Sommerfeld effect and unitarity

Ryosuke Sato

大阪大学
THE UNIVERSITY OF OSAKA

K. Blum, R. Sato, T. R. Slatyer, 1603.01383, JCAP 06 (2016) 021
A. Parikh, R. Sato, T. R. Slatyer, 2410.18168

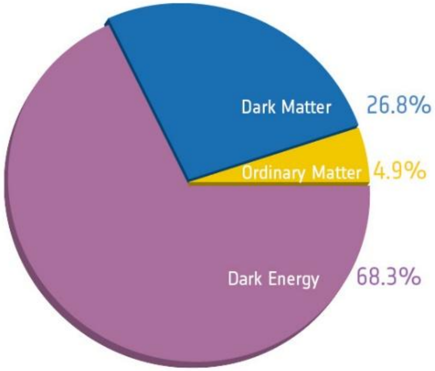
2025. 6. 19 @ The Frontier of Particle Physics:
Exploring Muons, Quantum Science and the Cosmos

Plan

1. Dark matter and Sommerfeld effect
2. Sommerfeld effect and unitarity

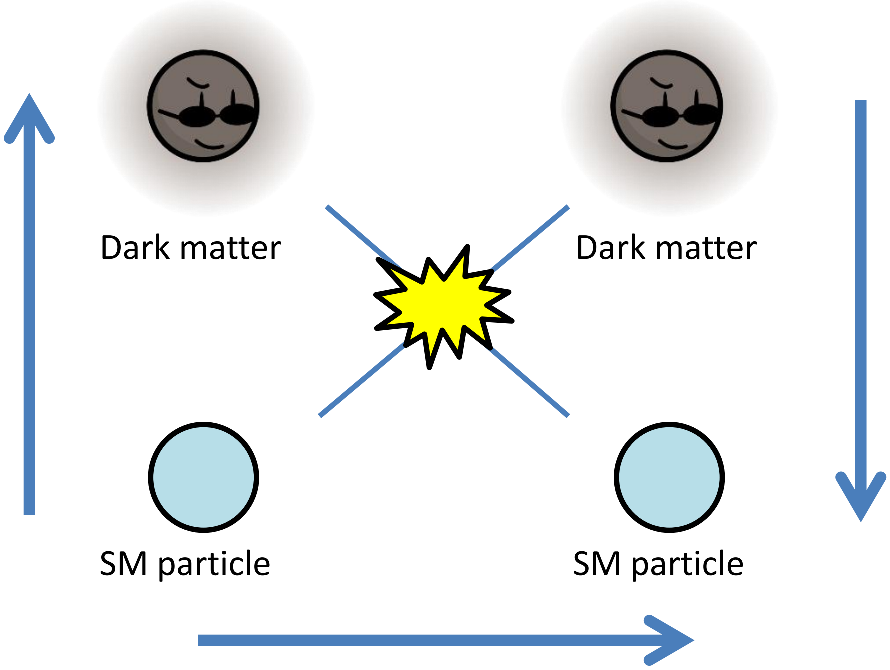
Dark matter and its annihilation

~ 27% of our current universe is made of dark matter
Many evidences but only through gravitational effects.



[Planck collaboration]

Dark Matter could have some interaction with SM particles :



DM pair creation

DM annihilation

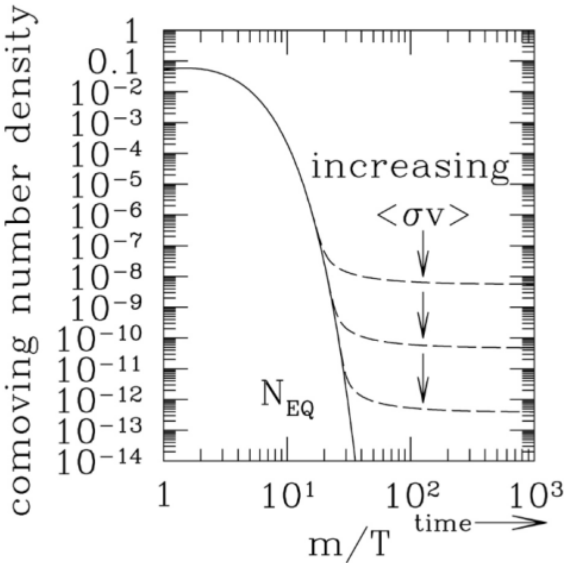
Elastic scattering w/ SM particles

Dark matter annihilation cross section

- **Freeze-out scenario**

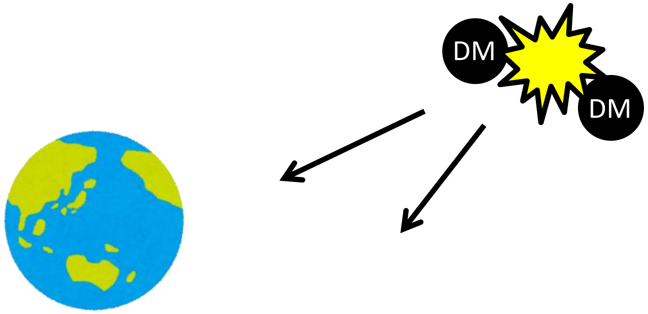
  Relic abundance today

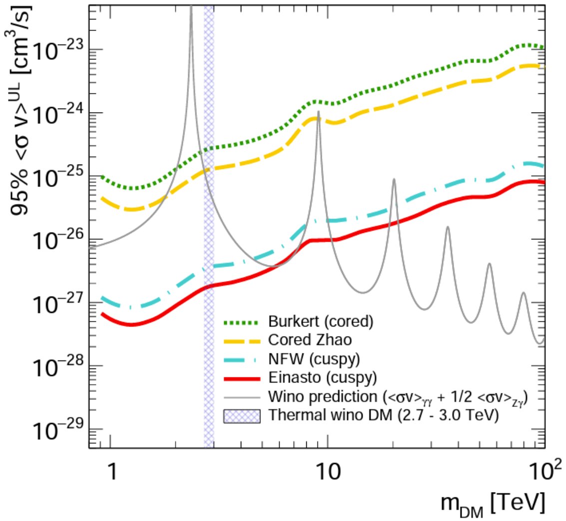
$$\rightarrow \quad \Omega_{\mathrm{DM}} h^2 \sim 0.1 \times \frac{3 \times 10^{-26} cm^3/s}{\sigma v}$$



- **Indirect detection**

  High energy cosmic ray from DM annihilation



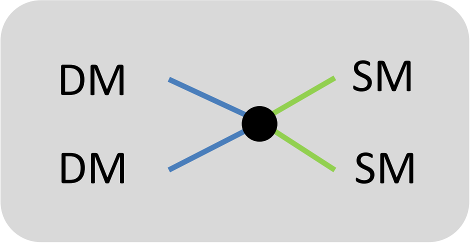


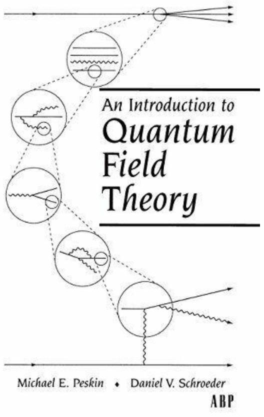
[MAGIC, 2212.10527]

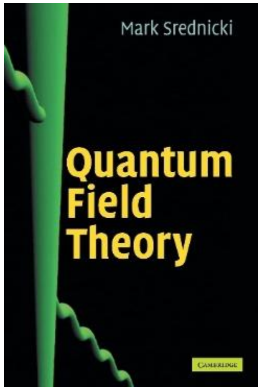
- …

How to calculate σv

If the coupling is <~ 1, perturbative calculation is efficient.







etc…

Typical example (higgs portal) :

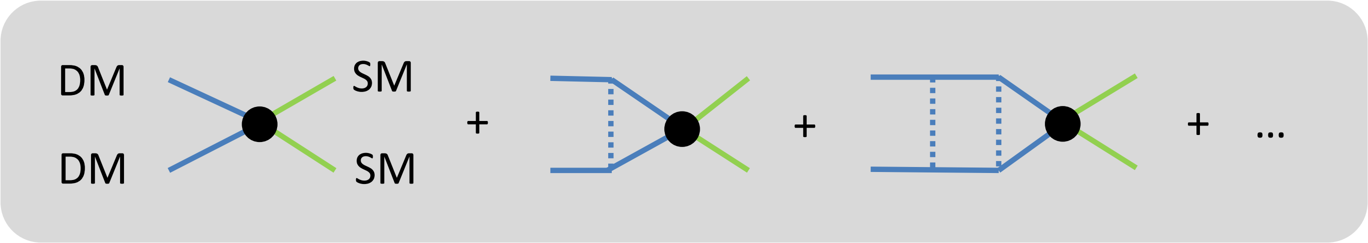
$$L \ni -\frac{\lambda}{4}\phi^2 h^2 \quad \Rightarrow \quad iM(\phi\phi \to hh) = -i\lambda \quad \Rightarrow \quad \sigma v_{rel} = \frac{\lambda^2}{64\pi^2 m_\phi^2}\sqrt{1-\frac{m_h^2}{m_\phi^2}}$$

How to calculate σv

Even if the coupling is $<\sim 1$, non-perturbative resummation is required,

if dark matter couples to a light force mediator boson

$(m_{\text{boson}} \ll m_{\text{DM}})$



M $\quad\quad \sim M\dfrac{\alpha}{v}$ $\quad\quad \sim M\left(\dfrac{\alpha}{v}\right)^2$

ex)
- Wino / Higgsino dark matter
- SU(2) 5-plet dark matter
- …

Sommerfeld effect

[Sommerfeld (1931)]
[Hisano, Matsumoto, Nojiri (2003)]

Strategy

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]
See also [Agrawal, Parikh, Reece (2020)]

We are interested in DM annihilation at non-relativistic regime.

- Freeze-out ($T \simeq m/20$)
- $v \simeq 10^{-3}c$ in galaxy

Schroedinger equation provides effective description!

$$E\psi = -\frac{1}{2\mu}\nabla^2\psi + V(x)\psi$$

$$V(r) = \begin{cases} V_{\text{short}}(r) & (r < a) \quad \text{complex} \\ V_{\text{long}}(r) & (r \geq a) \quad \text{real} \end{cases}$$

Annihilation etc.

Exchange of light boson(s)

ex) $V(r) \sim u\,\delta^3(x) + \dfrac{\alpha}{r}e^{-mr}$

How to calculate DM annihilation

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

Two body scattering problem (à la undergrad quantum mechanics)

$$E\psi = -\frac{1}{2\mu}\nabla^2\psi + V(x)\psi \quad \text{with} \quad \psi \to e^{ipz} + f(\theta)\frac{e^{ikr}}{r}$$

Flux of probability

$$\vec{j}(x) = \frac{1}{\mu}\mathrm{Im}\left[\psi^*(x)\vec{\nabla}\psi(x)\right] \quad \Rightarrow$$

"Non-conservation" of probability

$$\vec{\nabla}\cdot\vec{j}(x) = 2\mathrm{Im}V(x)|\psi(x)|^2$$

$$= 2\mathrm{Im}V_{\mathrm{short}}(x)|\psi(x)|^2$$

How to calculate DM annihilation

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

Two body scattering problem (à la undergrad quantum mechanics)

$$E\psi = -\frac{1}{2\mu}\nabla^2\psi + V(x)\psi$$ with $$\psi \to e^{ipz} + f(\theta)\frac{e^{ikr}}{r}$$

Definition of cross section

$$\sigma \times j_{in} = -\frac{dP}{dt}$$

Flux of incoming wave

$$j_{in} = \frac{1}{\mu}\mathrm{Im}\left[\psi_{\mathrm{in}}^*(x)\vec{\nabla}\psi_{\mathrm{in}}(x)\right] = \frac{p}{\mu} = v$$

Rate of annihilation

$$-\frac{dP}{dt} = \int_{r<a} d^3x\ 2\mathrm{Im}V_{\mathrm{short}}(x)|\psi(x)|^2$$

How to calculate DM annihilation

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

Two body scattering problem (à la undergrad quantum mechanics)

$$E\psi = -\frac{1}{2\mu}\nabla^2\psi + V(x)\psi \qquad \text{with} \qquad \psi \to e^{ipz} + f(\theta)\frac{e^{ikr}}{r}$$

Annihilation cross section

$$\sigma v = \int_{r<a} d^3x\, 2\mathrm{Im}V_{\mathrm{short}}(x)|\psi(x)|^2$$

How to calculate DM annihilation

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

$$\sigma v = \int_{r<a} d^3x\, 2\mathrm{Im}V_{\mathrm{short}}(x)|\psi(x)|^2 \Rightarrow \simeq \int_{r<a} d^3x\, 2\mathrm{Im}V_{\mathrm{short}}(x)\left|\psi_{\mathrm{long}}(x)\right|^2$$

$\psi \simeq \psi_{\mathrm{long}}$ (a.k.a. Distorted Wave Born Approximation)

This should be OK as long as σv is not so large…
(will come back to this point soon)

s. t. $\left[-\frac{1}{2\mu}\nabla^2 + V_{\mathrm{long}}(x) - E\right]\psi_{\mathrm{long}} = 0$

s-wave case:

$$\sigma v \simeq \underbrace{\left|\psi_{\mathrm{long}}(0)\right|^2}_{\text{Enhancement factor}} \times \int_{r<a} d^3x\, 2\mathrm{Im}V_{\mathrm{short}}(x)$$

How to calculate DM annihilation

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

$$\sigma v = \int_{r<a} d^3x \, 2\mathrm{Im}V_{\mathrm{short}}(x)|\psi(x)|^2 \Rightarrow \simeq \int_{r<a} d^3x \, 2\mathrm{Im}V_{\mathrm{short}}(x)\left|\psi_{\mathrm{long}}(x)\right|^2$$

$$\psi \simeq \psi_{\mathrm{long}}$$

(a.k.a. Distorted Wave Born Approximation)

This should be OK as long as σv is not so large…
(will come back to this point soon)

s. t. $\left[-\frac{1}{2\mu}\nabla^2 + V_{\mathrm{long}}(x) - E\right]\psi_{\mathrm{long}} = 0$

s-wave case:

$$\sigma v \simeq \left|\psi_{\mathrm{long}}(0)\right|^2 \times (\sigma v)_0$$

Enhancement factor

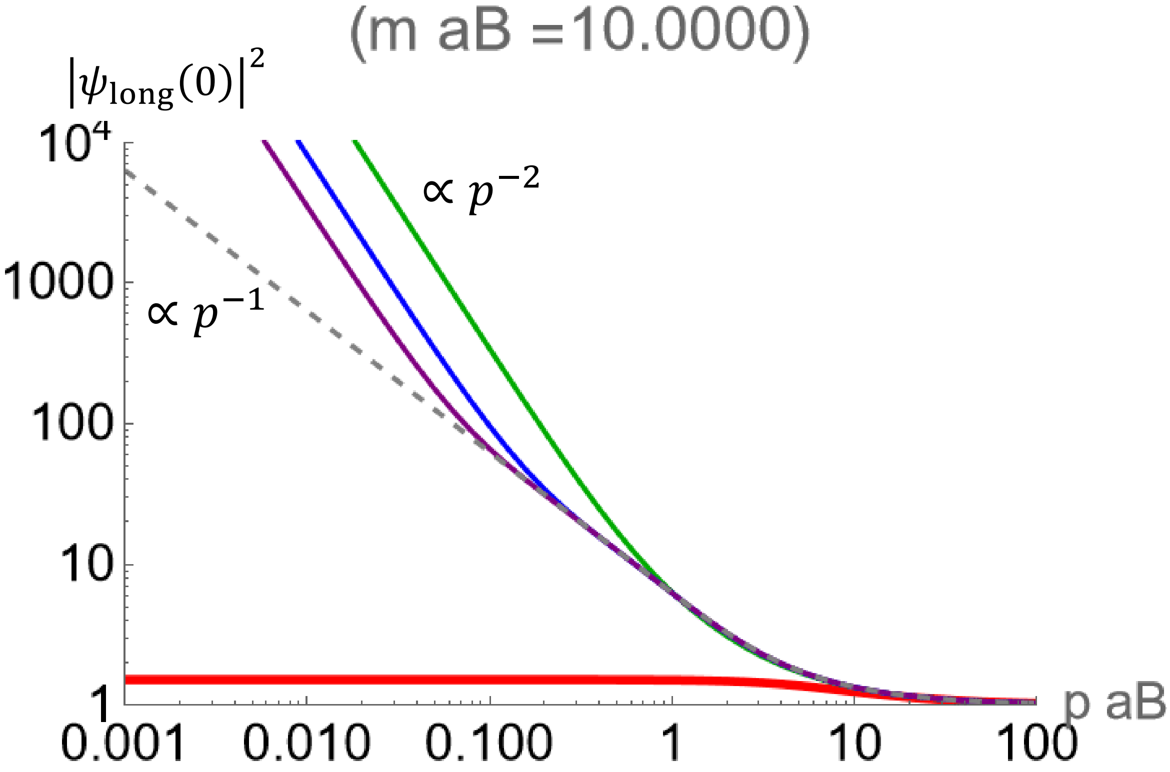
Annihilation cross section
w/o long-range force

[Hisano, Matsumoto, Nojiri (2002)]
[Arkani-hamed, Finkbeiner, Slatyer, Weiner (2008)]
etc

Sommerfeld factor

$$V(r) = -\frac{\alpha}{r}\exp(-mr)$$ Bohr radius : $a_B \equiv \frac{1}{\alpha\mu}$

- σv enhances when $m < \frac{1}{a_B}$ & $p < \frac{1}{a_B}$
- At some specific value of $m a_B$, σv violently enhances



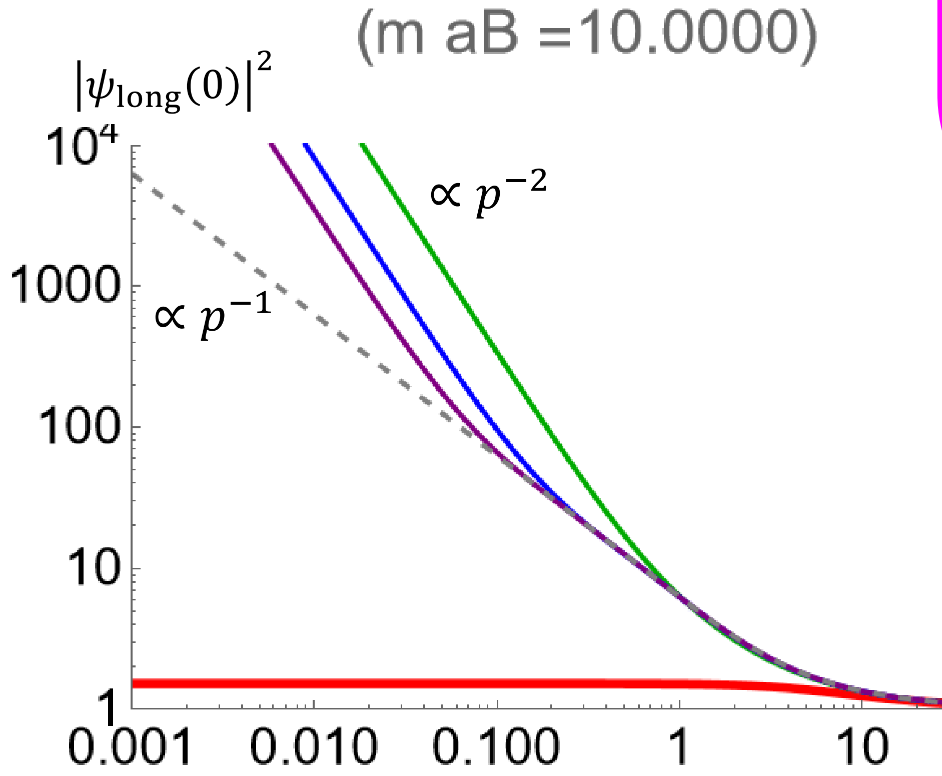
$$|\psi_{\rm long}(0)|^2 = \frac{2\pi}{pa_B}\frac{1}{1-\exp\left(-\frac{2\pi}{pa_B}\right)}$$

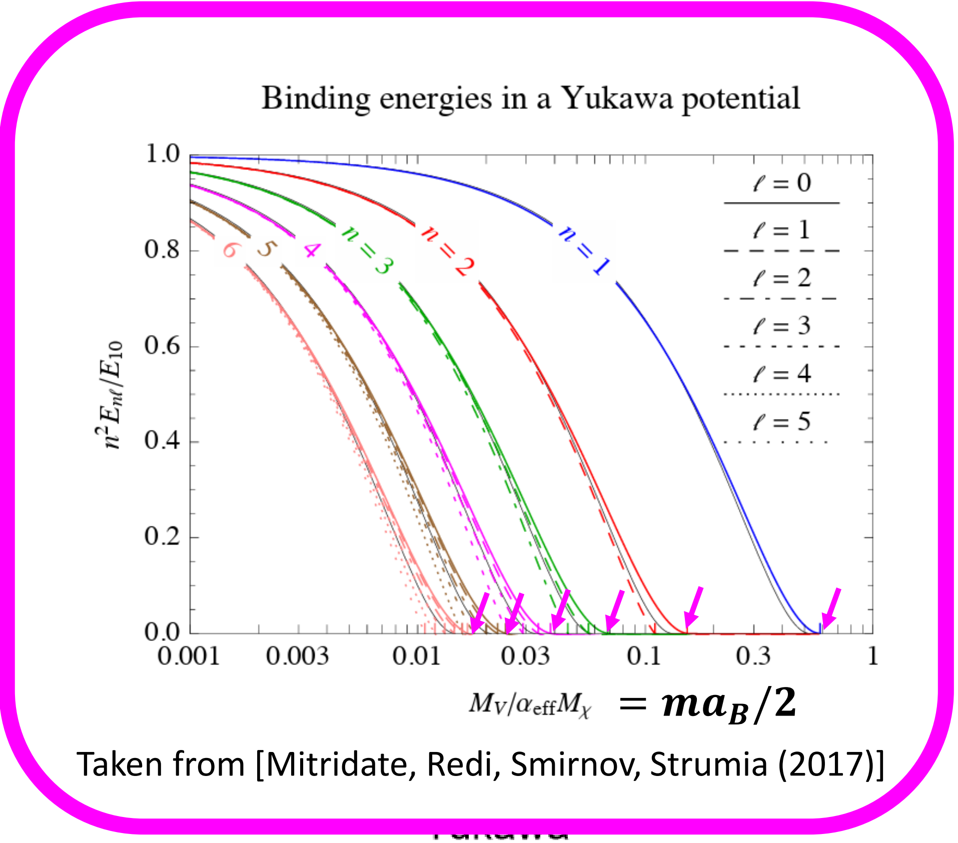
Sommerfeld factor

$$V(r) = -\frac{\alpha}{r}\exp(-mr)$$

Bohr radi

- σv enhances wh
- At some specific





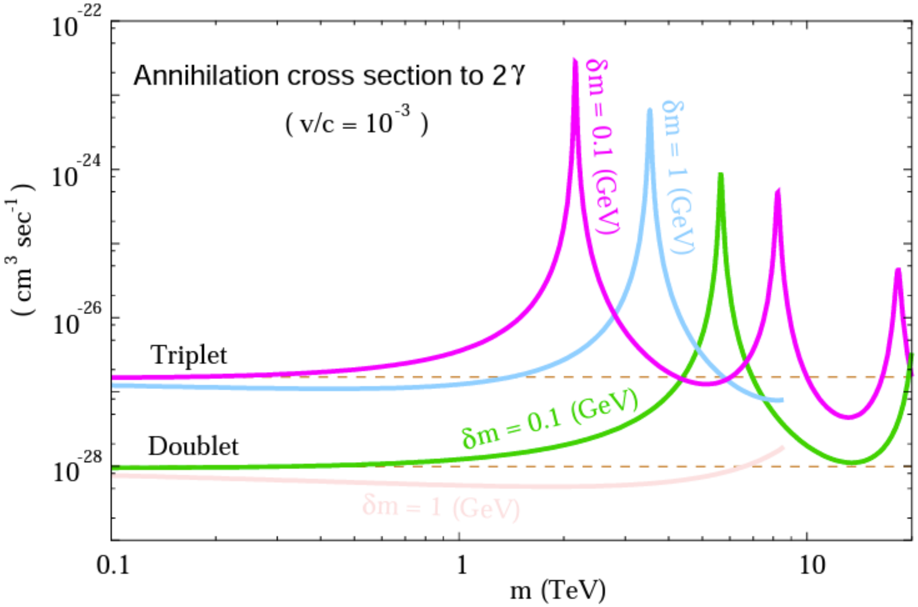
$= \boldsymbol{ma_B/2}$

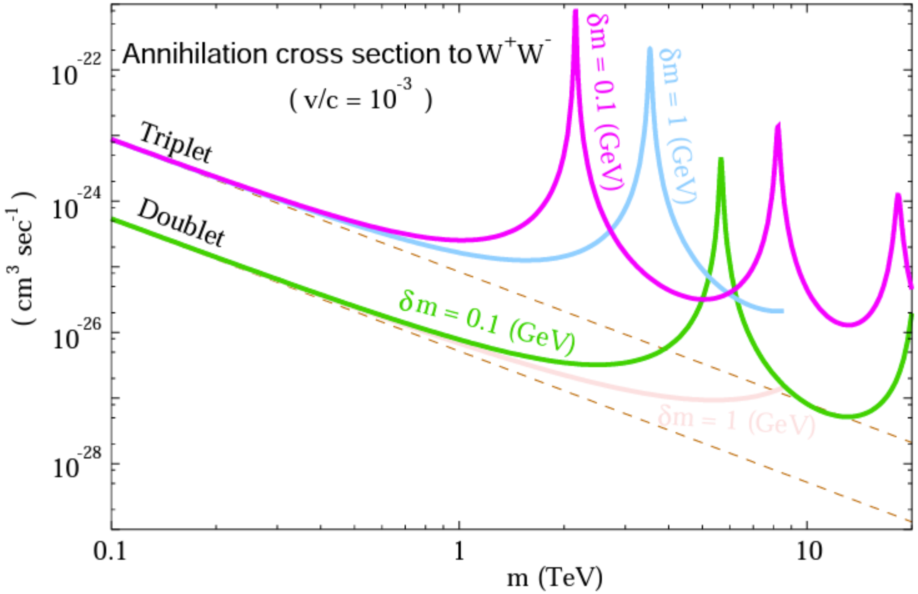
Taken from [Mitridate, Redi, Smirnov, Strumia (2017)]

$$|\psi_{\mathrm{long}}(0)|^2 = \frac{2\pi}{pa_B}\frac{1}{1-\exp\left(-\frac{2\pi}{pa_B}\right)}$$

Wino / Higgsino DM in SUSY

Dashed lines : usual perturbative calculation (w/o Sommerfeld effect)
Colored curves : non-perturbative calculation (w/ Sommerfeld effect)





[Hisano, Matsumoto, Nojiri (2003)]

Huge difference!

Plan

1. Dark matter and Sommerfeld effect
2. Sommerfeld effect and unitarity

Unitarity bound?

$$\boldsymbol{\sigma} = \boldsymbol{\sigma_0} \times \boldsymbol{S(v)}$$

LO cross section : σ_0

Enhancement factor : $S(v) = |\psi(\mathbf{0})|^2$

σ_0 and $S(v)$ are irrelevant each other.

$\Updownarrow$

Unitarity bound on s-wave : $\sigma \leq \frac{\pi}{p^2}$

[Griest, Kamionkowski (1992)]
[Landau-Lifshits's textbook]

Problematic situations

1. Large $\sigma_0 v$
2. At zero energy resonance $(S(v) \propto v^{-2} \rightarrow \sigma \propto v^{-3})$
   (for s-wave)

Unitarity boun

$$\boldsymbol{\sigma} = \boldsymbol{\sigma_0} \times \boldsymbol{S(v)}$$

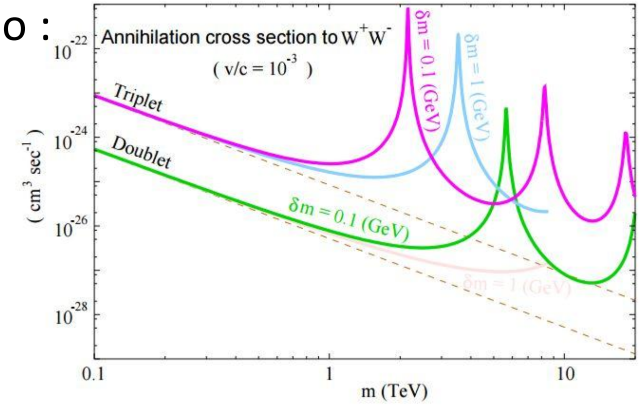
LO cros

Enhan

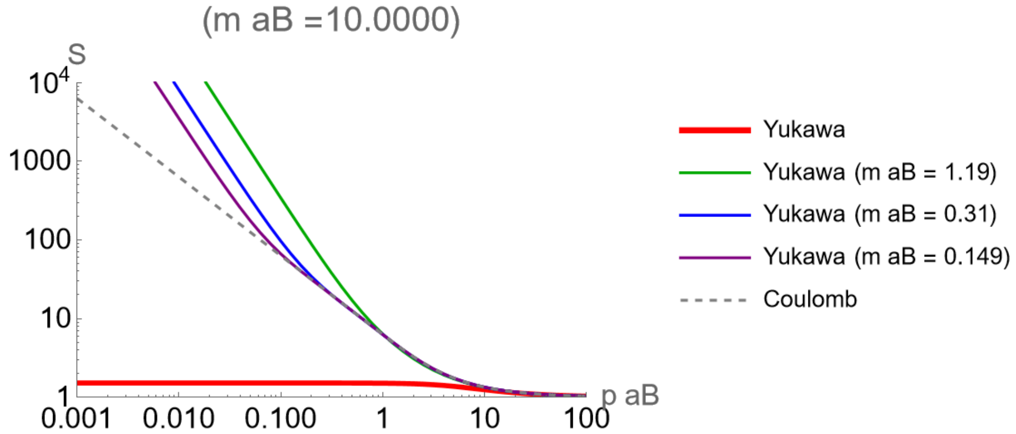
σ_0 and $S(v)$

Unitarity bou

Wino / Higgsino :



Yukawa potential :



Problematic situations

1. Large $\sigma_0 v$

2. At zero energy resonance $(S(v) \propto v^{-2} \to \sigma \propto v^{-3})$
(for s-wave)

Unitarity bound vs. zero-energy resonance

Annihilation cross section on zero-energy resonance

$$\left|\psi_{\mathrm{long}}\right|^2 \propto p^{-2} \quad \Longrightarrow \quad \sigma_{ann,s} \;=\; \frac{1}{v} \times \left|\psi_{long}\right|^2 \times (\sigma v)_0 \;\propto\; \frac{1}{p^3}$$

(for s-wave)

violates unitarity bound at small p

Partial wave expansion in two body scattering

$$E\psi = -\frac{1}{2\mu}\nabla^2\psi + V(x)\psi$$

with $\psi \;\to\; e^{ipz} \;+\; f(\theta)\frac{e^{ikr}}{r} \;=\; \sum_{\ell} P_\ell(\cos\theta)\frac{S_\ell e^{ipr} - (-1)^\ell e^{-ipr}}{2ipr}$

Annihilation cross section

$$\sigma_{ann} = \frac{\pi}{p^2}\sum_{\ell}(2\ell+1)(1-|S_\ell|^2) \quad \Longrightarrow \quad \sigma_{ann,s} \le \frac{\pi}{p^2}$$

Need to solve Schroedinger eq, seriously

Going back to cross section formula…

$$\sigma v = \int_{r<a} d^3x \; 2\mathrm{Im}V_{\mathrm{short}}(x)|\psi(x)|^2 \Rightarrow \simeq \int_{r<a} d^3x \; 2\mathrm{Im}V_{\mathrm{short}}(x)\left|\psi_{\mathrm{long}}(x)\right|^2$$

$\psi \simeq \psi_{\mathrm{long}}$ (a.k.a. Distorted Wave Born Approximation)

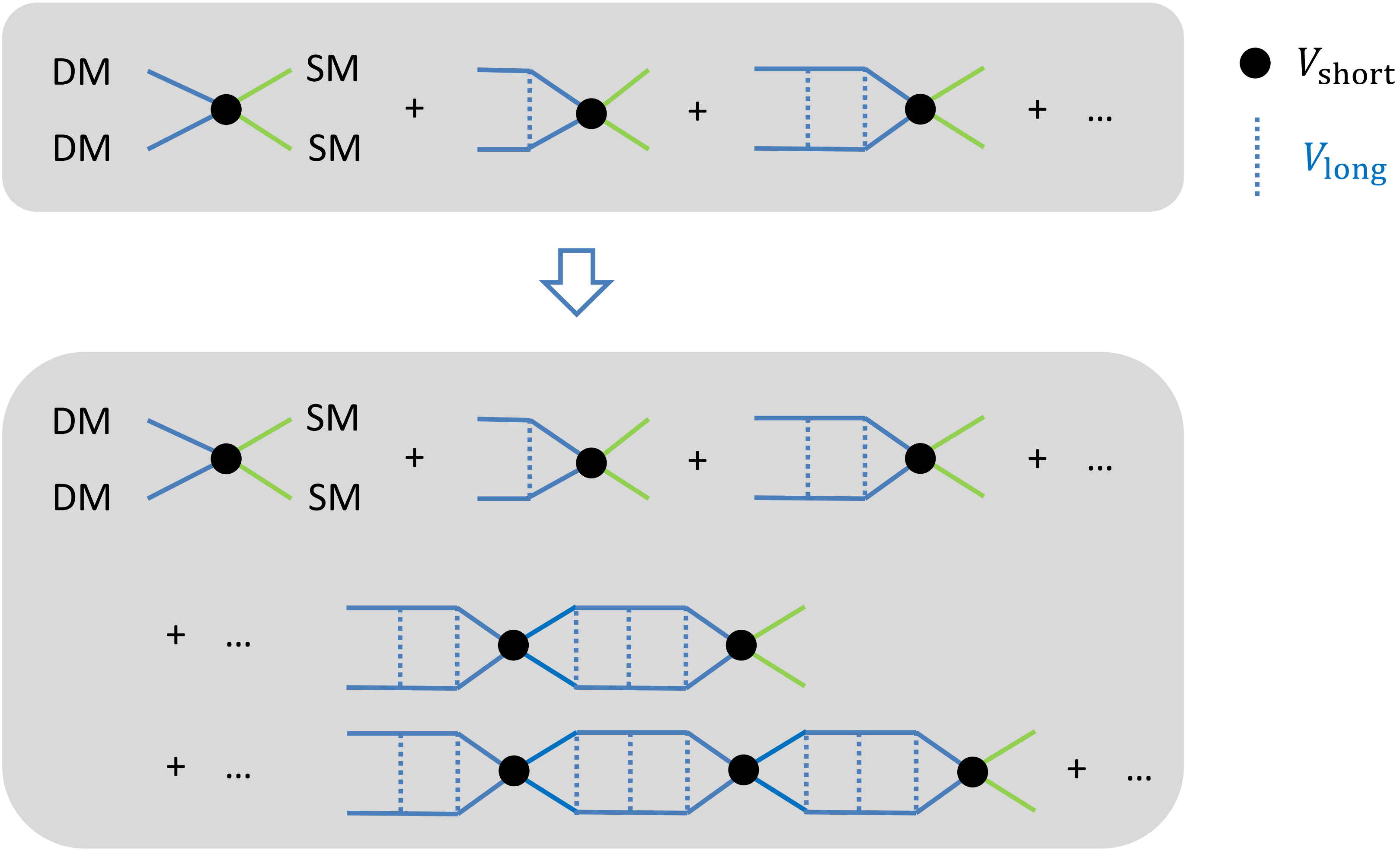
This should be OK as long as σv is not so large…
(will come back to this point soon)

s. t. $\left[-\frac{1}{2\mu}\nabla^2 + V_{\mathrm{long}}(x) - E\right]\psi_{\mathrm{long}} = 0$

ψ is quite different from ψ_{long} if σv is large!

Need to solve Schroedinger eq, seriously

Diagramatic interpretation



S-matrix from Schroedinger eq.

What we need: $\psi = \sum_{\ell} P_{\ell}(\cos\theta) \dfrac{(-1)^{\ell} \chi_{\ell}(r)}{pr}$

Schroedinger eq.

$$\left[-\frac{1}{2\mu}\nabla^2 + V(r) - \frac{p^2}{2\mu}\right]\psi(r) = 0 \quad \Longrightarrow \quad \left[-\frac{1}{2\mu}\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{2\mu r^2} + V(r) - \frac{p^2}{2\mu}\right]\chi_{\ell}(r) = 0$$

$$V(r) = \begin{cases} V_{\text{short}}(r) & (r < a) \quad \text{complex} \\ V_{\text{long}}(r) & (r \geq a) \quad \text{real} \end{cases}$$

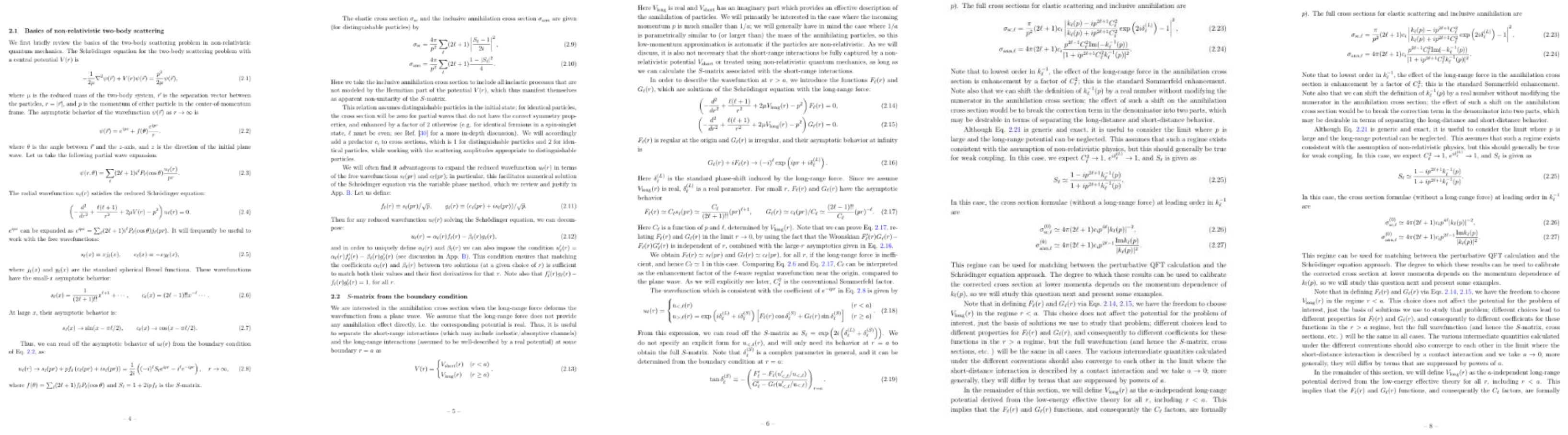
Boundary condition at $r \to \infty$

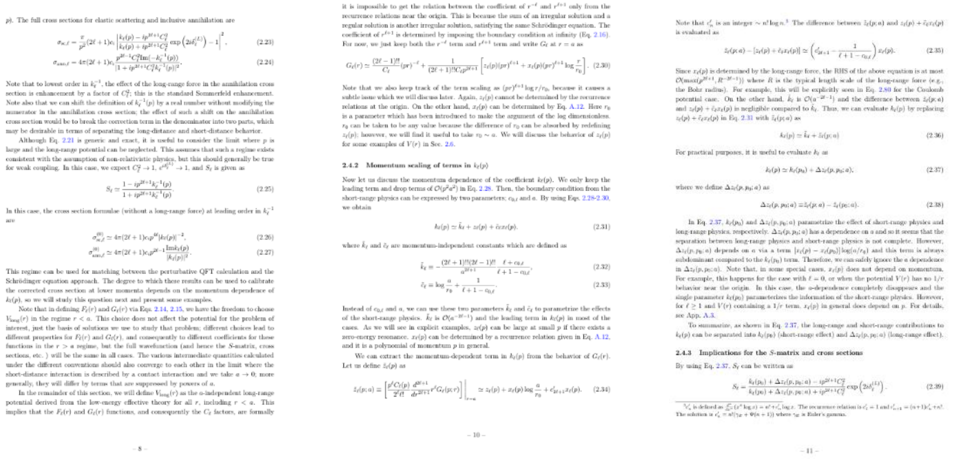
$$\chi_{\ell}(r) \quad \to \quad \frac{S_{\ell} \exp(ipr) \; - \; \exp(-ipr)}{2i}$$

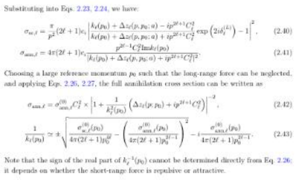
Boundary condition at $r = a$ (condition for short-range effect)

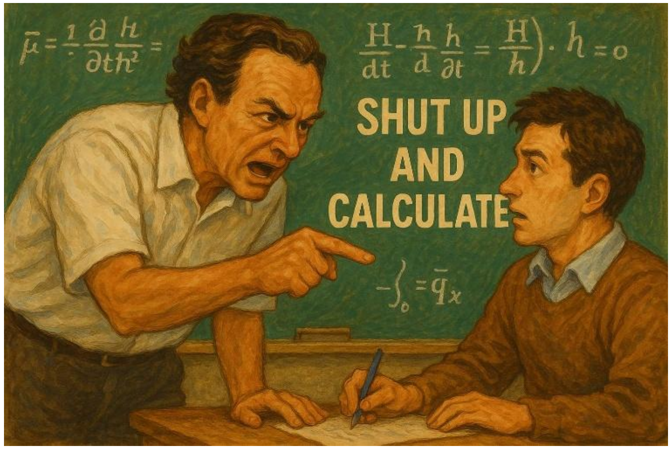
$\chi_{\ell}{}'(r)/\chi_{\ell}(r)$ at $r = a$ is p-independent

Straightforward (but lengthy) calculation









S-matrix

We obtain S-matrix for each ℓ as

$$S_\ell \simeq \underbrace{\exp\left(2i\delta_\ell^{(L)}(p)\right)}_{\text{Phase-shift by long-range force}} \times \underbrace{\frac{k_{\ell,0} + z_\ell(p) - ip^{2\ell+1}C_\ell^2(p)}{k_{\ell,0} + z_\ell(p) + ip^{2\ell+1}C_\ell^2(p)}}_{\text{Relevant part for annihilation}}$$

a single complex parameter : $k_{\ell,0}$

Three real functions : $C_\ell^2(p)$, $z_\ell(p)$, $\delta_\ell(p)$

S-matrix

We obtain S-matrix for each ℓ as

$$S_\ell \simeq \underbrace{\exp\left(2i\delta_\ell^{(L)}(p)\right)}_{\text{Phase-shift by long-range force}} \times \underbrace{\frac{k_{\ell,0}+z_\ell(p)-ip^{2\ell+1}C_\ell^2(p)}{k_{\ell,0}+z_\ell(p)+ip^{2\ell+1}C_\ell^2(p)}}_{\text{Relevant part for annihilation}}$$

Annihilation cross section :

Unitarity bound ✓

$$\sigma_{ann,\ell} = \frac{\pi}{p^2}(2\ell+1)(1-|S_\ell|^2) < \frac{(2\ell+1)\pi}{p^2}$$

S-matrix

We obtain S-matrix for each ℓ as

$$S_\ell \simeq \underbrace{\exp\left(2i\delta_\ell^{(L)}(p)\right)}_{\text{Phase-shift by long-range force}} \times \underbrace{\frac{k_{\ell,0} + z_\ell(p) - ip^{2\ell+1}C_\ell^2(p)}{k_{\ell,0} + z_\ell(p) + ip^{2\ell+1}C_\ell^2(p)}}_{\text{Relevant part for annihilation}}$$

Annihilation cross section :

$$\sigma_{ann,\ell} = \frac{\pi}{p^2}(2\ell+1)(1-|S_\ell|^2) \quad \overset{\text{Unitarity bound ✓}}{<} \quad \frac{(2\ell+1)\pi}{p^2}$$

$$= \frac{\pi}{p^2}(2\ell+1) \times 4\mathrm{Re}\left[\frac{ip^{2\ell+1}C_\ell^2}{k_{\ell,0}}\right] \times \left|1 + \frac{z_\ell + ip^{2\ell+1}C_\ell^2}{k_{\ell,0}}\right|^{-2}$$

S-matrix

We obtain S-matrix for each ℓ as

$$S_\ell \simeq \underbrace{\exp\left(2i\delta_\ell^{(L)}(p)\right)}_{\text{Phase-shift by long-range force}} \times \underbrace{\frac{k_{\ell,0} + z_\ell(p) - ip^{2\ell+1}C_\ell^2(p)}{k_{\ell,0} + z_\ell(p) + ip^{2\ell+1}C_\ell^2(p)}}_{\text{Relevant part for annihilation}}$$

Annihilation cross section :

Unitarity bound ✓

$$\sigma_{ann,\ell} = \frac{\pi}{p^2}(2\ell+1)(1-|S_\ell|^2) < \frac{(2\ell+1)\pi}{p^2}$$

$$= 4\pi(2\ell+1)p^{2\ell-1}\,\mathrm{Im}\left[-\frac{1}{k_{\ell,0}}\right] \times C_\ell^2 \times \left|1+\frac{z_\ell + ip^{2\ell+1}C_\ell^2}{k_{\ell,0}}\right|^{-2}$$

Annihilation cross section
w/o long-range force

Conventional
Sommerfeld factor

Correction factor

Examples

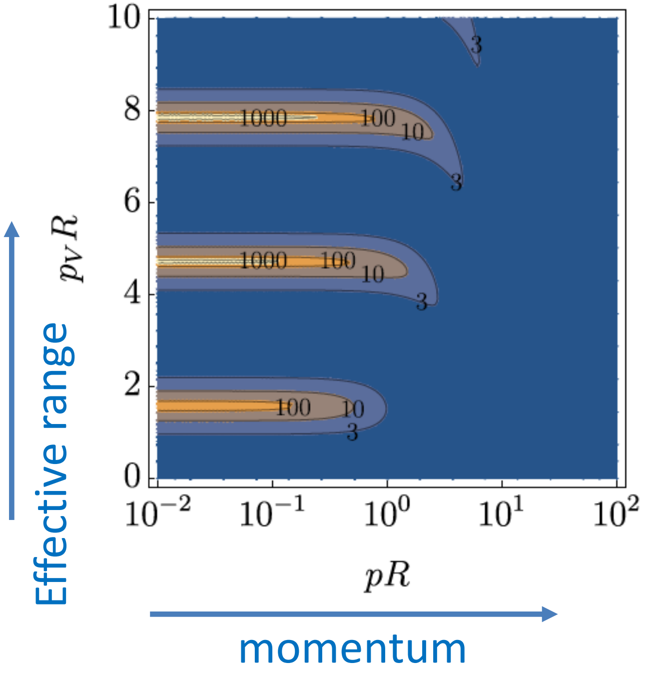
[Parikh, Sato, Slatyer (2024)]

Spherical well potential

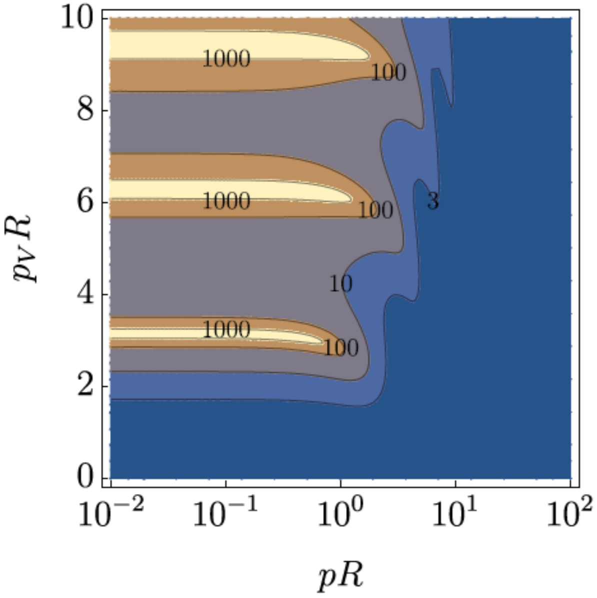
$$V(r) = -\frac{p_V^2}{2\mu}\theta(R-r)$$

(Conventional) Sommerfeld factor : C_ℓ^2

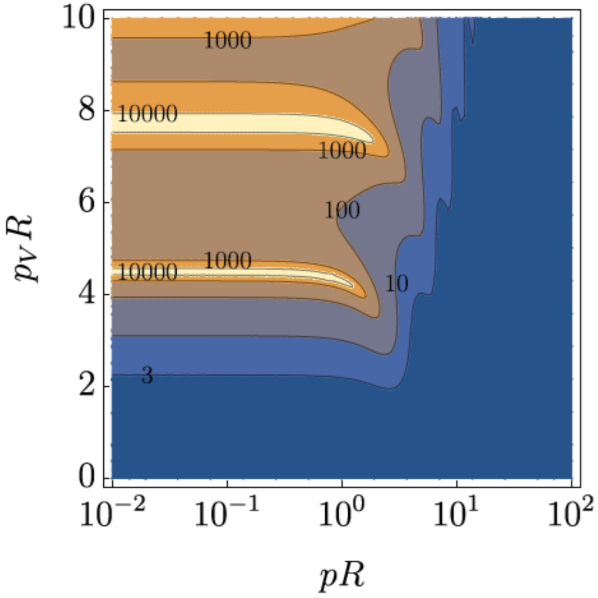
S-wave



P-wave



D-wave



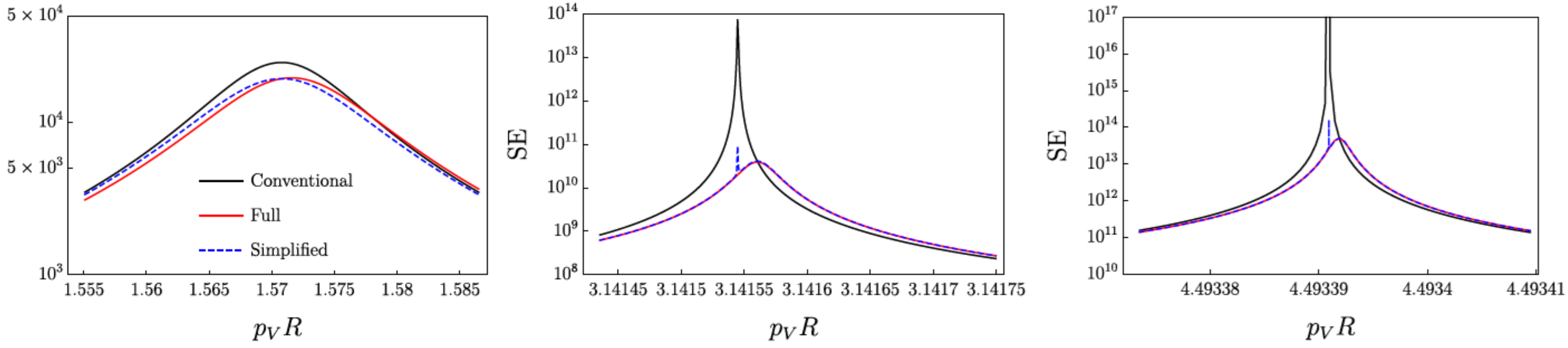
Examples

[Parikh, Sato, Slatyer (2024)]

Spherical well potential

$$V(r) = -\frac{p_V^2}{2\mu}\theta(R-r)$$

$$\frac{\sigma_{ann}}{\sigma_{ann,\mathrm{LO}}}$$



Enhancement factor

Effective range

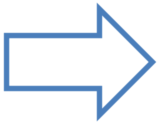
Examples

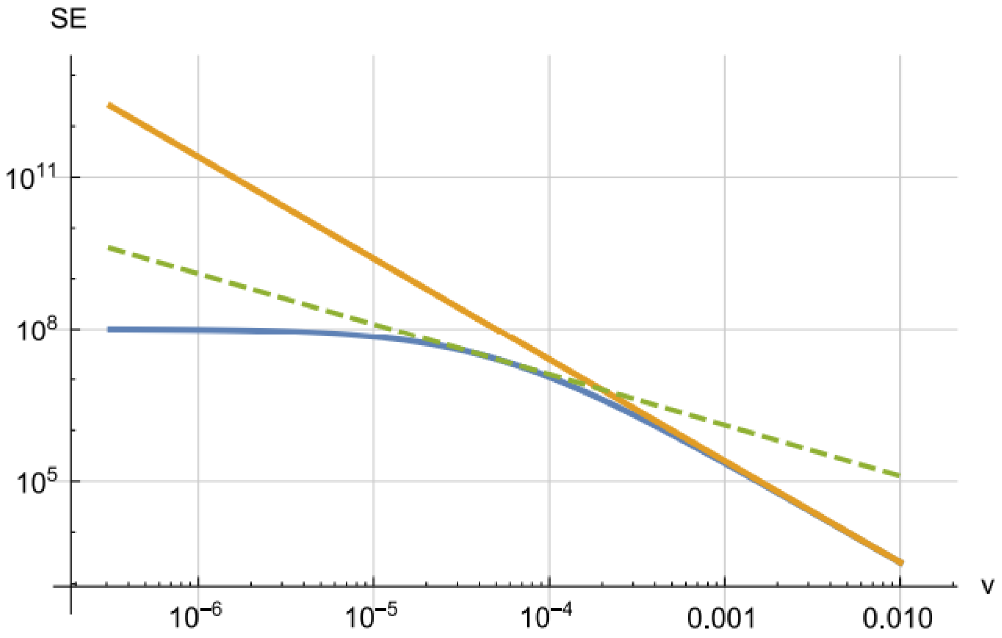
[Blum, Sato, Slatyer (2016)]

Hulthen potential : $V(r) = -\frac{\alpha m_* e^{-m_* r}}{1 - e^{-m_* r}}$ (Good approximation of $V(r) = -\frac{\alpha e^{-mr}}{r}, m_* = \frac{\pi^2}{6} m$)

$\alpha = 1, \qquad \sigma v = \frac{1}{32\pi M^2}, \qquad \sigma_{sc} = \frac{\mu^2}{4\pi}(\sigma v)^2$

$m_* = 0.0625M$





Yellow : usual formula

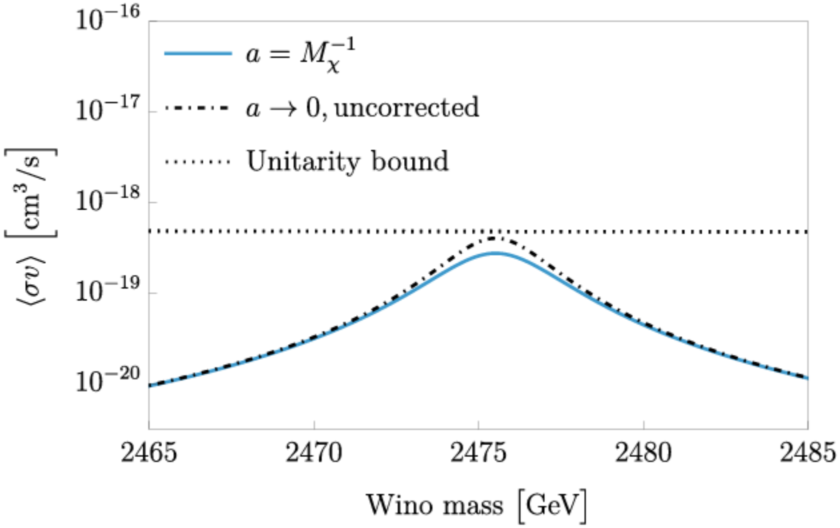
Blue : our formula

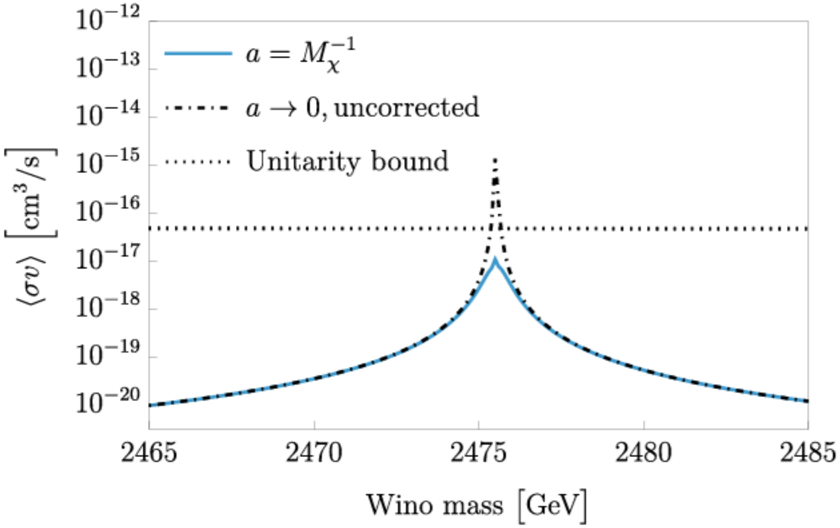
Green dotted : Unitarity bound

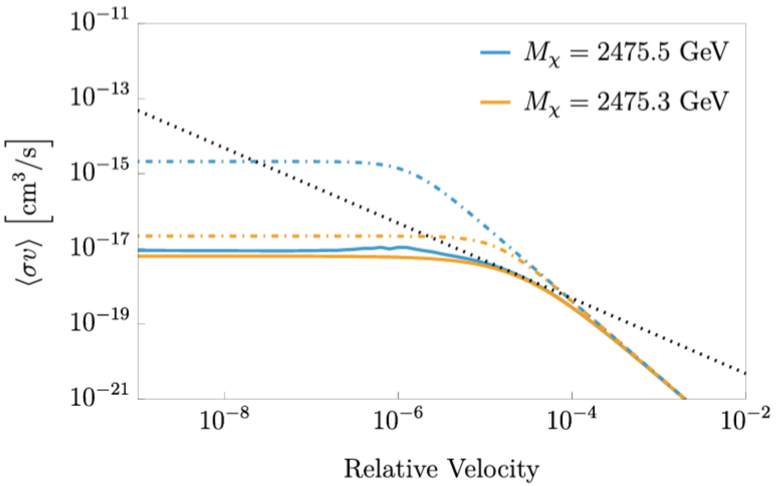
Examples

[Parikh, Sato, Slatyer (2024)]

Wino



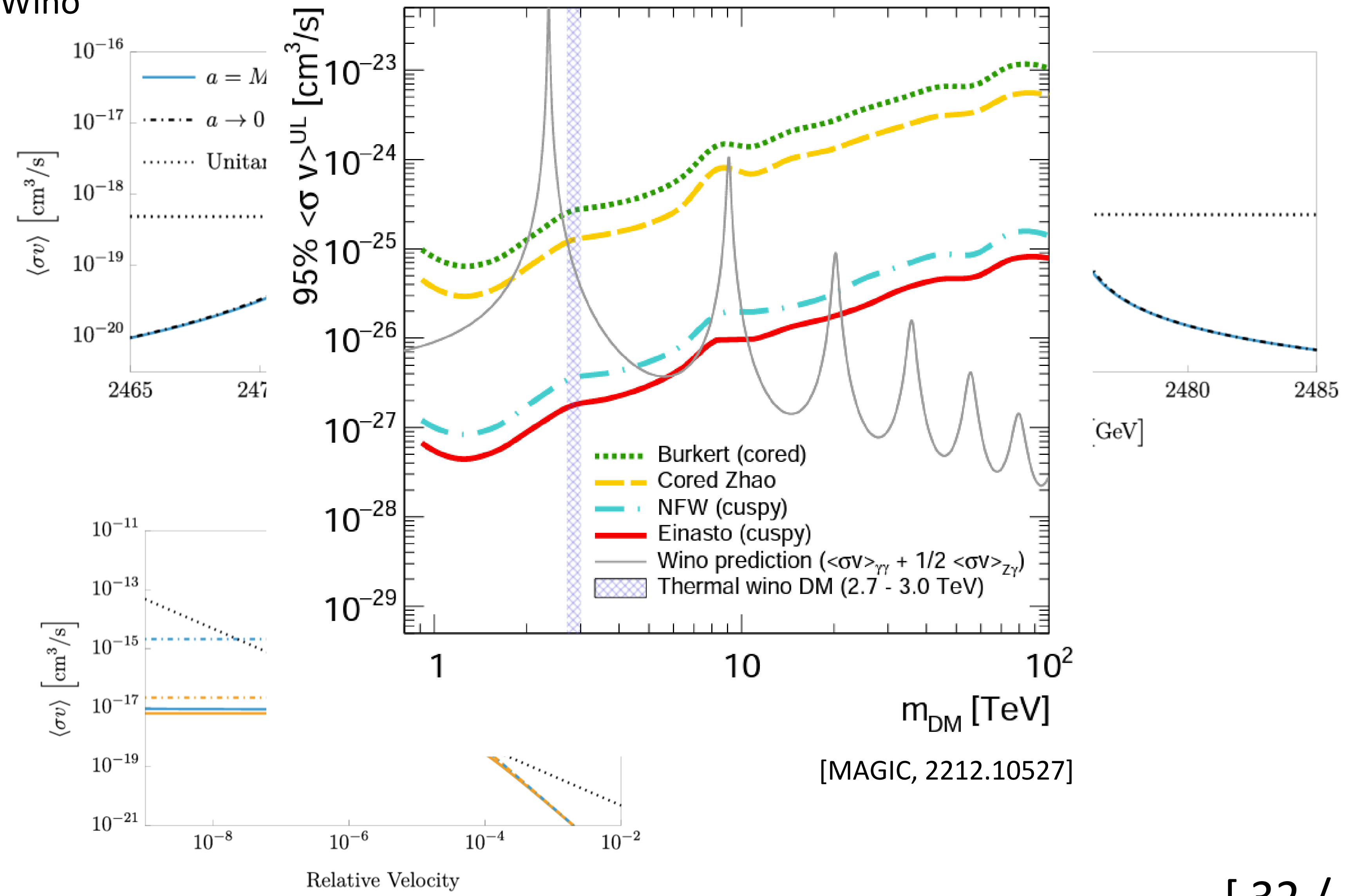




Examples

[Parikh, Sato, Slatyer (2024)]

Wino



[MAGIC, 2212.10527]

Summary & Outlook

- Annihilation cross section is important for DM phenomenology
- Schroedinger equation can treat long-range force by a light mediator
- The effect of annihilation can be treated as potential with complex coefficient
- This formulation is consistent with unitarty of QM.
- Can be relevant for large LO annihilation cross section (higher mass)

Backup

Wave function w/ long-range force

$$\left[-\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{r^2}+2\mu V(r)-p^2\right]H_\ell^+(r)=0, \qquad H_\ell^+(r)\to(-i)^\ell\exp\left(ipr+\delta_\ell^{(L)}\right)$$

$$F_\ell(r)\ \equiv\ \mathrm{Im}H_\ell^+\ \simeq\ C_\ell p^{\ell+1}\times\left[\underbrace{\frac{r^{\ell+1}}{(2\ell+1)\,!!}}_{\text{Leading term}}+\ \dots\right]$$

$$G_\ell(r)\ \equiv\ \mathrm{Re}H_\ell^+\ \simeq\ \frac{1}{C_\ell p^\ell}\times\left[\underbrace{\frac{(2\ell-1)\,!!}{r^\ell}}_{\text{(basically) leading term}}+\ \dots\ +\underbrace{z_\ell(p)\frac{r^{\ell+1}}{(2\ell+1)\,!!}}_{\text{Sizable in some cases}}+\ \dots\ \right]$$

On zero energy resonance,

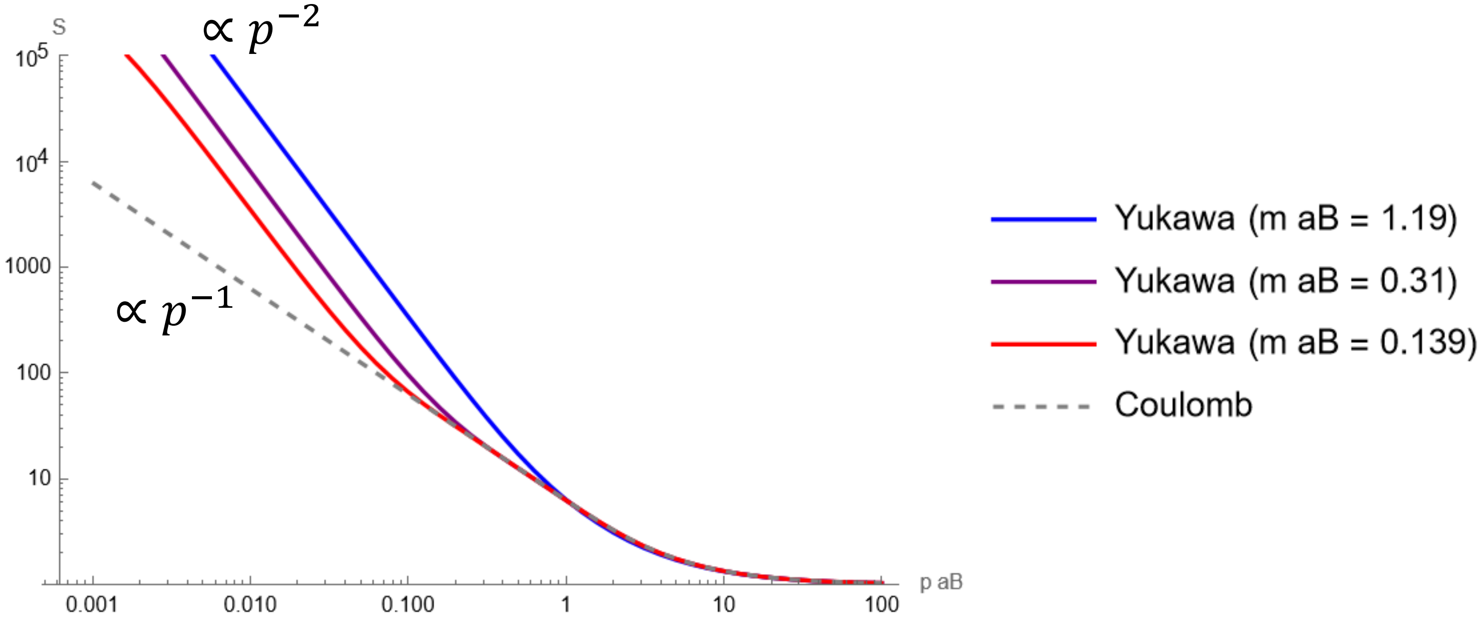
$$C_\ell^2(p)\ \propto\ \left\{\begin{array}{ll} p^{-2} & (\ell=0) \\ p^{-4} & (\ell\geq 1)\end{array}\right. \qquad z_\ell(p)\ \propto\ \left\{\begin{array}{ll} p^{0} & (\ell=0) \\ p^{-2} & (\ell\geq 1)\end{array}\right.$$

See also [Kamada, Kuwahara, Patel (2023)]

Resonant points

At some specific points ($ma_B = 1.19,\ \ 0.31,\ \ 0.139, \ldots$), $\left|\psi_{\text{long}}\right|^2 \propto \dfrac{1}{p^2}$

(zero energy resonance : bound state with zero binding energy)



$$\sigma = \frac{1}{v} \times \left|\psi_{long}\right|^2 \times (\sigma v)_0 \propto \frac{1}{p^3}$$

Zero energy resonance

Schroedinger eq

$$\left[-\frac{1}{2\mu}\frac{d^2}{dr^2}+V(r)-\frac{p^2}{2\mu}\right]\chi(r)=0,$$

Boundary cond.

$$\chi(r) \;\to\; \frac{S\exp(ipr) \;-\; \exp(-ipr)}{2i}$$

Bound state w/ $E=-\frac{\kappa^2}{2m}$

$$\chi(r) \;\to\; \exp(-\kappa r)$$

⇨

pole in $S(k)$ at $p=i\kappa$

$$\to \quad S=e^{2i\delta}=-\frac{p+i\kappa}{p-i\kappa}$$

$$\to \quad \sin\delta=-\frac{\kappa}{p}$$

$$\psi(\vec{x})=\frac{\chi(r)}{pr}$$

$$\left|\psi_{\text{long}}(0)\right|^2 \;\propto\; \frac{\chi^2(0)}{p^2}=\frac{\sin^2\delta}{p^2}=\frac{1}{p^2+\kappa^2}$$

Solution of Schroedinger eq.

[Blum, Sato, Slatyer (2016)]
[Parikh, Sato, Slatyer (2024)]

Schroedinger equation:

$$\left[-\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{r^2}+2\mu V(r)-p^2\right]u_\ell(r)=0, \qquad V(r) \;=\; \begin{cases} V_{\text{short}}(r) & (r<a) \quad \text{complex} \\ V_{\text{long}}(r) & (r\geq a) \quad \text{real} \end{cases}$$

Solution of Schroedinger eq.

[Blum, Sato, Slatyer (2016)]
[Parikh, Sato, Slatyer (2024)]

Schroedinger equation:

$$\left[-\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{r^2}+2\mu V(r)-p^2\right]u_\ell(r)=0, \qquad V(r) = \begin{cases} V_{\text{short}}(r) & (r<a) \quad \text{complex} \\ V_{\text{long}}(r) & (r \geq a) \quad \text{real} \end{cases}$$

u_ℓ should be a linear combination of
- $F_\ell(r)$ (regular solution)
- $G_\ell(r)$ (irregular solution)

at $r > a$

$$\left[-\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{r^2}+2\mu V_{long}(r)-p^2\right]F_\ell(r)=0 \quad \& \quad \left[-\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{r^2}+2\mu V_{long}(r)-p^2\right]G_\ell(r)=0$$

	$r \simeq 0$	$r \to \infty$
$F_\ell(r)$	$\frac{C_\ell}{(2\ell+1)\,!!}(pr)^{\ell+1}$	$\sin\left(pr+\delta_\ell^{(L)}-\frac{\pi\ell}{2}\right)$
$G_\ell(r)$	$\frac{(2\ell-1)\,!!}{C_\ell}(pr)^{-\ell}$	$\cos\left(pr+\delta_\ell^{(L)}-\frac{\pi\ell}{2}\right)$

$C_\ell \simeq 1,\ \ \delta_\ell^{(L)} \simeq 0$ for $V_{long}(r) \simeq 0$

Solution of Schroedinger eq.

[Blum, Sato, Slatyer (2016)]
[Parikh, Sato, Slatyer (2024)]

Schroedinger equation:

$$\left[-\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{r^2}+2\mu V(r)-p^2\right]u_\ell(r)=0, \qquad V(r) = \begin{cases} V_{\text{short}}(r) & (r<a) \quad \text{complex} \\ V_{\text{long}}(r) & (r\geq a) \quad \text{real} \end{cases}$$

u_ℓ should be a linear combination of

- $F_\ell(r)$ (regular solution)
- $G_\ell(r)$ (irregular solution)

at $r > a$

$$u_\ell(r) = \begin{cases} u_{\ell,<}(r) & (r<a) \\ \exp\left(i\delta_\ell^{(L)}+i\delta_\ell^{(S)}\right)\left[\cos\delta_\ell^{(S)}F_\ell(r)+\sin\delta_\ell^{(S)}G_\ell(r)\right] & (r\geq a) \end{cases}$$

$$\rightarrow \quad \frac{1}{2i}\left((-i)^\ell \exp\left(2i\delta_\ell^{(L)}+2i\delta_\ell^{(S)}\right)e^{ipr}-i^\ell e^{-ipr}\right)$$

Solution of Schroedinger eq.

[Blum, Sato, Slatyer (2016)]
[Parikh, Sato, Slatyer (2024)]

Schroedinger equation:

$$\left[-\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{r^2}+2\mu V(r)-p^2\right]u_\ell(r)=0, \qquad V(r) = \begin{cases} V_{\mathrm{short}}(r) \quad (r<a) & \text{complex} \\ V_{\mathrm{long}}(r) \quad (r\geq a) & \text{real} \end{cases}$$

u_ℓ should be a linear combination of

- $F_\ell(r)$ (regular solution)
- $G_\ell(r)$ (irregular solution)

at $r > a$

$$u_\ell(r) = \begin{cases} u_{\ell,<}(r) & (r<a) \\ \exp\left(i\delta_\ell^{(L)}+i\delta_\ell^{(S)}\right)\left[\cos\delta_\ell^{(S)}F_\ell(r)+\sin\delta_\ell^{(S)}G_\ell(r)\right] & (r\geq a) \end{cases}$$

$$\rightarrow \quad \frac{1}{2i}\left(\underbrace{(-i)^\ell \exp\left(2i\delta_\ell^{(L)}+2i\delta_\ell^{(S)}\right)e^{ipr}}_{\text{Outgoing wave}} - \underbrace{i^\ell e^{-ipr}}_{\text{Incoming wave}}\right)$$

Solution of Schroedinger eq.

[Blum, Sato, Slatyer (2016)]
[Parikh, Sato, Slatyer (2024)]

Schroedinger equation:

$$\left[-\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{r^2}+2\mu V(r)-p^2\right]u_\ell(r)=0, \qquad V(r) = \begin{cases} V_{\rm short}(r) & (r<a) \quad \text{complex} \\ V_{\rm long}(r) & (r\geq a) \quad \text{real} \end{cases}$$

u_ℓ should be a linear combination of

- $F_\ell(r)$ (regular solution)
- $G_\ell(r)$ (irregular solution)

at $r > a$

$$u_\ell(r) = \begin{cases} u_{\ell,<}(r) & (r<a) \\ \exp\left(i\delta_\ell^{(L)}+i\delta_\ell^{(S)}\right)\left[\cos\delta_\ell^{(S)}F_\ell(r)+\sin\delta_\ell^{(S)}G_\ell(r)\right] & (r\geq a) \end{cases}$$

$$\to \frac{1}{2i}\left(\underbrace{(-i)^\ell \underbrace{\exp\left(2i\delta_\ell^{(L)}+2i\delta_\ell^{(S)}\right)}_{S_\ell} e^{ipr}}_{\text{Outgoing wave}} - \underbrace{i^\ell e^{-ipr}}_{\text{Incoming wave}}\right)$$

S-matrix

[Blum, Sato, Slatyer (2016)]
[Parikh, Sato, Slatyer (2024)]

$$u_\ell(r) = \begin{cases} u_{\ell,<}(r) & (r < a) \\ \exp\left(i\delta_\ell^{(L)} + i\delta_\ell^{(S)}\right)\left[\cos\delta_\ell^{(S)} F_\ell(r) + \sin\delta_\ell^{(S)} G_\ell(r)\right] & (r \geq a) \end{cases}$$

$$\to \quad \frac{1}{2i}\left((-i)^\ell \underbrace{\exp\left(2i\delta_\ell^{(L)} + 2i\delta_\ell^{(S)}\right)}_{S_\ell} e^{ipr} - i^\ell e^{-ipr}\right)$$

Outgoing wave

Incoming wave

S-matrix

[Blum, Sato, Slatyer (2016)]
[Parikh, Sato, Slatyer (2024)]

$$u_\ell(r) = \begin{cases} u_{\ell,<}(r) & (r < a) \\ \exp\left(i\delta_\ell^{(L)} + i\delta_\ell^{(S)}\right)\left[\cos\delta_\ell^{(S)} F_\ell(r) + \sin\delta_\ell^{(S)} G_\ell(r)\right] & (r \geq a) \end{cases}$$

$$\rightarrow \frac{1}{2i}\left((-i)^\ell \underbrace{\exp\left(2i\delta_\ell^{(L)} + 2i\delta_\ell^{(S)}\right)}_{S_\ell} e^{ipr} - i^\ell e^{-ipr}\right)$$

Outgoing wave

Incoming wave

$$\frac{u_\ell{}'}{u_\ell} \text{ is continuous at } r = a \quad \Rightarrow \quad \frac{u_{\ell,<}{}'}{u_{\ell,<}} = \frac{\cos\delta_\ell^{(S)} F_\ell' + \sin\delta_\ell^{(S)} G_\ell'}{\cos\delta_\ell^{(S)} F_\ell + \sin\delta_\ell^{(S)} G_\ell}$$

S-matrix

[Blum, Sato, Slatyer (2016)]
[Parikh, Sato, Slatyer (2024)]

$$u_\ell(r) = \begin{cases} u_{\ell,<}(r) & (r < a) \\ \exp\left(i\delta_\ell^{(L)} + i\delta_\ell^{(S)}\right)\left[\cos\delta_\ell^{(S)} F_\ell(r) + \sin\delta_\ell^{(S)} G_\ell(r)\right] & (r \geq a) \end{cases}$$

$$\rightarrow \frac{1}{2i}\left((-i)^\ell \underbrace{\exp\left(2i\delta_\ell^{(L)} + 2i\delta_\ell^{(S)}\right)}_{S_\ell} e^{ipr} - i^\ell e^{-ipr}\right)$$

Outgoing wave

Incoming wave

$\frac{u_\ell{}'}{u_\ell}$ is continuous at $r = a$ ⇒

$$\frac{u_{\ell,<}{}'}{u_{\ell,<}} = \frac{F_\ell' + \tan\delta_\ell^{(S)} G_\ell'}{F_\ell + \tan\delta_\ell^{(S)} G_\ell}$$

S-matrix

[Blum, Sato, Slatyer (2016)]
[Parikh, Sato, Slatyer (2024)]

$$u_\ell(r) = \begin{cases} u_{\ell,<}(r) & (r < a) \\ \exp\left(i\delta_\ell^{(L)} + i\delta_\ell^{(S)}\right)\left[\cos\delta_\ell^{(S)} F_\ell(r) + \sin\delta_\ell^{(S)} G_\ell(r)\right] & (r \geq a) \end{cases}$$

$$\rightarrow \quad \frac{1}{2i}\left((-i)^\ell \underbrace{\exp\left(2i\delta_\ell^{(L)} + 2i\delta_\ell^{(S)}\right)}_{S_\ell} e^{ipr} - i^\ell e^{-ipr}\right)$$

Outgoing wave

Incoming wave

$\frac{u_\ell{}'}{u_\ell}$ is continuous at $r = a$

$$\tan\delta_\ell^{(S)} = -\frac{F_\ell' - F_\ell\left(u_{\ell,<}'/u_{\ell,<}\right)}{G_\ell' - G_\ell\left(u_{\ell,<}'/u_{\ell,<}\right)}$$

$$S_\ell \quad = \quad \exp\left(2i\delta_\ell^{(L)}\right) \times \exp\left(2i\delta_\ell^{(S)}\right)$$

S-matrix

[Blum, Sato, Slatyer (2016)]
[Parikh, Sato, Slatyer (2024)]

$$u_\ell(r) = \begin{cases} u_{\ell,<}(r) & (r < a) \\ \exp\left(i\delta_\ell^{(L)} + i\delta_\ell^{(S)}\right)\left[\cos\delta_\ell^{(S)} F_\ell(r) + \sin\delta_\ell^{(S)} G_\ell(r)\right] & (r \geq a) \end{cases}$$

$$\to \quad \frac{1}{2i}\left((-i)^\ell \underbrace{\exp\left(2i\delta_\ell^{(L)} + 2i\delta_\ell^{(S)}\right)}_{S_\ell} e^{ipr} - i^\ell e^{-ipr} \right)$$

Outgoing wave

Incoming wave

$\frac{u_\ell{}'}{u_\ell}$ is continuous at $r = a$ ⇒

$$\tan\delta_\ell^{(S)} = -\frac{F_\ell' - F_\ell\left(u_{\ell,<}'/u_{\ell,<}\right)}{G_\ell' - G_\ell\left(u_{\ell,<}'/u_{\ell,<}\right)}$$

$$S_\ell = \exp\left(2i\delta_\ell^{(L)}\right) \times \frac{1 + i\tan\delta_\ell^{(S)}}{1 - i\tan\delta_\ell^{(S)}}$$

S-matrix

[Blum, Sato, Slatyer (2016)]
[Parikh, Sato, Slatyer (2024)]

$$S_\ell = \exp\left(2i\delta_\ell^{(L)}\right) \times \frac{1 + i\tan\delta_\ell^{(S)}}{1 - i\tan\delta_\ell^{(S)}} \qquad \tan\delta_\ell^{(S)} = -\frac{F'_\ell - F_\ell\left(u'_{\ell,<}/u_{\ell,<}\right)}{G'_\ell - G_\ell\left(u'_{\ell,<}/u_{\ell,<}\right)}$$

S-matrix

[Blum, Sato, Slatyer (2016)]
[Parikh, Sato, Slatyer (2024)]

$$S_\ell \;=\; \exp\left(2i\delta_\ell^{(L)}\right) \times \frac{1 + i\tan\delta_\ell^{(S)}}{1 - i\tan\delta_\ell^{(S)}}$$

$$\tan\delta_\ell^{(S)} = -\frac{F_\ell' - F_\ell\left(u_{\ell,<}'/u_{\ell,<}\right)}{G_\ell' - G_\ell\left(u_{\ell,<}'/u_{\ell,<}\right)}$$

Solutions for $V(r) = V_{long}(r)$

$$F_\ell(r) \;\simeq\; \frac{C_\ell}{(2\ell+1)\,!!}(pr)^{\ell+1}$$

$$G_\ell(r) \simeq \frac{(2\ell-1)\,!!}{C_\ell}(pr)^{\ell+1}$$

S-matrix

[Blum, Sato, Slatyer (2016)]
[Parikh, Sato, Slatyer (2024)]

$$S_\ell \;=\; \exp\left(2i\delta_\ell^{(L)}\right) \times \frac{1 + i\tan\delta_\ell^{(S)}}{1 - i\tan\delta_\ell^{(S)}}$$

$$\tan\delta_\ell^{(S)} = -p^{2\ell+1} C_\ell^2 \frac{f_\ell' - f_\ell \left(u_{\ell,<}'/u_{\ell,<}\right)}{g_\ell' - g_\ell \left(u_{\ell,<}'/u_{\ell,<}\right)}$$

Solutions for $V(r) = V_{long}(r)$

$$f_\ell(r) \;\equiv\; \frac{F_\ell}{C_\ell p^{\ell+1}} \;\simeq\; \frac{r^{\ell+1}}{(2\ell+1)!!}$$

$$g_\ell(r) \;\equiv\; \frac{C_\ell}{p^\ell} G_\ell \;\simeq\; \frac{(2\ell-1)!!}{r^\ell}$$

S-matrix

[Blum, Sato, Slatyer (2016)]
[Parikh, Sato, Slatyer (2024)]

$$S_\ell \; = \; \exp\left(2i\delta_\ell^{(L)}\right) \times \frac{1 + i\tan\delta_\ell^{(S)}}{1 - i\tan\delta_\ell^{(S)}}$$

$$\tan\delta_\ell^{(S)} = -p^{2\ell+1}C_\ell^2 \frac{f_\ell' - f_\ell\left(u_{\ell,<}'/u_{\ell,<}\right)}{g_\ell' - g_\ell\left(u_{\ell,<}'/u_{\ell,<}\right)}$$

Solutions for $V(r) = V_{long}(r)$

$$f_\ell(r) \; \equiv \; \frac{F_\ell}{C_\ell p^{\ell+1}} \; \simeq \; \frac{r^{\ell+1}}{(2\ell+1)\,!!}$$

$$g_\ell(r) \; \equiv \; \frac{C_\ell}{p^\ell} G_\ell \; \simeq \; \frac{(2\ell-1)\,!!}{r^\ell}$$

$$k_\ell(p) \; \equiv \; -\frac{p^{2\ell+1}C_\ell^2}{\tan\delta_\ell^{(S)}} \; = \; \frac{g_\ell' - g_\ell\left(u_{\ell,<}'/u_{\ell,<}\right)}{f_\ell' - f_\ell\left(u_{\ell,<}'/u_{\ell,<}\right)}$$

$k_\ell(p)$ is *almost* independent on p
(will be discussed later)

S-matrix

[Blum, Sato, Slatyer (2016)]
[Parikh, Sato, Slatyer (2024)]

$$S_\ell \;=\; \exp\left(2i\delta_\ell^{(L)}\right) \times \frac{1 + i\tan\delta_\ell^{(S)}}{1 - i\tan\delta_\ell^{(S)}}$$

$$\tan\delta_\ell^{(S)} = -p^{2\ell+1} C_\ell^2 \frac{f_\ell' - f_\ell\left(u_{\ell,<}'/u_{\ell,<}\right)}{g_\ell' - g_\ell\left(u_{\ell,<}'/u_{\ell,<}\right)}$$

Solutions for $V(r) = V_{long}(r)$

$$f_\ell(r) \;\equiv\; \frac{F_\ell}{C_\ell p^{\ell+1}} \;\simeq\; \frac{r^{\ell+1}}{(2\ell+1)\,!!}$$

$$g_\ell(r) \;\equiv\; \frac{C_\ell}{p^\ell} G_\ell \;\simeq\; \frac{(2\ell-1)\,!!}{r^\ell}$$

$$k_\ell(p) \;\equiv\; -\frac{p^{2\ell+1} C_\ell^2}{\tan\delta_\ell^{(S)}} \;=\; \frac{g_\ell' - g_\ell\left(u_{\ell,<}'/u_{\ell,<}\right)}{f_\ell' - f_\ell\left(u_{\ell,<}'/u_{\ell,<}\right)}$$

$u_{\ell,<}(r)$: p independent
$f_\ell(r)$: p independent
$g_\ell(r)$: *almost* p independent

⇨ $k_\ell(p)$ is *almost* independent on p
(will be discussed later)

$$S_\ell \;=\; \exp\left(2i\delta_\ell^{(L)}\right) \times \frac{k_\ell(p) - i p^{2\ell+1} C_\ell^2}{k_\ell(p) + i p^{2\ell+1} C_\ell^2}$$

S-matrix

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

$$S_\ell = \exp\left(2i\delta_\ell^{(L)}\right) \times \frac{k_\ell(p) - ip^{2\ell+1}C_\ell^2}{k_\ell(p) + ip^{2\ell+1}C_\ell^2}$$

$$k_\ell(p) = \frac{g_\ell' - g_\ell\left(u_{\ell,<}'/u_{\ell,<}\right)}{f_\ell' - f_\ell\left(u_{\ell,<}'/u_{\ell,<}\right)}$$

S-matrix

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

$$S_\ell \;=\; \exp\left(2i\delta_\ell^{(L)}\right) \times \frac{k_\ell(p) - ip^{2\ell+1}C_\ell^2}{k_\ell(p) + ip^{2\ell+1}C_\ell^2}$$

$$k_\ell(p) \;=\; \frac{g_\ell' - g_\ell\left(u_{\ell,<}'/u_{\ell,<}\right)}{f_\ell' - f_\ell\left(u_{\ell,<}'/u_{\ell,<}\right)}$$

Solutions for $V(r) = V_{long}(r)$

$$f_\ell(r) \;\simeq\; \frac{r^{\ell+1}}{(2\ell+1)\,!!} + \cdots$$

$$g_\ell(r) \;\simeq\; \frac{(2\ell-1)\,!!}{r^\ell} + \cdots + z_\ell(p)\frac{r^{\ell+1}}{(2\ell+1)\,!!} + \cdots$$

S-matrix

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

$$S_\ell = \exp\left(2i\delta_\ell^{(L)}\right) \times \frac{k_\ell(p) - ip^{2\ell+1}C_\ell^2}{k_\ell(p) + ip^{2\ell+1}C_\ell^2}$$

$$k_\ell(p) = \frac{g'_\ell - g_\ell\left(u'_{\ell,<}/u_{\ell,<}\right)}{f'_\ell - f_\ell\left(u'_{\ell,<}/u_{\ell,<}\right)}$$

Solutions for $V(r) = V_{long}(r)$

$$f_\ell(r) \simeq \underbrace{\frac{r^{\ell+1}}{(2\ell+1)\,!!}}_{\text{Leading term}} + \ldots$$

$$g_\ell(r) \simeq \underbrace{\frac{(2\ell-1)\,!!}{r^\ell}}_{\text{(basically) leading term}} + \ldots + \underbrace{z_\ell(p)\frac{r^{\ell+1}}{(2\ell+1)\,!!}}_{\text{Sizable in some cases}} + \ldots$$

S-matrix

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

$$S_\ell \;=\; \exp\left(2i\delta_\ell^{(L)}\right) \times \frac{k_\ell(p) - ip^{2\ell+1}C_\ell^2}{k_\ell(p) + ip^{2\ell+1}C_\ell^2}$$

$$k_\ell(p) \;=\; \frac{g_\ell' - g_\ell\left(u_{\ell,<}'/u_{\ell,<}\right)}{f_\ell' - f_\ell\left(u_{\ell,<}'/u_{\ell,<}\right)}$$

Solutions for $V(r) = V_{long}(r)$

$$f_\ell(r) \;\simeq\; \frac{r^{\ell+1}}{(2\ell+1)\,!!} + \ldots$$

Leading term

$$g_\ell(r) \;\simeq\; \frac{(2\ell-1)\,!!}{r^\ell} + \ldots + z_\ell(p)\frac{r^{\ell+1}}{(2\ell+1)\,!!} + \ldots$$

(basically) leading term

Sizable in some cases

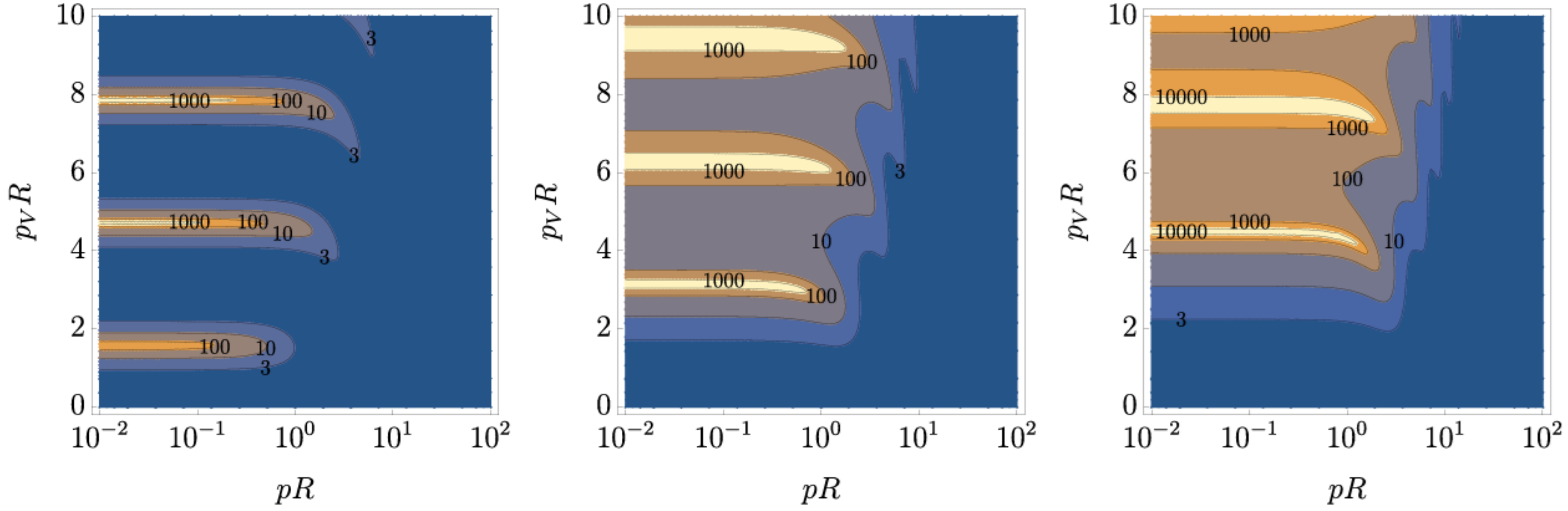
$$k_\ell(p) \;\simeq\; k_{\ell,0} + z_\ell(p)$$

(basically)
Leading term

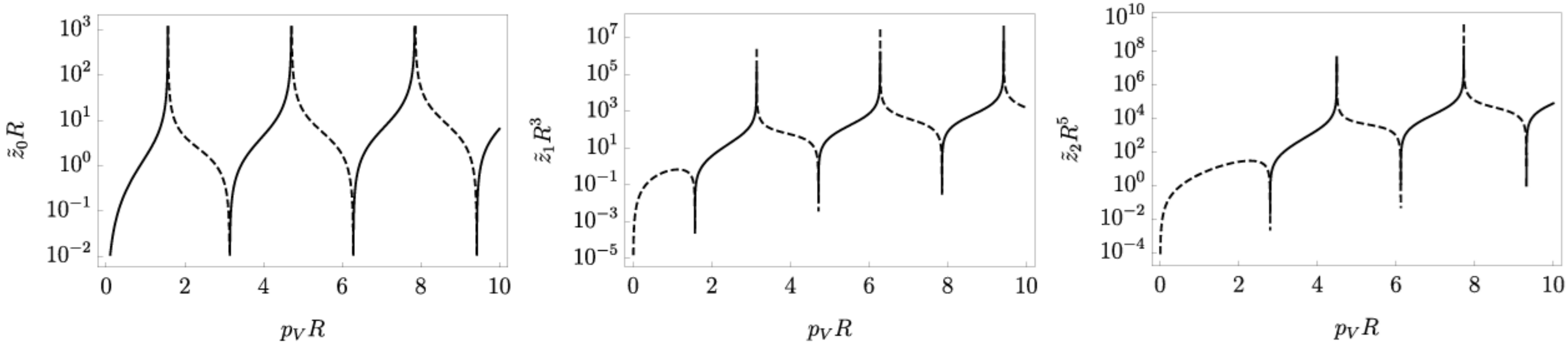
Sizable in some cases

$$S_\ell \;\simeq\; \exp\left(2i\delta_\ell^{(L)}\right) \times \frac{k_{\ell,0} + z_\ell(p) - ip^{2\ell+1}C_\ell^2}{k_{\ell,0} + z_\ell(p) + ip^{2\ell+1}C_\ell^2}$$

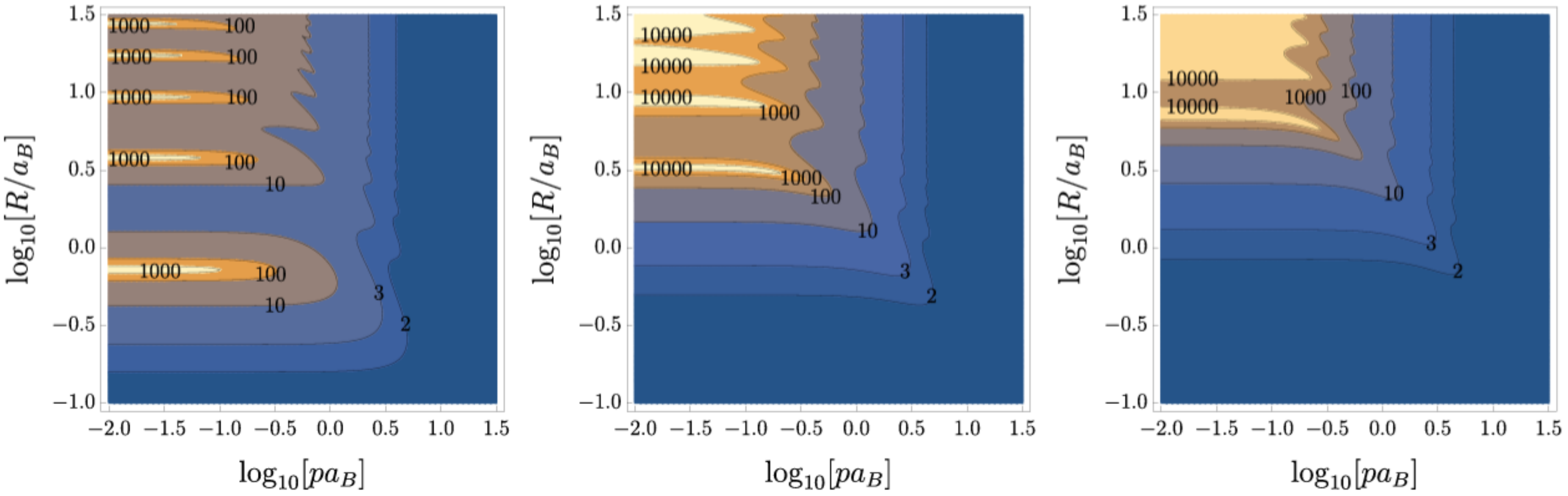
SE for Spherical-well potential



Z function for Spherical-well potential



SE for finite range Coulomb potential



Z function for finite range Coulomb potential