Rinaldi et al., PRX Quantum 3 (2022) 010324]

Computational Resources

How hard is it?

Bosonic Model

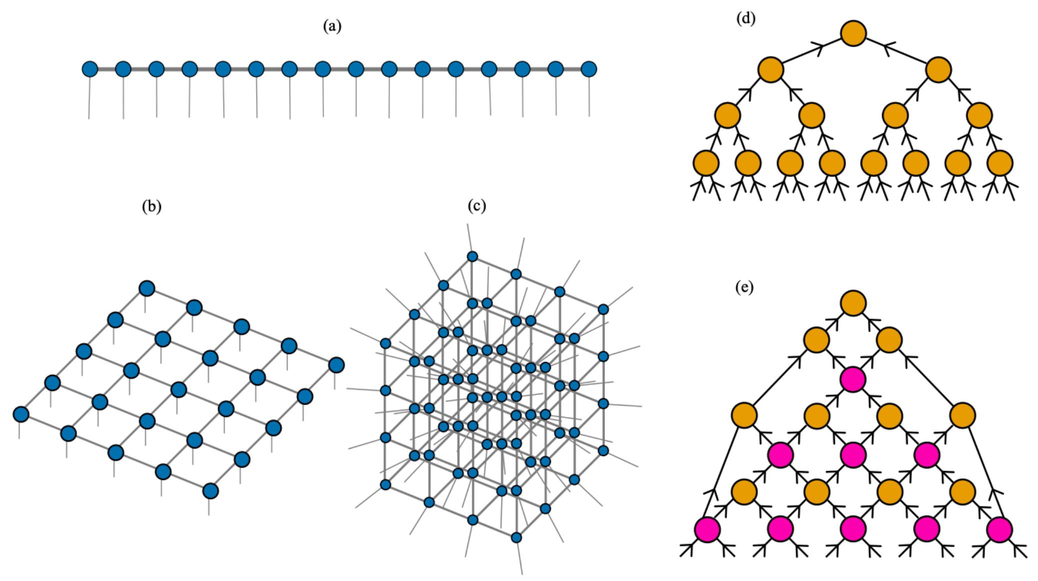
D=2	HT	VQE	DL	MC
N=2	✅	Λ = 2,4	✅	✅
N=3	‼️	❌	✅	✅
N>3	❌	❌	✅	✅

Supersymmetric Model

D=2	HT	VQE	DL	MC
N=2	✅	Λ = 2	✅	✅
N=3	‼️	❌	✅	✅
N>3	❌	❌	✅	✅

[tensors.net, https://tensornetwork.org/]

[Berezutskii et al., arxiv:2503.08626]



Tensor Network Methods

Quick Introduction

- In **quantum many-body systems**, TN are used as a parsimonious representation of the amplitudes of the wave function
- TN elements can be optimized to compress correlated wave functions
- Algorithms based on TN allow for representing ground states and for time-evolving arbitrary states
- TN with finite computational resources are **inherently approximate**
- In quantum computing, TN are the **leading "classical" simulation** methods

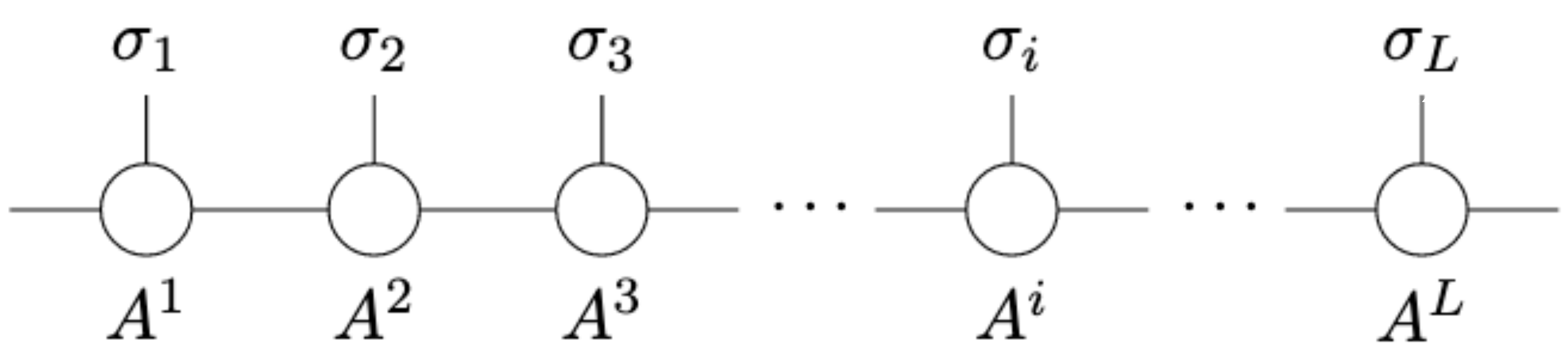
[tensors.net, https://tensornetwork.org/]

[Brehm et al., JHEP 09 (2025) 116]

Matrix Product States

Ingredients

$$|\psi\rangle = \sum_{\{\sigma_i,\alpha_i\}} A^{1,\sigma_1}_{\alpha_0\alpha_1} A^{2,\sigma_2}_{\alpha_1\alpha_2} \cdots A^{L-1,\sigma_{L-1}}_{\alpha_{L-2}\alpha_{L-1}} A^{L,\sigma_L}_{\alpha_{L-1}\alpha_0} |\sigma_1\sigma_2\cdots\sigma_{L-1}\sigma_L\rangle,$$



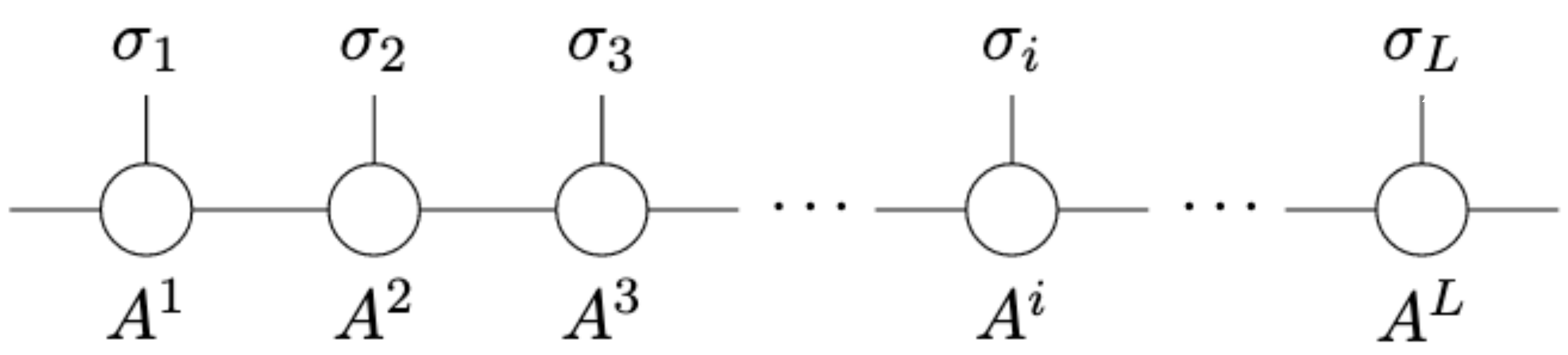
- L sites → total number of bosons ($\sim N^2 D$)
- Sites are d dimensional → dimension of each bosonic Hilbert space Λ
- D is the *bond dimension* → computational cost of DMRG:

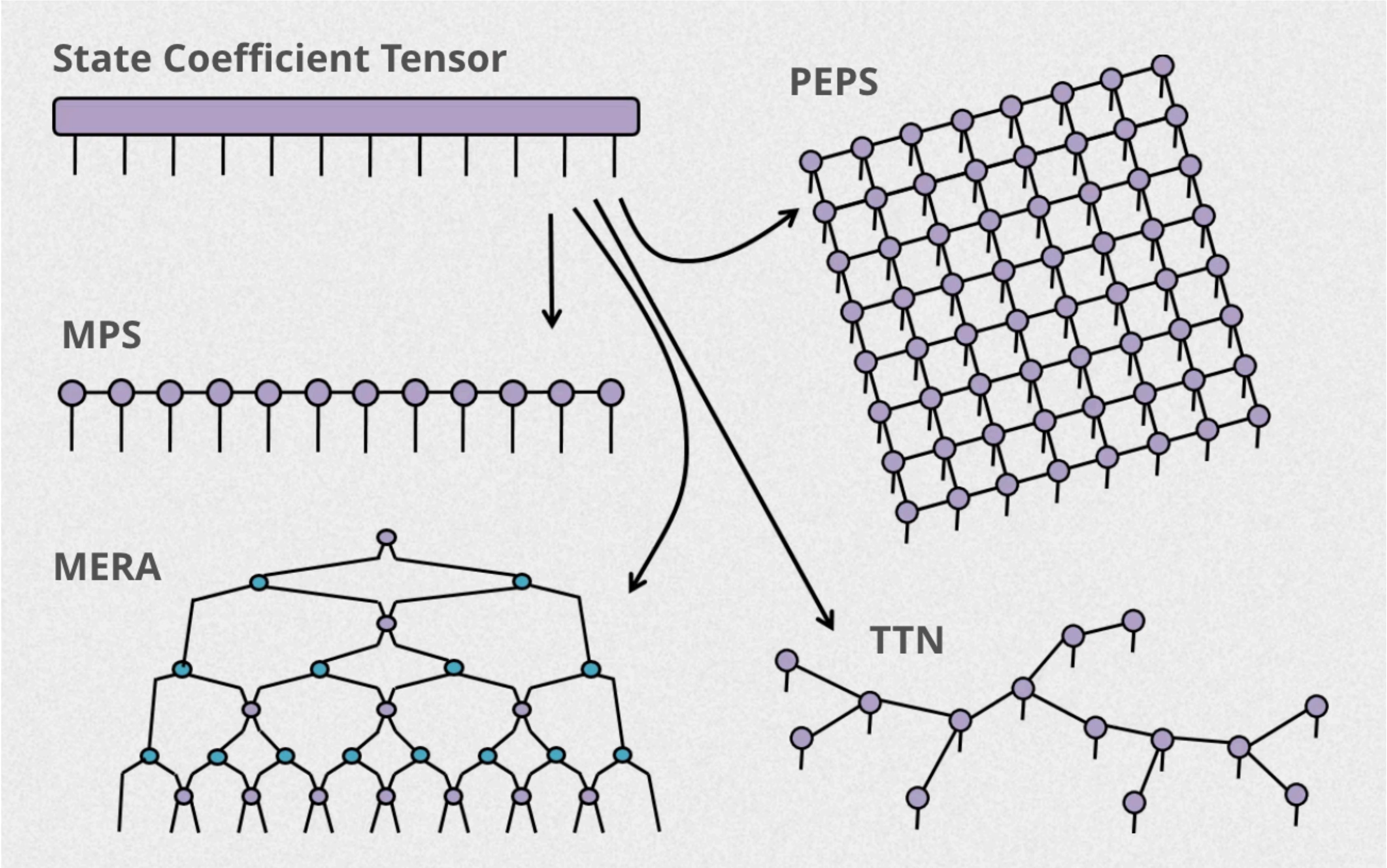
$$\mathcal{O}\left(2LD^3_{\max}\chi d + 2LD^2_{\max}\chi^2 d^2 + LD^3_{\max}\chi d^2\right)$$

Matrix Product States

Ingredients

$$|\psi\rangle = \sum_{\{\sigma_i,\alpha_i\}} A^{1,\sigma_1}_{\alpha_0\alpha_1} A^{2,\sigma_2}_{\alpha_1\alpha_2} \cdots A^{L-1,\sigma_{L-1}}_{\alpha_{L-2}\alpha_{L-1}} A^{L,\sigma_L}_{\alpha_{L-1}\alpha_0} |\sigma_1\sigma_2\cdots\sigma_{L-1}\sigma_L\rangle,$$





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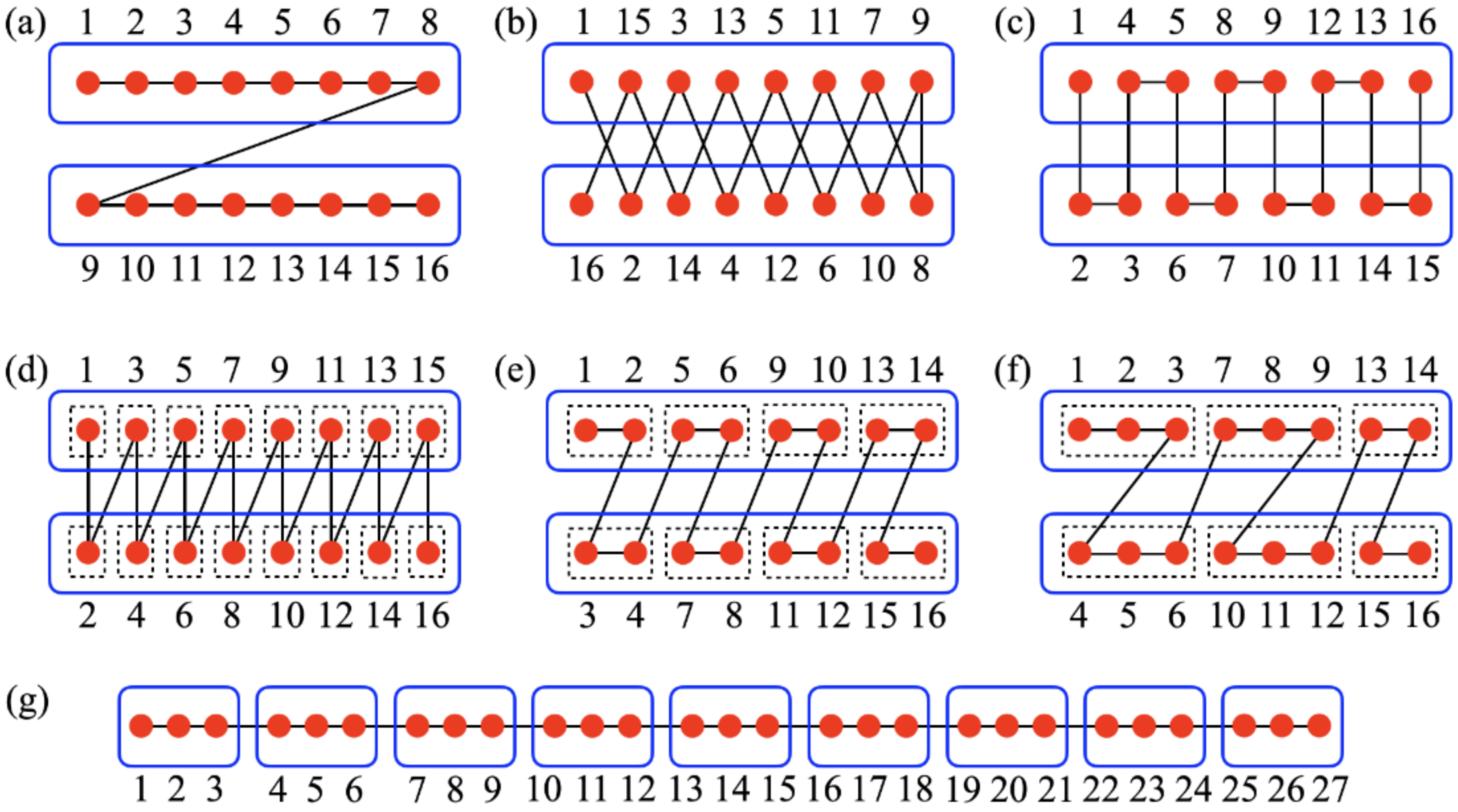
[Brehm et al., JHEP 09 (2025) 116]

Matrix Product States

Layouts

SU(3), D=2

SU(2), D=9

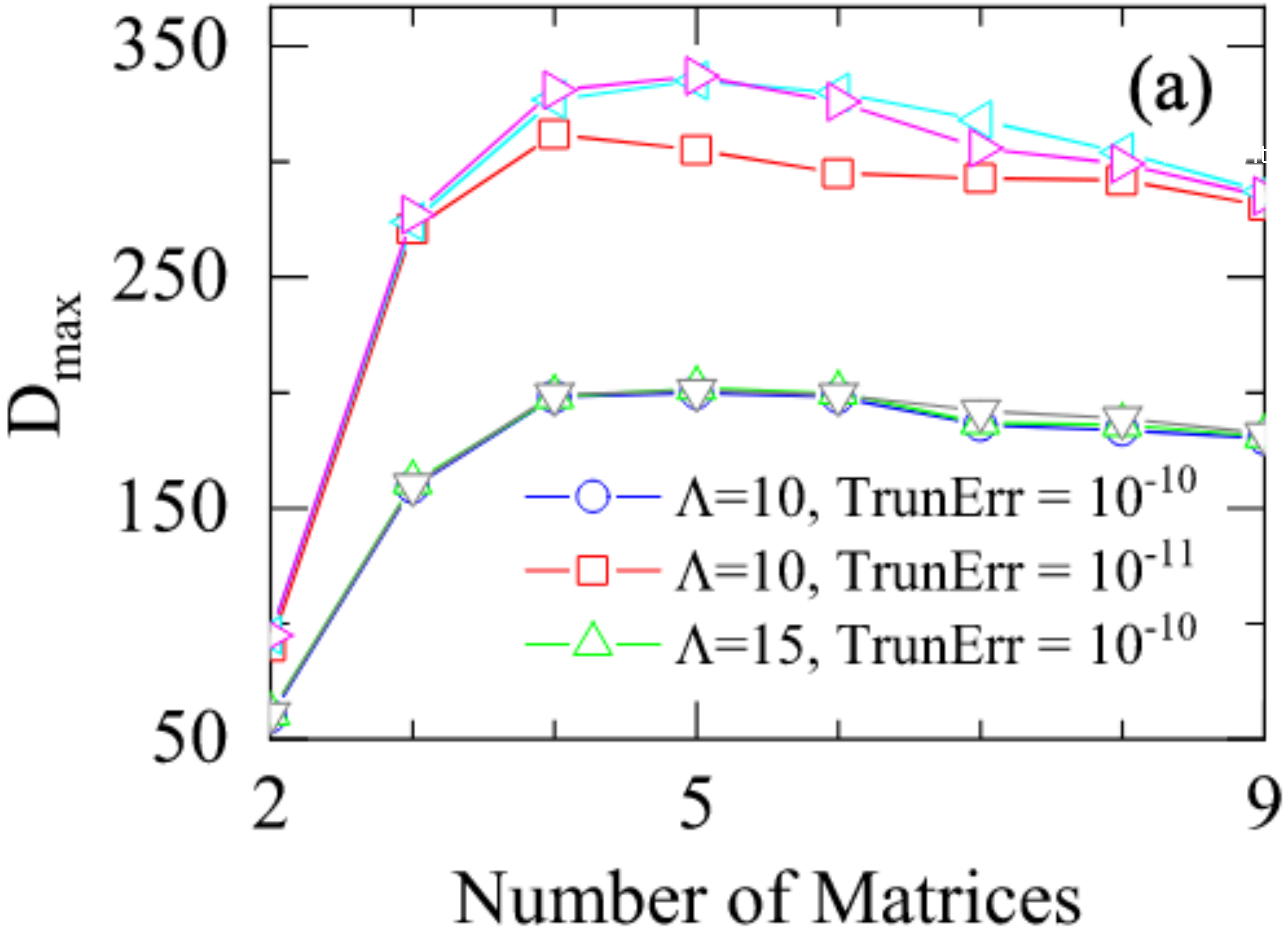


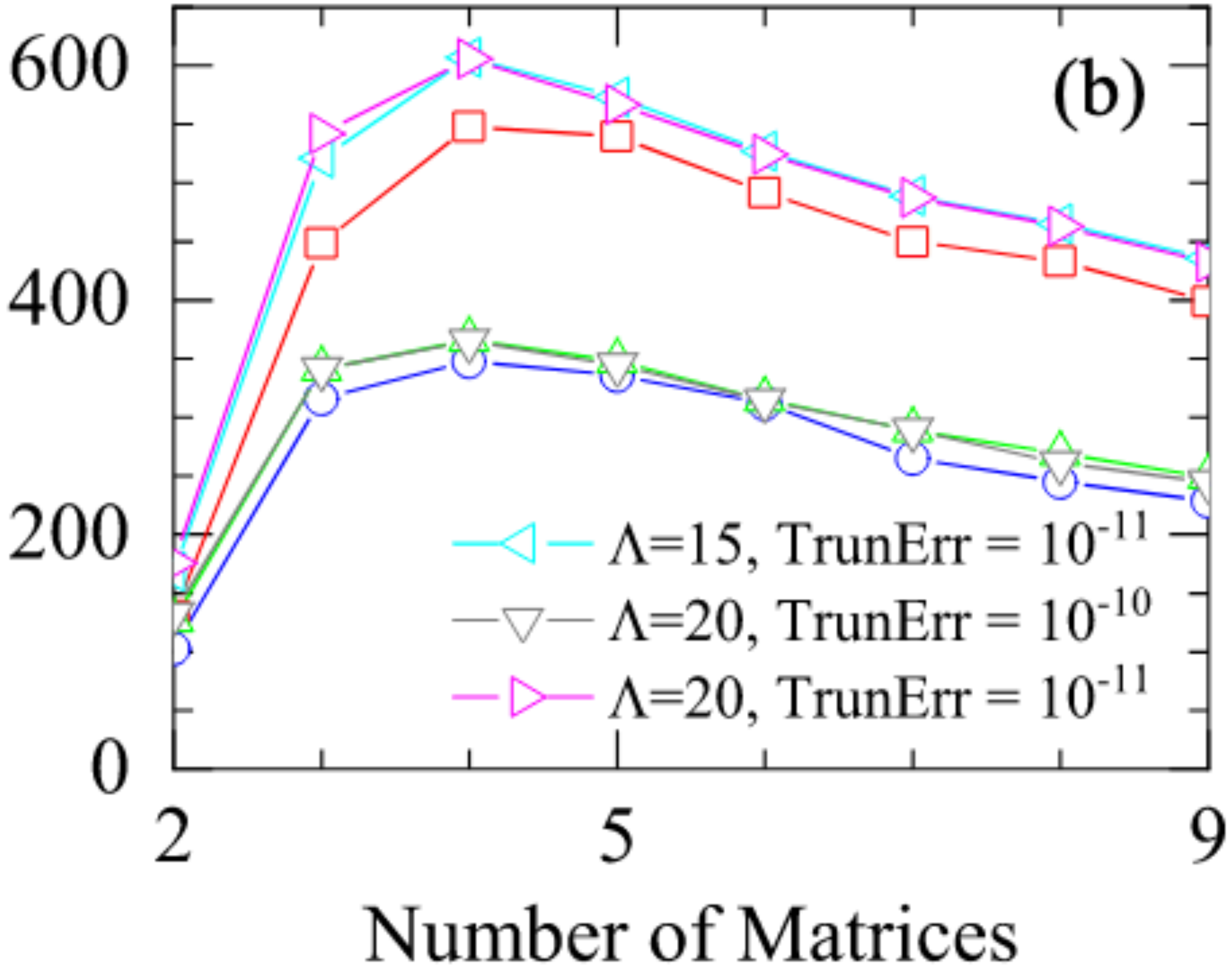
Results

SU(2), D Matrices (Bosonic)

g=0.5

g=1.0





[Brehm et al., JHEP 09 (2025) 116]

Results

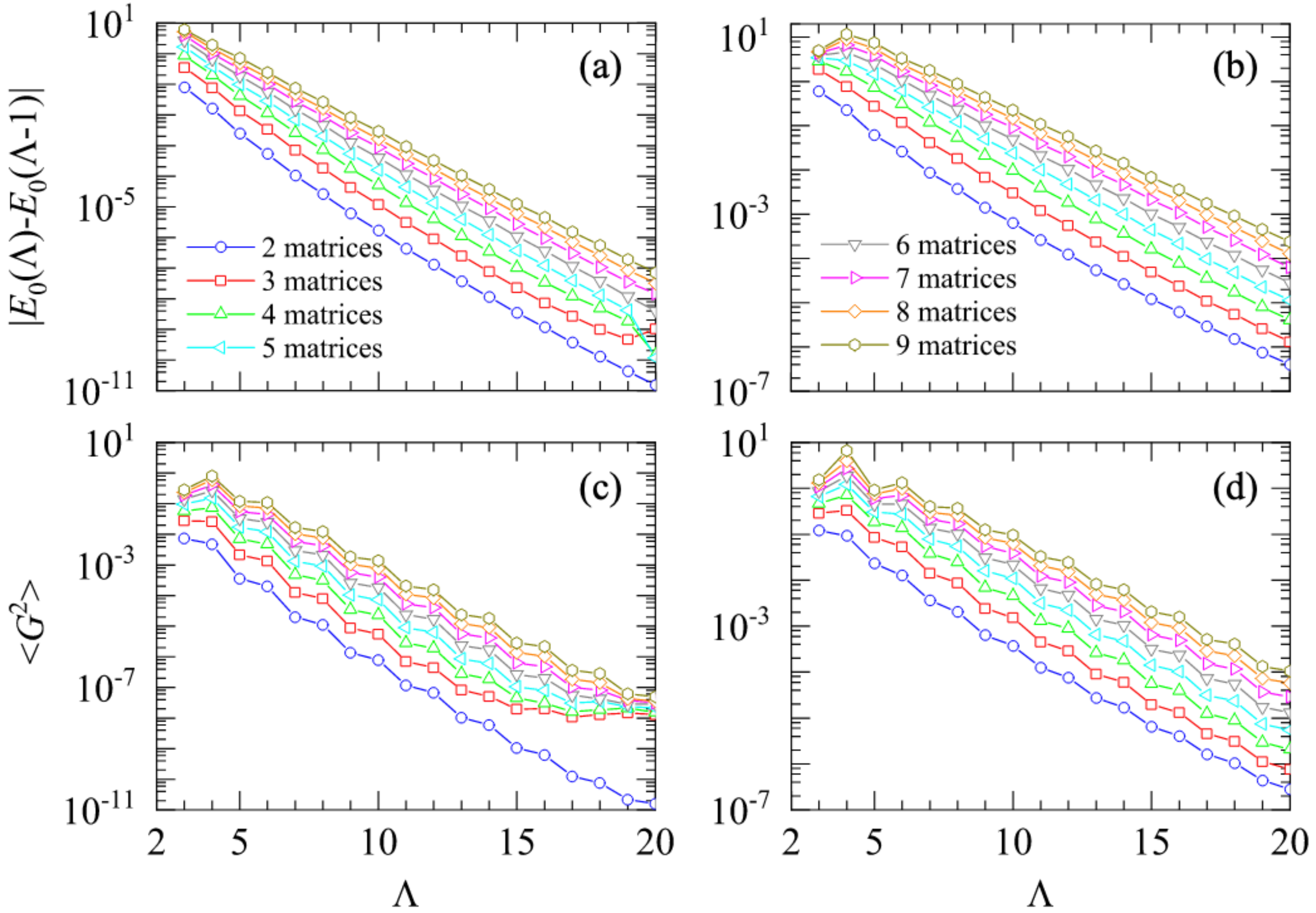
SU(2), D Matrices (Bosonic), Λ → ∞

g=0.5

g=1.0

Ground state energy

Gauge invariance violation



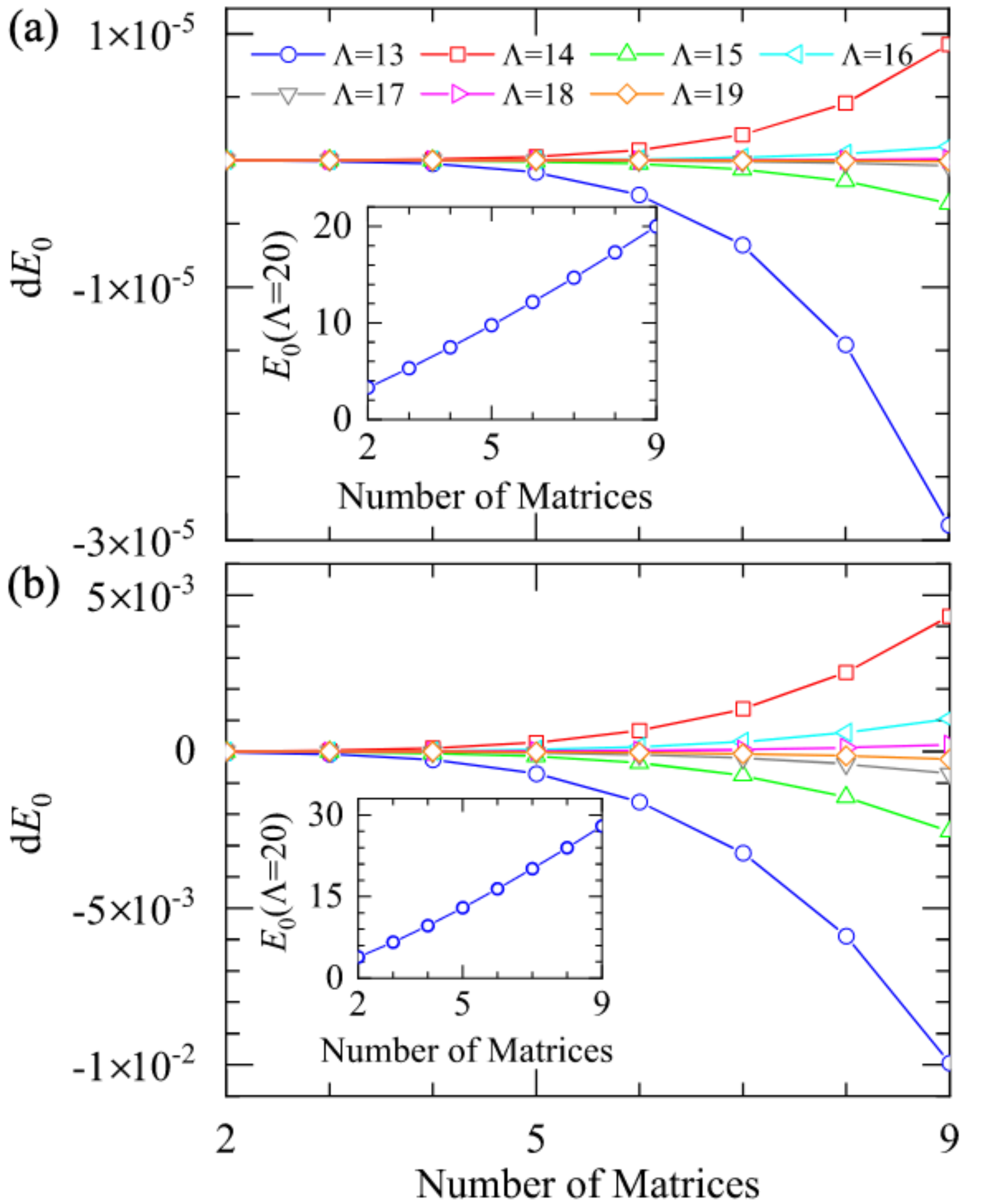
iTensor

[Brehm et al., JHEP 09 (2025) 116]

Results

SU(2), D Matrices (Bosonic), Λ → ∞

Energy convergence



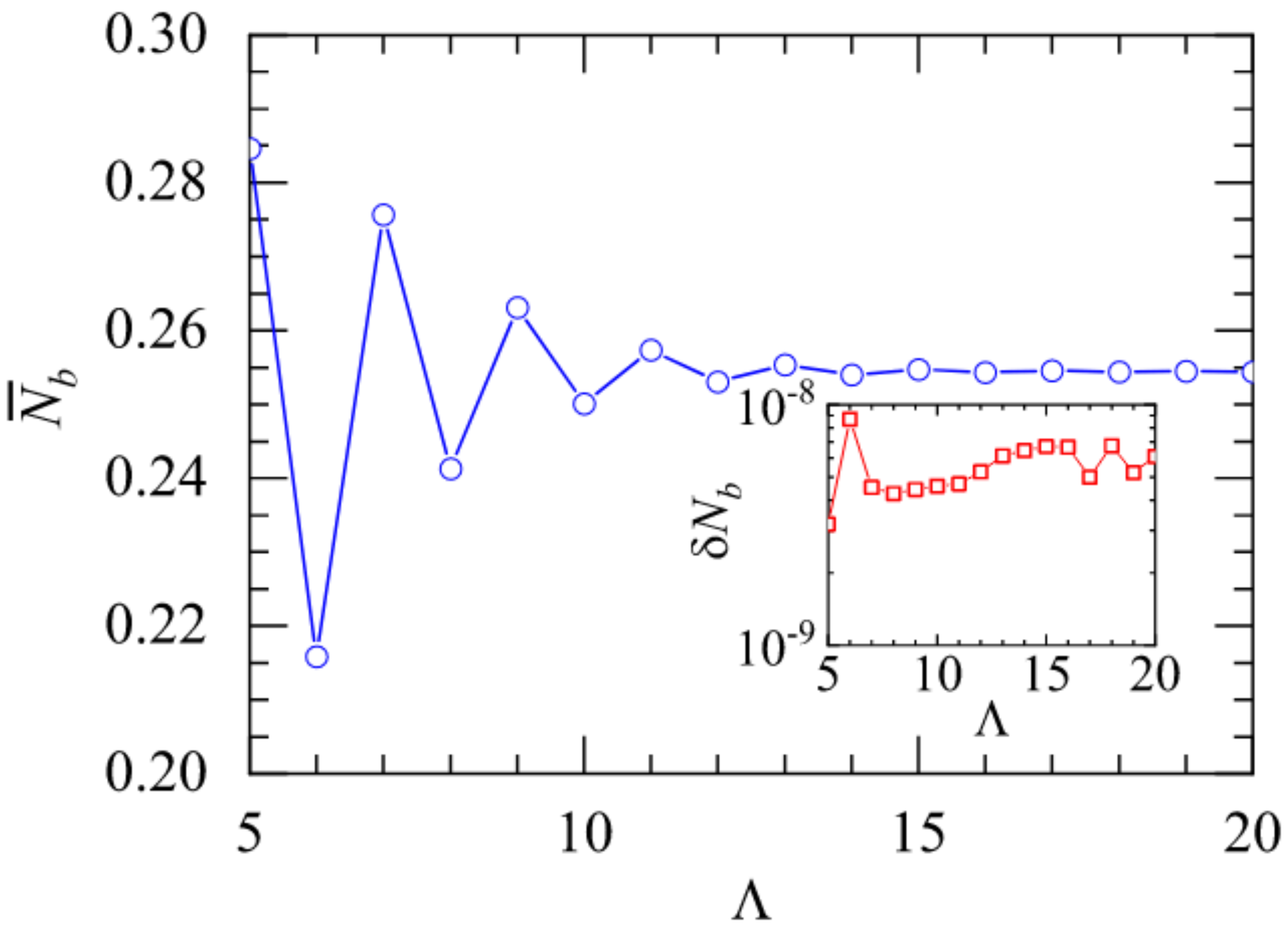
g=0.5

g=1.0

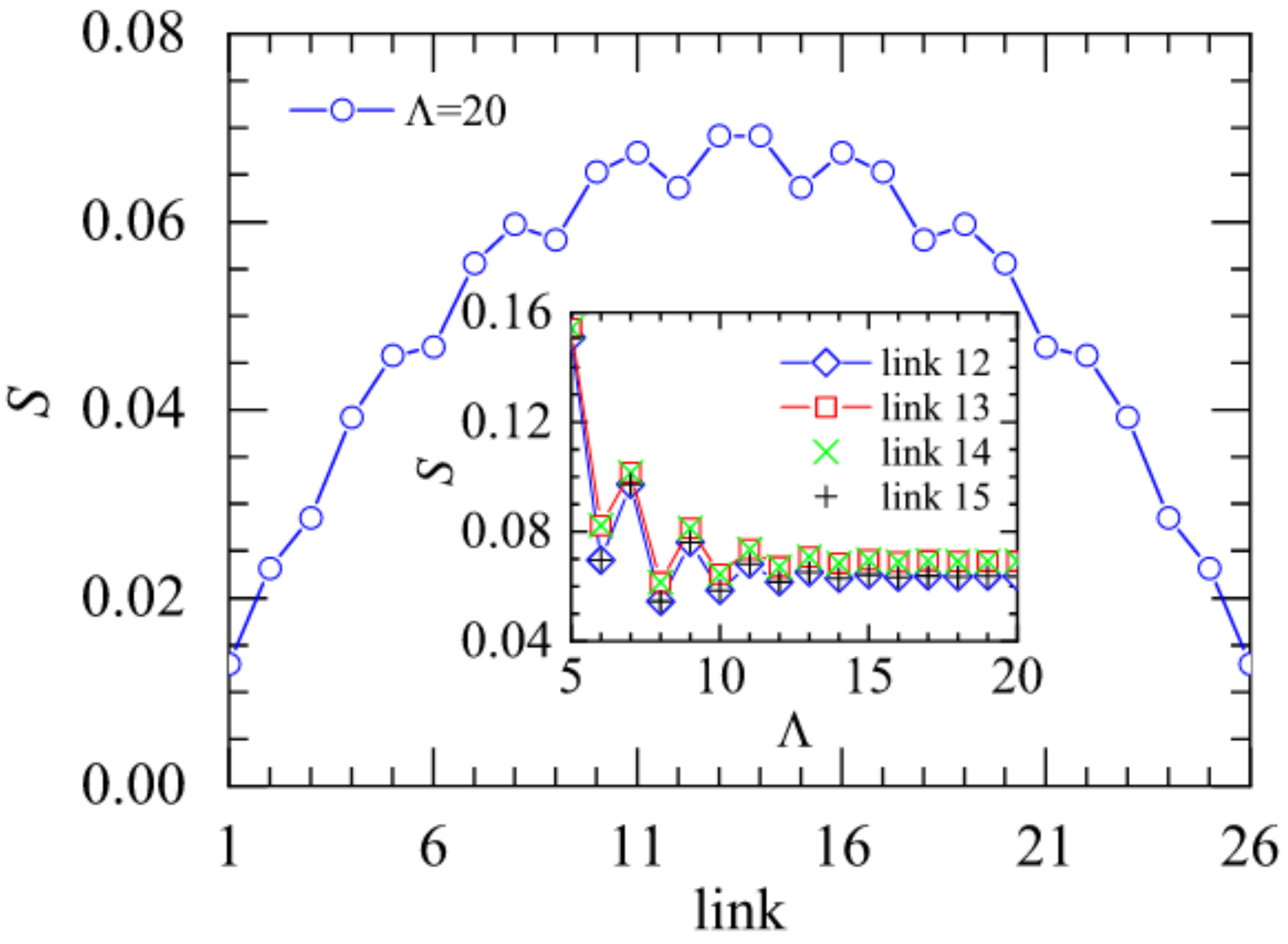
Results

SU(2), 9 Matrices (Bosonic BFSS)

m=1.0 g=1.0



Average boson occupation number



Entanglement Entropy

[Brehm et al., JHEP 09 (2025) 116]

Results

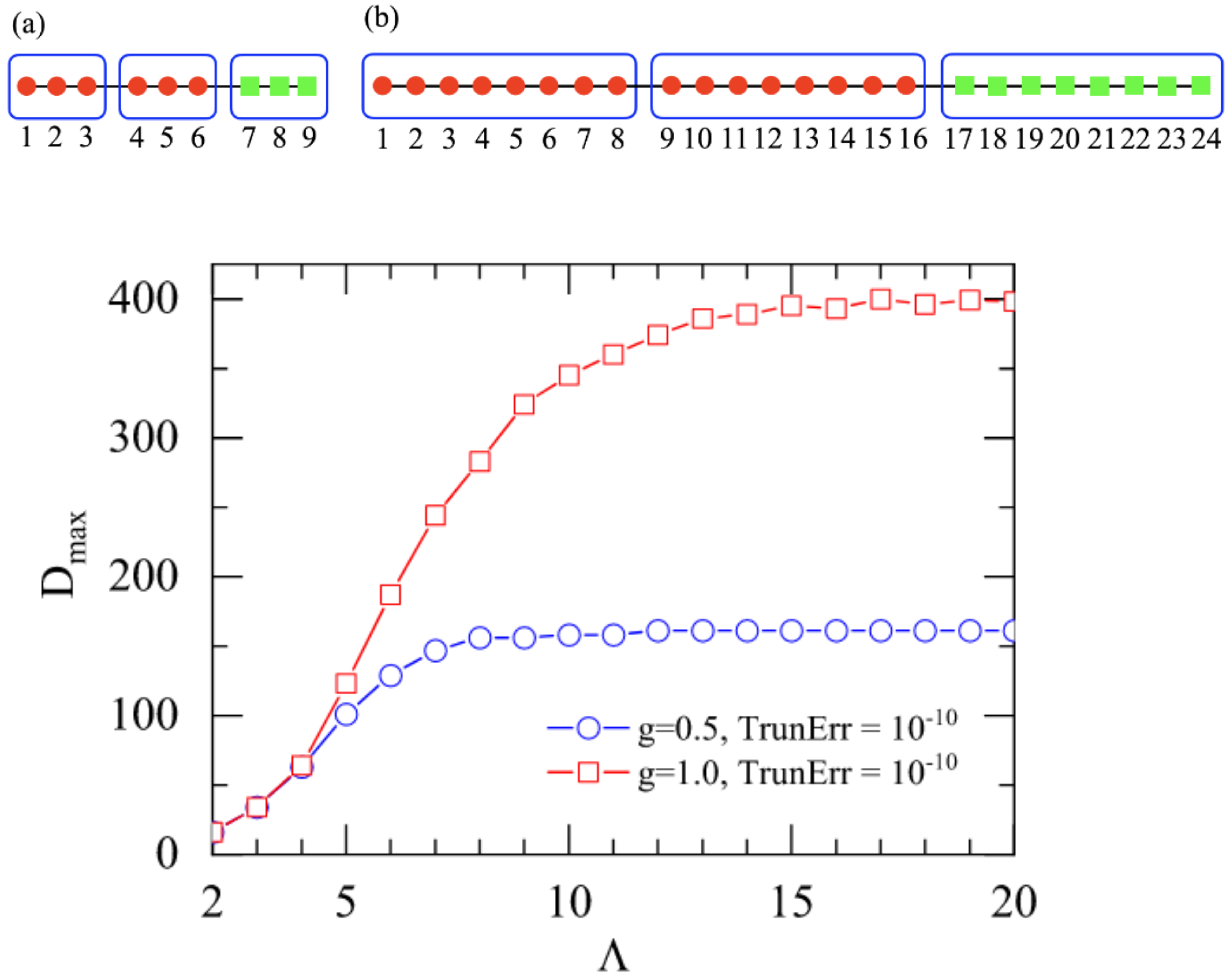
SU(N), 2 Matrices (Bosonic)

g^2N=1.0

[Brehm et al., JHEP 09 (2025) 116]

Results

SU(2), 2 Matrices (SUSY)

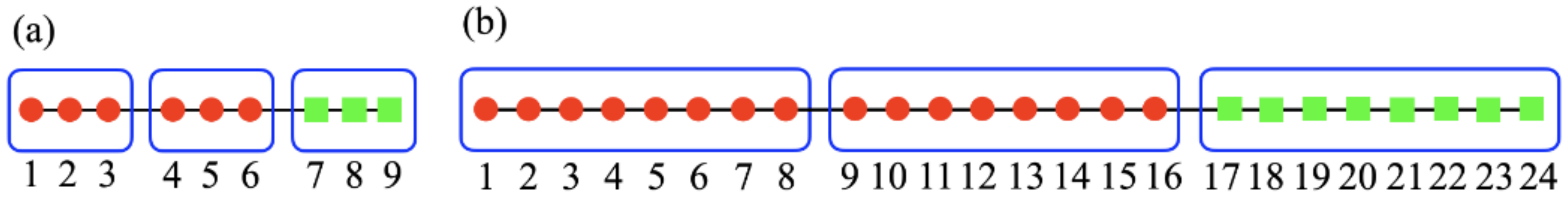


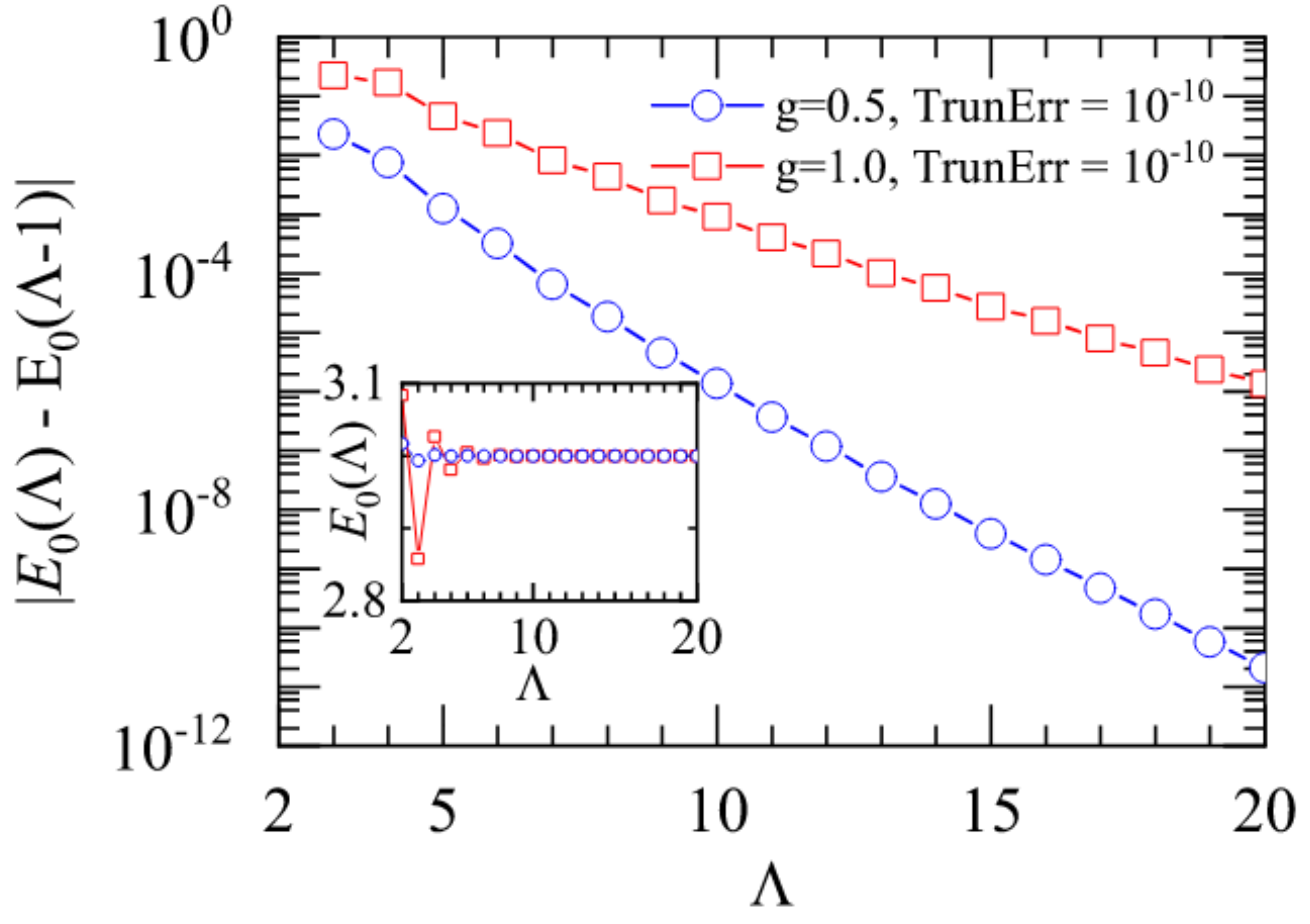
[iTensor](#)

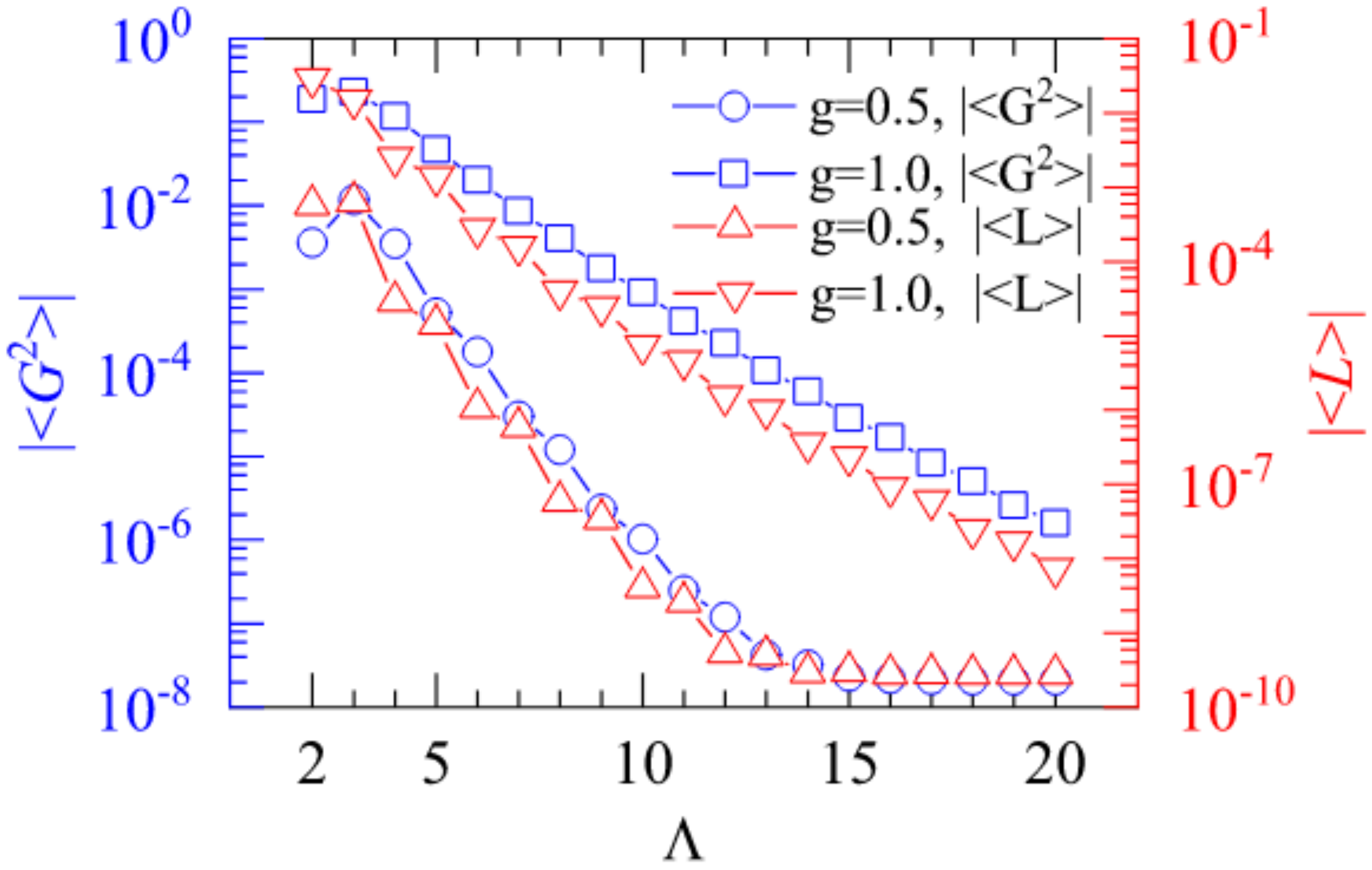
[Brehm et al., JHEP 09 (2025) 116]

Results

SU(2), 2 Matrices (SUSY)







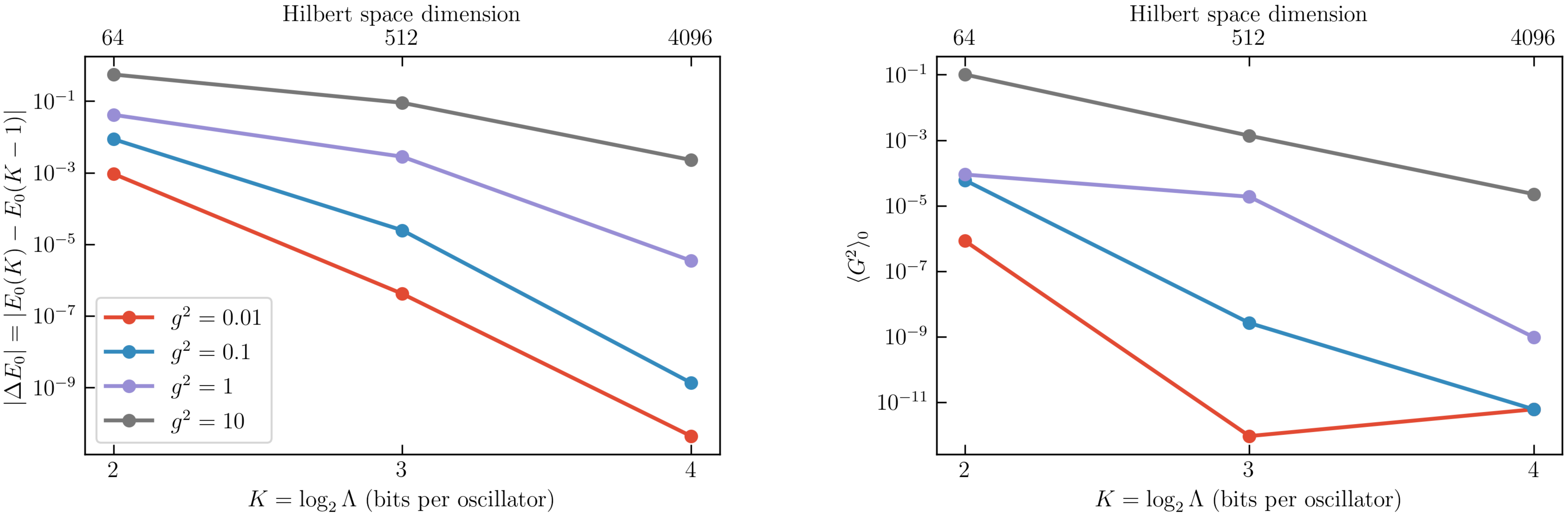
Conclusions

and outlook

- ✓ Quantum simulations may be used for addressing **Quantum Gravity** problems, using the holographic duality
- ✦ Hybrid quantum-classical algorithms can be used on current quantum hardware to study small-size matrix models. Difficult to scale.
- ✦ Tensor Network methods allow an efficient representation of the ground state of matrix models up to larger N and number of matrices.
- ➡ Study dynamics of D0-brane bunches in “small” systems: black hole formation
- ➡ Study entanglement entropy of matrix ground states in different “configurations”
- ➡ Use matrix models to benchmark how quantum computers improve with time

Time Evolution

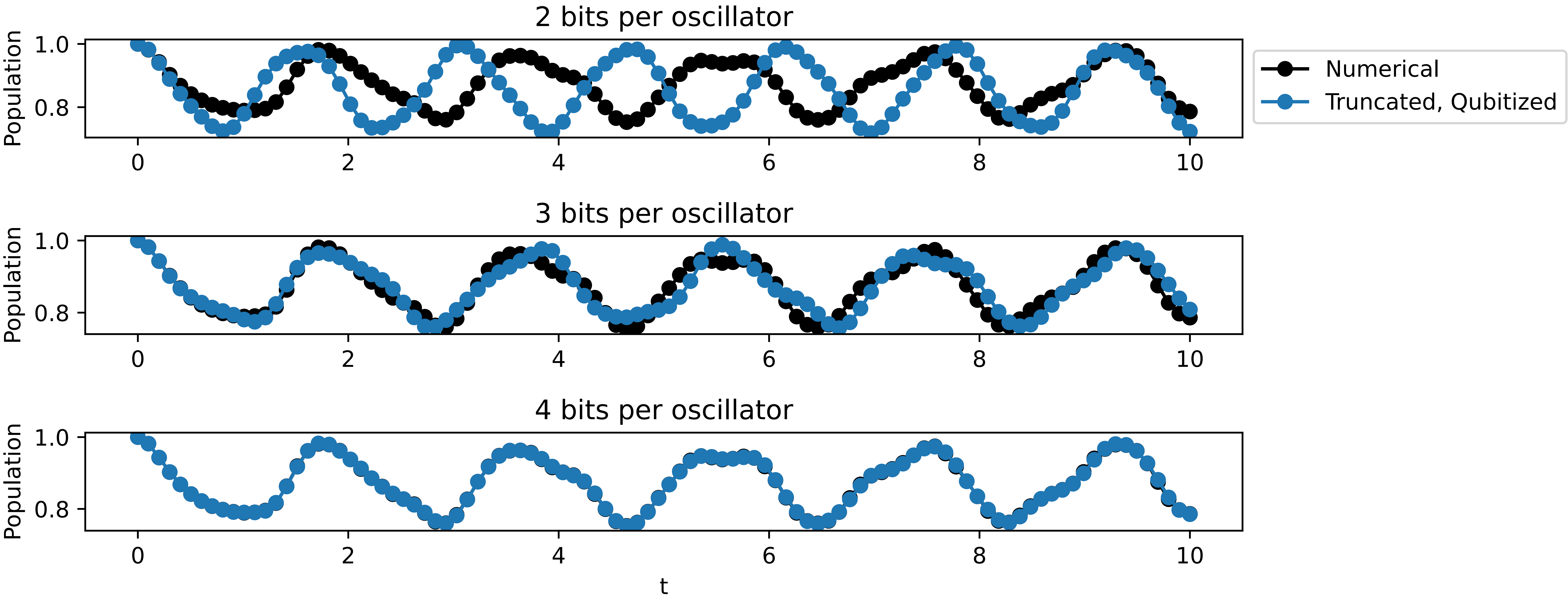
Of SU(2) 1 Matrix model on Quantum Computers



In preparation

Time Evolution

Of SU(2) 1 Matrix model on Quantum Computers



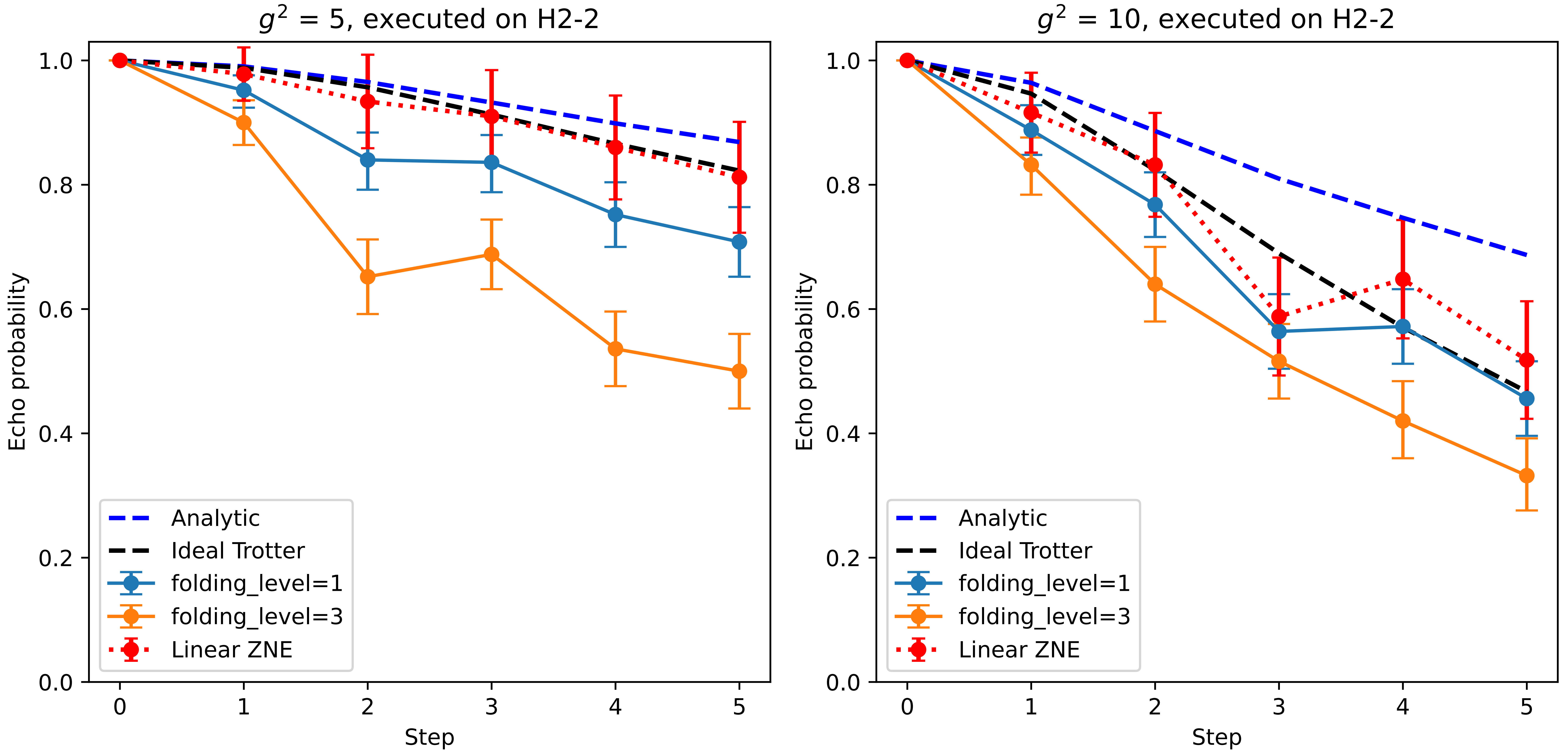
In preparation

Time Evolution

Of SU(2) 1 Matrix model on Quantum Computers

$\log_2 \Lambda^3$ = 6 qubits

$|\langle 0 | e^{-i\hat{H}t} | 0 \rangle|^2$

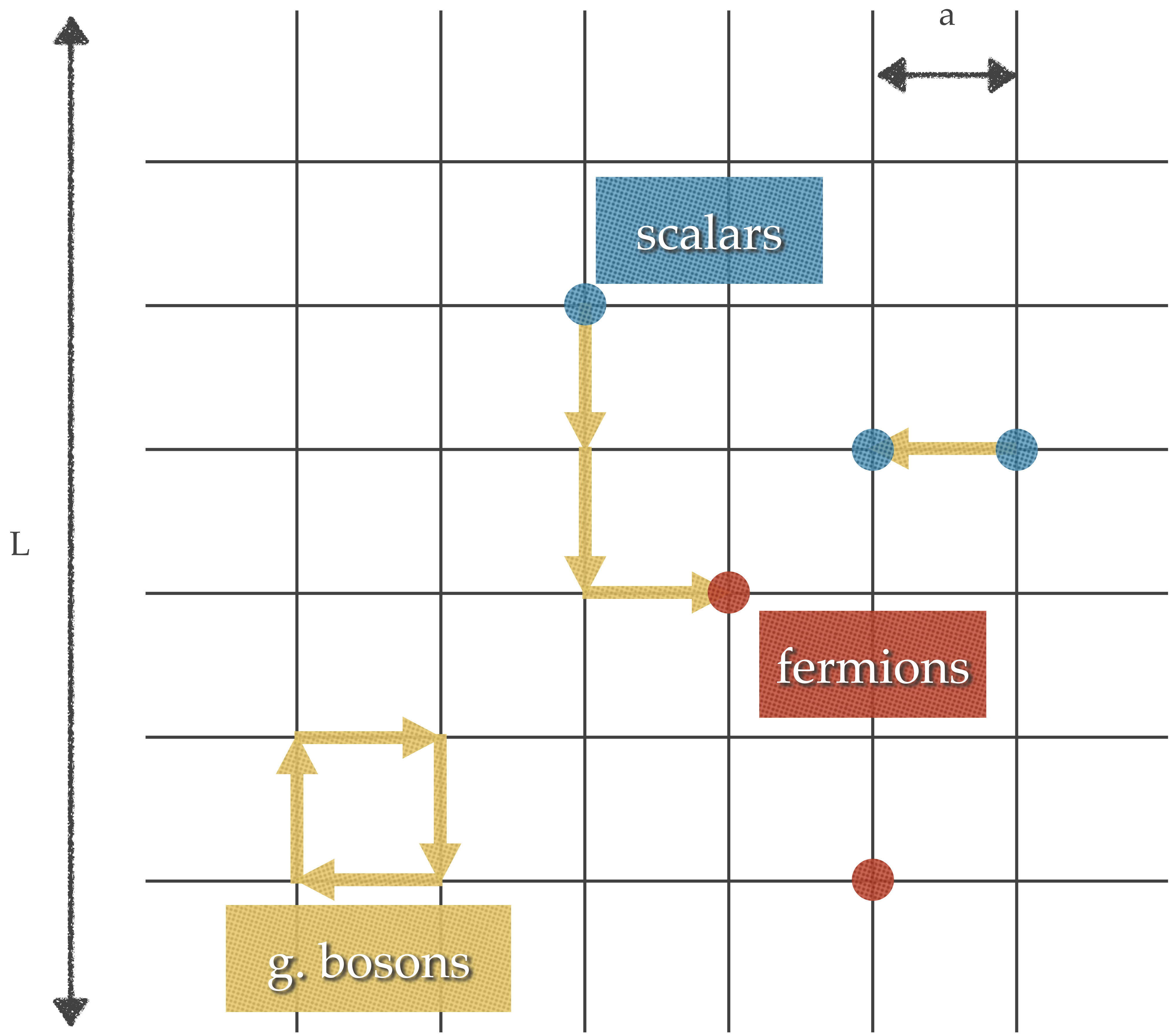


In preparation

Questions?

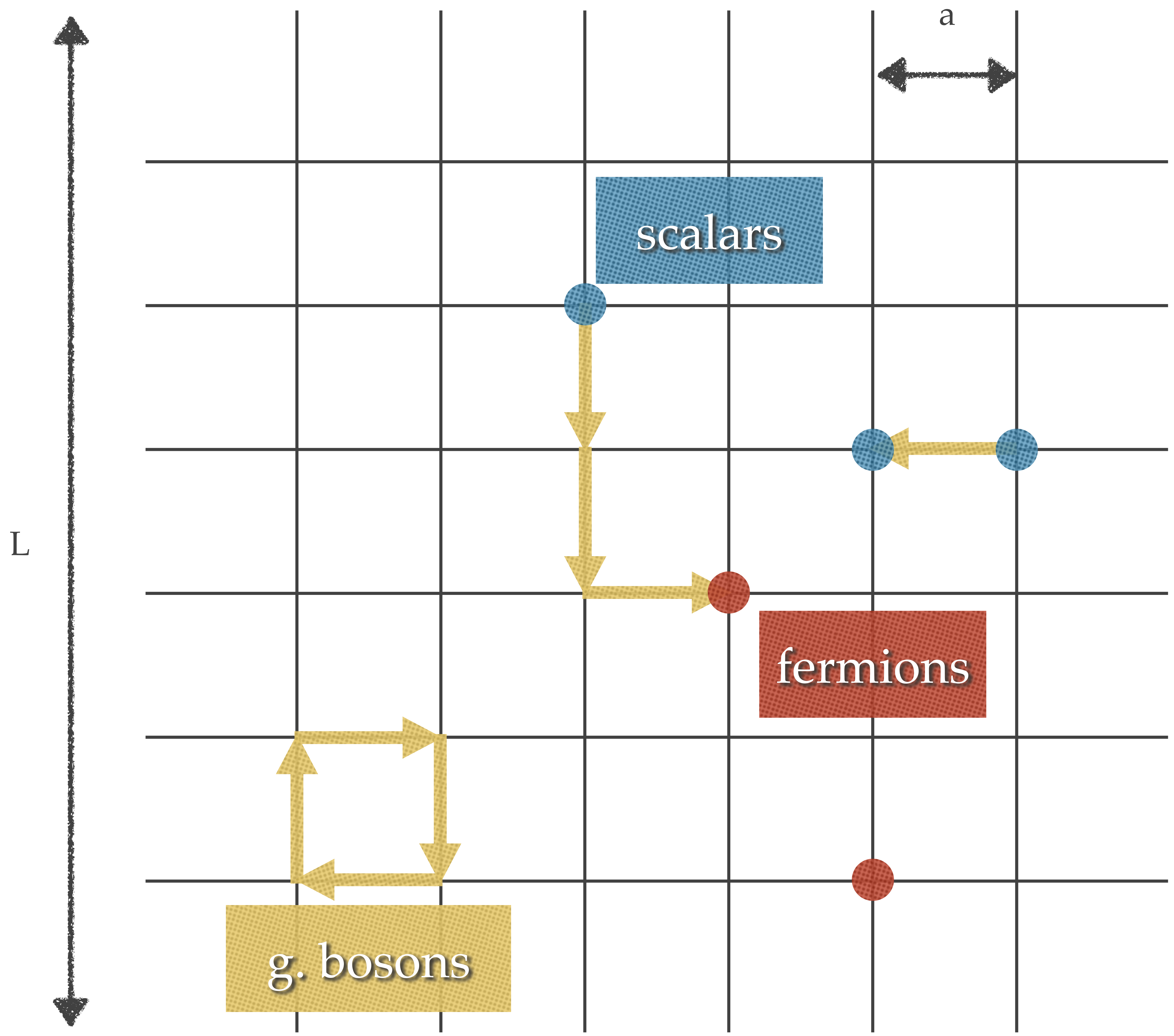
Path Integral Monte Carlo

Lattice Gauge Theory Primer



Path Integral Monte Carlo

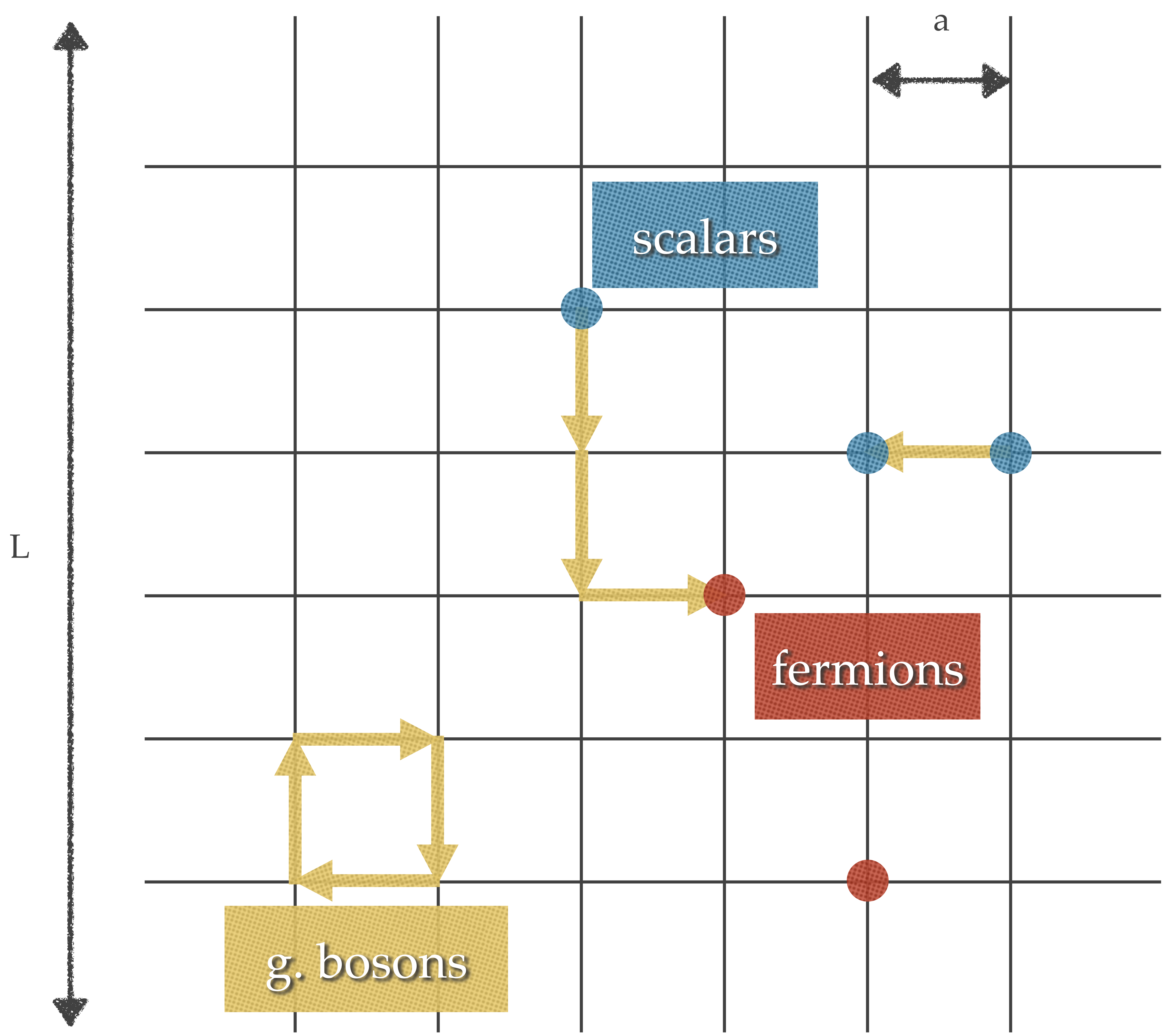
Lattice Gauge Theory Primer



- Discretize space and time
  - lattice spacing “a”
  - lattice size “L”

Path Integral Monte Carlo

Lattice Gauge Theory Primer



- Discretize space and time
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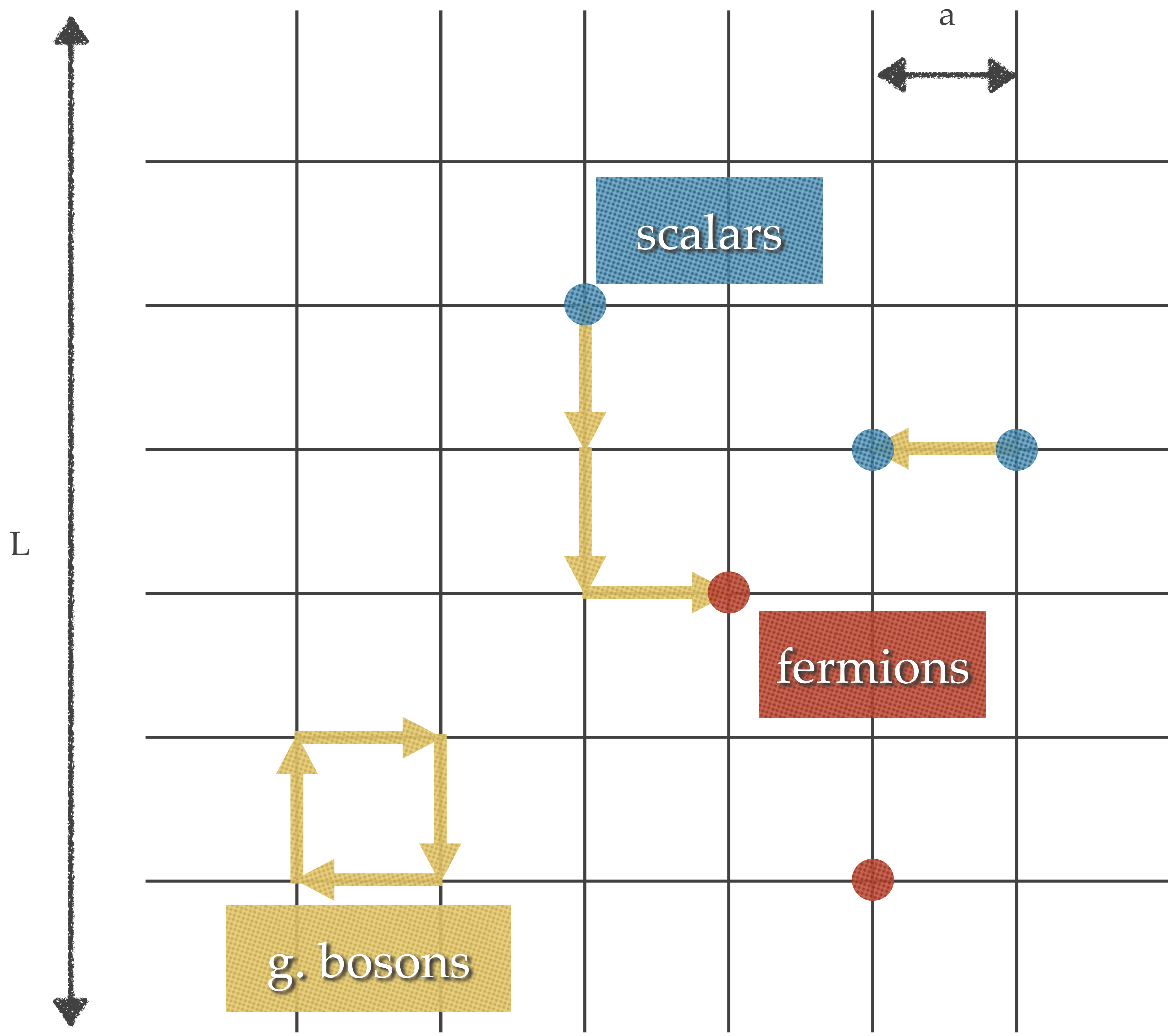
- Keep all d.o.f. of the theory
  - not a model!
  - no simplifications

Path Integral Monte Carlo

Lattice Gauge Theory Primer



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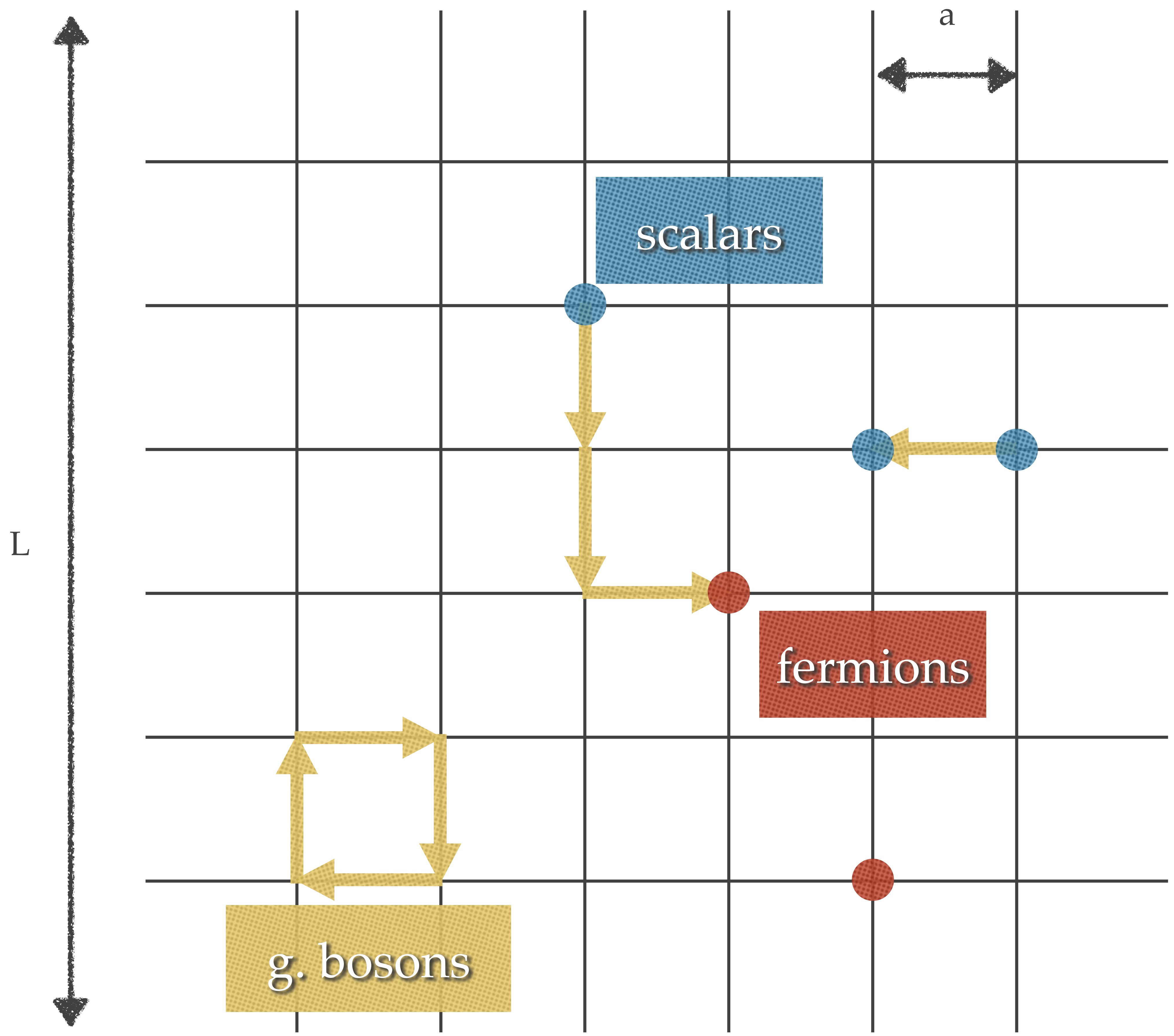
  - Monte Carlo sampling
  - use supercomputers

Path Integral Monte Carlo

Lattice Gauge Theory Primer



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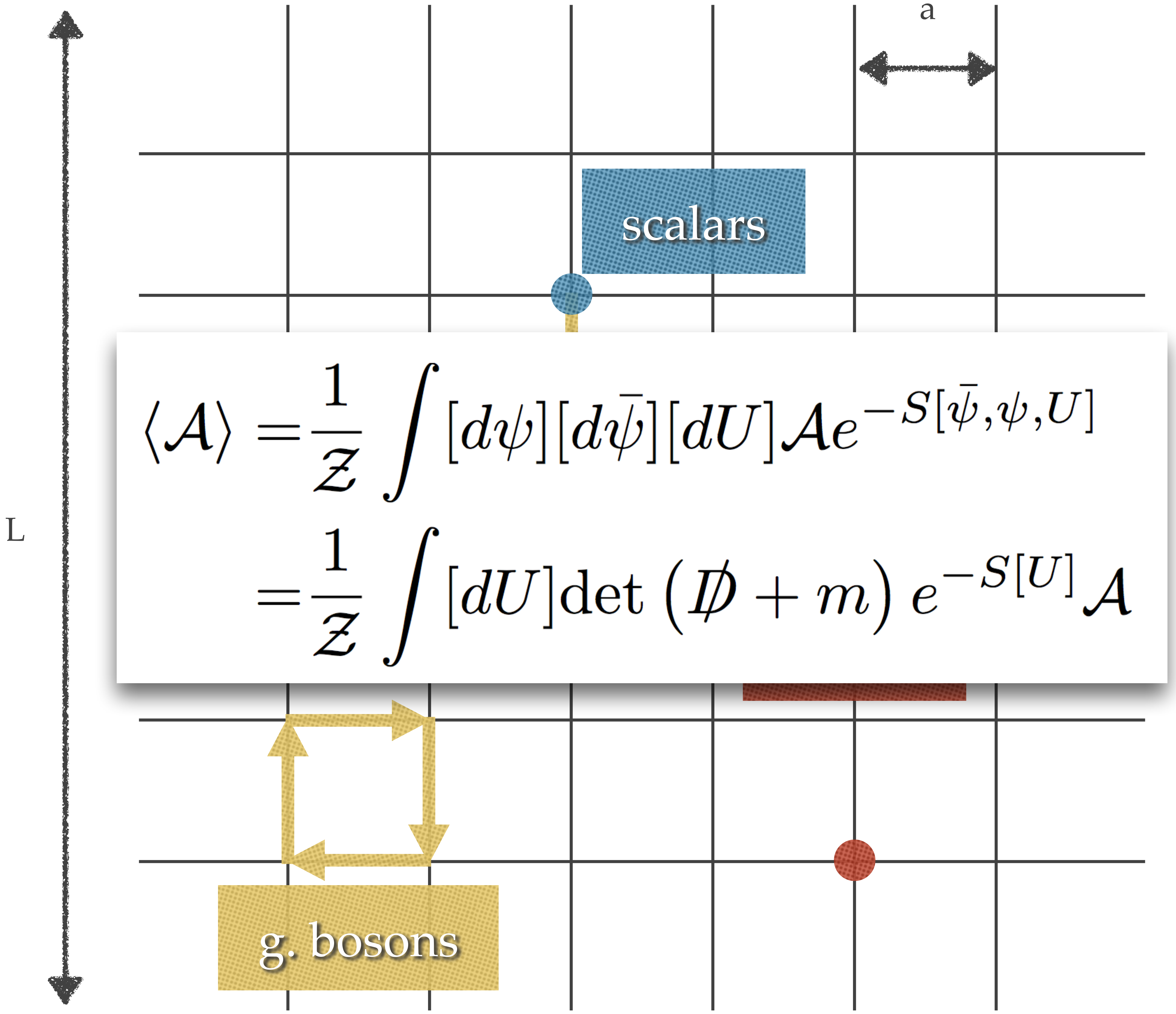
- Precisely quantifiable and improvable errors
  - Systematic
  - Statistical

Path Integral Monte Carlo

Lattice Gauge Theory Primer



$$\langle \mathcal{A} \rangle = \frac{1}{\mathcal{Z}} \int [d\psi][d\bar{\psi}][dU] \mathcal{A} e^{-S[\bar{\psi},\psi,U]}$$

$$= \frac{1}{\mathcal{Z}} \int [dU] \det\left(\not{D} + m\right) e^{-S[U]} \mathcal{A}$$

- Discretize space and time
  - lattice spacing “a”
  - lattice size “L”
- Keep all d.o.f. of the theory
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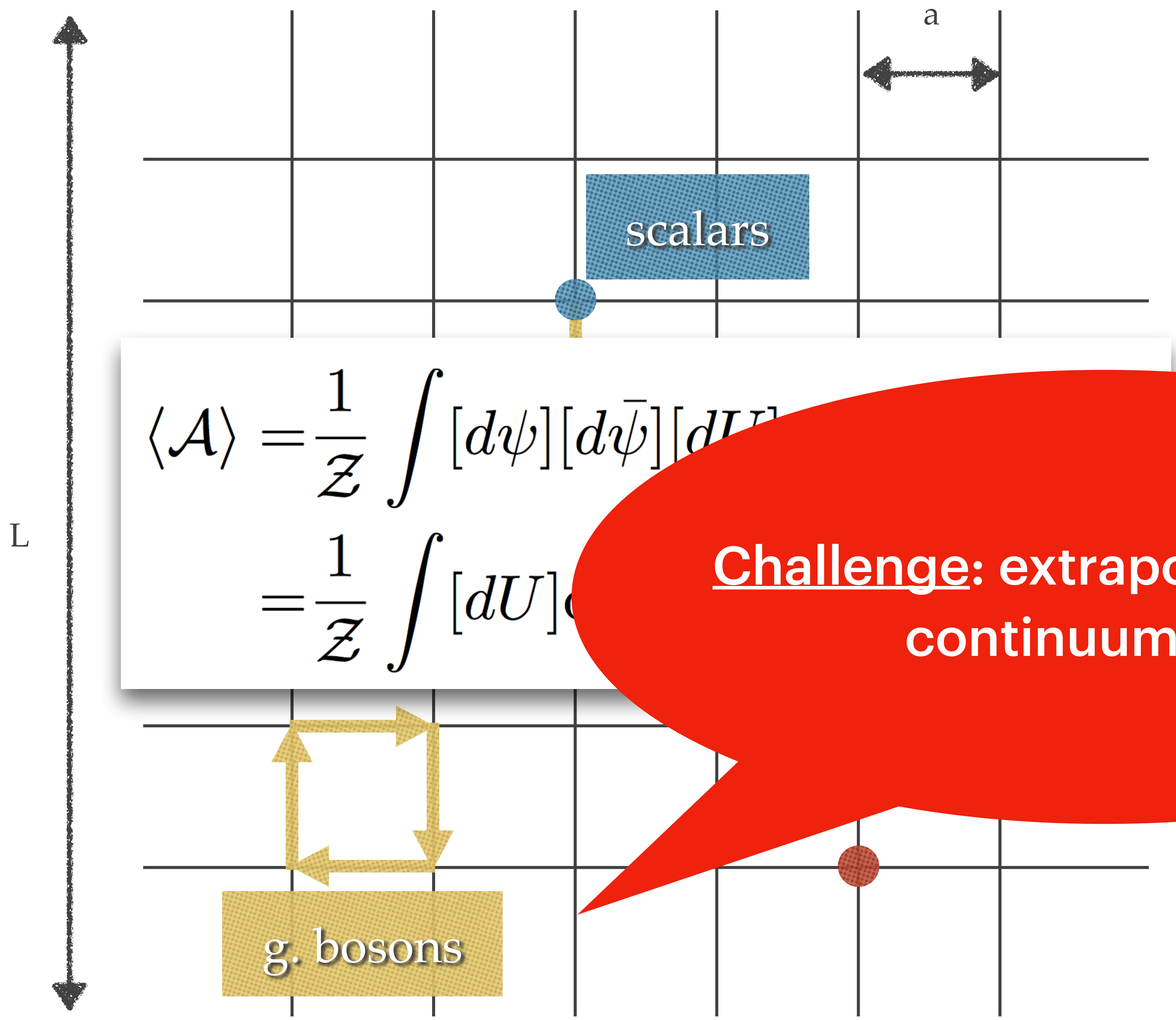
Path Integral Monte Carlo

Lattice Gauge Theory Primer



$$\langle \mathcal{A} \rangle = \frac{1}{\mathcal{Z}} \int [d\psi][d\bar{\psi}][dU] \ldots$$

$$= \frac{1}{\mathcal{Z}} \int [dU] \ldots$$

Challenge: extrapolation to the continuum limit

- Discretize space and time
  - lattice spacing "a"
  - lattice size "L"
- Keep all d.o.f. of the theory
  - not a model!
  - no simplifications
- …menable to numerical …hods
  - Monte Carlo sampling
  - use supercomputers
- Precisely quantifiable and improvable errors
  - Systematic
  - Statistical

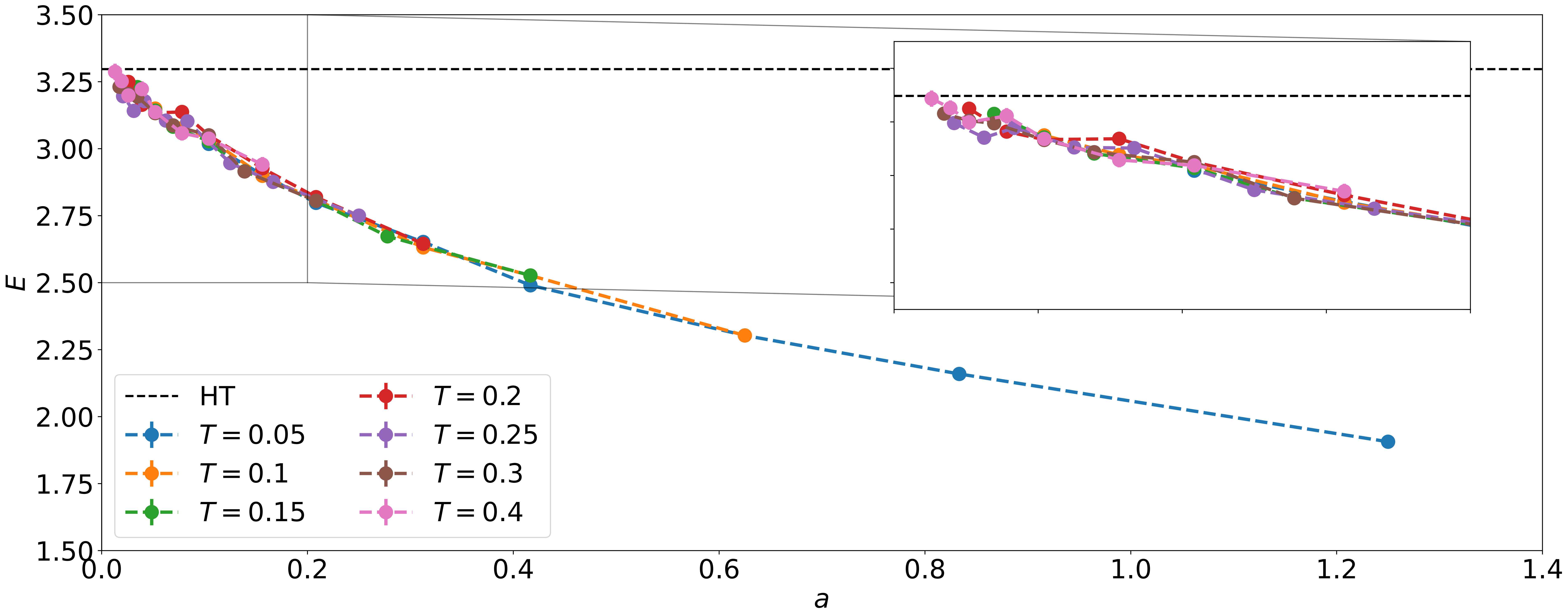
Results

Small-scale: N=2, D=2

No truncation Λ

Parameters:
- Temperature: T
- Number of lattice sites: N_t

Observables:
- Energy



https://github.com/erinaldi/bmn2-lattice

[Rinaldi et al., arxiv:2108.02942]

Results

Small-scale: N=2, D=2

No truncation Λ

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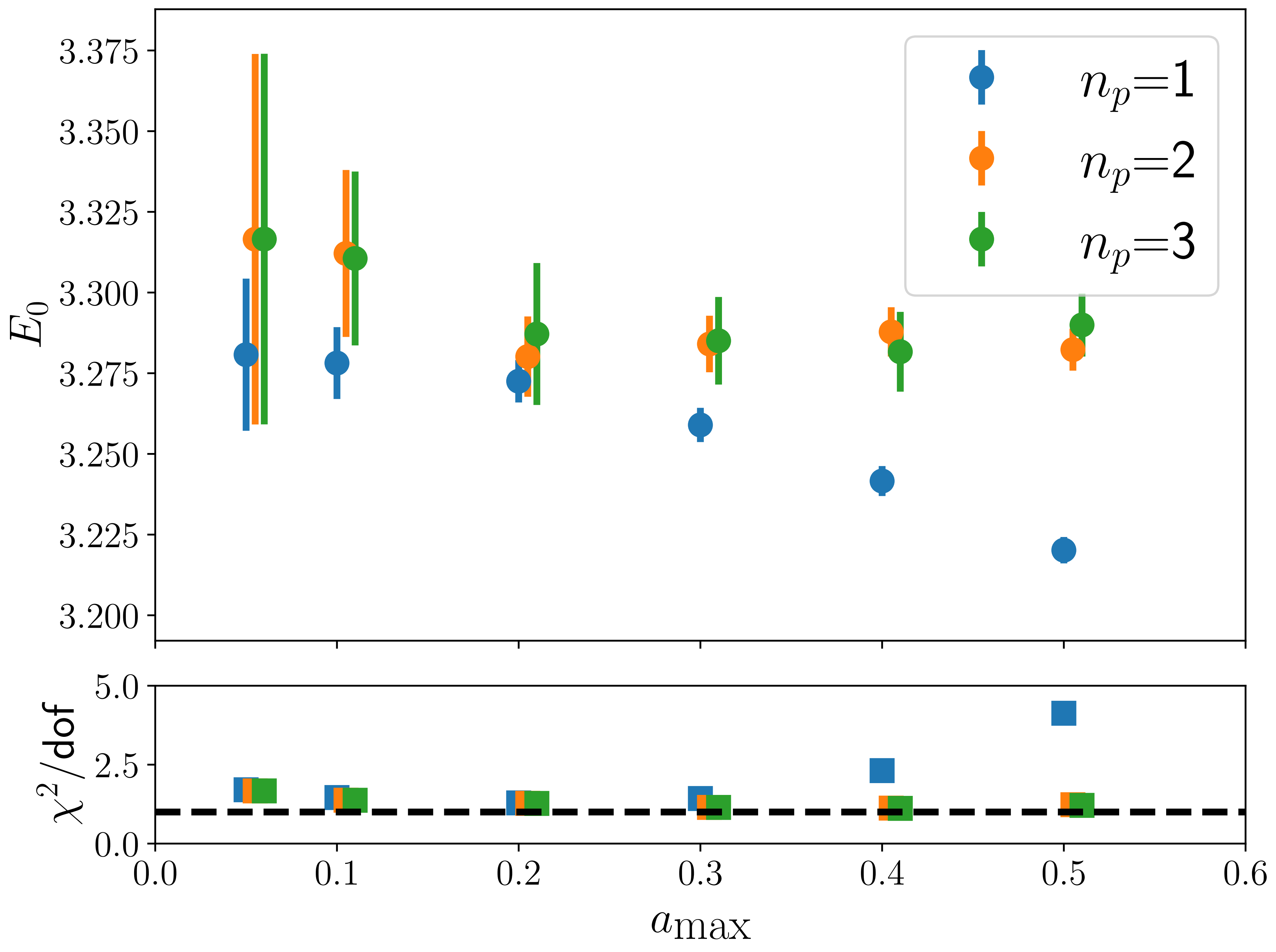
Global Extrapolation



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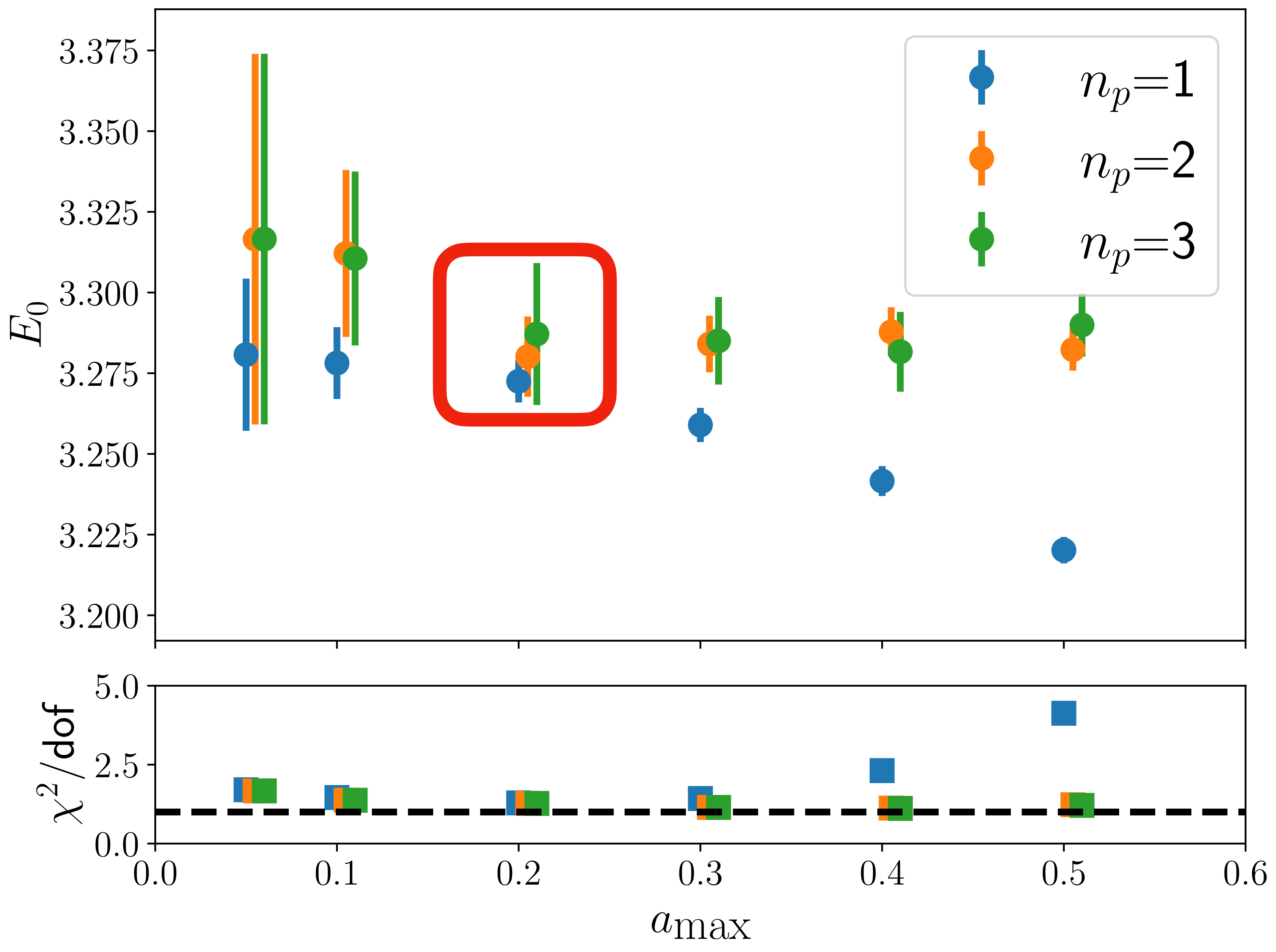

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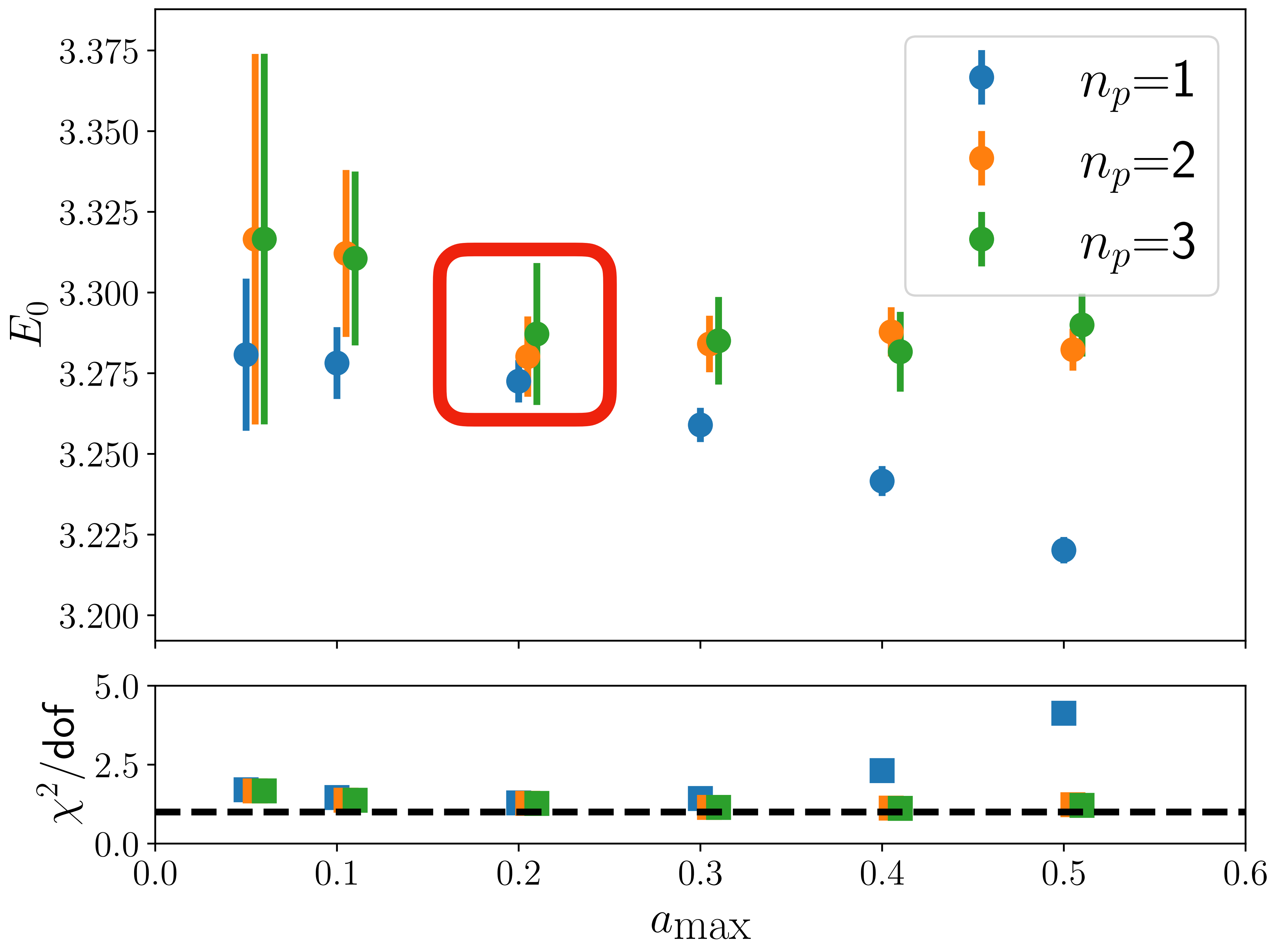
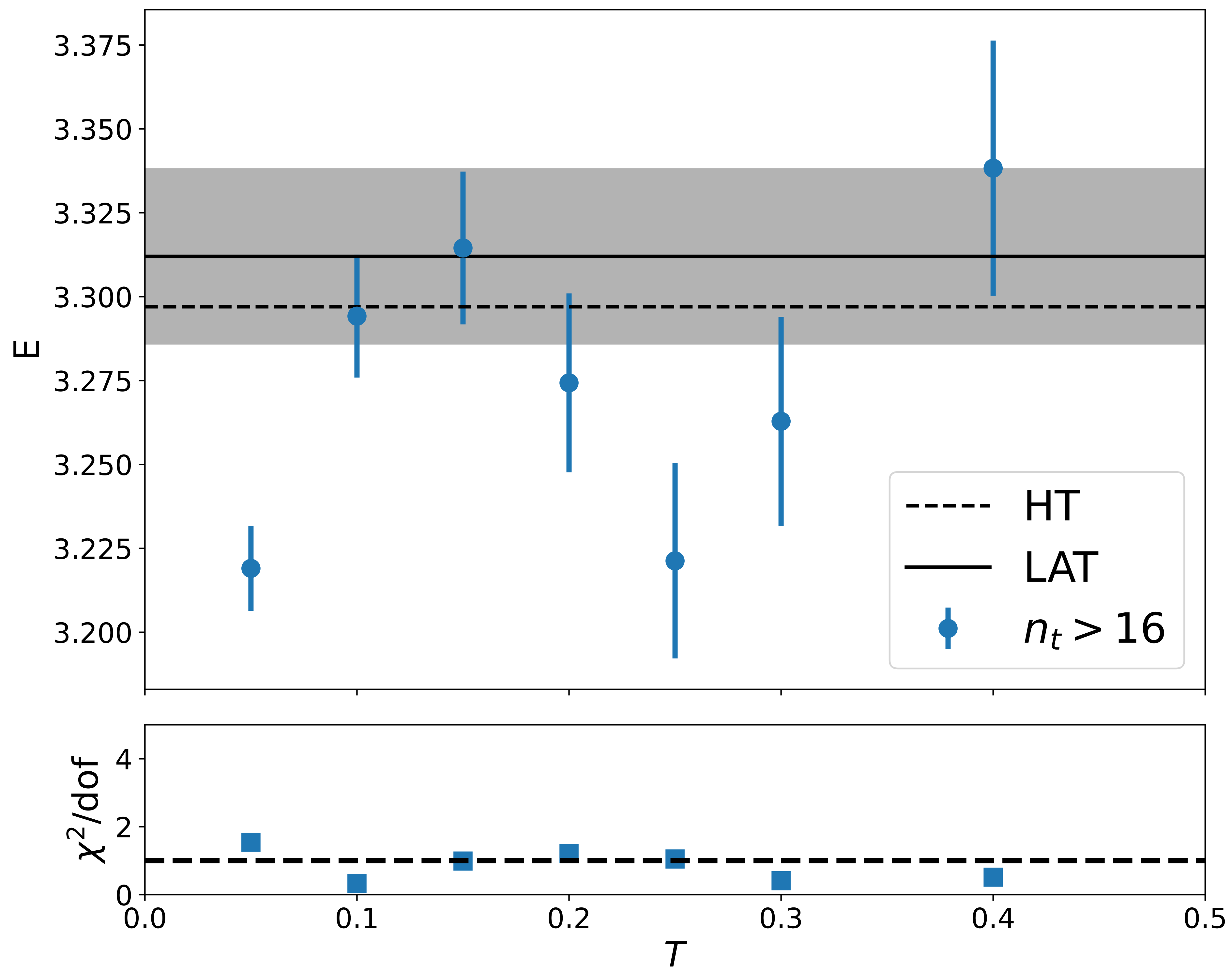


Local Extrapolation



Results

Small-scale: N=2, D=2

No truncation Λ

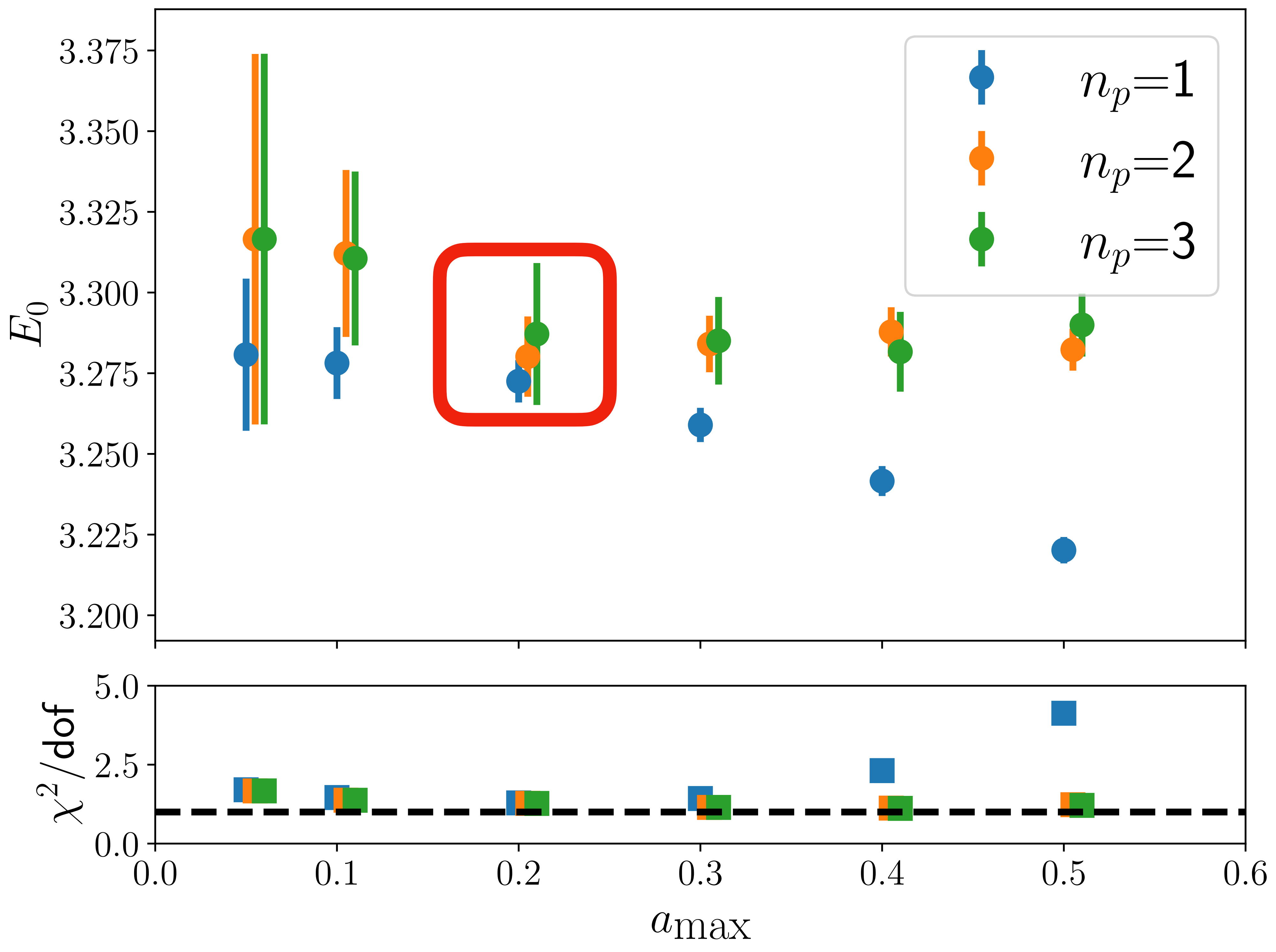
Parameters:

- Temperature: T
- Number of lattice sites: N_t

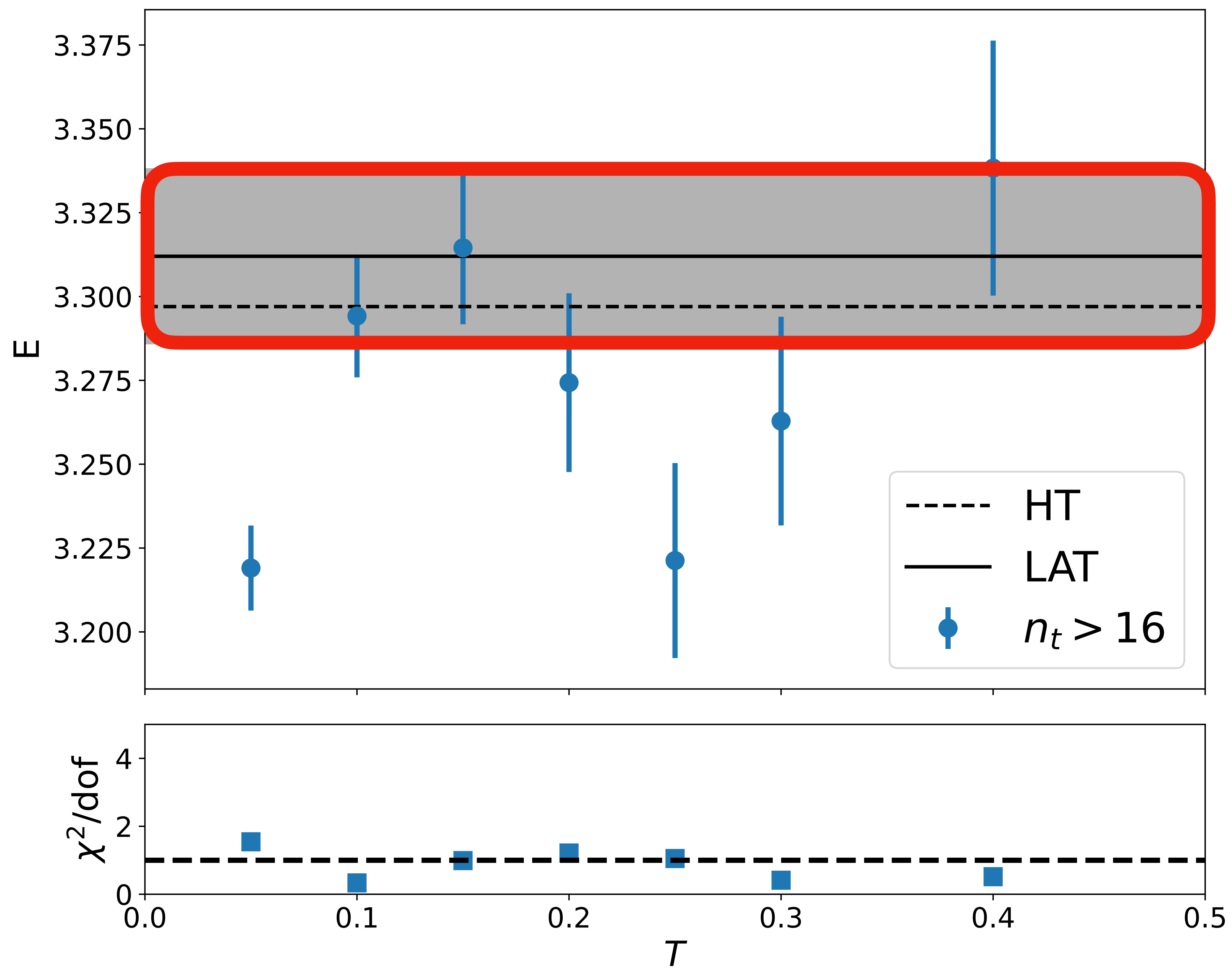
Observables:

- Energy

Global Extrapolation



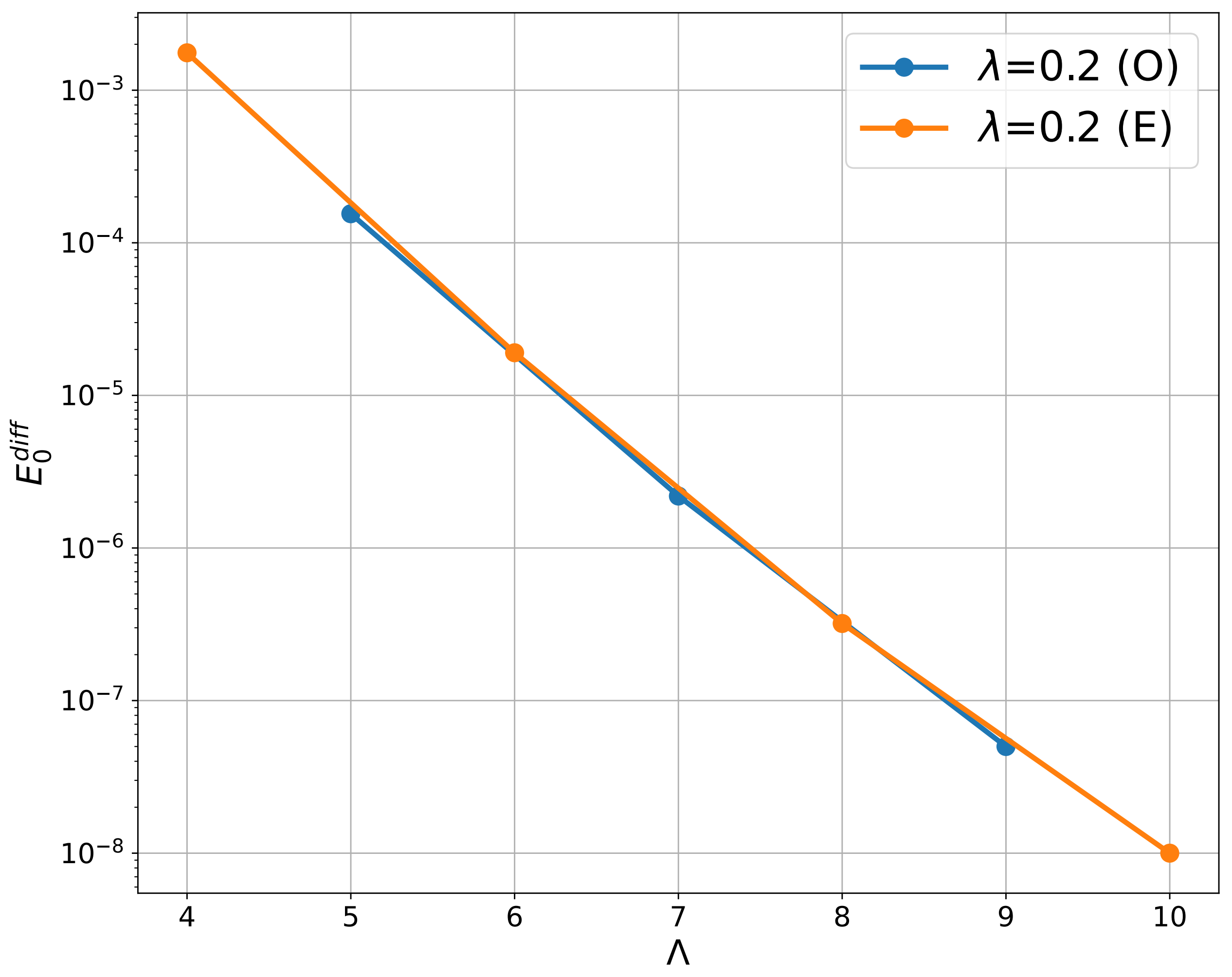
Local Extrapolation

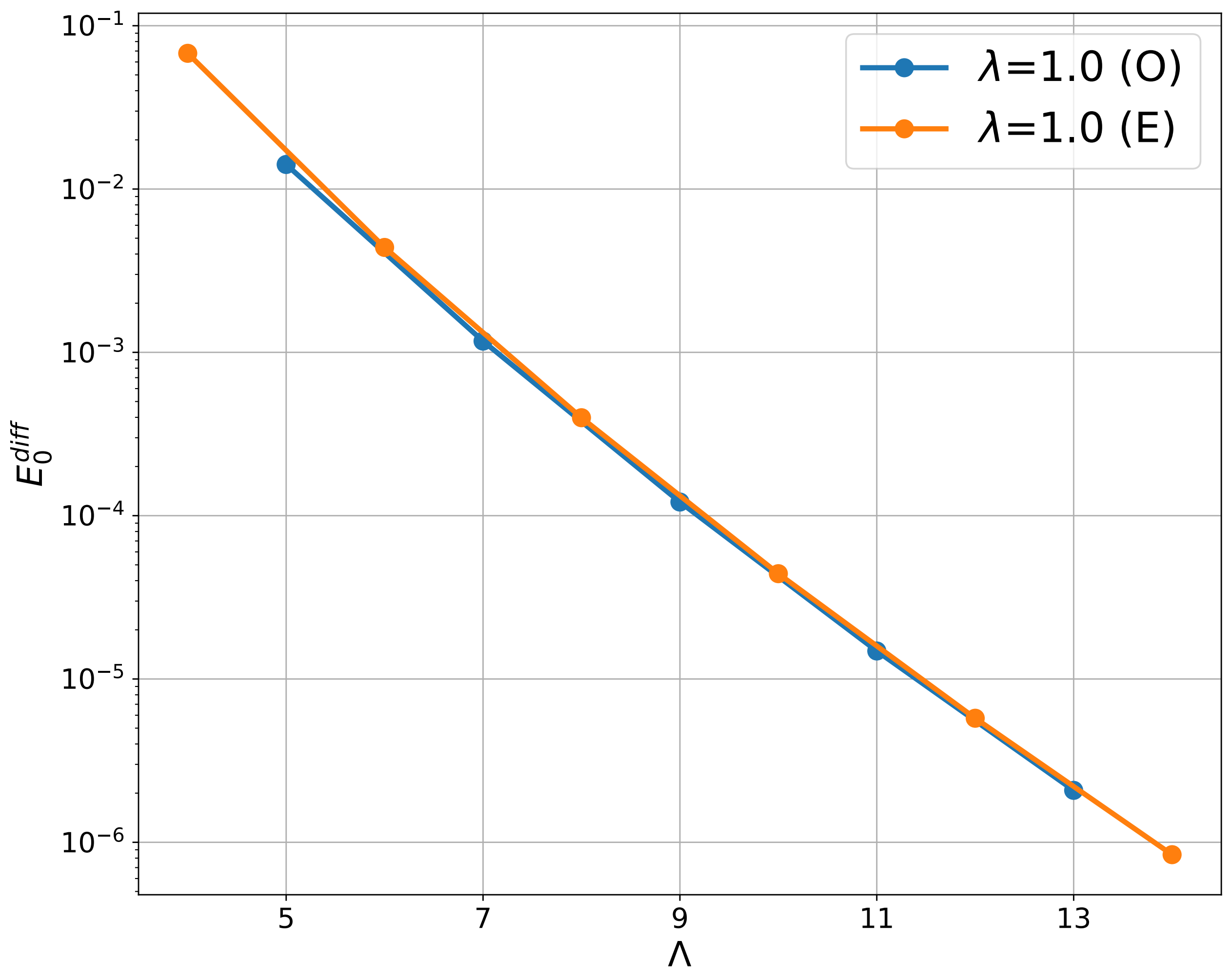


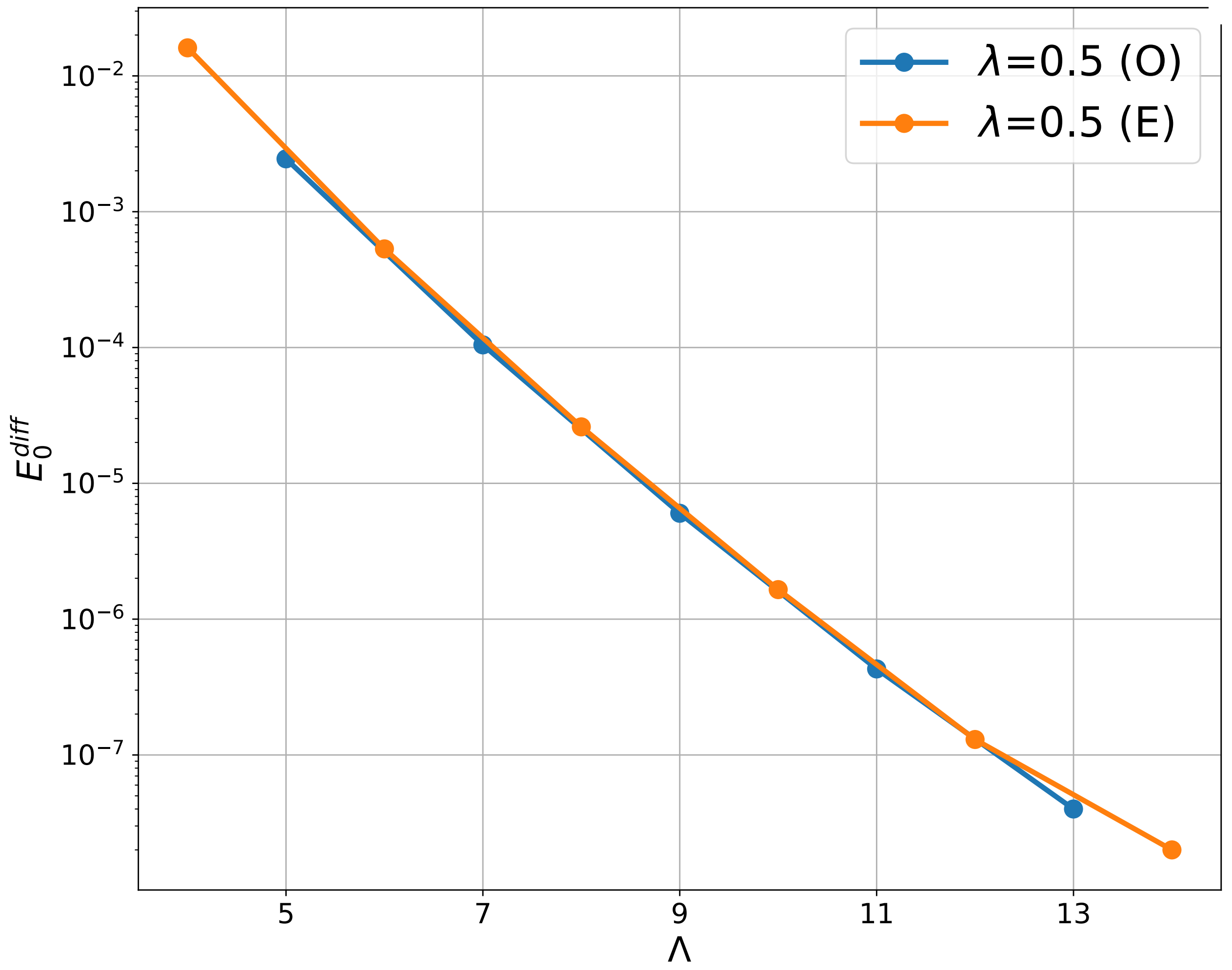
[Rinaldi et al., PRX Quantum 3 (2022) 010324]

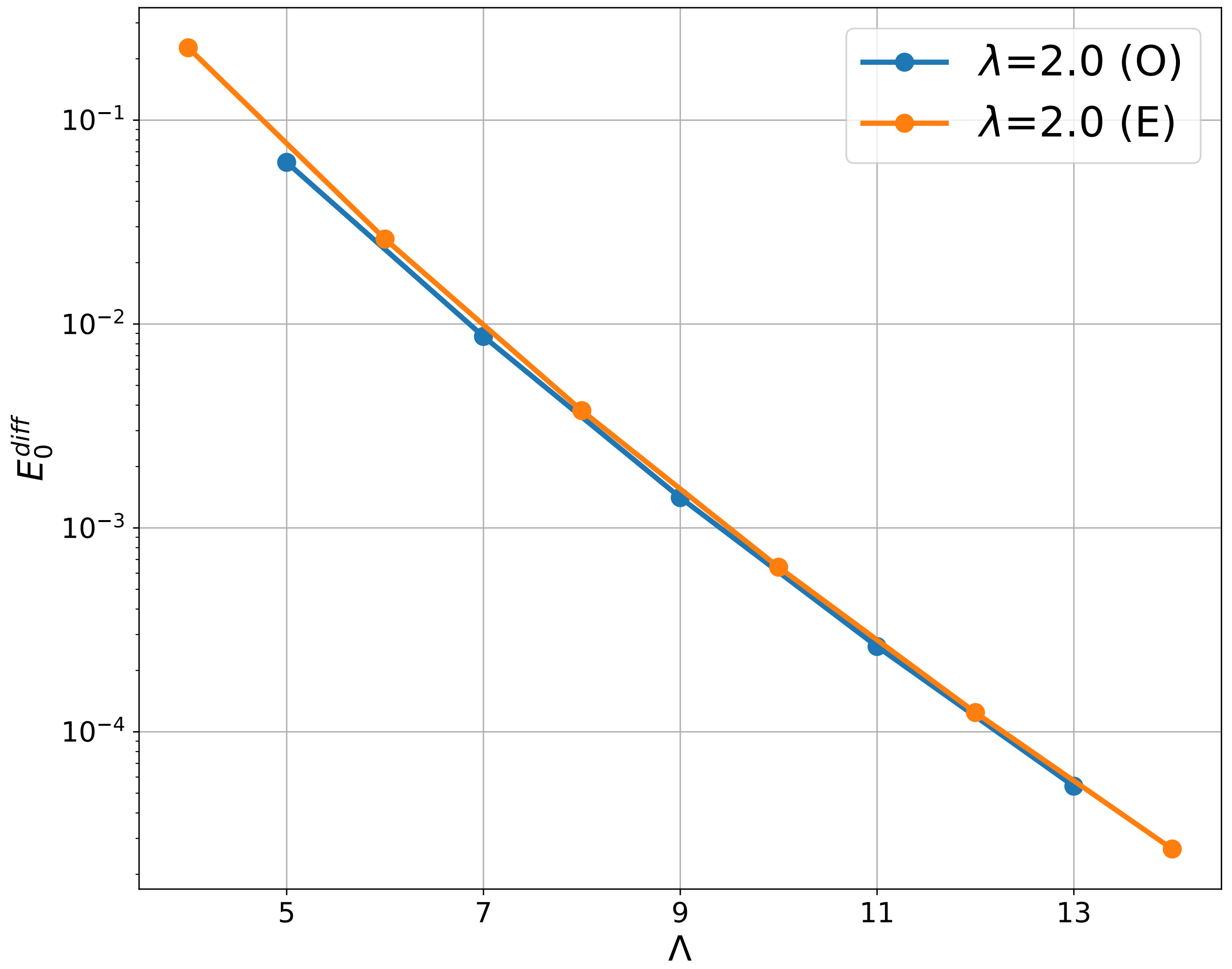
Energy difference

Dependence on Λ









[Rinaldi et al., PRX Quantum 3 (2022) 010324]

Comparison

Benchmarking different methods

Bosonic Model

SU(2)

	$\lambda = 0.5$	$\lambda = 1.0$	$\lambda = 2.0$
$E_{0,\mathrm{HT}}$	3.297	3.516	3.855
$E_{0,\mathrm{DL}}$	3.302(2)	3.519(2)	3.857(3)
$E_{0,\mathrm{MC}}$	3.312(26)	3.497(33)	3.847(30)
$E_{0,\mathrm{VQE}}$	3.309	3.547	3.933

SU(3)

	$\lambda = 0.5$	$\lambda = 1.0$	$\lambda = 2.0$
$E_{0,\mathrm{DL}}$	8.824(7)	9.432(7)	10.426(8)
$E_{0,\mathrm{MC}}$	8.836(38)	9.381(38)	10.236(41)

Supersymmetric Model

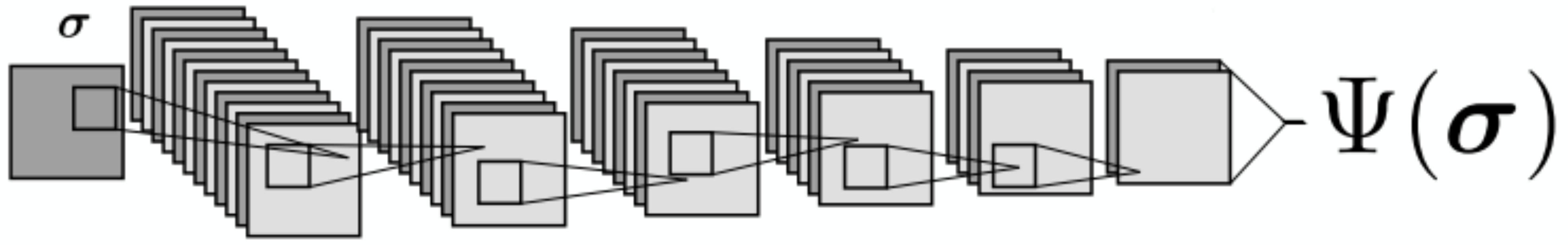
SU(2)

	$\lambda = 0.5$	$\lambda = 1.0$	$\lambda = 2.0$
$E_{0,\mathrm{HT}}$	0.000	0.000	0.000
$E_{0,\mathrm{DL}}$	0.009(5)	0.014(6)	0.034(7)
$E_{0,\mathrm{VQE}}$	0.027	0.079	0.177

[Carleo and Troyer, Science 2017]

Deep Learning

Variational Quantum Monte Carlo with Neural Quantum States



$$\psi_\theta(X) = \langle X \mid \psi_\theta \rangle$$

$$E_\theta \equiv \left\langle \psi_\theta \middle| \hat{H} \middle| \psi_\theta \right\rangle = \int dX \left| \psi_\theta(X) \right|^2 \cdot \frac{\left\langle X \middle| \hat{H} \middle| \psi_\theta \right\rangle}{\psi_\theta(X)} = \mathbf{E}_{X \sim \left|\psi_\theta\right|^2} \left[\epsilon_\theta(X) \right]$$

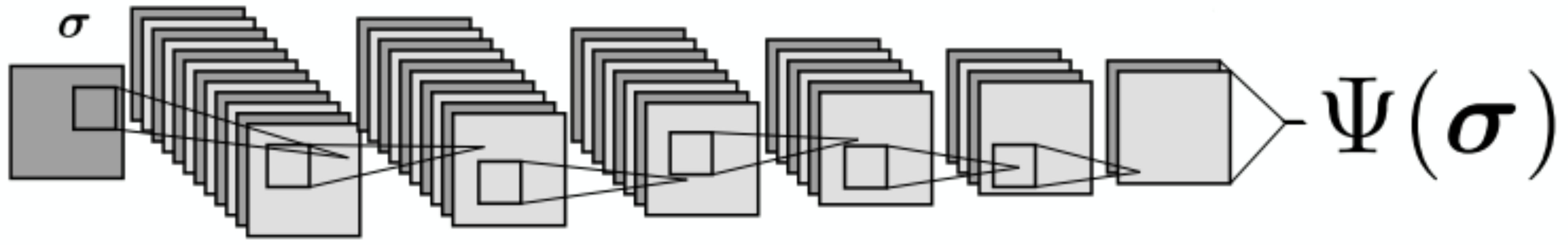
$$\nabla_\theta E_\theta = \mathbf{E}_{X \sim \left|\psi_\theta\right|^2} \left[\nabla_\theta \epsilon_\theta(X) \right] + \mathbf{E}_{X \sim \left|\psi_\theta\right|^2} \left[\epsilon_\theta(X) \, \nabla_\theta \ln \left| \psi_\theta \right|^2 \right]$$

$$\theta' = \theta - \beta \nabla_\theta E_\theta$$

[Carleo and Troyer, Science 2017]

Deep Learning

Variational Quantum Monte Carlo with Neural Quantum States



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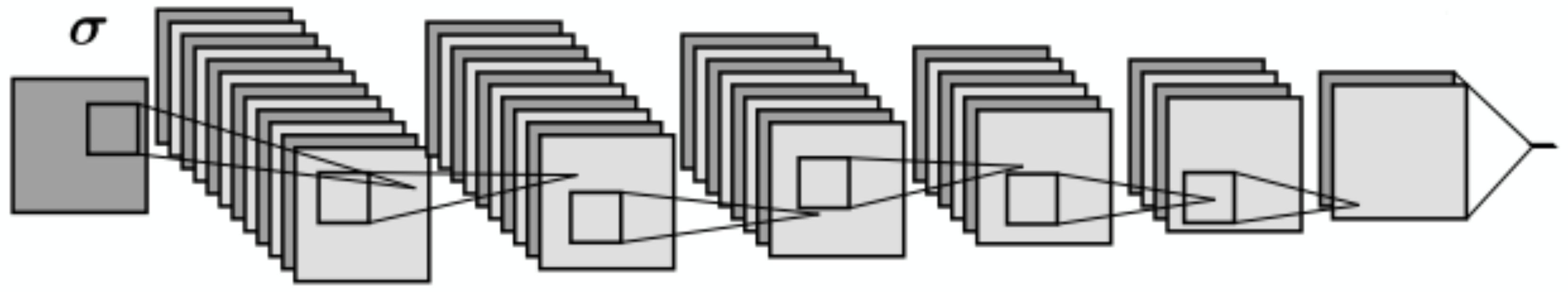
NQS → Variational Ansatz for |Φ〉 → Evaluation of cost function → E(θ) → Optimize parameters → θ*

[Carleo and Troyer, Science 2017]

Deep Learning

Variational Quantum Monte Carlo with Neural Quantum States



$\Psi(\boldsymbol{\sigma})$

$$\psi_\theta(X) = \langle X \mid \psi_\theta \rangle$$

Choice of Neural Network Architecture

$$E_\theta \equiv \langle \psi_\theta | \hat{H} | \psi_\theta \rangle = \int dX \left| \psi_\theta(X) \right|^2 \cdot \frac{\langle X | \hat{H} | \psi_\theta \rangle}{\psi_\theta(X)} = \mathbf{E}_{X \sim |\psi_\theta|^2} \left[\epsilon_\theta(X) \right]$$

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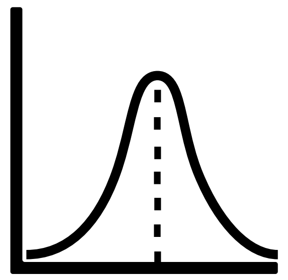
NQS → Variational Ansatz for |Φ⟩

Evaluation of cost function → E(θ)

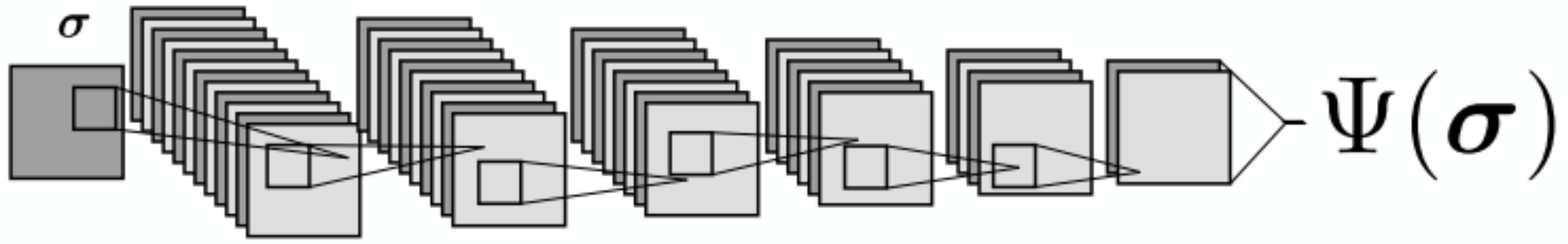
Optimize parameters → θ*

[Carleo and Troyer, Science 2017]



Deep Learning

Variational Quantum Monte Carlo with Neural Quantum States



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Choice of Monte Carlo Sampling

$$\nabla_\theta E_\theta = \mathbf{E}_{X \sim |\psi_\theta|^2} \left[\nabla_\theta \epsilon_\theta(X) \right] + \mathbf{E}_{X \sim |\psi_\theta|^2} \left[\epsilon_\theta(X) \nabla_\theta \ln \left| \psi_\theta \right|^2 \right]$$

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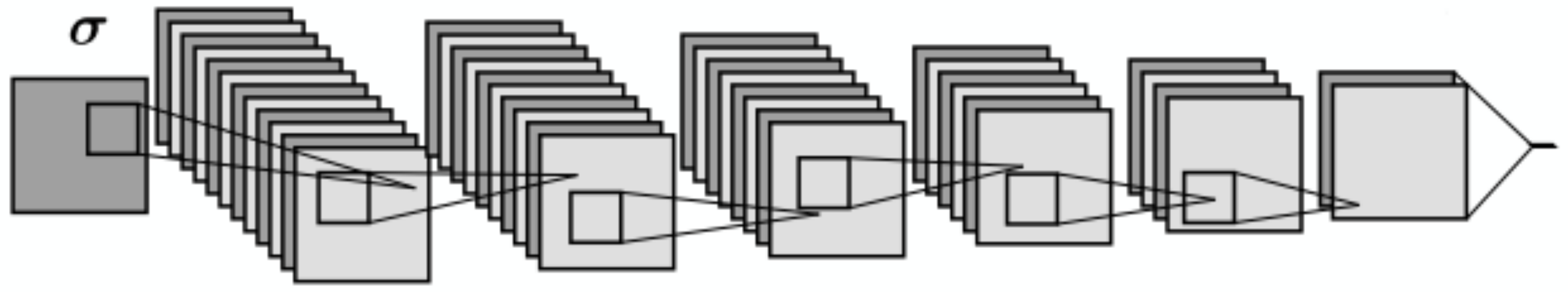
Deep Learning

Variational Quantum Monte Carlo with Neural Quantum States



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$$\nabla_\theta E_\theta = \mathbf{E}_{X \sim |\psi_\theta|^2} \left[\nabla_\theta \epsilon_\theta(X) \right] + \mathbf{E}_{X \sim |\psi_\theta|^2} \left[\epsilon_\theta(X) \nabla_\theta \ln \left| \psi_\theta \right|^2 \right]$$

$$\theta' = \theta - \beta \nabla_\theta E_\theta$$

Choice of learning algorithm

NQS → Variational Ansatz for |Φ〉

Evaluation of cost function → E(θ)

Optimize parameters → θ*

Neural quantum state

Small-scale: N=2, D=2

Wave function

$$\psi(X) = |\psi(X)| e^{i\theta(X)}$$

Neural quantum state

Small-scale: N=2, D=2

Wave function

$$\psi(X) = |\psi(X)|\, e^{i\theta(X)} \longrightarrow |\psi(X)| = \sqrt{p_\theta(X)}$$

[Rinaldi et al., PRX Quantum 3 (2022) 010324]

Neural quantum state

Small-scale: N=2, D=2

Wave function

$$\psi(X) = |\psi(X)|\, e^{i\theta(X)} \longrightarrow |\psi(X)| = \sqrt{p_\theta(X)}$$

$$p_\theta(X) = p\left(x_1; F_\theta^0\right) p\left(x_2; F_\theta^1\left(x_1\right)\right) p\left(x_3; F_\theta^2\left(x_1, x_2\right)\right)\ldots$$

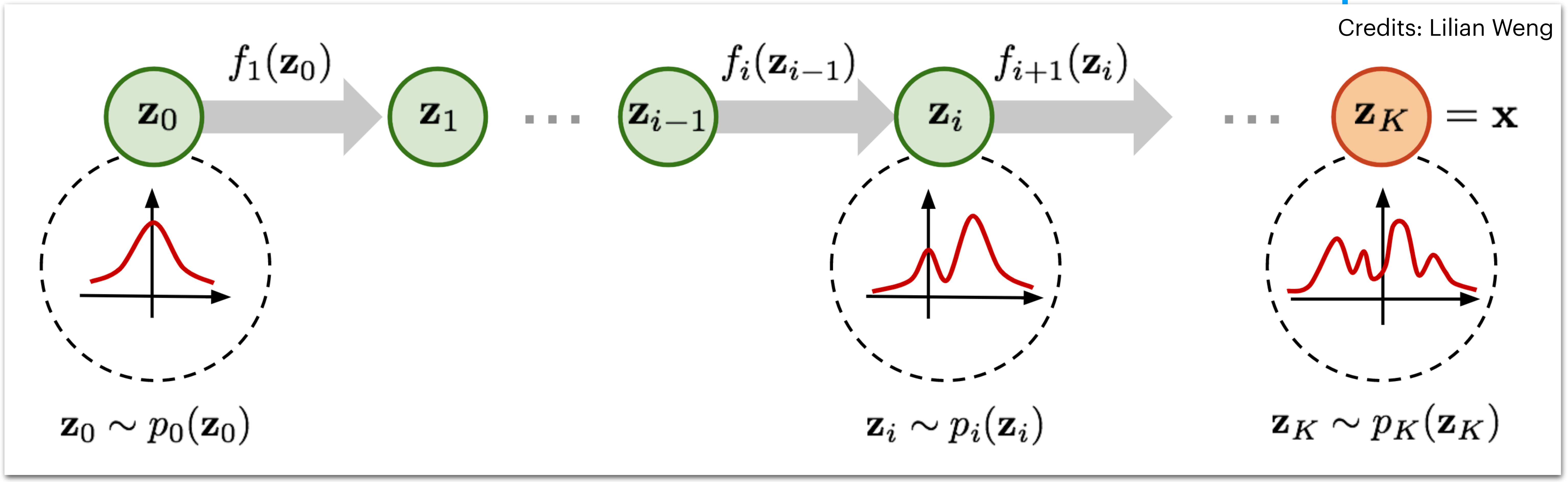
Autoregressive Flow

Neural quantum state

Small-scale: N=2, D=2

Wave function

$$\psi(X) = |\psi(X)|\, e^{i\theta(X)} \longrightarrow |\psi(X)| = \sqrt{p_\theta(X)}$$

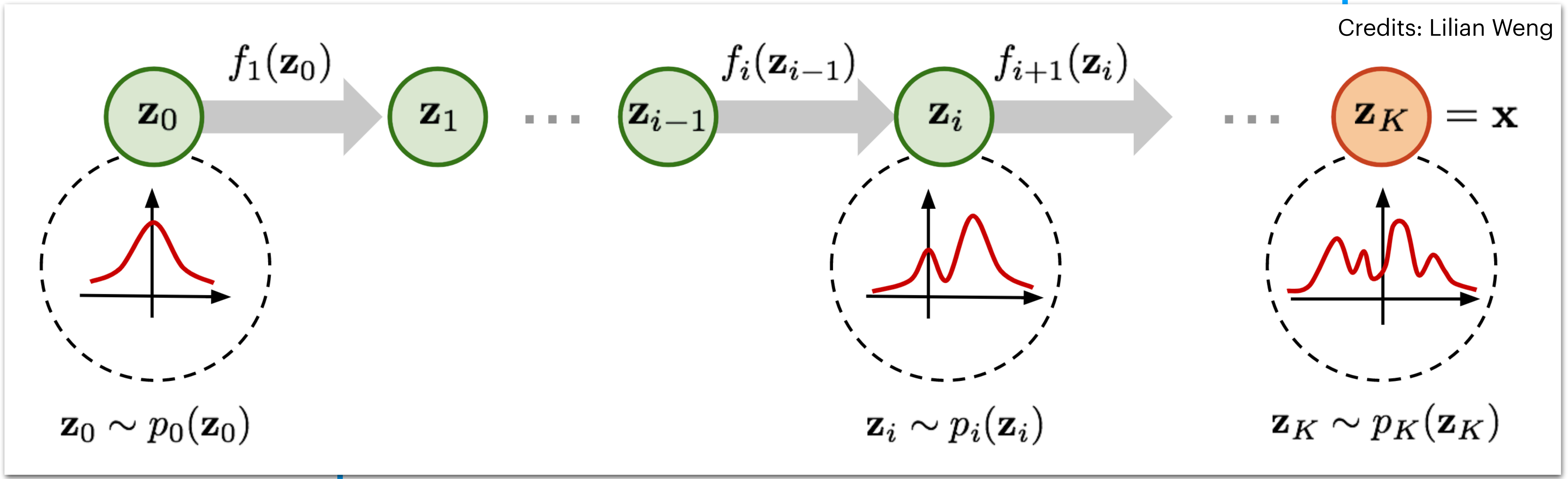


Autoregressive Flow

Neural quantum state

Small-scale: N=2, D=2

Wave function $\psi(X) = |\psi(X)|e^{i\theta(X)}$ → $|\psi(X)| = \sqrt{p_\theta(X)}$



Autoregressive Flow

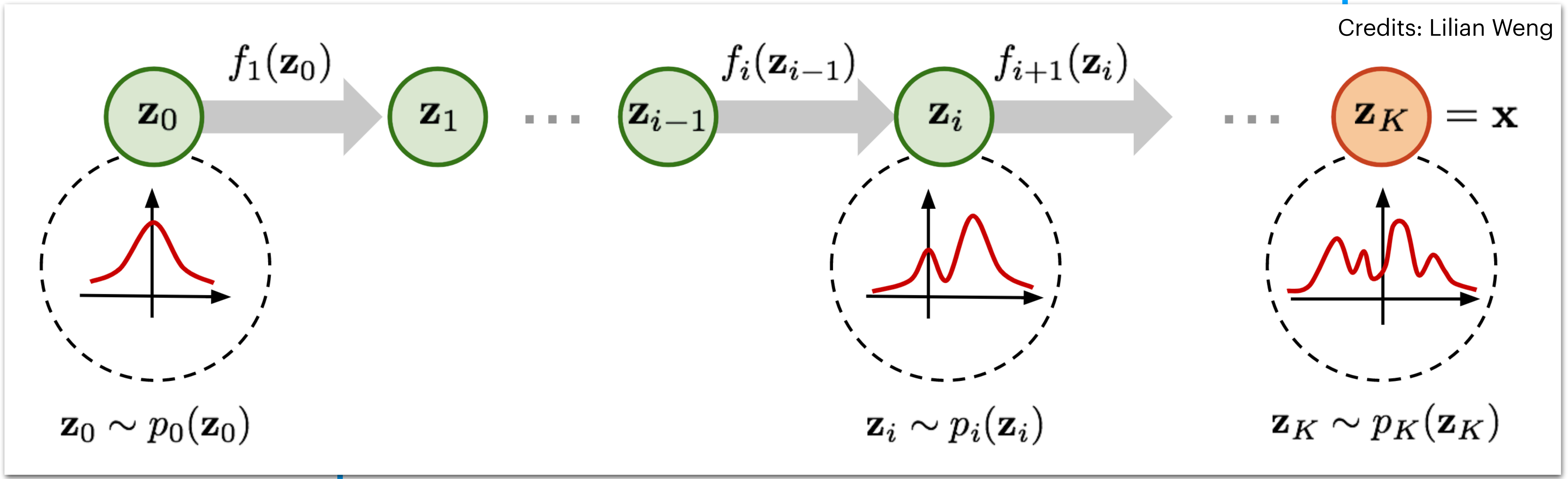
Prob. distribution $p\left(x_1; F_\theta^0\right)$

Neural quantum state

Small-scale: N=2, D=2

Wave function

$$\psi(X) = |\psi(X)| e^{i\theta(X)} \longrightarrow |\psi(X)| = \sqrt{p_\theta(X)}$$



Autoregressive Flow

Parametrization

Prob. distribution

$$p\left(x_1; F_\theta^0\right) \longrightarrow F_\theta^i = A_\theta^{i,m} \circ \tanh \circ A_\theta^{i,m-1} \circ \tanh \circ \cdots \circ A_\theta^{i,2} \circ \tanh \circ A_\theta^{i,1}$$

$$A_\theta^{i,a}(\vec{x}) = M_\theta^{i,a} \cdot \vec{x} + \vec{b}_\theta^{i,a}$$

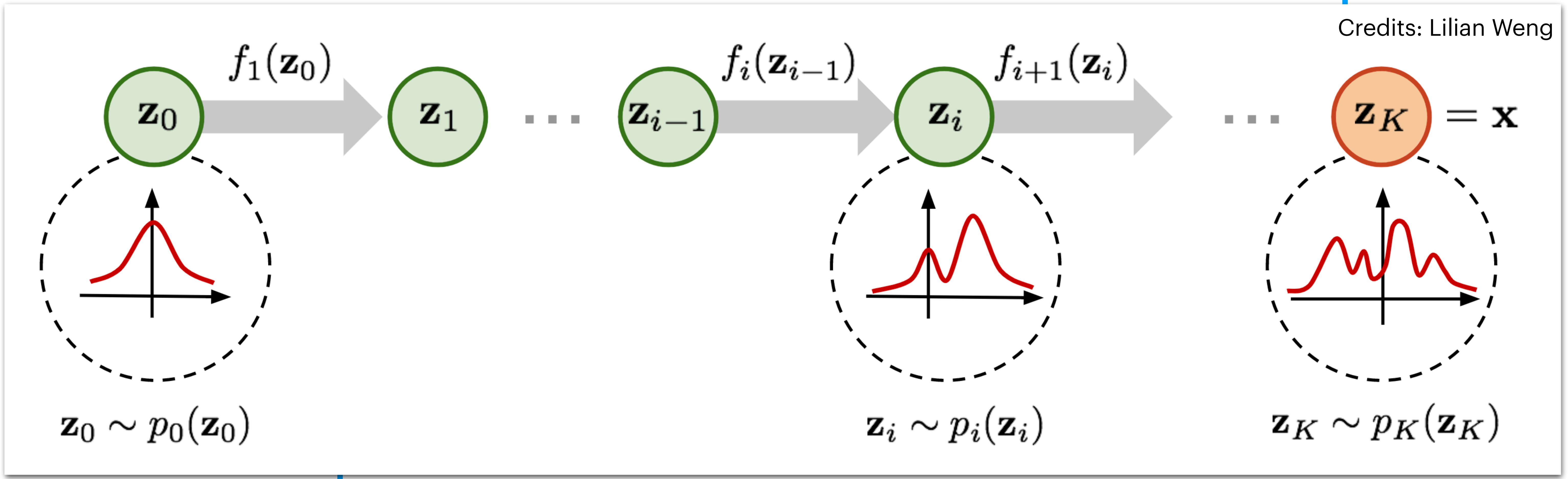
Neural quantum state

Small-scale: N=2, D=2

No truncation Λ

Wave function

$$\psi(X) = |\psi(X)|e^{i\theta(X)} \longrightarrow |\psi(X)| = \sqrt{p_\theta(X)}$$



Autoregressive Flow

Parametrization

Prob. distribution

$$p\left(x_1; F_\theta^0\right) \longrightarrow F_\theta^i = A_\theta^{i,m} \circ \tanh \circ A_\theta^{i,m-1} \circ \tanh \circ \cdots \circ A_\theta^{i,2} \circ \tanh \circ A_\theta^{i,1}$$

$$A_\theta^{i,a}(\vec{x}) = M_\theta^{i,a} \cdot \vec{x} + \vec{b}_\theta^{i,a}$$

https://github.com/hanxzh94/minimal_BMN_MQM

[Rinaldi et al., PRX Quantum 3 (2022) 010324]

Neural quantum state

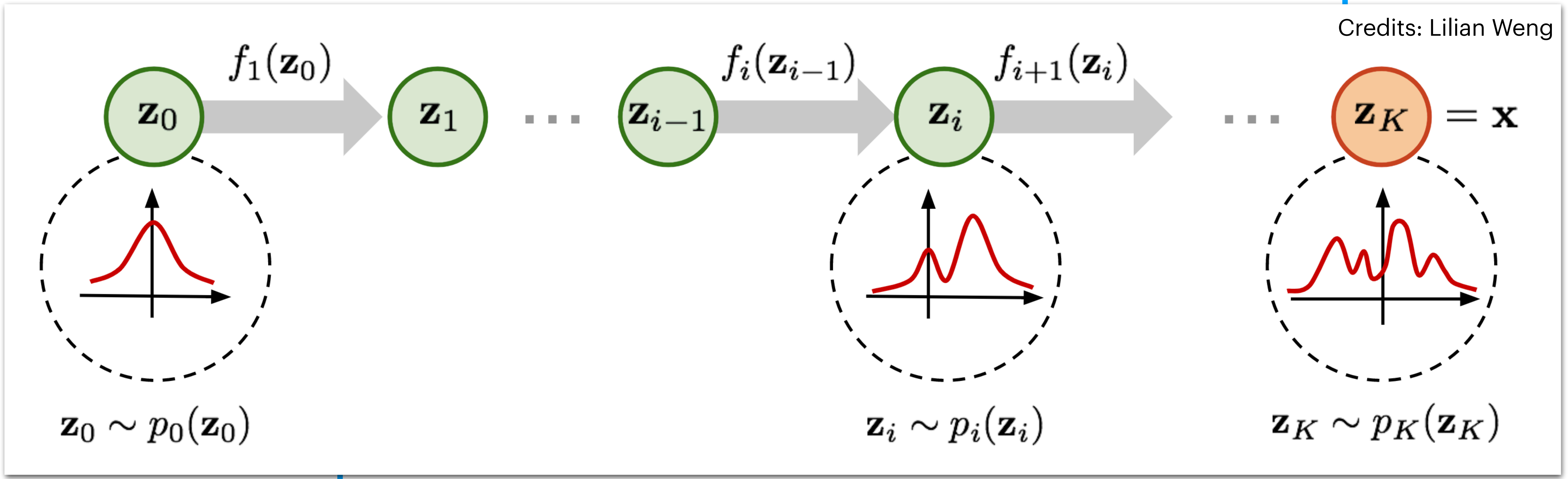
Small-scale: N=2, D=2

No truncation Λ

Direct Sampling

Wave function

$$\psi(X) = |\psi(X)|\, e^{i\theta(X)} \longrightarrow |\psi(X)| = \sqrt{p_\theta(X)}$$



Autoregressive Flow

Parametrization

Prob. distribution

$$p\left(x_1; F_\theta^0\right) \longrightarrow F_\theta^i = A_\theta^{i,m} \circ \tanh \circ A_\theta^{i,m-1} \circ \tanh \circ \cdots \circ A_\theta^{i,2} \circ \tanh \circ A_\theta^{i,1}$$

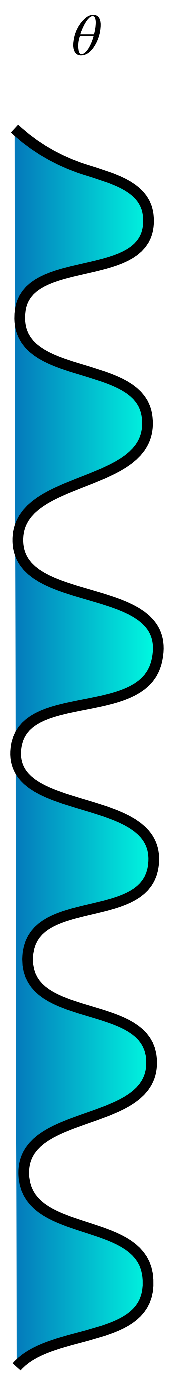
$$A_\theta^{i,a}(\vec{x}) = M_\theta^{i,a} \cdot \vec{x} + \vec{b}_\theta^{i,a}$$

Optimization

Small-scale: N=2, D=2

$$|\psi(X)| = \sqrt{p_\theta(X)}$$

$$X = [x_1, \ldots, x_6]$$



NQS → Variational Ansatz for |Φ 〉

Optimization

Small-scale: N=2, D=2

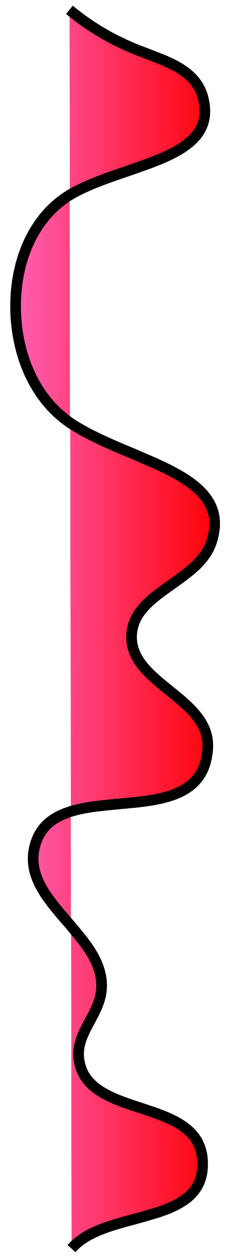
θ

$$|\psi(X)| = \sqrt{p_\theta(X)}$$

$X = [x_1, \ldots, x_6]$

$$E_{\theta'} = \mathbf{E}_{X \sim \left|\psi_{\theta'}\right|^2} \left[\epsilon_{\theta'}(X)\right]$$

θ'



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Evaluation of cost function → E(θ)

https://github.com/hanxzh94/minimal_BMN_MQM

[Rinaldi et al., PRX Quantum 3 (2022) 010324]

Optimization

Small-scale: N=2, D=2

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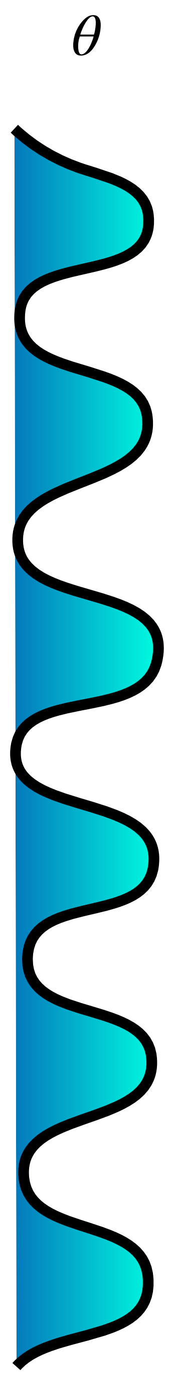
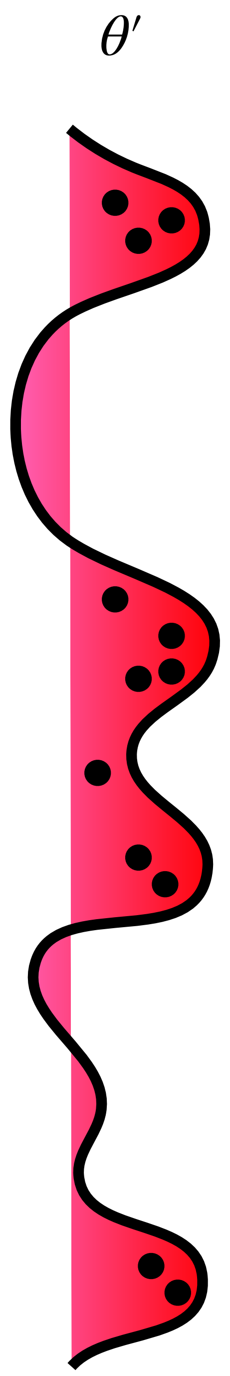


$$E_{\theta'} = \mathbf{E}_{X \sim |\psi_{\theta'}|^2} \left[\epsilon_{\theta'}(X)\right]$$



NQS → Variational Ansatz for |Φ〉

Evaluation of cost function → E(θ)

Optimization

Small-scale: N=2, D=2

θ

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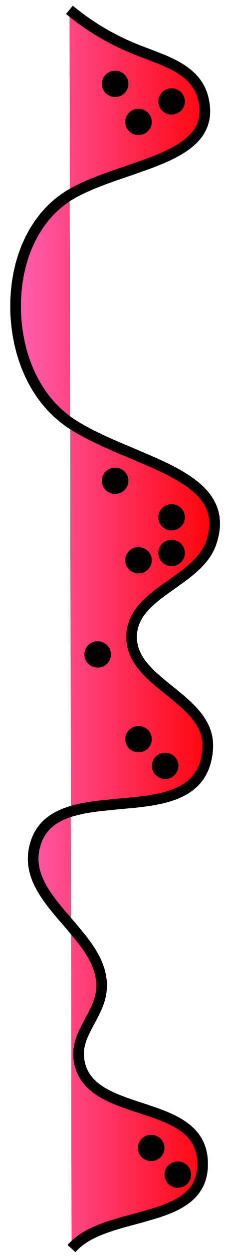
→

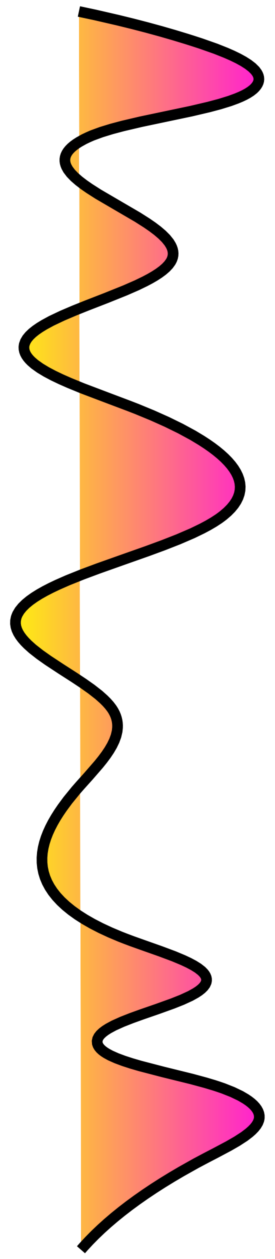
θ'

$$E_{\theta'} = \mathbf{E}_{X \sim |\psi_{\theta'}|^2}\left[\epsilon_{\theta'}(X)\right]$$



→

$\theta^\star$



$$|\text{Ground State}\rangle = \psi_{\theta^\star}(X)$$

$$E_{\theta^\star} = \mathbf{E}_{X \sim |\psi_{\theta^\star}|^2}\left[\epsilon_{\theta^\star}(X)\right]$$

NQS → Variational Ansatz for |Φ〉 → Evaluation of cost function → E(θ) → Optimize parameters → θ*

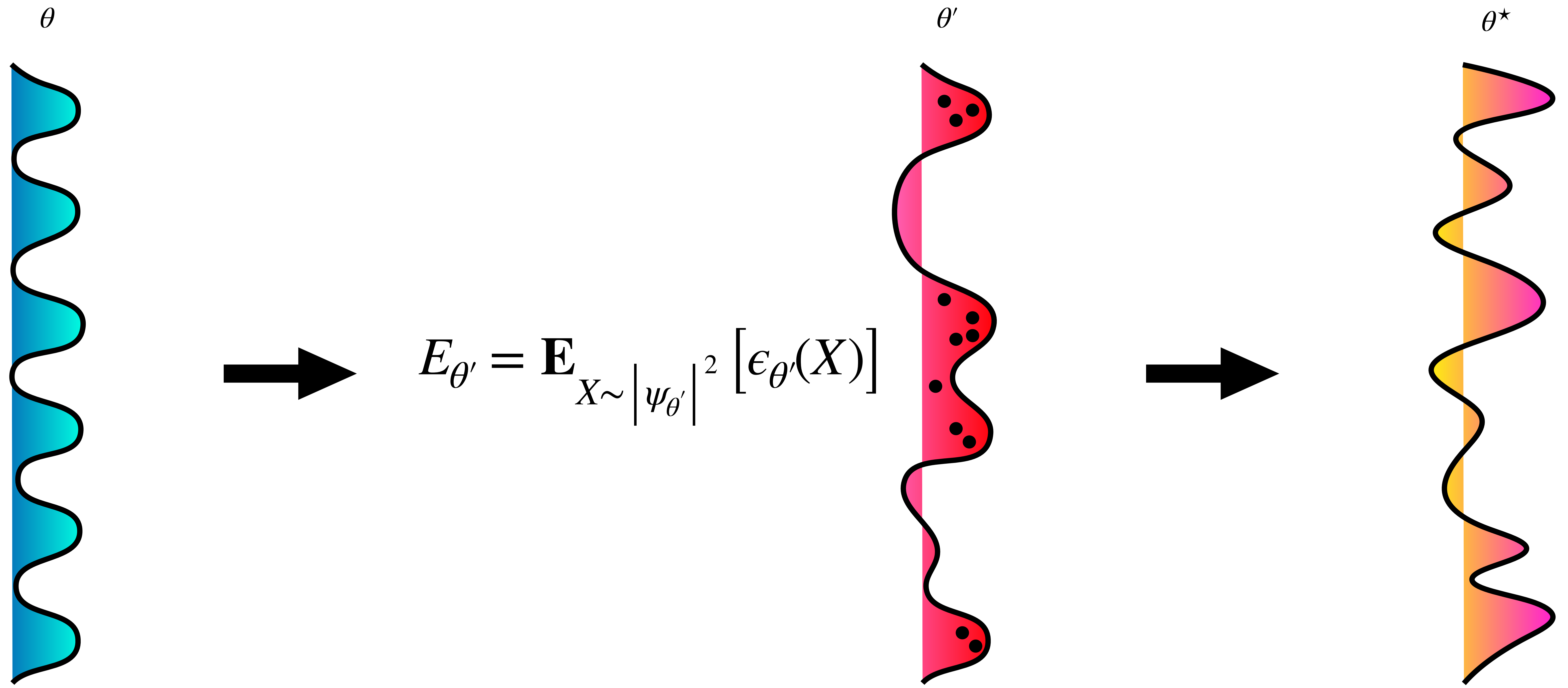
[Rinaldi et al., PRX Quantum 3 (2022) 010324]

Optimization

Small-scale: N=2, D=2

$$|\psi(X)| = \sqrt{p_\theta(X)}$$

$$X = [x_1, \ldots, x_6]$$



$$E_{\theta'} = \mathbf{E}_{X \sim |\psi_{\theta'}|^2} \left[\epsilon_{\theta'}(X) \right]$$

$$|\text{Ground State}\rangle = \psi_{\theta^\star}(X)$$

$$E_{\theta^\star} = \mathbf{E}_{X \sim |\psi_{\theta^\star}|^2} \left[\epsilon_{\theta^\star}(X) \right]$$

Results

Small-scale: N=2, D=2

$$\hat{H}' = \hat{H} + c \sum \hat{G}_\alpha^2$$

	$c = 0$			
λ	0.2	0.5	1.0	2.0
$E_{0,\mathrm{var}}$	3.137(2)	3.299(2)	3.518(2)	3.856(3)
G^2_{var}	0.0028(4)	0.0059(6)	0.0062(7)	0.0122(9)
$E_{0,\mathrm{singlet}}$	3.135(2)	3.297(2)	3.520(2)	3.859(3)

	$c = 10$			
λ	0.2	0.5	1.0	2.0
$E_{0,\mathrm{var}}$	3.137(2)	3.309(2)	3.545(3)	3.912(3)
G^2_{var}	0.00011(8)	0.00019(7)	0.00022(11)	0.00021(8)
$E_{0,\mathrm{singlet}}$	3.139(2)	3.307(2)	3.544(3)	3.908(3)

Results

Small-scale: N=2, D=2

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dependence on hidden layer units α

α	1	2	5	10	20	50	HT (exact)
$\lambda = 0.2$	3.137(2)	3.137(2)	3.140(2)	3.138(2)	3.137(2)	3.135(2)	3.134
$\lambda = 0.5$	3.313(2)	3.312(2)	3.308(2)	3.307(2)	3.302(2)	3.305(2)	3.297
$\lambda = 1.0$	3.544(3)	3.544(2)	3.541(3)	3.528(2)	3.519(2)	3.520(2)	3.516
$\lambda = 2.0$	3.914(3)	3.910(3)	3.892(3)	3.872(3)	3.857(3)	3.859(3)	3.854

https://github.com/hanxzh94/minimal_BMN_MQM

[Rinaldi et al., PRX Quantum 3 (2022) 010324]

Results

Small-scale: N=2, D=2

$$\hat{H}' = \hat{H} + c \sum \hat{G}_\alpha^2$$

	$c = 0$			
λ	0.2	0.5	1.0	2.0
$E_{0,\mathrm{var}}$	3.137(2)	3.299(2)	3.518(2)	3.856(3)
G^2_{var}	0.0028(4)	0.0059(6)	0.0062(7)	0.0122(9)
$E_{0,\mathrm{singlet}}$	3.135(2)	3.297(2)	3.520(2)	3.859(3)

	$c = 10$			
λ	0.2	0.5	1.0	2.0
$E_{0,\mathrm{var}}$	3.137(2)	3.309(2)	3.545(3)	3.912(3)
G^2_{var}	0.00011(8)	0.00019(7)	0.00022(11)	0.00021(8)
$E_{0,\mathrm{singlet}}$	3.139(2)	3.307(2)	3.544(3)	3.908(3)

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$\lambda = 1.0$	3.544(3)	3.544(2)	3.541(3)	3.528(2)	3.519(2)	3.520(2)	3.516
$\lambda = 2.0$	3.914(3)	3.910(3)	3.892(3)	3.872(3)	3.857(3)	3.859(3)	3.854

Results

Supersymmetric: N=2, D=2

dependence on hidden layer units α

$\lambda = 1.0$

α	1	2	5	10	20	50	HT (exact)
H	0.058(6)	0.053(6)	0.041(6)	0.031(6)	0.014(6)	0.005(6)	0.000
G^2	0.007(8)	−0.008(8)	0.014(8)	0.007(9)	0.022(9)	0.012(9)	0.000
M	−0.0003(3)	−0.0004(3)	−0.0001(4)	0.0001(4)	−0.0003(5)	−0.0001(4)	0.0000
F	0.1844(6)	0.1833(6)	0.1895(6)	0.1922(6)	0.1946(7)	0.1935(7)	0.2034

Strong coupling

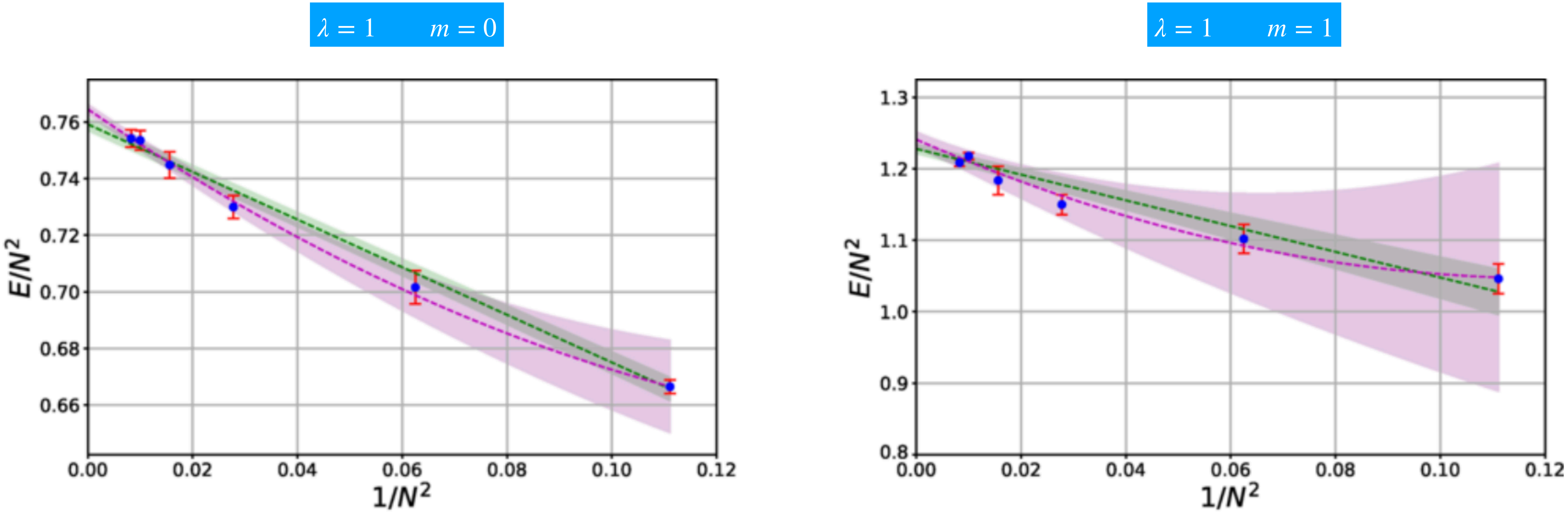
	$\lambda = 0.5$	$\lambda = 1.0$	$\lambda = 2.0$
H	0.009(5)	0.014(6)	0.034(7)
G^2	0.010(6)	0.022(9)	0.038(14)
M	−0.0002(3)	−0.0003(5)	0.0006(7)
F	0.1224(4)	0.1946(7)	0.2729(9)

https://github.com/hanxzh94/minimal_BMN_MQM

[Bodendorfer et al., Phys. Rev. D 112, 046010]

Results

Large N limit: N=3,4,6,8,10,11 with D=2



Extrapolated results are not compatible with the Path Integral Monte Carlo ones