

Sommerfeld effect and unitarity

Ryosuke Sato



K. Blum, R. Sato, T. R. Slatyer, 1603.01383, JCAP 06 (2016) 021

A. Parikh, R. Sato, T. R. Slatyer, 2410.18168

2025. 6. 19 @ The Frontier of Particle Physics:

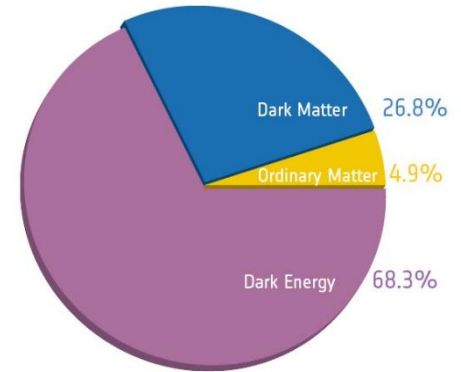
Exploring Muons, Quantum Science and the Cosmos

Plan

1. Dark matter and Sommerfeld effect
2. Sommerfeld effect and unitarity

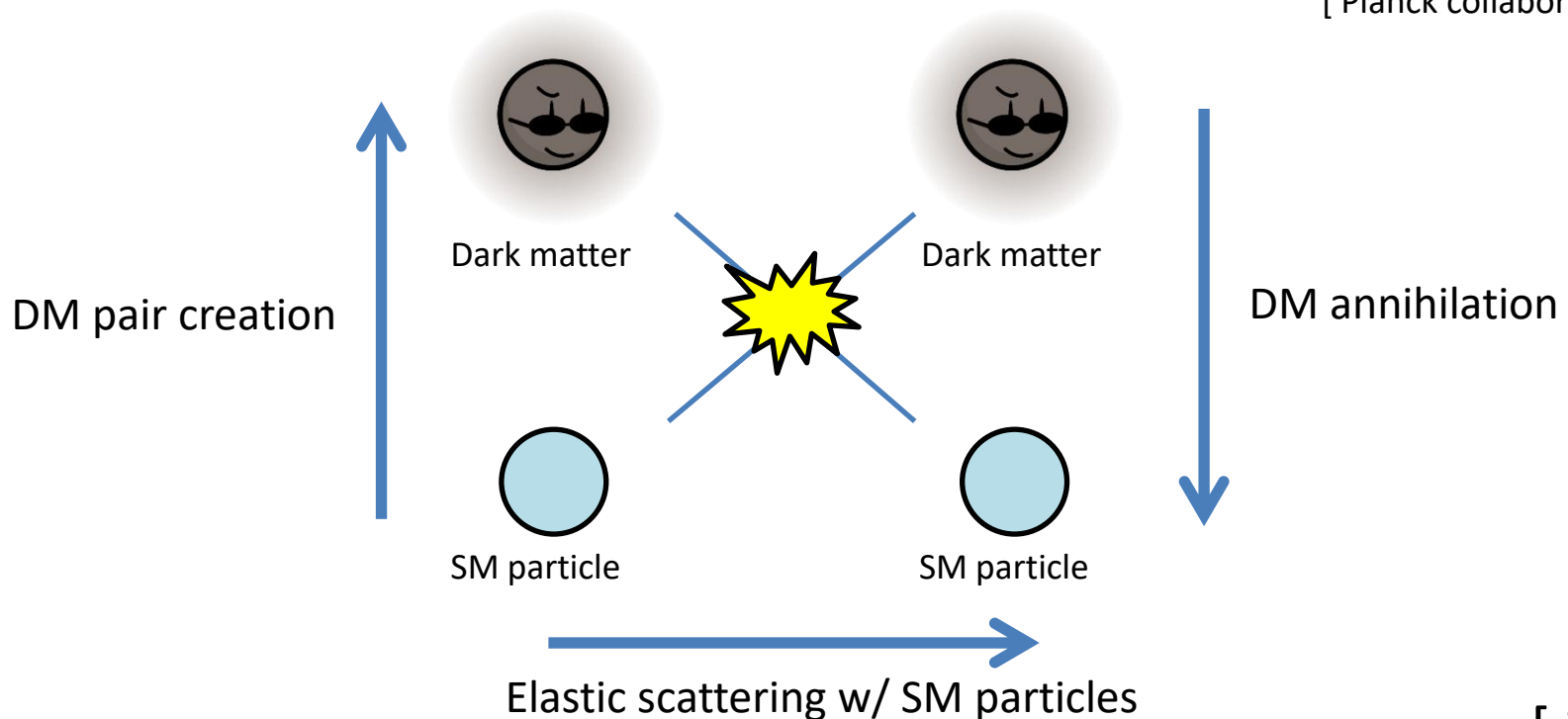
Dark matter and its annihilation

~ 27% of our current universe is made of **dark matter**
Many evidences but only through gravitational effects.



[Planck collaboration]

Dark Matter could have **some interaction with SM particles** :



Dark matter annihilation cross section

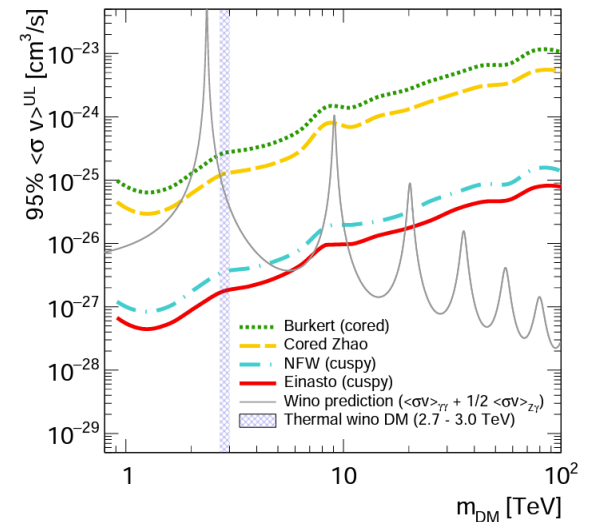
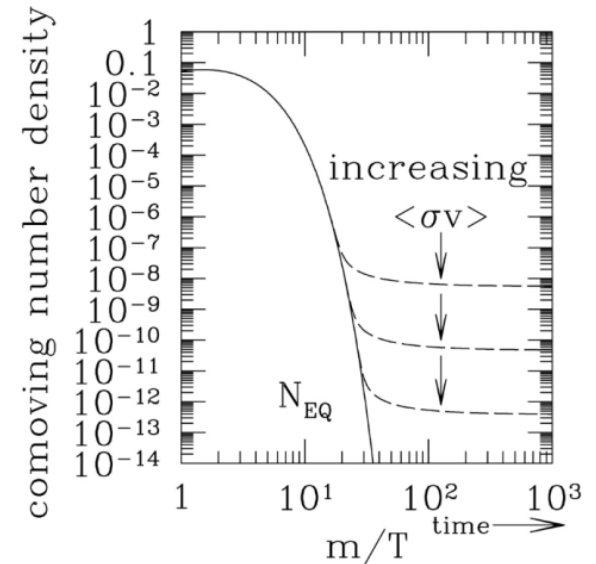
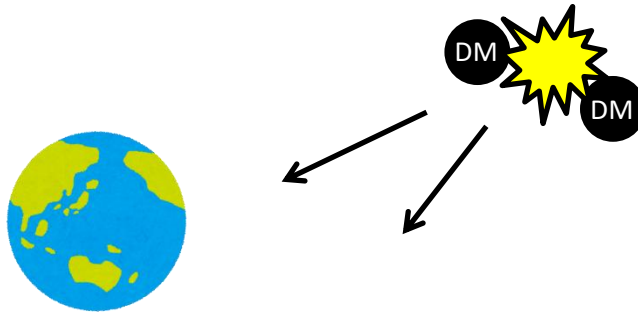
- Freeze-out scenario

Relic abundance today

$$\rightarrow \Omega_{\text{DM}} h^2 \sim 0.1 \times \frac{3 \times 10^{-26} \text{ cm}^3/\text{s}}{\sigma v}$$

- Indirect detection

High energy cosmic ray from DM annihilation

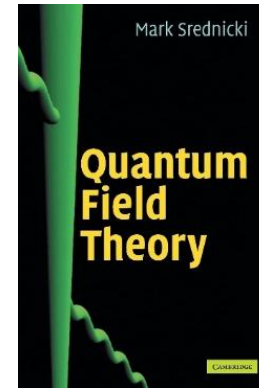
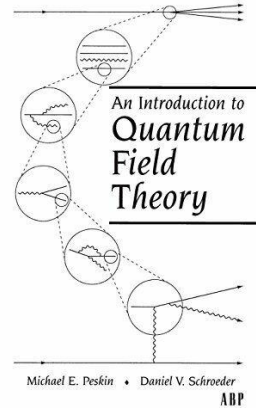
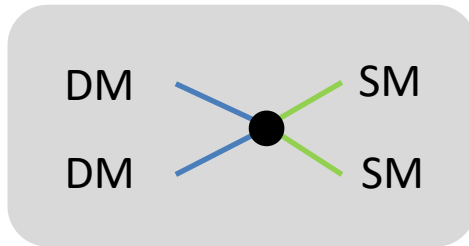


[MAGIC, 2212.10527]

- ...

How to calculate σv

If the coupling is $< \sim 1$, perturbative calculation is efficient.



etc...

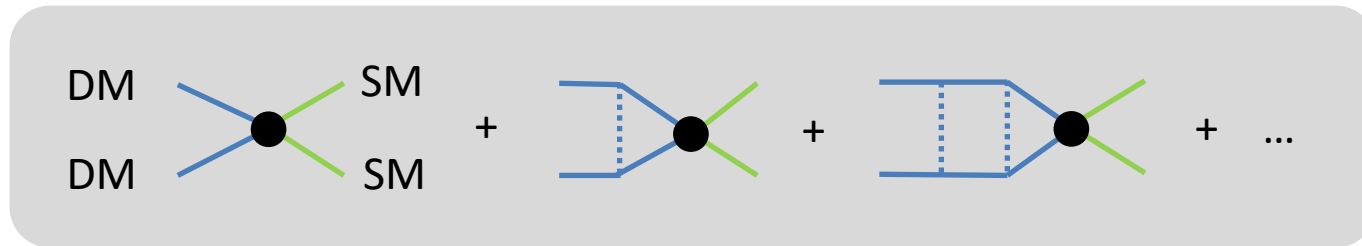
Typical example (higgs portal) :

$$L \ni -\frac{\lambda}{4}\phi^2 h^2 \quad \Rightarrow \quad iM(\phi\phi \rightarrow hh) = -i\lambda \quad \Rightarrow \quad \sigma v_{rel} = \frac{\lambda^2}{64\pi^2 m_\phi^2} \sqrt{1 - \frac{m_h^2}{m_\phi^2}}$$

How to calculate σv

Even if the coupling is $< \sim 1$, **non-perturbative resummation** is required,
if dark matter couples to **a light force mediator boson**

$$(m_{\text{boson}} \ll m_{\text{DM}})$$



$$M$$

$$\sim M \frac{\alpha}{v}$$

$$\sim M \left(\frac{\alpha}{v} \right)^2$$

- Wino / Higgsino dark matter
- ex) • SU(2) 5-plet dark matter
- ...

Sommerfeld effect

[Sommerfeld (1931)]
[Hisano, Matsumoto, Nojiri (2003)]

Strategy

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]
See also [Agrawal, Parikh, Reece (2020)]

We are interested in DM annihilation at **non-relativistic regime**.

- Freeze-out ($T \simeq m/20$)
- $v \simeq 10^{-3}c$ in galaxy

Schroedinger equation provides **effective description**!

$$E\psi = -\frac{1}{2\mu}\nabla^2\psi + V(x)\psi$$

$$V(r) = \begin{cases} V_{\text{short}}(r) & (r < a) & \text{complex} \\ & \text{Annihilation etc.} \\ V_{\text{long}}(r) & (r \geq a) & \text{real} \\ & \text{Exchange of light boson(s)} \end{cases}$$

$$\text{ex) } V(r) \sim u \delta^3(x) + \frac{\alpha}{r} e^{-mr}$$

How to calculate DM annihilation

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

Two body scattering problem (à la undergrad quantum mechanics)

$$E\psi = -\frac{1}{2\mu}\nabla^2\psi + V(x)\psi \quad \text{with} \quad \psi \rightarrow e^{ipz} + f(\theta)\frac{e^{ikr}}{r}$$

Flux of probability

$$\vec{j}(x) = \frac{1}{\mu}\text{Im}[\psi^*(x)\vec{\nabla}\psi(x)]$$



“Non-conservation” of probability

$$\begin{aligned}\vec{\nabla} \cdot \vec{j}(x) &= 2\text{Im}V(x)|\psi(x)|^2 \\ &= 2\text{Im}V_{\text{short}}(x)|\psi(x)|^2\end{aligned}$$

How to calculate DM annihilation

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

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$$E\psi = -\frac{1}{2\mu}\nabla^2\psi + V(x)\psi \quad \text{with} \quad \psi \rightarrow e^{ipz} + f(\theta)\frac{e^{ikr}}{r}$$

Definition of cross section

$$\sigma \times j_{in} = -\frac{dP}{dt}$$

Flux of incoming wave

$$j_{in} = \frac{1}{\mu} \text{Im}[\psi_{in}^*(x) \vec{\nabla} \psi_{in}(x)] = \frac{p}{\mu} = v$$

Rate of annihilation

$$-\frac{dP}{dt} = \int_{r < a} d^3x \, 2\text{Im} V_{\text{short}}(x) |\psi(x)|^2$$

How to calculate DM annihilation

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

Two body scattering problem (à la undergrad quantum mechanics)

$$E\psi = -\frac{1}{2\mu}\nabla^2\psi + V(x)\psi \quad \text{with} \quad \psi \rightarrow e^{ipz} + f(\theta)\frac{e^{ikr}}{r}$$

Annihilation cross section

$$\sigma v = \int_{r < a} d^3x \, 2\text{Im}V_{\text{short}}(x)|\psi(x)|^2$$

How to calculate DM annihilation

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

$$\sigma v = \int_{r < a} d^3x \, 2\text{Im}V_{\text{short}}(x) |\psi(x)|^2 \quad \Rightarrow \quad \simeq \int_{r < a} d^3x \, 2\text{Im}V_{\text{short}}(x) |\psi_{\text{long}}(x)|^2$$

$$\psi \simeq \psi_{\text{long}} \quad (\text{a.k.a. Distorted Wave Born Approximation})$$

This should be OK as long as σv is not so large...
(will come back to this point soon)

$$\text{s. t.} \quad \left[-\frac{1}{2\mu} \nabla^2 + V_{\text{long}}(x) - E \right] \psi_{\text{long}} = 0$$

s-wave case:

$$\sigma v \simeq |\psi_{\text{long}}(0)|^2 \times \int_{r < a} d^3x \, 2\text{Im}V_{\text{short}}(x)$$

Enhancement factor

How to calculate DM annihilation

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

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s. t. $\left[-\frac{1}{2\mu} \nabla^2 + V_{\text{long}}(x) - E \right] \psi_{\text{long}} = 0$

s-wave case:

$$\sigma v \simeq |\psi_{\text{long}}(0)|^2 \times (\sigma v)_0$$

Enhancement factor

Annihilation cross section
w/o long-range force

[Hisano, Matsumoto, Nojiri (2002)]

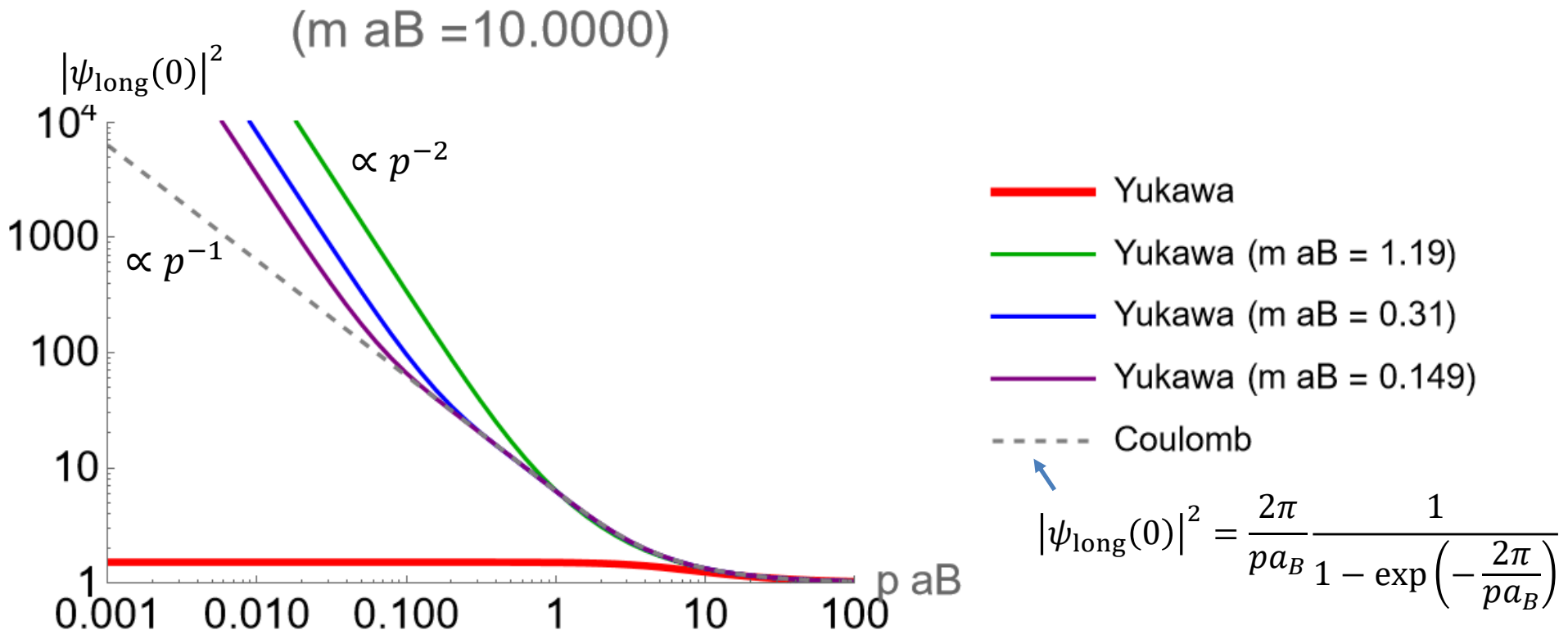
[Arkani-hamed, Finkbeiner, Slatyer, Weiner (2008)]
etc

Sommerfeld factor

$$V(r) = -\frac{\alpha}{r} \exp(-mr)$$

$$\text{Bohr radius : } a_B \equiv \frac{1}{\alpha\mu}$$

- σv enhances when $m < \frac{1}{a_B}$ & $p < \frac{1}{a_B}$
- At some specific value of ma_B , σv violently enhances



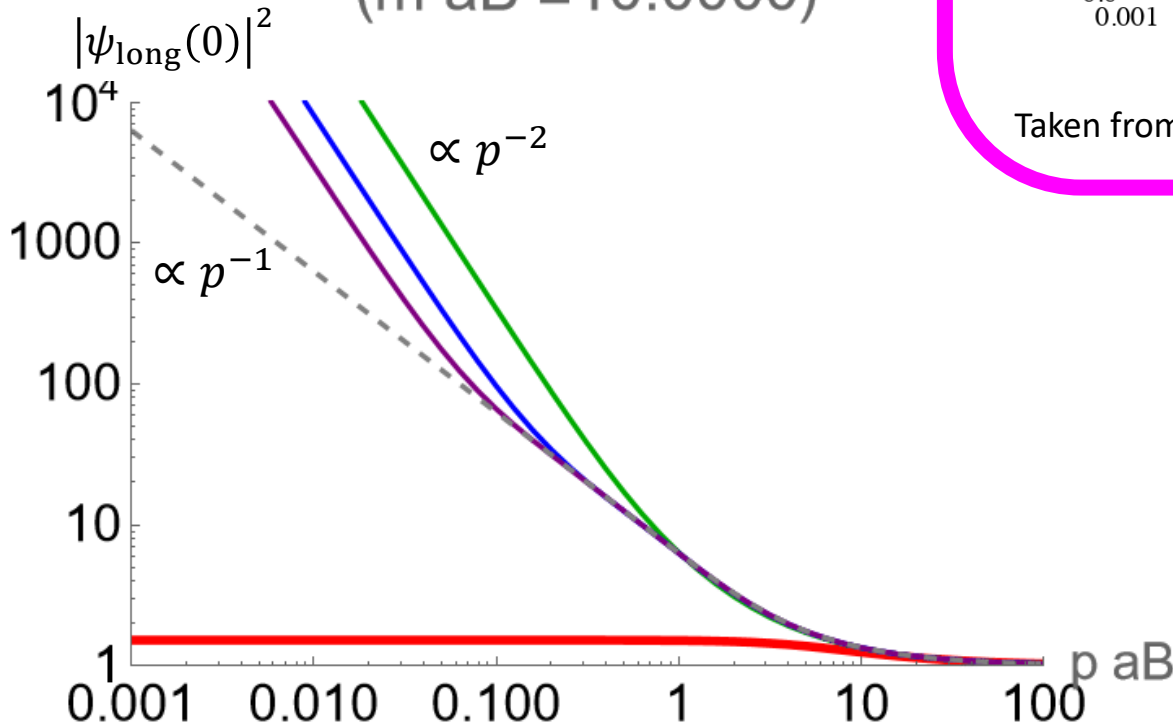
Sommerfeld factor

$$V(r) = -\frac{\alpha}{r} \exp(-mr)$$

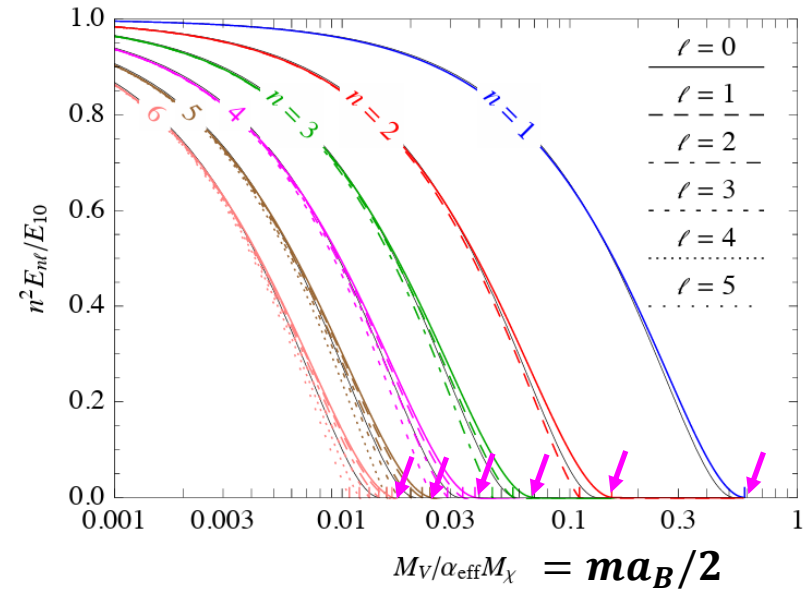
Bohr radii

- σv enhances wh
- At some specific

(m aB = 10.0000)



Binding energies in a Yukawa potential



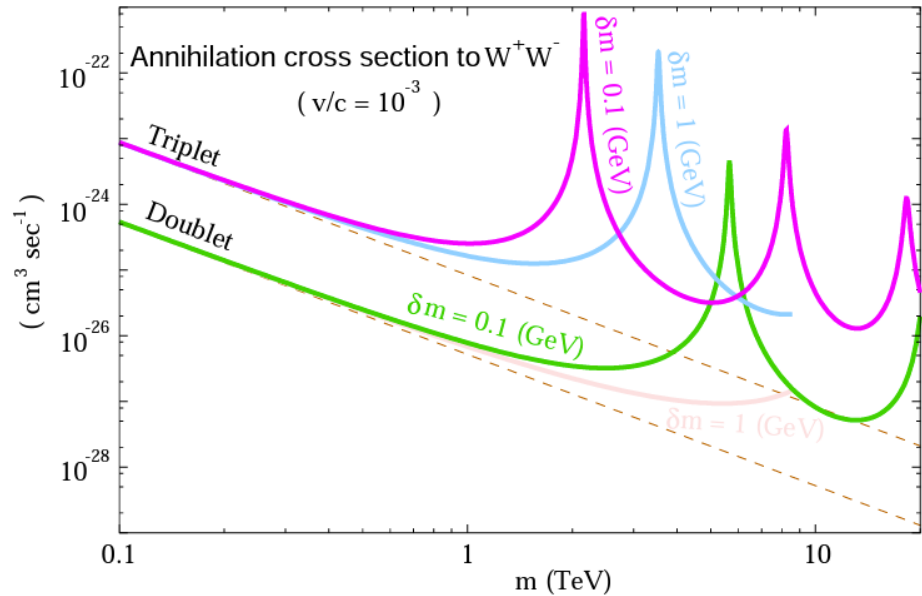
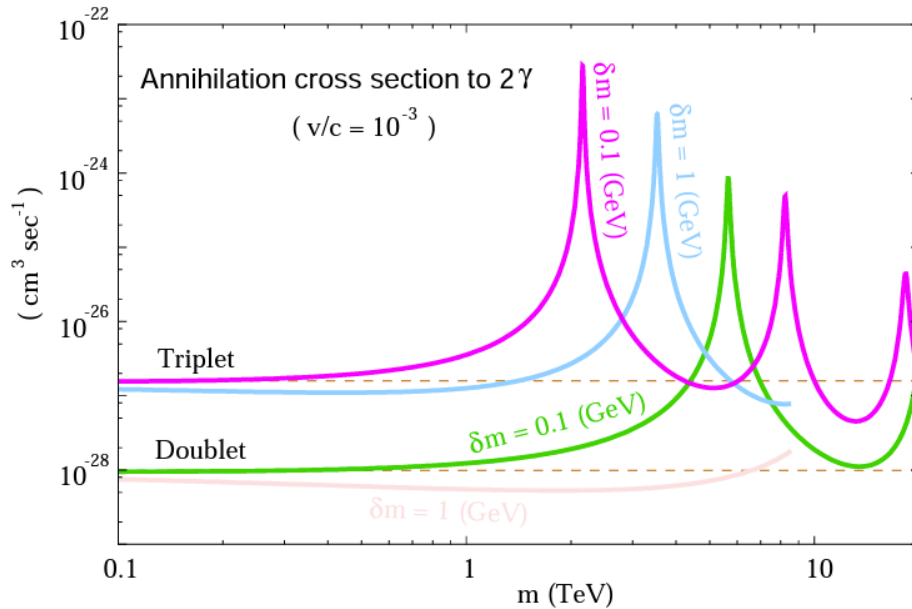
Taken from [Mitridate, Redi, Smirnov, Strumia (2017)]

- Yukawa (m aB = 1.19)
- Yukawa (m aB = 0.31)
- Yukawa (m aB = 0.149)
- - - Coulomb

$$|\psi_{\text{long}}(0)|^2 = \frac{2\pi}{pa_B} \frac{1}{1 - \exp\left(-\frac{2\pi}{pa_B}\right)}$$

Wino / Higgsino DM in SUSY

Dashed lines : usual perturbative calculation (w/o Sommerfeld effect)
Colored curves : non-perturbative calculation (w/ Sommerfeld effect)



[Hisano, Matsumoto, Nojiri (2003)]

Huge difference!

Plan

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Unitarity bound?

$$\sigma = \sigma_0 \times S(v)$$

LO cross section :

$$\sigma_0$$

Enhancement factor :

$$S(v) = |\psi(\mathbf{0})|^2$$

σ_0 and $S(v)$ are irrelevant each other.



Unitarity bound on s-wave : $\sigma \leq \frac{\pi}{p^2}$

[Griest, Kamionkowski (1992)]
[Landau-Lifshits's textbook]

Problematic situations

1. Large $\sigma_0 v$
2. At zero energy resonance ($S(v) \propto v^{-2} \rightarrow \sigma \propto v^{-3}$)
(for s-wave)

Unitarity bound

$$\sigma = \sigma_0 \times S(v)$$

LO cross

Enhanc

σ_0 and $S(v)$

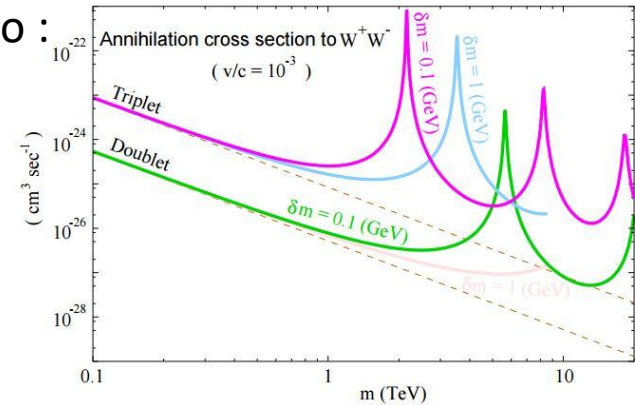
Unitarity bound

Problematic situations

1. Large $\sigma_0 v$

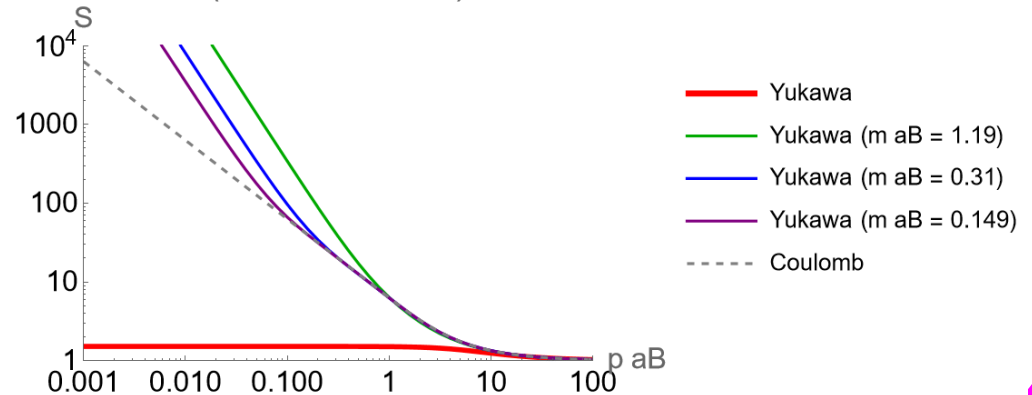
2. At zero energy resonance ($S(v) \propto v^{-2} \rightarrow \sigma \propto v^{-3}$)
(for s-wave)

Wino / Higgsino :



Yukawa potential :

(m aB = 10.0000)



Unitarity bound vs. zero-energy resonance

Annihilation cross section on zero-energy resonance

$$|\psi_{\text{long}}|^2 \propto p^{-2} \quad (\text{for s-wave}) \quad \Rightarrow \quad \sigma_{\text{ann},s} = \frac{1}{v} \times |\psi_{\text{long}}|^2 \times (\sigma v)_0 \propto \frac{1}{p^3}$$

Partial wave expansion in two body scattering

$$E\psi = -\frac{1}{2\mu} \nabla^2 \psi + V(x)\psi$$

$$\text{with } \psi \rightarrow e^{ipz} + f(\theta) \frac{e^{ikr}}{r} = \sum_{\ell} P_{\ell}(\cos \theta) \frac{S_{\ell} e^{ipr} - (-1)^{\ell} e^{-ipr}}{2ipr}$$

Annihilation cross section

$$\sigma_{\text{ann}} = \frac{\pi}{p^2} \sum_{\ell} (2\ell + 1) (1 - |S_{\ell}|^2)$$



$$\sigma_{\text{ann},s} \leq \frac{\pi}{p^2}$$

violates
unitarity bound
at small p

Need to solve Schroedinger eq, seriously

Going back to cross section formula...

$$\sigma v = \int_{r < a} d^3x \, 2\text{Im}V_{\text{short}}(x) |\psi(x)|^2 \quad \Rightarrow \quad \approx \int_{r < a} d^3x \, 2\text{Im}V_{\text{short}}(x) |\psi_{\text{long}}(x)|^2$$

$\psi \simeq \psi_{\text{long}}$ (a.k.a. Distorted Wave Born Approximation)

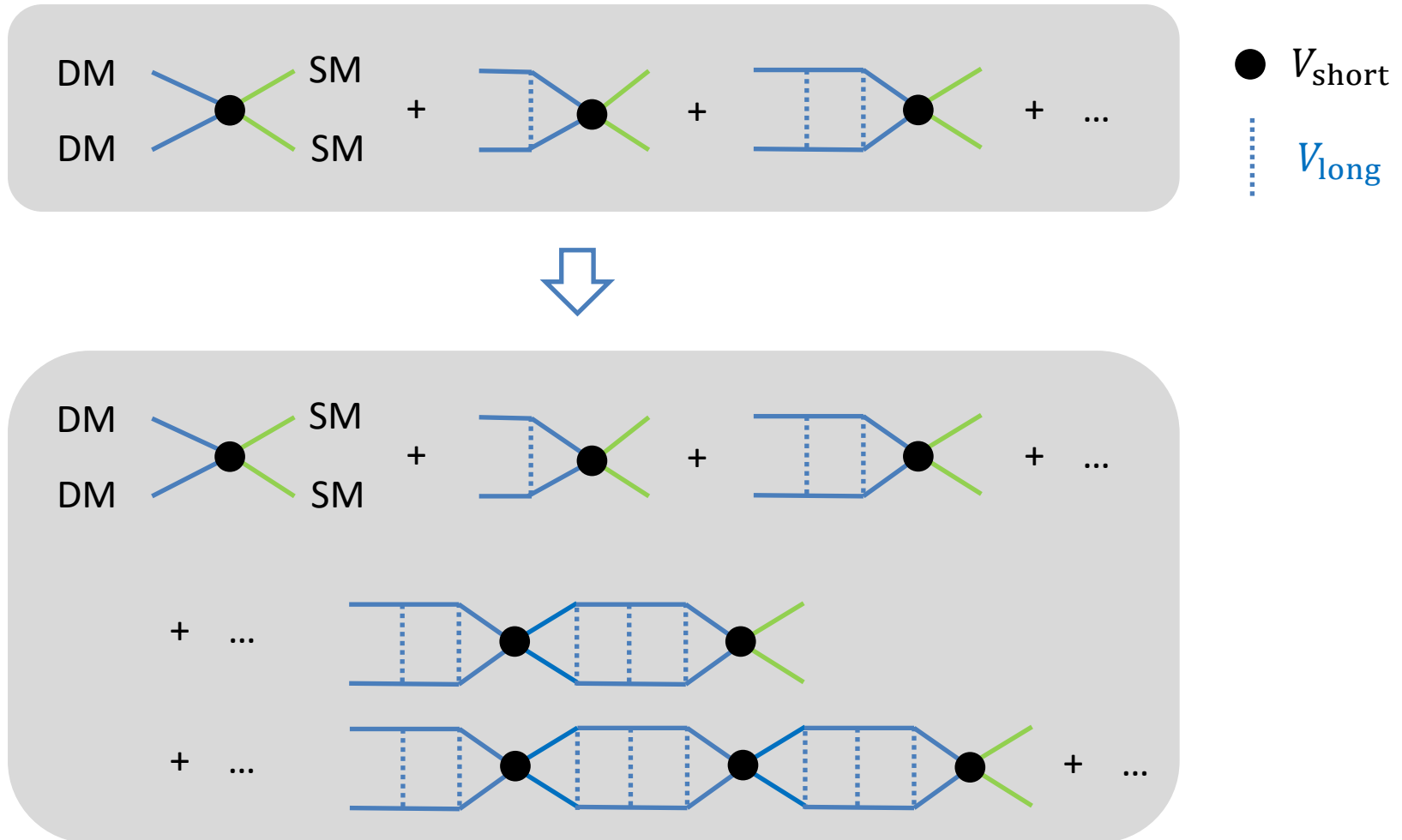
This should be OK **as long as σv is not so large...**
(will come back to this point soon)

s. t. $\left[-\frac{1}{2\mu} \nabla^2 + V_{\text{long}}(x) - E \right] \psi_{\text{long}} = 0$

ψ is quite different from ψ_{long} if σv is large!

Need to solve Schroedinger eq, seriously

Diagrammatic interpretation



S-matrix from Schroedinger eq.

What we need: $\psi = \sum_{\ell} P_{\ell}(\cos \theta) \frac{(-1)^{\ell} \chi_{\ell}(r)}{pr}$

Schroedinger eq.

$$\left[-\frac{1}{2\mu} \nabla^2 + V(r) - \frac{p^2}{2\mu} \right] \psi(r) = 0 \quad \Rightarrow \quad \left[-\frac{1}{2\mu} \frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{2\mu r^2} + V(r) - \frac{p^2}{2\mu} \right] \chi_{\ell}(r) = 0$$

$$V(r) = \begin{cases} V_{\text{short}}(r) & (r < a) & \text{complex} \\ V_{\text{long}}(r) & (r \geq a) & \text{real} \end{cases}$$

Boundary condition at $r \rightarrow \infty$

$$\chi_{\ell}(r) \rightarrow \frac{S_{\ell} \exp(ipr) - \exp(-ipr)}{2i}$$

Boundary condition at $r = a$ (condition for short-range effect)

$$\chi_{\ell}'(r)/\chi_{\ell}(r) \text{ at } r = a \text{ is } p\text{-independent}$$

Straightforward (but lengthy) calculation

2.1. Derivation of non-relativistic two-body scattering

We first briefly review the basis of the two-body scattering problem in non-relativistic quantum mechanics. The Schrödinger equation for the two-body scattering problem with a central potential $V(r)$ is

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi(r) + V(r)\psi(r) = E\psi(r), \quad (2.1)$$

where μ is the reduced mass of the two-body system, r is the separation vector between the particles, $r = |\mathbf{r}|$, and μ is the momentum of either particle in the center of momentum frame. The asymptotic behavior of the wavefunction $\psi(r)$ for $r \rightarrow \infty$ is

$$\psi(r) \rightarrow e^{i\mathbf{p} \cdot \mathbf{r}} + f(\theta) \frac{e^{i\mathbf{p} \cdot \mathbf{r}}}{r}, \quad (2.2)$$

where θ is the angle between $\mathbf{r}$ and the $\mathbf{z}$ -axis, and $\mathbf{p}$ is the direction of the initial plane wave. The standard partial wave expansion is

$$\psi(r, \theta) = \sum_{\ell=0}^{\infty} (2\ell+1) f_{\ell}(\theta) \cos\theta P_{\ell}(\cos\theta). \quad (2.3)$$

The radial wavefunction $u_{\ell}(r)$ satisfies the reduced Schrödinger equation:

$$-\frac{\hbar^2}{2\mu} \frac{d^2 u_{\ell}}{dr^2} + V(r) u_{\ell}(r) = E u_{\ell}(r). \quad (2.4)$$

E can be equivalently as $E = \frac{\hbar^2 k^2}{2\mu} = \sum_{\ell=0}^{\infty} (2\ell+1) f_{\ell}(0) \cos\theta P_{\ell}(\cos\theta)$. It will generally be useful to work with the wavefunction

$$u_{\ell}(r) = \frac{1}{k} \frac{d}{dr} \left(\frac{r}{k} \frac{d}{dr} \right) \psi(r, \theta) = -i k r \psi(r, \theta). \quad (2.5)$$

where $u_{\ell}(r)$ and $g_{\ell}(r)$ are the standard asymptotic radial functions. These wavefunctions have a useful asymptotic behavior:

$$u_{\ell}(r) \sim \frac{1}{k} \frac{d}{dr} \left(\frac{r}{k} \frac{d}{dr} \right) e^{i\mathbf{p} \cdot \mathbf{r}} \sim e^{i\mathbf{p} \cdot \mathbf{r}} - i k r \psi(r, \theta), \quad (2.6)$$

At large r , this asymptotic behavior is

$$u_{\ell}(r) \sim \frac{1}{k} \frac{d}{dr} \left(\frac{r}{k} \frac{d}{dr} \right) e^{i\mathbf{p} \cdot \mathbf{r}} \sim e^{i\mathbf{p} \cdot \mathbf{r}} - i k r \psi(r, \theta). \quad (2.7)$$

Thus, we can read off the asymptotic behavior of $u_{\ell}(r)$ from the boundary condition in Eq. 2.5 as

$$u_{\ell}(r) \rightarrow u_{\ell}(r) + i k r \psi(r, \theta) \sim \frac{1}{k} \frac{d}{dr} \left(\frac{r}{k} \frac{d}{dr} \right) e^{i\mathbf{p} \cdot \mathbf{r}} \sim e^{i\mathbf{p} \cdot \mathbf{r}} - i k r \psi(r, \theta), \quad (2.8)$$

where $P_{\ell}(x) = \sum_{m=0}^{\ell} (2\ell+1) f_{\ell}(m) \cos\theta P_{\ell}(\cos\theta)$ and $S_{\ell} = 1 + 2i k r \psi(r, \theta)$ is the S -matrix.

2.2. The full cross section for elastic scattering and inclusive annihilation

$$\sigma_{\text{el}} = \frac{\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) \left| \frac{f_{\ell}(0) + i k r \psi(r, \theta)}{f_{\ell}(0) + i k r \psi(r, \theta)} \right|^2 \exp(2i\mathbf{p} \cdot \mathbf{r}) \cdot \frac{1}{k^2}. \quad (2.9)$$

Note that to lowest order in k^2 , the effect of the long-range force in the annihilation cross section is enhanced by a factor of C^2 , this is the standard Sommerfeld enhancement. Note also that we can shift the definition of k^2 by a real number without modifying the momentum in the annihilation cross section; the effect of such a shift on the annihilation cross section would be to break the conservation terms in the denominator into two parts, which may be desirable in terms of separating the long-distance and short-distance behavior.

Although Eq. 2.9 is generic and exact, it is useful to consider the limit where p is large and the long-range potential can be neglected. This assumes that such a regime exists consistent with the assumption of non-relativistic physics, but this should generally be true for weak coupling. In this case, we expect $C^2 \rightarrow 1$, $e^{i\mathbf{p} \cdot \mathbf{r}} \rightarrow 1$, and S_{ℓ} is given as

$$S_{\ell} = \frac{1 - \frac{g^2 \mu^2}{4\pi} \frac{1}{k^2}}{1 + \frac{g^2 \mu^2}{4\pi} \frac{1}{k^2}}. \quad (2.10)$$

$$\sigma_{\text{el}} = \frac{\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) \left| \frac{f_{\ell}(0) + i k r \psi(r, \theta)}{f_{\ell}(0) + i k r \psi(r, \theta)} \right|^2 \exp(2i\mathbf{p} \cdot \mathbf{r}) \cdot \frac{1}{k^2}. \quad (2.11)$$

This regime can be used for matching between the perturbative QFT calculation and the Schrödinger equation approach. The degree to which these results can be used to calculate the corrected cross section at lower moments depends on the momentum dependence of k_{eff} , so we will study this question next and present some examples.

Note that in defining $f_{\ell}(r)$ and $G_{\ell}(r)$ via Eqs. 2.14, 2.15, we have the freedom to choose $V_{\text{ann}}(r)$ in the regime $r < a$. This choice does not affect the potential for the problem of interest, just the basis of solutions we use to study that problem; different choices lead to different properties for $f_{\ell}(r)$ and $G_{\ell}(r)$, and consequently to different coefficients for these functions in the $r > a$ regime, but the full wavefunction (and hence the S -matrix, cross section, etc.) will be the same in all cases. The various intermediate quantities calculated under the different conventions should also converge to each other in the limit where the short-distance interaction is described by a contact interaction and we take $a \rightarrow 0$ more generally, they will differ by terms that are suppressed by powers of a .

In the remainder of this section, we will define $V_{\text{ann}}(r)$ as the a -independent long-range potential derived from the low-energy effective theory for all r , including $r < a$. This implies that the $f_{\ell}(r)$ and $G_{\ell}(r)$ functions, and consequently the C_{ℓ} factors, are formally

$$S_{\ell} = \frac{1 - \frac{g^2 \mu^2}{4\pi} \frac{1}{k^2}}{1 + \frac{g^2 \mu^2}{4\pi} \frac{1}{k^2}}. \quad (2.12)$$

$$\sigma_{\text{el}} = \frac{\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) \left| \frac{f_{\ell}(0) + i k r \psi(r, \theta)}{f_{\ell}(0) + i k r \psi(r, \theta)} \right|^2 \exp(2i\mathbf{p} \cdot \mathbf{r}) \cdot \frac{1}{k^2}. \quad (2.13)$$

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$$S_{\ell} = \frac{1 - \frac{g^2 \mu^2}{4\pi} \frac{1}{k^2}}{1 + \frac{g^2 \mu^2}{4\pi} \frac{1}{k^2}}. \quad (2.16)$$

$$\sigma_{\text{el}} = \frac{\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) \left| \frac{f_{\ell}(0) + i k r \psi(r, \theta)}{f_{\ell}(0) + i k r \psi(r, \theta)} \right|^2 \exp(2i\mathbf{p} \cdot \mathbf{r}) \cdot \frac{1}{k^2}. \quad (2.17)$$

This regime can be used for matching between the perturbative QFT calculation and the Schrödinger equation approach. The degree to which these results can be used to calculate the corrected cross section at lower moments depends on the momentum dependence of k_{eff} , so we will study this question next and present some examples.

Note that in defining $f_{\ell}(r)$ and $G_{\ell}(r)$ via Eqs. 2.14, 2.15, we have the freedom to choose $V_{\text{ann}}(r)$ in the regime $r < a$. This choice does not affect the potential for the problem of interest, just the basis of solutions we use to study that problem; different choices lead to different properties for $f_{\ell}(r)$ and $G_{\ell}(r)$, and consequently to different coefficients for these functions in the $r > a$ regime, but the full wavefunction (and hence the S -matrix, cross section, etc.) will be the same in all cases. The various intermediate quantities calculated under the different conventions should also converge to each other in the limit where the short-distance interaction is described by a contact interaction and we take $a \rightarrow 0$ more generally, they will differ by terms that are suppressed by powers of a .

In the remainder of this section, we will define $V_{\text{ann}}(r)$ as the a -independent long-range potential derived from the low-energy effective theory for all r , including $r < a$. This implies that the $f_{\ell}(r)$ and $G_{\ell}(r)$ functions, and consequently the C_{ℓ} factors, are formally

$$S_{\ell} = \frac{1 - \frac{g^2 \mu^2}{4\pi} \frac{1}{k^2}}{1 + \frac{g^2 \mu^2}{4\pi} \frac{1}{k^2}}. \quad (2.18)$$

$$\sigma_{\text{el}} = \frac{\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) \left| \frac{f_{\ell}(0) + i k r \psi(r, \theta)}{f_{\ell}(0) + i k r \psi(r, \theta)} \right|^2 \exp(2i\mathbf{p} \cdot \mathbf{r}) \cdot \frac{1}{k^2}. \quad (2.19)$$

This regime can be used for matching between the perturbative QFT calculation and the Schrödinger equation approach. The degree to which these results can be used to calculate the corrected cross section at lower moments depends on the momentum dependence of k_{eff} , so we will study this question next and present some examples.

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$$S_{\ell} = \frac{1 - \frac{g^2 \mu^2}{4\pi} \frac{1}{k^2}}{1 + \frac{g^2 \mu^2}{4\pi} \frac{1}{k^2}}. \quad (2.20)$$

$$\sigma_{\text{el}} = \frac{\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) \left| \frac{f_{\ell}(0) + i k r \psi(r, \theta)}{f_{\ell}(0) + i k r \psi(r, \theta)} \right|^2 \exp(2i\mathbf{p} \cdot \mathbf{r}) \cdot \frac{1}{k^2}. \quad (2.21)$$

This regime can be used for matching between the perturbative QFT calculation and the Schrödinger equation approach. The degree to which these results can be used to calculate the corrected cross section at lower moments depends on the momentum dependence of k_{eff} , so we will study this question next and present some examples.

Note that in defining $f_{\ell}(r)$ and $G_{\ell}(r)$ via Eqs. 2.14, 2.15, we have the freedom to choose $V_{\text{ann}}(r)$ in the regime $r < a$. This choice does not affect the potential for the problem of interest, just the basis of solutions we use to study that problem; different choices lead to different properties for $f_{\ell}(r)$ and $G_{\ell}(r)$, and consequently to different coefficients for these functions in the $r > a$ regime, but the full wavefunction (and hence the S -matrix, cross section, etc.) will be the same in all cases. The various intermediate quantities calculated under the different conventions should also converge to each other in the limit where the short-distance interaction is described by a contact interaction and we take $a \rightarrow 0$ more generally, they will differ by terms that are suppressed by powers of a .

In the remainder of this section, we will define $V_{\text{ann}}(r)$ as the a -independent long-range potential derived from the low-energy effective theory for all r , including $r < a$. This implies that the $f_{\ell}(r)$ and $G_{\ell}(r)$ functions, and consequently the C_{ℓ} factors, are formally

$$S_{\ell} = \frac{1 - \frac{g^2 \mu^2}{4\pi} \frac{1}{k^2}}{1 + \frac{g^2 \mu^2}{4\pi} \frac{1}{k^2}}. \quad (2.22)$$

$$\sigma_{\text{el}} = \frac{\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) \left| \frac{f_{\ell}(0) + i k r \psi(r, \theta)}{f_{\ell}(0) + i k r \psi(r, \theta)} \right|^2 \exp(2i\mathbf{p} \cdot \mathbf{r}) \cdot \frac{1}{k^2}. \quad (2.23)$$

The elastic cross section σ_{el} and the inclusive annihilation cross section σ_{ann} are given (for distinguishable particles) by

$$\sigma_{\text{el}} = \frac{\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) \left| \frac{f_{\ell}(0) + i k r \psi(r, \theta)}{f_{\ell}(0) + i k r \psi(r, \theta)} \right|^2 \exp(2i\mathbf{p} \cdot \mathbf{r}) \cdot \frac{1}{k^2}. \quad (2.24)$$

$$\sigma_{\text{ann}} = \frac{\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) \left| \frac{f_{\ell}(0) + i k r \psi(r, \theta)}{f_{\ell}(0) + i k r \psi(r, \theta)} \right|^2 \exp(2i\mathbf{p} \cdot \mathbf{r}) \cdot \frac{1}{k^2}. \quad (2.25)$$

Here we take the inclusive annihilation cross section to include all inclusive processes that are not excluded by the flux factor of the potential $V(r)$, which then includes interference or apparent non-unitarity of the S -matrix.

The relation among distinguishable particles in the initial state, the identical particles, the cross section will be more or less partial waves do not have the correct symmetry properties, and enhanced by a factor of C^2 of otherwise (e.g. for identical fermions in a spin-singlet state, C must be even, so that $[S]$ for a more in-depth discussion). We will accordingly add a prefactor C to cross sections, which is 1 for distinguishable particles and 2 for identical particles, while working with the scattering equations appropriate to distinguishable particles.

We will often find it advantageous to expand the reduced wavefunction $u_{\ell}(r)$ in terms of the free wavefunction $u_{\ell}^{(0)}(r)$ and $u_{\ell}^{(1)}(r)$ in particular, the full wavefunction $u_{\ell}(r)$ is the solution of the Schrödinger equation via the variational principle, which we review and justify in App. B. Let us define

$$f_{\ell}(r) = u_{\ell}(r)/\sqrt{k}, \quad G_{\ell}(r) = u_{\ell}(r)/\sqrt{k}, \quad (2.26)$$

Then for any reduced wavefunction $u_{\ell}(r)$ solving the Schrödinger equation, we can decompose

$$u_{\ell}(r) = u_{\ell}^{(0)}(r) + u_{\ell}^{(1)}(r). \quad (2.27)$$

and is useful to separately define $u_{\ell}^{(0)}(r)$ and $u_{\ell}^{(1)}(r)$ as we can also impose the coefficient $C_{\ell} = u_{\ell}^{(0)}(r)/u_{\ell}^{(1)}(r)$ (see discussion in App. B). This condition means that matching the coefficient C_{ℓ} to $u_{\ell}^{(0)}(r)$ and $u_{\ell}^{(1)}(r)$ between two solutions (at $r = a$) is sufficient to match both their values and their first derivatives (at $r = a$). Note also that $f_{\ell}(r)/G_{\ell}(r) = 1$ for $r \rightarrow \infty$.

2.2.1 Momentum scaling of the boundary condition

We start from the annihilation cross section when the long-range force defines the wavefunction from a plane wave. We assume that the long-range force does not provide any additional effect directly, i.e. the corresponding potential is zero. Thus, it is useful to separate the short-range interactions (which can include radiative corrections) and the long-range interactions (assumed to be well-described by a real potential) at some boundary $r = a$ as

$$V(r) = \begin{cases} V_{\text{ann}}(r) & (r < a) \\ 0 & (r \geq a) \end{cases}. \quad (2.28)$$

It is impossible to get the relation between the coefficient of r^{-4} and r^{-2} from the momentum relations near the origin. This is because the case of an irregular solution and a regular solution is another irregular solution, satisfying the same Schrödinger equation. The coefficient of r^{-2} is determined by imposing the boundary condition at infinity (Eq. 2.30). For now, we just keep both the r^{-4} term and r^{-2} term and write C_{ℓ} as $r \rightarrow \infty$ as

$$C_{\ell} = \frac{(2\ell+1)k^2}{4\pi} \frac{1}{k^2} \frac{1}{(2\ell+1)k^2} \frac{1}{k^2} \left[\frac{f_{\ell}(0) + i k r \psi(r, \theta)}{f_{\ell}(0) + i k r \psi(r, \theta)} \right] \exp(2i\mathbf{p} \cdot \mathbf{r}) \cdot \frac{1}{k^2}. \quad (2.29)$$

Note that we also keep track of the term scaling as $|p|^2 \log|p|$, because it comes a subleading when we will discuss later. Again, C_{ℓ} cannot be determined by the momentum relations at the origin. On the other hand, C_{ℓ} can be determined by Eq. 2.12. Here C_{ℓ} is a parameter which has been introduced to make the argument of the full wavefunction $u_{\ell}(r)$ can be taken to be any value because the difference of C_{ℓ} can be absorbed by redefining $u_{\ell}(r)$, however, we will find it useful to take $C_{\ell} \rightarrow \infty$. We will discuss the behavior of C_{ℓ} for some examples of $V(r)$ in Sec. 2.6.

2.2.2 Momentum scaling of terms in $u_{\ell}(r)$

Now let us discuss the momentum dependence of the coefficient k_{eff} . We only keep the leading term and drop terms of $O(p^2/a^2)$ in Eq. 2.28. Thus, the boundary condition from the short-range physics can be expressed by two parameters, k_{eff} and a . By using Eqs. 2.28-2.30, we obtain

$$k_{\text{eff}}(p) = k + i k p + i k p^2. \quad (2.31)$$

where k_{eff} and k_{eff} are momentum-independent constants which are defined as

$$k_{\text{eff}} = \frac{(2\ell+1)k^2}{4\pi} \frac{1}{k^2} \frac{1}{(2\ell+1)k^2} \frac{1}{k^2} \left[\frac{f_{\ell}(0) + i k r \psi(r, \theta)}{f_{\ell}(0) + i k r \psi(r, \theta)} \right] \exp(2i\mathbf{p} \cdot \mathbf{r}) \cdot \frac{1}{k^2}. \quad (2.32)$$

$$k_{\text{eff}} = \frac{(2\ell+1)k^2}{4\pi} \frac{1}{k^2} \frac{1}{(2\ell+1)k^2} \frac{1}{k^2} \left[\frac{f_{\ell}(0) + i k r \psi(r, \theta)}{f_{\ell}(0) + i k r \psi(r, \theta)} \right] \exp(2i\mathbf{p} \cdot \mathbf{r}) \cdot \frac{1}{k^2}. \quad (2.33)$$

Instead of k_{eff} and a , we can use three parameters k_{eff} and k_{eff} to parameterize the effects of the short-range physics. k_{eff} is $O(p^{-2})$ and the leading term k_{eff} is real in the case of $a \rightarrow 0$. As we will see in explicit examples, k_{eff} can be large at small a if there exists a non-zero momentum, k_{eff} can be determined by a reference relation given in Eqs. A.15, and it is a polynomial of momentum p in general.

We can extract the momentum-dependent term in k_{eff} from the behavior of $C_{\ell}(r)$. Let us define k_{eff} as

$$k_{\text{eff}}(p) = \frac{1}{2\pi} \left[\frac{f_{\ell}(0) + i k r \psi(r, \theta)}{f_{\ell}(0) + i k r \psi(r, \theta)} \right] \exp(2i\mathbf{p} \cdot \mathbf{r}) \cdot \frac{1}{k^2}. \quad (2.34)$$

$k_{\text{eff}}(p) = \frac{1}{2\pi} \left[\frac{f_{\ell}(0) + i k r \psi(r, \theta)}{f_{\ell}(0) + i k r \psi(r, \theta)} \right] \exp(2i\mathbf{p} \cdot \mathbf{r}) \cdot \frac{1}{k^2}$

$k_{\text{eff}}(p) = \frac{1}{2\pi} \left[\frac{f_{\ell}(0) + i k r \psi(r, \theta)}{f_{\ell}(0) + i k r \psi(r, \theta)} \right] \exp(2i\mathbf{p} \cdot \mathbf{r}) \cdot \frac{1}{k^2}$

$k_{\text{eff}}(p) = \frac{1}{2\pi} \left[\frac{f_{\ell}(0) + i k r \psi(r, \theta)}{f_{\ell}(0) + i k r \psi(r, \theta)} \right] \exp(2i\mathbf{p} \cdot \mathbf{r}) \cdot \frac{1}{k^2}$

$k_{\text{eff}}(p) = \frac{1}{2\pi} \left[\frac{f_{\ell}(0) + i k r \psi(r, \theta)}{f_{\ell}(0) + i k r \psi(r, \theta)} \right] \exp(2i\mathbf{p} \cdot \mathbf{r}) \cdot \frac{1}{k^2}$

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$k_{\text{eff}}(p) = \frac{1}{2\pi} \left[\frac{f_{\ell}(0) + i k r \psi(r, \theta)}{f_{\ell}(0) + i k r \psi(r, \theta)} \right] \exp(2i\mathbf{p} \cdot \mathbf{r}) \cdot \frac{1}{k^2}$

k_{eff}

S-matrix

We obtain S-matrix for each ℓ as

$$S_\ell \simeq \underbrace{\exp\left(2i\delta_\ell^{(L)}(p)\right)}_{\substack{\text{Phase-shift} \\ \text{by long-range force}}} \times \underbrace{\frac{k_{\ell,0} + z_\ell(p) - ip^{2\ell+1}c_\ell^2(p)}{k_{\ell,0} + z_\ell(p) + ip^{2\ell+1}c_\ell^2(p)}}_{\text{Relevant part for annihilation}}$$

a single complex parameter : $k_{\ell,0}$

Three real functions : $c_\ell^2(p)$, $z_\ell(p)$, $\delta_\ell(p)$

S-matrix

We obtain S-matrix for each ℓ as

$$S_\ell \simeq \underbrace{\exp\left(2i\delta_\ell^{(L)}(p)\right)}_{\text{Phase-shift by long-range force}} \times \underbrace{\frac{k_{\ell,0} + z_\ell(p) - ip^{2\ell+1}c_\ell^2(p)}{k_{\ell,0} + z_\ell(p) + ip^{2\ell+1}c_\ell^2(p)}}_{\text{Relevant part for annihilation}}$$

Annihilation cross section :

Unitarity bound ✓

$$\sigma_{ann,\ell} = \frac{\pi}{p^2} (2\ell + 1)(1 - |S_\ell|^2) < \frac{(2\ell + 1)\pi}{p^2}$$

S-matrix

We obtain S-matrix for each ℓ as

$$S_\ell \simeq \underbrace{\exp\left(2i\delta_\ell^{(L)}(p)\right)}_{\text{Phase-shift by long-range force}} \times \underbrace{\frac{k_{\ell,0} + z_\ell(p) - ip^{2\ell+1}C_\ell^2(p)}{k_{\ell,0} + z_\ell(p) + ip^{2\ell+1}C_\ell^2(p)}}_{\text{Relevant part for annihilation}}$$

Annihilation cross section :

Unitarity bound ✓

$$\begin{aligned}\sigma_{ann,\ell} &= \frac{\pi}{p^2} (2\ell + 1)(1 - |S_\ell|^2) < \frac{(2\ell + 1)\pi}{p^2} \\ &= \frac{\pi}{p^2} (2\ell + 1) \times 4\text{Re}\left[\frac{ip^{2\ell+1}C_\ell^2}{k_{\ell,0}}\right] \times \left|1 + \frac{z_\ell + ip^{2\ell+1}C_\ell^2}{k_{\ell,0}}\right|^{-2}\end{aligned}$$

S-matrix

We obtain S-matrix for each ℓ as

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Annihilation cross section :

Unitarity bound ✓

$$\begin{aligned} \sigma_{ann,\ell} &= \frac{\pi}{p^2} (2\ell + 1)(1 - |S_\ell|^2) < \frac{(2\ell + 1)\pi}{p^2} \\ &= 4\pi(2\ell + 1)p^{2\ell-1} \operatorname{Im}\left[-\frac{1}{k_{\ell,0}}\right] \times C_\ell^2 \times \left|1 + \frac{z_\ell + ip^{2\ell+1}C_\ell^2}{k_{\ell,0}}\right|^{-2} \end{aligned}$$

Annihilation cross section
w/o long-range force

Conventional
Sommerfeld factor

Correction factor

Examples

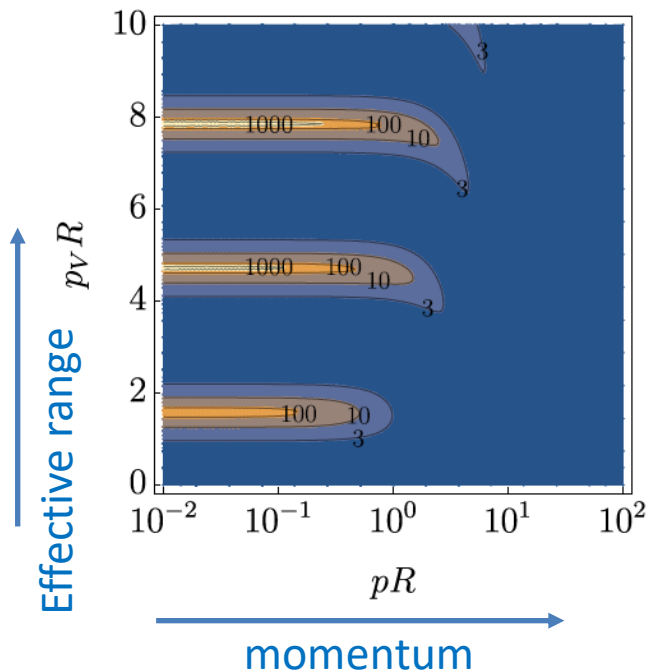
[Parikh, Sato, Slatyer (2024)]

Spherical well potential

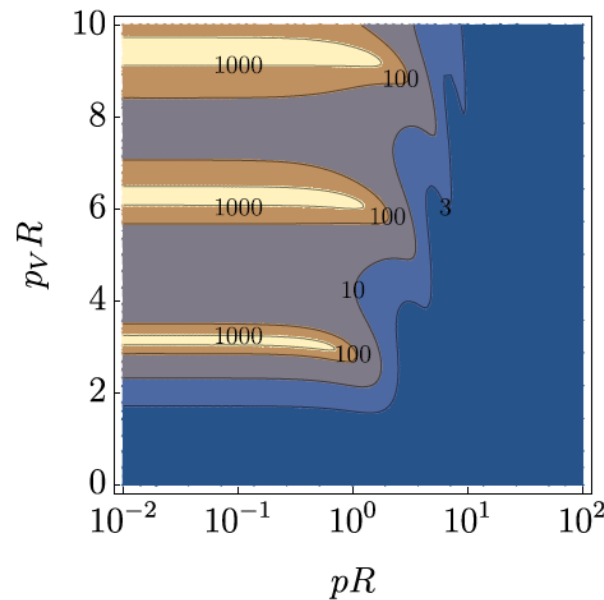
$$V(r) = -\frac{p_V^2}{2\mu} \theta(R - r)$$

(Conventional) Sommerfeld factor : C_ℓ^2

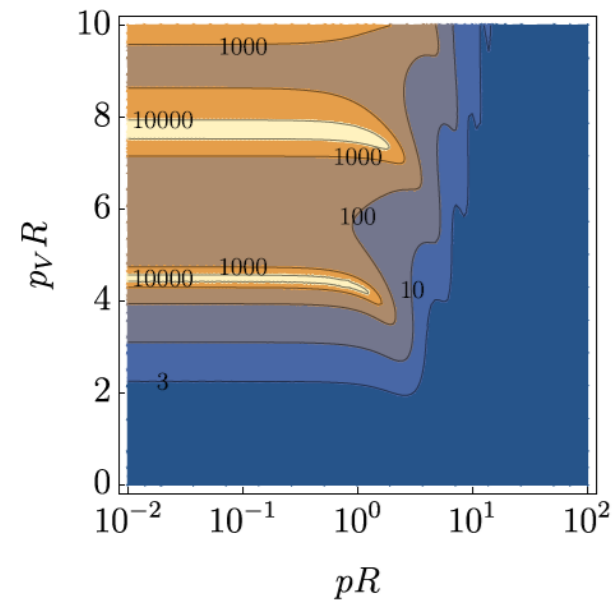
S-wave



P-wave



D-wave



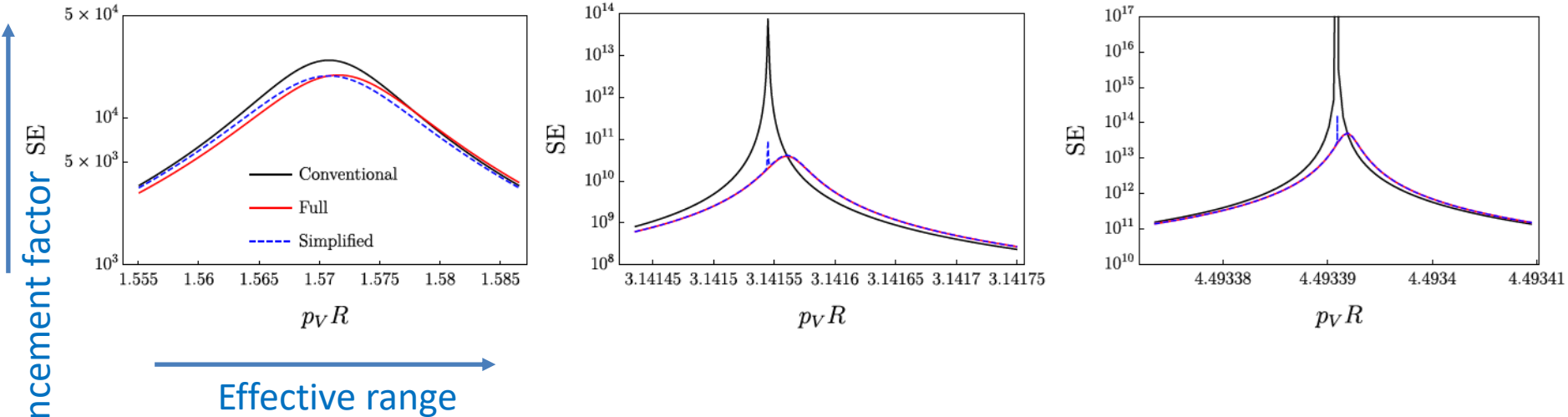
Examples

[Parikh, Sato, Slatyer (2024)]

Spherical well potential

$$V(r) = -\frac{p_V^2}{2\mu} \theta(R - r)$$

$$\frac{\sigma_{ann}}{\sigma_{ann,LO}}$$



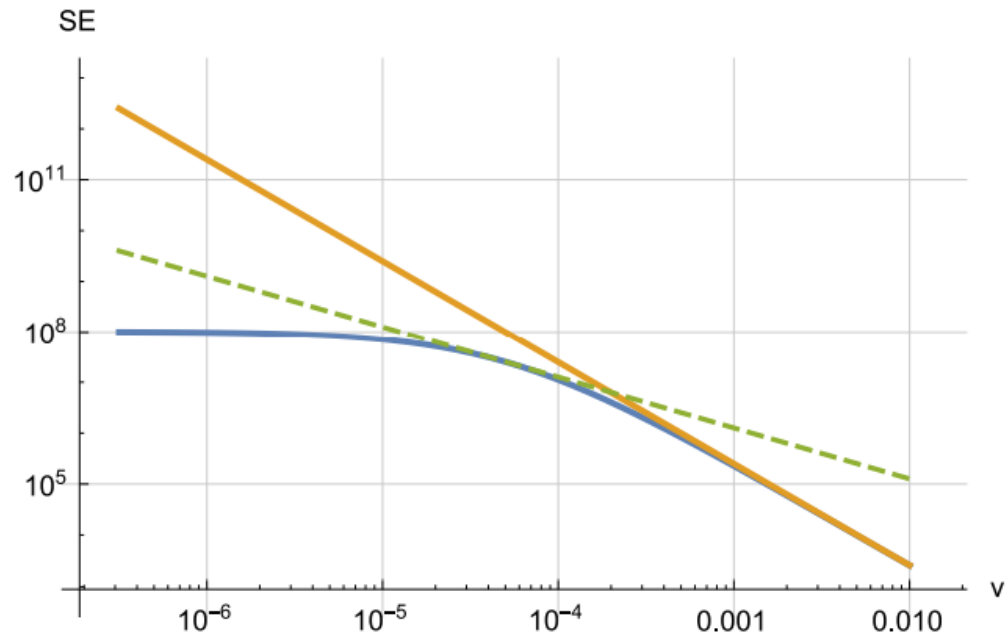
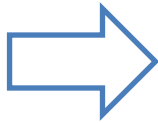
Examples

[Blum, Sato, Slatyer (2016)]

Hulthen potential : $V(r) = -\frac{\alpha m_* e^{-m_* r}}{1 - e^{-m_* r}}$ (Good approximation of $V(r) = -\frac{\alpha e^{-mr}}{r}$, $m_* = \frac{\pi^2}{6} m$)

$$\alpha = 1, \quad \sigma v = \frac{1}{32\pi M^2}, \quad \sigma_{sc} = \frac{\mu^2}{4\pi} (\sigma v)^2$$

$$m_* = 0.0625M$$



Yellow : usual formula

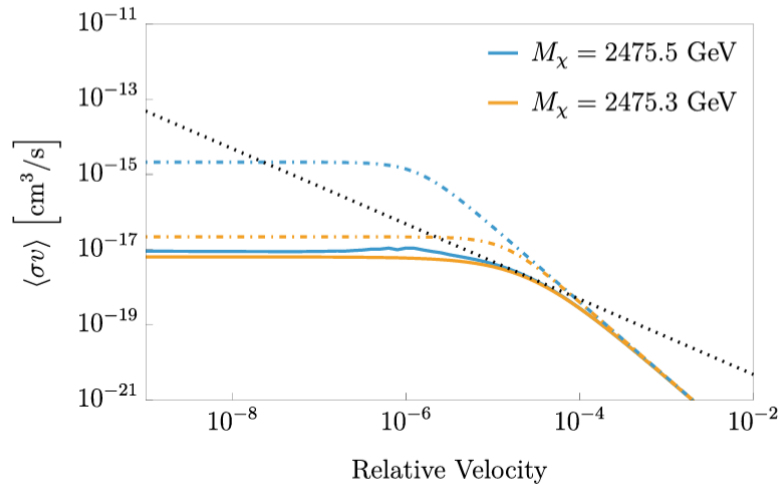
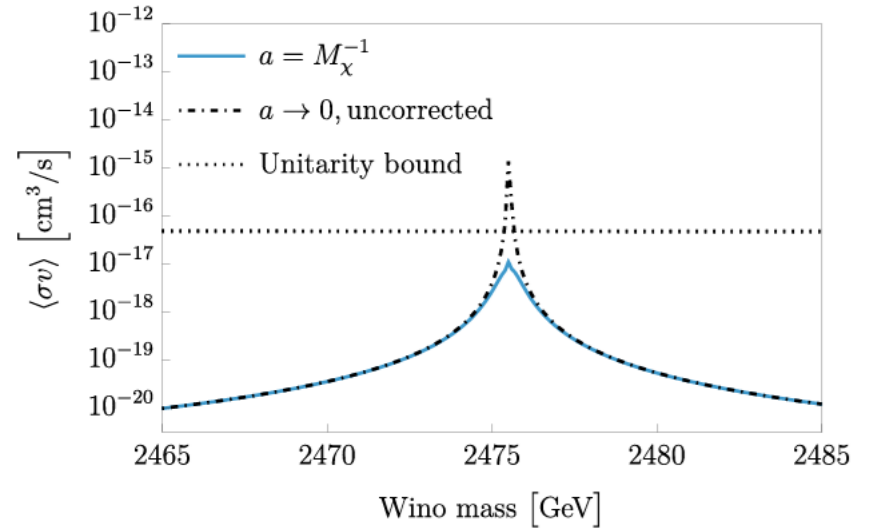
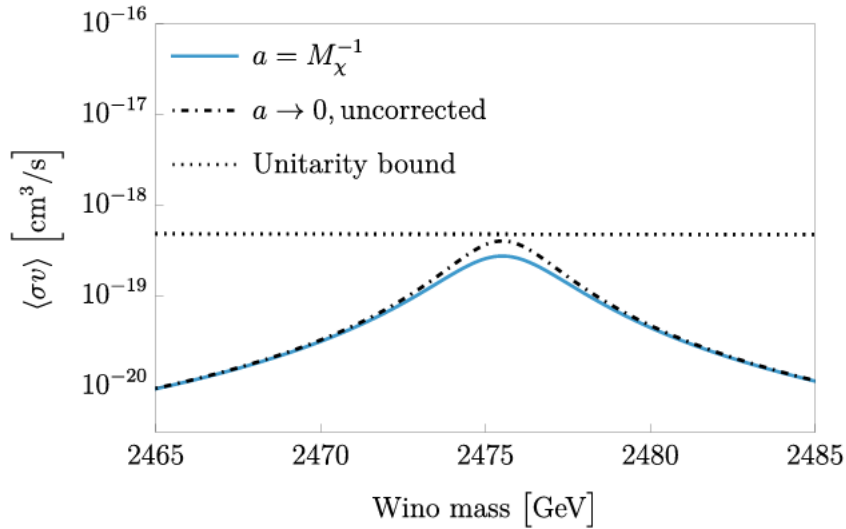
Blue : our formula

Green dotted : Unitarity bound

Examples

[Parikh, Sato, Slatyer (2024)]

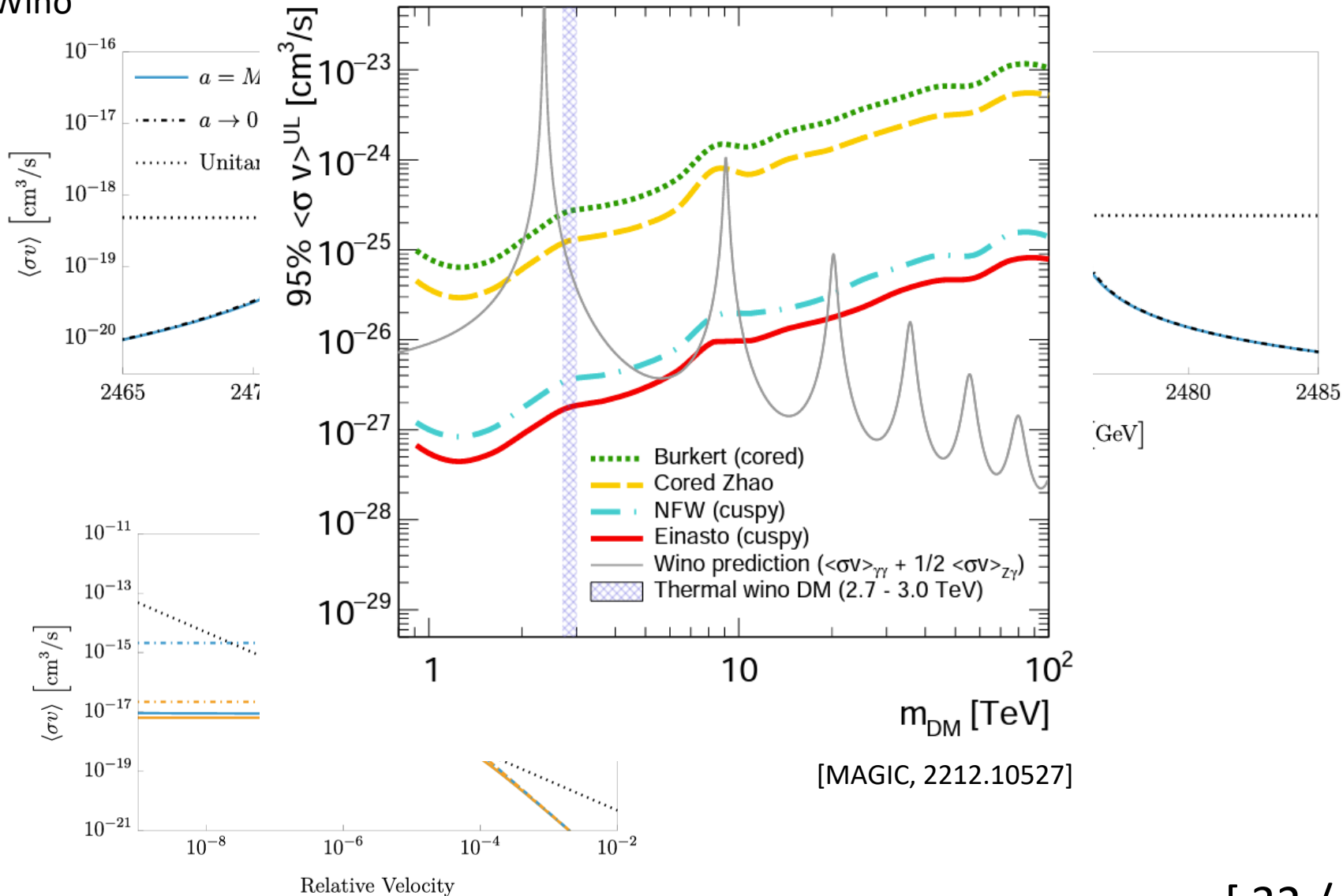
Wino



Examples

[Parikh, Sato, Slatyer (2024)]

Wino



Summary & Outlook

- Annihilation cross section is important for DM phenomenology
- Schroedinger equation can treat long-range force by a light mediator
- The effect of annihilation can be treated as potential with complex coefficient
- This formulation is consistent with unitarity of QM.
- Can be relevant for large LO annihilation cross section (higher mass)

Backup

Wave function w/ long-range force

$$\left[-\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + 2\mu V(r) - p^2 \right] H_\ell^+(r) = 0, \quad H_\ell^+(r) \rightarrow (-i)^\ell \exp(ipr + \delta_\ell^{(L)})$$

$$F_\ell(r) \equiv \text{Im} H_\ell^+ \simeq C_\ell p^{\ell+1} \times \left[\frac{r^{\ell+1}}{(2\ell+1)!!} + \dots \right]$$

Leading term

$$G_\ell(r) \equiv \text{Re} H_\ell^+ \simeq \frac{1}{C_\ell p^\ell} \times \left[\frac{(2\ell-1)!!}{r^\ell} + \dots + z_\ell(p) \frac{r^{\ell+1}}{(2\ell+1)!!} + \dots \right]$$

(basically) leading term

Sizable in some cases

On zero energy resonance,

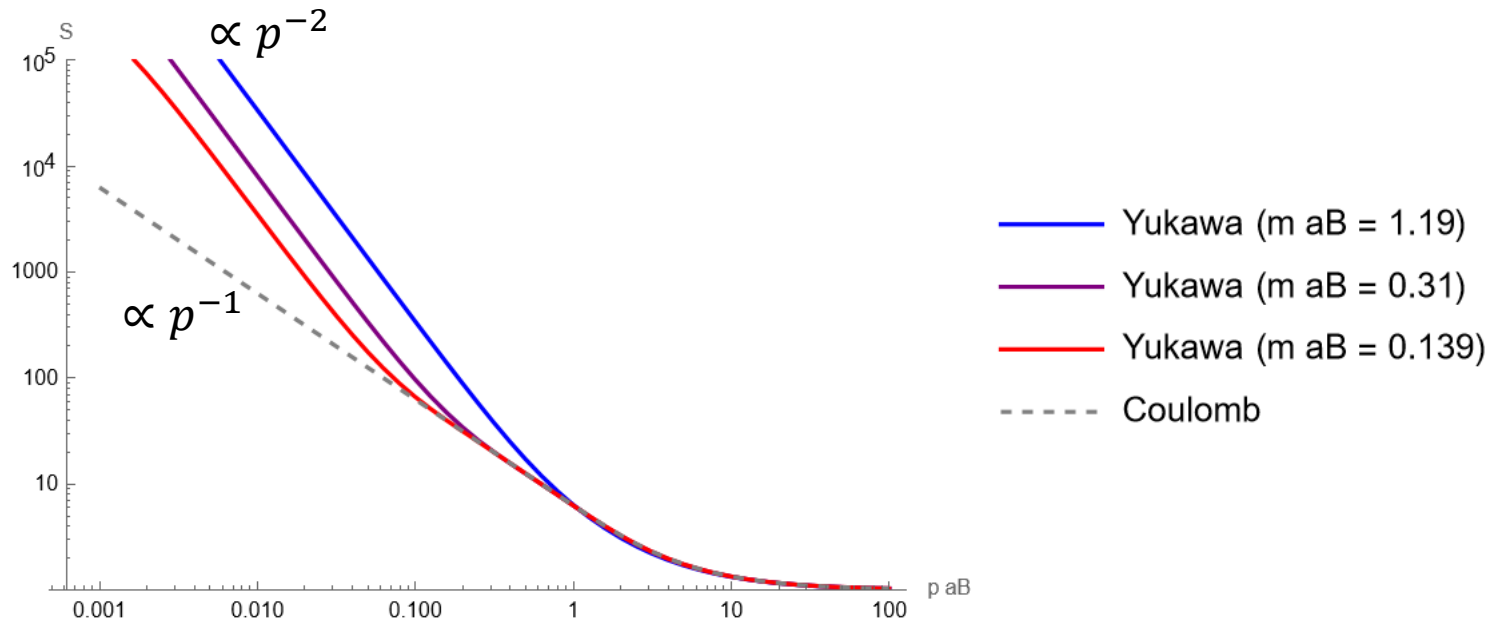
$$C_\ell^2(p) \propto \begin{cases} p^{-2} & (\ell = 0) \\ p^{-4} & (\ell \geq 1) \end{cases} \quad z_\ell(p) \propto \begin{cases} p^0 & (\ell = 0) \\ p^{-2} & (\ell \geq 1) \end{cases}$$

See also [Kamada, Kuwahara, Patel (2023)]

Resonant points

At some specific points ($ma_B = 1.19, 0.31, 0.139, \dots$), $|\psi_{\text{long}}|^2 \propto \frac{1}{p^2}$

(zero energy resonance : bound state with zero binding energy)



$$\sigma = \frac{1}{v} \times |\psi_{\text{long}}|^2 \times (\sigma v)_0 \propto \frac{1}{p^3}$$

Zero energy resonance

Schroedinger eq

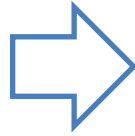
$$\left[-\frac{1}{2\mu} \frac{d^2}{dr^2} + V(r) - \frac{p^2}{2\mu} \right] \chi(r) = 0,$$

Boundary cond.

$$\chi(r) \rightarrow \frac{S \exp(ipr) - \exp(-ipr)}{2i}$$

Bound state w/ $E = -\frac{\kappa^2}{2m}$

$$\chi(r) \rightarrow \exp(-\kappa r)$$



pole in $S(k)$ at $p = i\kappa$

$$\rightarrow S = e^{2i\delta} = -\frac{p + i\kappa}{p - i\kappa}$$

$$\rightarrow \sin \delta = -\frac{\kappa}{p}$$

$$\psi(\vec{x}) = \frac{\chi(r)}{pr}$$

$$|\psi_{\text{long}}(0)|^2 \propto \frac{\chi^2(0)}{p^2} = \frac{\sin^2 \delta}{p^2} = \frac{1}{p^2 + \kappa^2}$$

Solution of Schroedinger eq.

[Blum, Sato, Slatyer (2016)]
[Parikh, Sato, Slatyer (2024)]

Schroedinger equation:

$$\left[-\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + 2\mu V(r) - p^2 \right] u_\ell(r) = 0,$$

$$V(r) = \begin{cases} V_{\text{short}}(r) & (r < a) & \text{complex} \\ V_{\text{long}}(r) & (r \geq a) & \text{real} \end{cases}$$

Solution of Schroedinger eq.

[Blum, Sato, Slatyer (2016)]
[Parikh, Sato, Slatyer (2024)]

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	$r \simeq 0$	$r \rightarrow \infty$
$F_\ell(r)$	$\frac{C_\ell}{(2\ell+1)!!} (pr)^{\ell+1}$	$\sin\left(pr + \delta_\ell^{(L)} - \frac{\pi\ell}{2}\right)$
$G_\ell(r)$	$\frac{(2\ell-1)!!}{C_\ell} (pr)^{-\ell}$	$\cos\left(pr + \delta_\ell^{(L)} - \frac{\pi\ell}{2}\right)$

$$C_\ell \simeq 1, \quad \delta_\ell^{(L)} \simeq 0 \text{ for } V_{\text{long}}(r) \simeq 0$$

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S-matrix

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$$\frac{u_\ell'}{u_\ell} \text{ is continuous at } r = a \Rightarrow \frac{u_{\ell,<}'}{u_{\ell,<}} = \frac{\cos \delta_\ell^{(S)} F_\ell' + \sin \delta_\ell^{(S)} G_\ell'}{\cos \delta_\ell^{(S)} F_\ell + \sin \delta_\ell^{(S)} G_\ell}$$

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$k_\ell(p)$ is *almost* independent on p
(will be discussed later)

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$u_{\ell,<}(r)$: p independent
 $f_\ell(r)$: p independent
 $g_\ell(r)$: *almost* p independent



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Leading term

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(basically) leading term

Sizable in some cases

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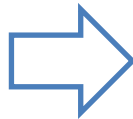
Leading term

$$g_\ell(r) \simeq \frac{(2\ell-1)!!}{r^\ell} + \dots + z_\ell(p) \frac{r^{\ell+1}}{(2\ell+1)!!} + \dots$$

(basically) leading term Sizable in some cases

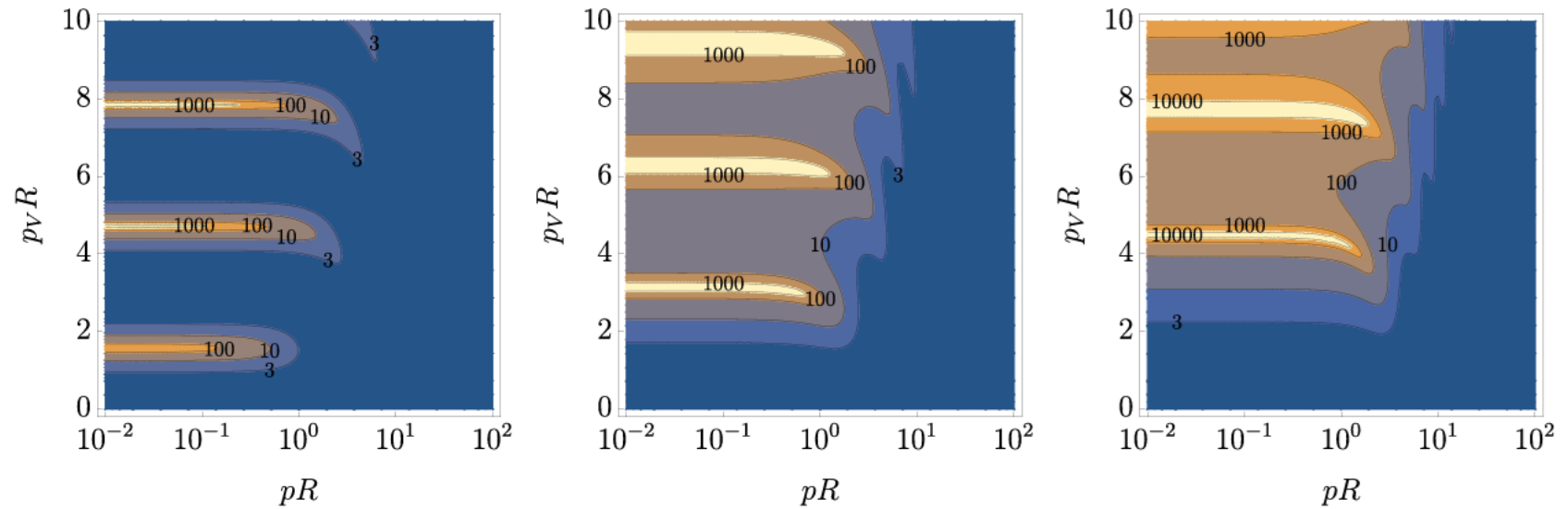
$$k_\ell(p) \simeq k_{\ell,0} + z_\ell(p)$$

(basically) Leading term Sizable in some cases

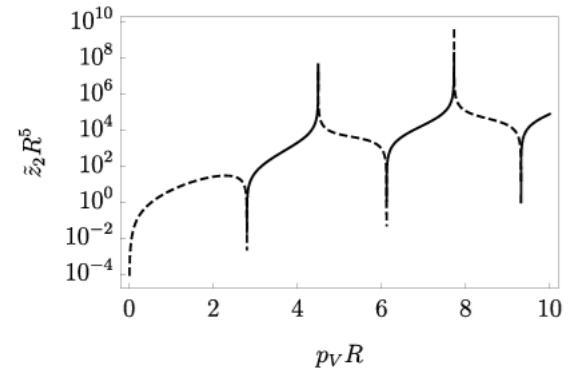
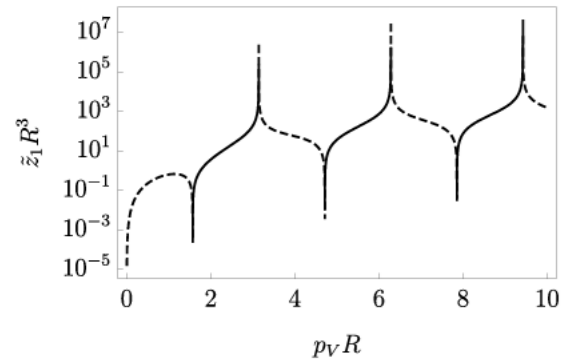
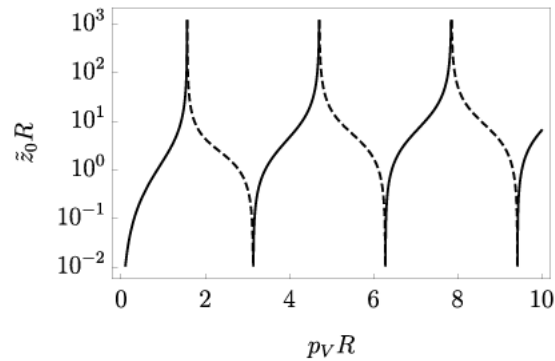


$$S_\ell \simeq \exp\left(2i\delta_\ell^{(L)}\right) \times \frac{k_{\ell,0} + z_\ell(p) - ip^{2\ell+1}C_\ell^2}{k_{\ell,0} + z_\ell(p) + ip^{2\ell+1}C_\ell^2}$$

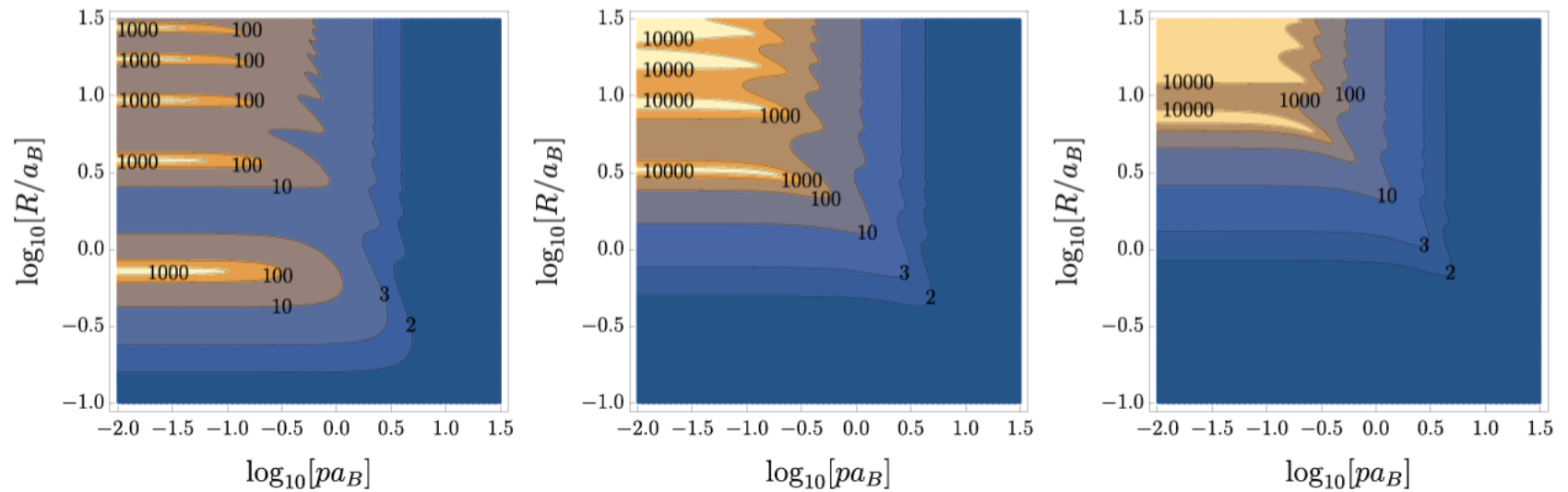
SE for Spherical-well potential



Z function for Spherical-well potential



SE for finite range Coulomb potential



Z function for finite range Coulomb potential

