Sommerfeld effect and unitarity

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K. Blum, R. Sato, T. R. Slatyer, 1603.01383, JCAP 06 (2016) 021 A. Parikh, R. Sato, T. R. Slatyer, 2410.18168

2025. 6. 19 @ The Frontier of Particle Physics: Exploring Muons, Quantum Science and the Cosmos

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Plan

1. Dark matter and Sommerfeld effect

2. Sommerfeld effect and unitarity

Dark matter and its annihilation

~ 27% of our current universe is made of dark matter Many evidences but only through gravitational effects.

Dark Matter could have some interaction with SM particles :





Dark matter annihilation cross section

• Freeze-out scenario

Relic abundance today

$$\rightarrow \Omega_{\rm DM} h^2 \sim 0.1 \times \frac{3 \times 10^{-26} cm^3/s}{\sigma v}$$

Indirect detection

High energy cosmic ray from DM annihilation





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How to calculate σv

If the coupling is $< \sim 1$, perturbative calculation is efficient.





Typical example (higgs portal) :

$$L \ni -\frac{\lambda}{4}\phi^2 h^2 \qquad \longrightarrow \qquad iM(\phi\phi \to hh) = -i\lambda \qquad \longrightarrow \qquad \sigma v_{rel} = \frac{\lambda^2}{64\pi^2 m_{\phi}^2} \sqrt{1 - \frac{m_h^2}{m_{\phi}^2}}$$

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How to calculate σv

Even if the coupling is $< \sim 1$, non-perturbative resummation is required, if dark matter couples to a light force mediator boson $(m_{\text{boson}} \ll m_{\text{DM}})$



- Wino / Higgsino dark matter
- ex) SU(2) 5-plet dark matter
 - ...

Sommerfeld effect

[Sommerfeld (1931)] [Hisano, Matsumoto, Nojiri (2003)]

Strategy

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)] See also [Agrawal, Parikh, Reece (2020)]

We are interested in DM annihilation at non-relativistic regime. • Freeze-out ($T \simeq m/20$) • $v \simeq 10^{-3}c$ in galaxy

Schroedinger equation provides effective description!

$$E\psi = -\frac{1}{2\mu}\nabla^2\psi + V(x)\psi$$

$$V(r) = \begin{cases} V_{\text{short}}(r) & (r < a) & \text{complex} \\ \text{Annihilation etc.} \\ V_{\text{long}}(r) & (r \ge a) & \text{real} \\ \text{Exchange of light boson(s)} \end{cases}$$

ex)
$$V(r) \sim u \delta^3(x) + \frac{\alpha}{r} e^{-mr}$$

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[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

Two body scattering problem (à la undergrad quantum mechanics)

$$E\psi = -\frac{1}{2\mu}\nabla^2\psi + V(x)\psi$$
 with $\psi \to e^{ipz} + f(\theta)\frac{e^{ikr}}{r}$

Flux of probability $\vec{j}(x) = \frac{1}{u} \operatorname{Im}[\psi^*(x) \vec{\nabla} \psi(x)]$

"Non-conservation" of probability

$$\vec{\nabla} \cdot \vec{j}(x) = 2 \mathrm{Im} V(x) |\psi(x)|^2$$

 $= 2 \text{Im}V_{\text{short}}(x) |\psi(x)|^2$

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

Two body scattering problem (à la undergrad quantum mechanics)

$$E\psi = -\frac{1}{2\mu}\nabla^2\psi + V(x)\psi$$
 with $\psi \to e^{ipz} + f(\theta)\frac{e^{ikr}}{r}$

Definition of cross section

$$\sigma \times j_{in} = -\frac{dP}{dt}$$

Flux of incoming wave

$$j_{in} = \frac{1}{\mu} \operatorname{Im} \left[\psi_{in}^*(x) \vec{\nabla} \psi_{in}(x) \right] = \frac{p}{\mu} = v$$

Rate of annihilation

$$-\frac{dP}{dt} = \int_{r < a} d^3x \ 2\mathrm{Im}V_{\mathrm{short}}(x)|\psi(x)|^2$$

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[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

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Two body scattering problem (à la undergrad quantum mechanics)

$$E\psi = -\frac{1}{2\mu}\nabla^2\psi + V(x)\psi$$
 with $\psi \to e^{ipz} + f(\theta)\frac{e^{i\kappa r}}{r}$

Annihilation cross section

$$\sigma v = \int_{r < a} d^3 x \, 2 \mathrm{Im} V_{\mathrm{short}}(x) |\psi(x)|^2$$

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[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

$$\sigma v = \int_{r < a} d^3x \, 2 \operatorname{Im} V_{\text{short}}(x) |\psi(x)|^2 \qquad \simeq \int_{r < a} d^3x \, 2 \operatorname{Im} V_{\text{short}}(x) |\psi_{\text{long}}(x)|^2$$

$$\psi \simeq \psi_{\text{long}} \quad \text{(a.k.a. Distorted Wave Born Approximation)}$$
This should be OK as long as σv is not so large...
(will come back to this point soon)
$$s. t. \left[-\frac{1}{2\mu} \nabla^2 + V_{\text{long}}(x) - E \right] \psi_{\text{long}} = 0$$

s-wave case:

$$\sigma v \simeq |\psi_{\text{long}}(0)|^2 \propto \int_{r < a} d^3 x \, 2 \text{Im} V_{\text{short}}(x)$$

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[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

$$\sigma v = \int_{r < a} d^3x \, 2 \operatorname{Im} V_{\text{short}}(x) |\psi(x)|^2 \qquad \simeq \int_{r < a} d^3x \, 2 \operatorname{Im} V_{\text{short}}(x) |\psi_{\text{long}}(x)|^2$$

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s-wave case:

$$\sigma v \simeq |\psi_{\text{long}}(0)|^2 \times (\sigma v)_0$$

Enhancement factor Annihilation

Annihilation cross section w/o long-range force

[Hisano, Matsumoto, Nojiri (2002)] [Arkani-hamed, Finkbeiner, Slatyer, Weiner (2008)] etc

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Sommerfeld factor

$$V(r) = -\frac{\alpha}{r} \exp(-mr) \qquad \text{Bohr radius : } a_B \equiv \frac{1}{\alpha \mu}$$

• σv enhances when $m < \frac{1}{a_B}$ & $p < \frac{1}{a_B}$

• At some specific value of ma_B , σv violently enhances





Wino / Higgsino DM in SUSY

Dashed lines :usual perturbative calculation(w/o Sommerfeld effect)Colored curves :non-perturbative calculation(w/ Sommerfeld effect)



[Hisano, Matsumoto, Nojiri (2003)]

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Huge difference!

1. Dark matter and Sommerfeld effect

2. Sommerfeld effect and unitarity

Unitarity bound?



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Problematic situations

- 1. Large $\sigma_0 v$
- 2. At zero energy resonance $(S(v) \propto v^{-2} \rightarrow \sigma \propto v^{-3})$ (for s-wave)



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Unitarity bound vs. zero-energy resonance

Annihilation cross section on zero-energy resonance

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Need to solve Schroedinger eq, seriously

Going back to cross section formula...

$$\sigma v = \int_{r < a} d^3 x \, 2 \mathrm{Im} V_{\mathrm{short}}(x) |\psi(x)|^2 \qquad \simeq \int_{r < a} d^3 x \, 2 \mathrm{Im} V_{\mathrm{short}}(x) |\psi_{\mathrm{long}}(x)|^2$$

$$\psi \simeq \psi_{\mathrm{long}} \quad \text{(a.k.a. Distorted Wave Born Approximation)}$$
This should be OK as long as σv is not so large...
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$$s. t. \left[-\frac{1}{2\mu} \nabla^2 + V_{\mathrm{long}}(x) - E \right] \psi_{\mathrm{long}} = 0$$

ψ is quite different from ψ_{long} if σv is large!

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Need to solve Schroedinger eq, seriously

Diagramatic interpretation



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S-matrix from Schroedinger eq.

What we need: $\psi = \sum_{\ell} P_{\ell}(\cos \theta) \frac{(-1)^{\ell} \chi_{\ell}(r)}{pr}$

Schroedinger eq.

$$V(r) = \begin{cases} V_{\text{short}}(r) & (r < a) & \text{complex} \\ V_{\text{long}}(r) & (r \ge a) & \text{real} \end{cases}$$

Boundary condition at $r \rightarrow \infty$

$$\chi_{\ell}(r) \rightarrow \frac{S_{\ell} \exp(ipr) - \exp(-ipr)}{2i}$$

Boundary condition at r = a (condition for short-range effect)

 $\chi_\ell'(r)/\chi_\ell(r)$ at r=a is p-independent

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Straightforward (but lengthy) calculation

2.1 Basics of non-relativistic two-body scattering

We first briefly review the basics of the two-body scattering problem in non-relativistic quantum mechanics. The Schrödinger equation for the two-body scattering problem with central potential V(r) is

 $-\frac{1}{2\mu}\nabla^2\psi(\vec{r}) + V(r)\psi(\vec{r}) = \frac{p^2}{2\mu}\psi(\vec{r}),$ where μ is the reduced mass of the two-body system, r' is the separation vector between the particles, r = |r|, and p is the momentum of either particle in the center-of-momentum frame. The asymptotic behavior of the wavefunction $\psi(r)$ or $r \to \infty$ is

 $\psi(\vec{r}) = e^{ipz} + f(\theta) \frac{e^{ipr}}{dt}$ (2.2)

ngle between \vec{r} and the z-axis, and z is the direction of the initial plane is the following partial wave expansion:

 $\psi(r, \theta) = \sum (2\ell + 1)i^{\ell}P_{\ell}(\cos \theta) \frac{u_{\ell}(r)}{m}$.

action $u_i(r)$ satisfies the reduced Schrödinger equation $\left(-\frac{d^2}{dr^2} + \frac{\ell(\ell + 1)}{r^2} + 2\mu V(r) - p^2\right)w(r) = 0.$

 e^{ips} can be expanded as $e^{ips}=\sum_\ell (2\ell+1) i^\ell P_\ell(\cos\theta) j_\ell(pr).$ It will irrepently be useful to work with the free wavefunctions:

 $s_l(x) = xj_l(x), \quad e_l(x) = -xy_l(x),$ (2.5)where $j_0(x)$ and $g_0(x)$ are the standard spherical Bessel functions. These wavefunctions have the small-x asymptotic behavior:

 $s_\ell(x) = \frac{1}{(2\ell + 1)!!} x^{\ell+1} + \cdots, \quad c_\ell(x) = (2\ell - 1)!! x^{-\ell} \cdots.$ (2.6)

At large x, their asymptotic behavior is: $s_t(x) \rightarrow \sin(x - \pi \ell/2), \quad c_t(x) \rightarrow \cos(x - \pi \ell/2).$ (2.7)

Thus, we can read off the asymptotic behavior of $w_i(r)$ from the boundary condition of Eq. 2.2, sec

 $u_{\ell}(r) \rightarrow s_{\ell}(pr) + pf_{\ell}\left(c_{\ell}(pr) + is_{\ell}(pr)\right) = \frac{1}{2i}\left((-i)^{\ell}S_{\ell}e^{iqr} - i^{\ell}e^{-iqr}\right), \quad r \rightarrow \infty,$ (2.8) where $f(\theta) = \sum_{\ell} (2\ell + 1) f_{\ell} P_{\ell}(\cos \theta)$ and $S_{\ell} = 1 + 2ipf_{\ell}$ is the S-matrix.

a). The full cross sections for elastic scattering and inclusive annihilation are

$\sigma_{\kappa,\ell} = \frac{\pi}{p^2} (2\ell + 1) c_i \left \frac{k_\ell(p) - ip^{2\ell+1}C_\ell^2}{k_\ell(p) + ip^{2\ell+1}C_\ell^2} \exp \left(2i\delta_\ell^{(L)} \right) - 1 \right ^2$,	(2.23)
$\sigma_{axe,\ell} = 4\pi [2\ell + 1)c_i \frac{p^{2\ell-1}C_\ell^2 Im[-k_\ell^{-1}(p)]}{ 1 + ip^{2\ell+1}C_\ell^2 k_\ell^{-1}(p) ^2}$	(2.24)

Note that to lowest order in k_0^{-1} , the effect of the long-range force in the annihilation cross section is enhancement by a factor of C_0^0 this is the standard Sommerfeld enhancement. Note also that see definition $k_0^{-1}(x)$ by a cut submitted with the main finite constant of $k_0^{-1}(x)$ by a cut submitted with the main finite constant $k_0^{-1}(x)$ by a cut submitted with the main finite constant $k_0^{-1}(x)$ by a cut submitted with $k_0^{-1}(x)$ and $k_0^{-1}(x)$ and kmay be desirable in terms of senarating the lang-distance and shart-distance behavior Although Eq. 2.21 is generic and exact, it is useful to consider the limit where p is harge and the long-range potential can be neglected. This assumes that such a regime exists consistent with the assumption of non-relativistic physics, but this should generally be true for weak coupling. In this case, we expect $C_t^2 \rightarrow 1$, $e^{b_t^{(1)}} \rightarrow 1$, and S_t is given as

 $S_{\ell} \simeq \frac{1 - ip^{2\ell+1}k_{\ell}^{-1}(p)}{1 + in^{2\ell+1}k_{\ell}^{-1}(n)}$.

In this case, the cross section formulae (without a long-range force) at leading order in k_{ℓ}^{-1}

$\sigma_{n\ell,\ell}^{(0)} \simeq 4\pi (2\ell + 1)c_{*}p^{4\ell} k_{\ell}(p) ^{-2}$,	(2.26)
$\sigma_{am,\ell}^{(0)} \simeq 4\pi (2\ell + 1)c_s p^{2\ell-1} \frac{\text{Im}k_\ell(p)}{ k_\ell(p) ^2}.$	(2.27)

(2.25)

This regime on its well be matching beyone the probability of the state bindings requires a special. The degree is which there reach as the start is calible as the state of sections, etc.) will be the same in all cases. The various intermediate quantities calculates under the different conventions should also converge to each other in the limit where the short-distance interaction is described by a contact interaction and we take $a \rightarrow 0$, more

generally, they will differ by terms that are suppressed by powers of a. In the remainder of this section, we will disfine $V_{long}(r)$ as the α -independent long-range potential derived from the low-energy effective theory for all r, including $r < \alpha$. This implies that the $F_{\ell}(r)$ and $G_{\ell}(r)$ functions, and consequently the C_{ℓ} factors, are formally

clustic cross section σ_{uc} and the inclusive annihilation cross section σ_{uon} are given inguidable particles) by $\sigma_{\mathcal{K}} = \frac{4\pi}{p^2} \sum (2\ell + 1) \left| \frac{S_{\ell} - 1}{2i} \right|^2,$

 $\sigma_{mn} = \frac{4\pi}{n^2} \sum (2\ell + 1) \frac{1 - |S_\ell|^2}{4}.$ (2.10)here we take the include amihilation cross section to include all inclusive processes that are or modeled by the Hermitian part of the potential V(r), which thus manifest themselves a supparent non-maintarity of the S-matrix. This relation assumes distinguishable particles in the initial state; for identical particles,

(2.9)

the cross section will be zero for martial waves that do not have the correct symmetry proour track we taken with weak particular particular states and a state interview is provided particular tests, and enhanced by a factor of 2 otherwise (e.g. for identical fermions in a sprin-single state, \ell amot be even; see Ref. [10] for a more in-depth discussion). We will accordingly add a perfactor ρ to cross sections, which is 1 for distinguishable particles and 2 for itam tical particles, while weaking with the scattering amplitudes appropriate to distinguishable ucces. We will often find it advantageous to expand the reduced wavefunction w(r) in terms

 $f_\ell(r) = s_\ell(pr)/\sqrt{p}, \qquad g_l(r) = (c_\ell(pr) + is_\ell(pr))/\sqrt{p}.$ (2.11)

Then for any reduced wavefunction $u_{\ell}(r)$ solving the Schrödinger equation, we can decom w(r) = w(r) h(r) - h(r)w(r),(2.12)

and in order to uniquely define $\alpha_i(r)$ and $\beta_i(r)$ we can also impose the condition $u_0^i(r) = \alpha_i(r)\beta_i^i(r) - \beta_i(r)\beta_0^i(r)$ (see discussion in App. B). This condition measure that matching the coefficients $\alpha_i(r)$ and $\beta_j(r)$ between two solutions (a a given choice of r) is sufficient to match both their values and their first derivatives for that r. Note also that $\beta_j(r)g(r)$ $f_1(r)q_2'(r) = 1$, for all r.

2.2 S-matrix from the boundary condition

We are interested in the annihilation cross section when the long-range force deforms the wavefunction from a plane wave. We assume that the long-range force does not provide any annihilation effect directly. I.e. the corresponding rotantial is reached in the section. ion effect directly, i.e. the corresponding potential is real. Thus, it is usefu s short-range interactions (which may include inelastic/absorptive channels mage interactions (assumed to be well-described by a real potential) at some

 $V(r) = \begin{cases} V_{\text{shart}}(r) & (r < a) \\ V_{\text{imag}}(r) & (r \ge a) \end{cases}$ (2.13)

it is impossible to get the relation between the coefficient of $r^{-\ell}$ and $r^{\ell+1}$ only from the It is impossible to get the reasons between the continuous at r^{-1} and r^{-1} only most time recurrence relations near the origin. This is because the sum of an integral solution and a regular solution is another integrales solution, satisfying the same Schrödinger equation. The coefficient of r^{L+1} is determined by imposing the boundary condition at infinity (Eq. 2.16) for zone, we just keep both the r^{-1} turn and r^{L+1} turn and write $G_{24} \neq r = a$ as

 $G_{\ell}(r) \simeq \frac{(2\ell-1)!!}{C_{\ell}}(pr)^{-\ell} + \frac{1}{(2\ell+1)!!C_{\ell}p^{2\ell+1}} \left[z_{\ell}(p)(pr)^{\ell+1} + x_{\ell}(p)(pr)^{\ell+1} \log \frac{r}{r_{0}}\right].$ (2.30)

Note that we also keep track of the term scaling as $(pr)^{\ell+1}\log \tau/r_0$, because it causes a while issue which we will discuss later. $Again, z_\ell(p)$ cannot be determined by the recurrance relations at the origin. On the other hand, $x_\ell(p)$ can be determined by Eq. A.12. Here r_0 is a parameter which has been introduced to make the argument of the log dimensionless. a can be taken to be any value because the difference of r_0 can be absorbed by redefinit $_2(p)$: however, we will find it useful to take $v_0 \sim a$. We will discuss the behavior of $z_2(p)$ for some examples of V(r) in Sec. 2.6

2.4.2 Momentum scaling of terms in $i_{\ell}(p)$

Now let us discuss the momentum dependence of the coefficient kr(p). We only keep the leading term and drop terms of $O(p^2a^2)$ in Eq. 2.28. Then, the boundary condition from the short-range physics can be expressed by two parameters; c_{0d} and a. By using Eqs. 2.28-2.30,

 $k_\ell(p) \simeq \tilde{k}_\ell + z_\ell(p) + \tilde{c}ex_\ell(p)$.

where \hat{k}_{ℓ} and \hat{c}_{ℓ} are momentum-independent constants which are defined as $\vec{k}_{\ell} = -\frac{(2\ell + 1)!!(2\ell - 1)!!}{a^{2\ell+1}} \frac{\ell + c_{0,\ell}}{\ell + 1 - \alpha_{\ell}}$ (2.32

 $\tilde{c}_\ell \equiv \log \frac{a}{r_0} + \frac{1}{\ell+1-c_{0,\ell}}.$ (2.33)

lasted of $c_{0,\ell}$ and a, we can use these two parameters \tilde{k}_ℓ and \tilde{c}_ℓ to parametrize the effects of the short-many playlies. R_k is $O(n^{-2k-1})$ and the leading term in k/(p) in most of the cases. A verse will use in explicit cancers, $x_i(p)$ ends they get an and p of there exists a zero-energy resonance. $x_i(p)$ can be determined by a recurrence relation given in Eq. A.12, and it is a phytocoil of anomation in a ingeneration. We can extract the momentum dependent term in $k_{\ell}(p)$ from the behavior of $G_{\ell}(r)$. Let us define $\bar{z}_i(p)$ as

 $\hat{z}_{l}(p; a) \equiv \left[\frac{p^{l}C_{l}(p)}{2^{l}l!}\frac{d^{2l+1}}{dt^{2l+1}}r^{l}G_{\ell}(p; \tau)\right]_{\tau=0} \simeq z_{l}(p) + x_{\ell}(p)\log\frac{a}{\tau_{0}} + c'_{2\ell+1}x_{\ell}(p).$ (2.34)

There $V_{\rm ext}$ such and $V_{\rm ext}$ have an equatory per think potential and even downtown for the submature of the World P point of the submature of the two instances measurements of the submature proton of the submat $G_{\ell}(r)$, which are solutions of the Schrödinger equation with the long-range force

 $\left(-\frac{d^{2}}{r^{2}}+\frac{\ell(\ell+1)}{r^{2}}+2\mu V_{keq}(r)-p^{2}\right)F_{\ell}(r)=0,$ (2.14) $\left(-\frac{d^2}{dr^2} + \frac{\ell(\ell + 1)}{r^2} + 2\mu V_{long}(r) - p^2\right)G_\ell(r) = 0.$ (2.15) $F_{\ell}(r)$ is regular at the origin and $G_{\ell}(r)$ is irregular, and their asymptotic behavior at infinity

$G_{\ell}(r) + iF_{\ell}(r) \rightarrow (-i)^{\ell} \exp \left(ipr + i\delta_{\ell}^{(L)}\right).$

(2.16)

(2.37)

(2.38)

(2.39)

Here $\delta_{\ell}^{(L)}$ is the standard plass-shift induced by the long-range force. Since we assume $V_{\text{start}}(r)$ is real, $\delta_{\ell}^{(L)}$ is a real parameter. For small r, $F_{\ell}(r)$ and $G_{\ell}(r)$ have the asymptotic

 $F_{\ell}(r) \simeq C_{\ell}s_{\ell}(pr) \simeq \frac{C_{\ell}}{(2\ell + 1)!!}(pr)^{\ell+1}, \quad G_{\ell}(r) \simeq c_{\ell}(pr)/C_{\ell} \simeq \frac{(2\ell - 1)!!}{C_{\ell}}(pr)^{-\ell}.$ (2.17) Here C_ℓ is a function of p and ℓ , determined by $V_{long}(r)$. Note that we can prove Eq. 2.17, re-lating $F_\ell(r)$ and $G_\ell(r)$ in the limit $r \rightarrow 0$, by using the fact that the Wronkian $F_\ell^*(r)G_\ell(r)$ – lating $F_i(r)$ and $G_i(r)$ in the limit $r \rightarrow 0$, by using the fact that the Wrontzian $F_i(r)G_i(r)$. $F_i(r)G'_i(r)$ is independent of r, combined with the large-r asymptotics given in Eq. 2.16. We obtain $F_i(r) \simeq m(pr)$ and $G_i(r) \simeq m(pr)$. See all r, if the long-range force is ineffi-cient, and hence $C_i \simeq 1$ in this case. Comparing Eq. 2.6 and Eq. 2.17, C_i can be interpreter the enhancement factor of the f-wave regular wavefunction near the origin, compared to the plane wave. As we will explicitly see later, C_T^2 is the conventional Sommerfeld factor. The wavefunction which is consistent with the coefficient of e^{-qr} in Eq. 2.8 is given by

 $u_{\ell}(r) = \begin{cases} u_{c,\ell}(r) & (r < a) \\ u_{>,\ell}(r) = \exp \left(i\delta_{\ell}^{(L)} + i\delta_{\ell}^{(S)}\right) \left[F_{\ell}(r)\cos\delta_{\ell}^{(S)} + G_{\ell}(r)\sin\delta_{\ell}^{(S)}\right] & (r \geq a) \end{cases}$ (2.18)

From this expression, we can read off the S-matrix as $S_i = \exp\left(2t\left(\frac{d_i}{d_i}^{(d_i)}-\frac{d_i}{d_i}^{(d_i)}\right)\right)$. We do not specify an explicit form for $u_{i,j}(r)$, and will only need its behavior $u_i = u_i$ to obtain the full S-matrix. Note that $\frac{d_i}{d_i}$ has complex parameter in general, and it can be determined from the boundary condition at r = u.

 $\tan \delta_{\ell}^{(S)} \equiv -\left(\frac{F_{\ell}' - F_{\ell}(u'_{c,\ell}/u_{c,\ell})}{G'_{\ell} - G_{\ell}(u'_{c,\ell}/u_{c,\ell})}\right)$ (2.19)

- 6 -inte that c_n^t is an integer $\sim n^2 \log n^3$. The difference between $\tilde{z}_\ell(p; n)$ and $z_\ell(p) + \tilde{c}_\ell x_\ell(p)$

 $\hat{z}_{\ell}(p; a) - [z_{\ell}(p) + \hat{c}_{\ell}x_{\ell}(p)] \simeq \left(c_{2\ell+1}^{\prime} - \frac{1}{\ell + 1 - c_{0,\ell}}\right)x_{\ell}(p).$ (2.35)

Since $r_0(q)$ is determined by the long range free the BUS of the action regarding is at most $O(max)P^{(0,1)}, R^{(0,1)})$, where R is the trypted length scale of the lange-maps free (e.g., the bolts ranks).⁽¹⁾ = R^{(0,1)} is completely have the peripher years in [5, 2, 5, 6 the Chardon potential case. On the orders hand, $\bar{k}_{\pm} \approx \Omega(m^{(0,1)})$ and the difference between $L_0(m)$ is $m(d) = \ell_1(q)/m$ is $\Omega(d)$ is negligible compared to \bar{k}_1 . Thus, we can evaluate $k_1(q)$ by replacing $m(d) = \ell_1(q)/m$ is $\Omega(d)$ with $L_0(q)$.

 $k_\ell(p) = \tilde{k}_\ell + \tilde{v}(p; a)$ For practical purposes, it is useful to evaluate k_i as

 $h_1(p) \simeq h_2(p_0) + \Delta \omega(p, p_0; q)$

where we define $\Delta z_i(p, p_i; a)$ as

$\Delta z_t(p, p_0; a) \equiv \tilde{z}_t(p; a) - \tilde{z}_t(p_0; a).$

In Eq. 2.37, $h_2(y_0)$ and $\Delta_{12}(y_1,y_0,z)$ parametrize the effect of short-range physics and large map physics, respectively. $\Delta_{12}(y_1,y_2,z)$ has a dependence on a ond so it seems that the sparation between long-range physics and between ang physics is not complete. Hence we can be a single strain of the strain $|x_1(x)-x_1(y_1)|_{22}|_{22}/|x_1|$ and this two is a large short holizamian compared to the $k_1(y_1)$ -me. Therefore, we must oblight physics are belowed to the strain $|x_1(x)-x_1(x_1)|_{22}$ and the strain is a large $x_1(x_1,x_2)$.

The second seco ingle parameter $k_\ell(p_0)$ parameterizes the information of the short-range physics. However, or $\ell \ge 1$ and V(r) containing a 1/r term, $x_\ell(p)$ in general does depend on p. For details, see App. A.S.

see App. A.S. To summarize, as shown in Eq. 2.37, the long-range and short-range contributions to $k_l(p)$ can be separated into $k_l(p_0)$ (short-range effect) and $\Delta z_l(p, p_l; a)$ (long-range effect).

2.4.3 Implications for the S-matrix and cross sections By mine Eq. 2.37, S₂ can be written as

 $S_{\ell} = \frac{k_{\ell}(p_0) + \Delta z_{\ell}(p, p_0; a) - ip^{2\ell+1}C_{\ell}^{2}}{k_{\ell}(p_0) + \Delta z_{\ell}(p, p_0; a) + ip^{2\ell+1}C_{\ell}^{2}} \exp\left(2i\delta_{\ell}^{(L)}\right).$

 $\overline{\gamma_{n_{1}}}$ is defined as $\int_{-\infty}^{\infty} (x^{n} \log x) = e^{x} + e_{n_{1}}^{2} \log x$. The securence relation is $e'_{n} = 1$ and $e'_{n+1} = (n+1)e'_{n} + n!$. The solution is $e'_{n} = n!(\gamma_{0} + Q(n+1))$ where γ_{0} is Eider's gaussia.

p). The full cross sections for elastic scattering and inclusive annihilation are

$\sigma_{sc,\ell} = \frac{\pi}{p^2} (2\ell + 1) c_i \left| \frac{k_\ell(p) - i p^{2\ell+1} C_\ell^2}{k_\ell(p) + i p^{2\ell+1} C_\ell^2} \exp\left(2i \delta_\ell^{(L)}\right) - 1 \right|^2,$ (2.23) $\sigma_{\mathrm{ana},\ell} = 4\pi (2\ell+1) c_t \frac{p^{2\ell-1} C_\ell^2 \mathrm{Im}(-k_\ell^{-1}(p))}{|1+ip^{2\ell+1} C_\ell^2 k_\ell^{-1}(p)|^2}.$ (2.24)

Note that to lowest order in k_c^{-1} , the effect of the long-range force in the annihilation cross Note that to loosest order in $k_{ij}^{(2)}$, the effect of the long-range force in the simulation cross-stering is exhaused by a factor of $c_{ij}^{(2)}$. But is the standard Summerful exhausement, interacting the simulation of the standard standard standard standard standard memorary in the simulations cross sections: the effect of suck a shift on the annihilation cross sections would be brock the correction true in the denomination into two parts, which may be a simulation of the longest the simulations and a data-d-fittener behavior. Although its part of the simulation of the longest the simulation of the longest behavior. large and the long-mange potential can be neglected. This assumes that such a regime s consistent with the assumption of non-relativistic physics, but this should generally be true for weak coupling. In this case, we expect $C_t^1 \rightarrow 1$, $e^{i t_{t_t}^{(1)}} \rightarrow 1$, and S_t is given as

 $S_{\ell} \simeq \frac{1 - ip^{2\ell+1}k_{\ell}^{-1}(p)}{1 + ip^{2\ell+1}k_{\ell}^{-1}(p)}$.

(2.25)

(2.25)

In this case, the cross section formulae (without a long-range force) at leading order in k_ℓ^{-1}

 $\sigma_{ac,\ell}^{(0)} \simeq 4\pi (2\ell+1)c_i p^{4\ell} |k_\ell(p)|^{-2},$ $\sigma_{stm,\ell}^{(0)} \simeq 4\pi (2\ell + 1)c_4 p^{2\ell-1} \frac{\text{Im}k_\ell(p)}{|k_\ell(p)|^2}$ (2.27)

This regime can be used for matching here en also permutations (QFT exhibitions and the Schödinger equation presents). The Schödinger equation presents the two restricts of the scheme the corrected coses see seen as the scheme matching of the scheme matching of the scheme s interest, just the basis of solutions we use to study that problem; different choices lead to different properties for $F_i(r)$ and $G_i(r)$, and consequently to different coefficients for these functions in the r > a regular, but the full avarlunction (and hence the S-matrix, cross Institution in the r > a rights, but the find wavefunction (and larses the so-merger, cross sources, e.e., v, will be how such of all coses. The outwork interfaces of the source of a local under the different convertions about his to converge to oxis duties in the limit where the about-function interaction is described by an context interaction and we like $a \rightarrow 0$. Thus, we have the source of the sources are well about the source how the limit of the source gravarity, they will differ by terms that are supposed by powers of a. It is the reasoning of different by the source of the source product of the source product of the source product and derived from the interactive well being the or $A_{\rm eff}$ of a star in the source of the source product of the source of the sourc

-8-

Substituting into Eqs. 2.23, 2.24, we have:

 $\sigma_{m,\ell} = \frac{\pi}{p^2} (2\ell + 1) c_i \left| \frac{k_\ell(p_0) + \Delta_{2\ell}(p, p_0; a) - i\rho^{2\ell+1}C_\ell^2}{k_\ell(p_0) + \Delta_{2\ell}(p, p_0; a) + i\rho^{2\ell+1}C_\ell^2} \exp\left(2i\delta_\ell^{(L)}\right) - 1 \right|^2, \quad (2.40)$ $\sigma_{am,\ell} = 4\pi (2\ell + 1) c_i \frac{p^{2\ell-1} C_\ell^2 \mathrm{Im} k_\ell(p_0)}{|k_\ell(p_0) + \Delta z_\ell(p, p_0; a) + i p^{2\ell+1} C_\ell^2|^2}$

Choosing a large reference momentum p_0 such that the long-range force can be neglected, and applying Eqs. 2.26, 2.27, the full annihilation cross section can be written as

 $\sigma_{ann,\ell} = \sigma_{ann,\ell}^{(0)} C_{\ell}^{2} \times \left| 1 + \frac{1}{k^{2}(m)} \left(\Delta z_{\ell}(p; p_{0}) + ip^{2\ell+1}C_{\ell}^{2} \right) \right|^{-2}$ (2.42) $-\frac{1}{k_l(p_0)} \simeq \pm \sqrt{\frac{\sigma_{ucl}^{(0)}(p_0)}{4\pi(2\ell+1)p_0^{(l)}}} - \left(\frac{\sigma_{ucl}^{(0)}(p_0)}{4\pi(2\ell+1)p_0^{(l)-1}}\right)^2 - i\frac{\sigma_{ucl}^{(0)}(p_0)}{4\pi(2\ell+1)p_0^{(l)-1}}.$ 12.433

Note that the sign of the real part of $k_{\ell}^{-1}(p_0)$ cannot be determined directly from Eq. 2.26

p). The full cross sections for elastic scattering and inclusive annihilation ar

 $e_{\kappa,\ell} = \frac{\pi}{p^2} (2\ell + 1)c_1 \left| \frac{k_l(p) - ip^{2\ell+1}C_l^2}{k_l(p) + ip^{2\ell+1}C_l^2} \exp\left(2i\delta_\ell^{(L)}\right) - 1 \right|^2$, (2.23) $\sigma_{ian,l} = 4\pi (2\ell + 1)c_{l} \frac{p^{2l-1}C_{l}^{2} lm(-k_{l}^{-1}(p))}{(1 + ip^{2\ell+1}C_{l}^{2}k_{l}^{-1}(p))^{2}}$ (2.24)

Note that to lowest order in k_{ℓ}^{-1} , the effect of the long-range force in the annihilation cross section is enhancement by a factor of C_2^2 ; this is the standard Sommerfeld enhancement Note also that we can shift the definition of $k_\ell^{-1}(p)$ by a real number without modifying th numerator in the annihilation cross section; the effect of such a shift on the annihilation interstore in the alumination (rows series), the relative scale alumin to maintain the main and the series of the

 $S_\ell \simeq \frac{1 - i p^{2\ell+1} k_\ell^{-1}(p)}{1 + i p^{2\ell+1} k_\ell^{-1}(p)}.$ (2.25)

In this case, the cross section formulae (without a long-range force) at leading order in k_i^{-1} - Mar. 1. 11-2

$$\sigma_{u\ell,\ell}^{(0)} \simeq 4\pi (2\ell + 1)c_0 p^{(\ell)} |k_\ell(p)|^{-\epsilon},$$
 (2.26)
 $\sigma_{un,\ell}^{(0)} \simeq 4\pi (2\ell + 1)c_0 p^{(\ell)-1} \frac{\mathrm{Im}k_\ell(p)}{|k_\ell(n)|^2}.$ (2.27)

This regime can be used for matching between the perturbative QFT calculation and the The regime can be used for interange periods in the periods of q is the same of the Schrödinger equation approach. The degree to which these results can be used to calibrate the corrected reconsistentian at lower momentum depends on the momentum dependence of $h_i(p)$, so we will study this question next and present some examples.

Note that in defining $F_{\ell}(r)$ and $G_{\ell}(r)$ via Eqs. 2.14, 2.15, we have the freedom to choose $V_{long}(r)$ in the regime r < a. This choice does not affect the potential for the problem of interest, just the basis of solutions we use to study that problem; different choices lead to introductions, for the value of meaning we are obtained in protons, university transmission of different properties for F(r) and $G_r(r)$, and consequently to different coefficients for these functions in the r > a regime, but the full wavefunction (and hence the S-matrix, errors sections, etc.) will be the same in all cases. The various intermediate quantities calculated writes, etc.) will be the main in all cases. The transmission intermediate quantities calculated indee the distribution of the strength of the strength of the distribution in the limit where the main of the distribution of the strength of the strength of the strength of the gravarily, they will affirst by rears that are suppressed by powers of a. In the running of this section, we will distribute of the science strength of the strength or main distribution of the strength or we distribution of the C factors, are formally major that the $f(r_{ij})$ and $f(r_{ij})$ functions, and consequently the C factors, are formally

$\frac{H}{dt} - \frac{h}{d} \frac{h}{\partial t} = \frac{H}{h} \cdot h = 0$ CALCULATE

[23/33]

We obtain S-matrix for each ℓ as



a single complex parameter : $k_{\ell,0}$

Three real functions : $C_{\ell}^2(p)$, $z_{\ell}(p)$, $\delta_{\ell}(p)$

[24/33]

We obtain S-matrix for each ℓ as



Annihilation cross section :

Unitarity bound ✓

$$\sigma_{ann,\ell} = \frac{\pi}{p^2} (2\ell + 1)(1 - |S_\ell|^2) < \frac{(2\ell + 1)\pi}{p^2}$$

[25 / 33]

We obtain S-matrix for each ℓ as



Annihilation cross section :

Unitarity bound ✓

$$\sigma_{ann,\ell} = \frac{\pi}{p^2} (2\ell+1)(1-|S_\ell|^2) < \frac{(2\ell+1)\pi}{p^2}$$
$$= \frac{\pi}{p^2} (2\ell+1) \times 4\operatorname{Re}\left[\frac{ip^{2\ell+1}C_\ell^2}{k_{\ell,0}}\right] \times \left|1+\frac{z_\ell+ip^{2\ell+1}C_\ell^2}{k_{\ell,0}}\right|^{-2}$$

[26/33]

We obtain S-matrix for each ℓ as



Annihilation cross section :

Unitarity bound ✓

$$\sigma_{ann,\ell} = \frac{\pi}{p^2} (2\ell+1)(1-|S_\ell|^2) < \frac{(2\ell+1)\pi}{p^2}$$
$$= 4\pi (2\ell+1)p^{2\ell-1} \operatorname{Im} \left[-\frac{1}{k_{\ell,0}} \right] \times C_\ell^2 \times \left[1 + \frac{z_\ell + ip^{2\ell+1}C_\ell^2}{k_{\ell,0}} \right]^{-2}$$

Annihilation cross section w/o long-range force

Conventional Sommerfeld factor

Correction factor

[27/33]



[Parikh, Sato, Slatyer (2024)]

Spherical well potential

$$V(r) = -\frac{p_V^2}{2\mu}\theta(R-r)$$

(Conventional) Sommerfeld factor : C_{ℓ}^2



[28/33]



[Parikh, Sato, Slatyer (2024)]

Spherical well potential

$$V(r) = -\frac{p_V^2}{2\mu}\theta(R-r)$$





[29/33]

Examples

[Blum, Sato, Slatyer (2016)]

Hulthen potential :
$$V(r) = -\frac{\alpha m_* e^{-m_* r}}{1 - e^{-m_* r}}$$
 (Good approximation of $V(r) = -\frac{\alpha e^{-mr}}{r}$, $m_* = \frac{\pi^2}{6}m$)

$$\alpha = 1$$
, $\sigma v = \frac{1}{32\pi M^2}$, $\sigma_{sc} = \frac{\mu^2}{4\pi} (\sigma v)^2$





Yellow : usual formula Blue : our formula Green dotted : Unitarity bound

[30/33]

Examples

[Parikh, Sato, Slatyer (2024)]

Wino





[31/33]

Examples

[Parikh, Sato, Slatyer (2024)]



Summary & Outlook

- Annihilation cross section is important for DM phenomenology
- Schroedinger equation can treat long-range force by a light mediator
- The effect of annihilation can be treated as potential with complex coefficient
- This formulation is consistent with unitarty of QM.
- Can be relevant for large LO annihilation cross section (higher mass)

Backup

Wave function w/ long-range force

$$\left[-\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + 2\mu V(r) - p^2\right] H_{\ell}^+(r) = 0, \qquad H_{\ell}^+(r) \to (-i)^{\ell} \exp\left(ipr + \delta_{\ell}^{(L)}\right)$$

$$F_{\ell}(r) \equiv \operatorname{Im} H_{\ell}^{+} \simeq C_{\ell} p^{\ell+1} \times \left[\frac{r^{\ell+1}}{(2\ell+1)!!} + \dots \right]$$

$$\operatorname{Leading term}$$

$$G_{\ell}(r) \equiv \operatorname{Re} H_{\ell}^{+} \simeq \frac{1}{C_{\ell} p^{\ell}} \times \left[\frac{(2\ell-1)!!}{r^{\ell}} + \dots + z_{\ell}(p) \frac{r^{\ell+1}}{(2\ell+1)!!} + \dots \right]$$

(basically) leading term

Sizable in some cases

On zero energy resonance,

$$C_{\ell}^{2}(p) \propto \begin{cases} p^{-2} \ (\ell = 0) \\ p^{-4} \ (\ell \ge 1) \end{cases} \qquad z_{\ell}(p) \propto \begin{cases} p^{0} \ (\ell = 0) \\ p^{-2} \ (\ell \ge 1) \end{cases}$$

See also [Kamada, Kuwahara, Patel (2023)]

Resonant points

At some specific points ($ma_B = 1.19, 0.31, 0.139, ...$), $|\psi_{\text{long}}|^2 \propto \frac{1}{p^2}$

(zero energy resonance : bound state with zero binding energy)



Zero energy resonance

Schroedinger eq

Boundary cond.

$$\left[-\frac{1}{2\mu}\frac{d^2}{dr^2} + V(r) - \frac{p^2}{2\mu}\right]\chi(r) = 0,$$

$$\chi(r) \rightarrow \frac{S \exp(ipr) - \exp(-ipr)}{2i}$$

Bound state w/ $E = -\frac{\kappa^2}{2m}$ $\chi(r) \rightarrow \exp(-\kappa r)$



pole in S(k) at $p = i\kappa$

$$\Rightarrow \quad S = e^{2i\delta} = -\frac{p + i\kappa}{p - i\kappa}$$
$$\Rightarrow \quad \sin \delta = -\frac{\kappa}{p}$$

$$\psi(\vec{x}) = \frac{\chi(r)}{pr}$$
 $|\psi_{\text{long}}(0)|^2 \propto \frac{\chi^2(0)}{p^2} = \frac{\sin^2 \delta}{p^2} = \frac{1}{p^2 + \kappa^2}$

[37]

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

Schroedinger equation:

$$\left[-\frac{d^{2}}{dr^{2}} + \frac{\ell(\ell+1)}{r^{2}} + 2\mu V(r) - p^{2}\right]u_{\ell}(r) = 0, \qquad V(r) = \begin{cases} V_{\text{short}}(r) & (r < a) & \text{complex} \\ V_{\text{long}}(r) & (r \ge a) & \text{real} \end{cases}$$

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

Schroedinger equation:

$$\begin{bmatrix} -\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + 2\mu V(r) - p^2 \end{bmatrix} u_{\ell}(r) = 0, \qquad V(r) = \begin{bmatrix} V_{\text{short}}(r) \quad (r < a) & \text{complex} \\ V_{\text{long}}(r) \quad (r \ge a) & \text{real} \end{bmatrix}$$

$$u_{\ell} \text{ should be a linear combination of } \begin{bmatrix} \bullet & F_{\ell}(r) \quad (\text{regular solution}) \\ \bullet & G_{\ell}(r) \quad (\text{irregular solution}) \\ \begin{bmatrix} -\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + 2\mu V_{\text{long}}(r) - p^2 \end{bmatrix} F_{\ell}(r) = 0 & \text{s.} \quad \begin{bmatrix} -\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + 2\mu V_{\text{long}}(r) - p^2 \end{bmatrix} G_{\ell}(r) = 0$$

$$\boxed{r \approx 0} \qquad r \rightarrow \infty$$

$$\boxed{F_{\ell}(r)} \qquad \frac{C_{\ell}}{(2\ell+1)!!} (pr)^{\ell+1} \qquad \sin\left(pr + \delta_{\ell}^{(L)} - \frac{\pi\ell}{2}\right)$$

$$G_{\ell}(r) \qquad \frac{(2\ell-1)!!}{C_{\ell}} (pr)^{-\ell} \qquad \cos\left(pr + \delta_{\ell}^{(L)} - \frac{\pi\ell}{2}\right)$$

 $C_\ell \simeq 1$, $\delta_\ell^{(L)} \simeq 0$ for $V_{long}(r) \simeq 0$

[39]

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

Schroedinger equation:

$$\begin{bmatrix} -\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + 2\mu V(r) - p^2 \end{bmatrix} u_{\ell}(r) = 0, \qquad V(r) = \begin{bmatrix} V_{\text{short}}(r) & (r < a) & \text{complex} \\ V_{\text{long}}(r) & (r \ge a) & \text{real} \end{bmatrix}$$
$$u_{\ell} \text{ should be a linear combination of } \begin{bmatrix} \bullet & F_{\ell}(r) & (\text{regular solution}) \\ \bullet & G_{\ell}(r) & (\text{irregular solution}) & \text{at } r > a \end{bmatrix}$$
$$u_{\ell}(r) = \begin{bmatrix} u_{\ell,<}(r) & (r < a) \\ \exp\left(i\delta_{\ell}^{(L)} + i\delta_{\ell}^{(S)}\right) \left[\cos\delta_{\ell}^{(S)}F_{\ell}(r) + \sin\delta_{\ell}^{(S)}G_{\ell}(r)\right] & (r \ge a) \end{bmatrix}$$

$$\rightarrow \quad \frac{1}{2i} \left((-i)^{\ell} \exp\left(2i\delta_{\ell}^{(L)} + 2i\delta_{\ell}^{(S)}\right) e^{ipr} - i^{\ell} e^{-ipr} \right)$$

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

Schroedinger equation:

$$\begin{bmatrix} -\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + 2\mu V(r) - p^2 \end{bmatrix} u_{\ell}(r) = 0, \qquad V(r) = \begin{bmatrix} V_{\text{short}}(r) & (r < a) & \text{complex} \\ V_{\text{long}}(r) & (r \ge a) & \text{real} \end{bmatrix}$$

$$u_{\ell} \text{ should be a linear combination of } \begin{bmatrix} \cdot & F_{\ell}(r) & (\text{regular solution}) \\ \cdot & G_{\ell}(r) & (\text{irregular solution}) \end{bmatrix} \text{ at } r > a$$

$$u_{\ell}(r) = \begin{bmatrix} u_{\ell,<}(r) & (r < a) \\ \exp\left(i\delta_{\ell}^{(L)} + i\delta_{\ell}^{(S)}\right) \left[\cos\delta_{\ell}^{(S)}F_{\ell}(r) + \sin\delta_{\ell}^{(S)}G_{\ell}(r)\right] & (r \ge a) \\ \rightarrow & \frac{1}{2i} \left((-i)^{\ell} \exp\left(2i\delta_{\ell}^{(L)} + 2i\delta_{\ell}^{(S)}\right)e^{ipr} - i^{\ell}e^{-ipr}\right)$$
Outgoing wave

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

Schroedinger equation:

[42]

$$u_{\ell}(r) = \begin{cases} u_{\ell,<}(r) & (r < a) \\ \exp\left(i\delta_{\ell}^{(L)} + i\delta_{\ell}^{(S)}\right) \left[\cos\delta_{\ell}^{(S)}F_{\ell}(r) + \sin\delta_{\ell}^{(S)}G_{\ell}(r)\right] & (r \ge a) \\ \rightarrow \frac{1}{2i} \left(\left(-i\right)^{\ell} \exp\left(2i\delta_{\ell}^{(L)} + 2i\delta_{\ell}^{(S)}\right)e^{ipr} - i^{\ell}e^{-ipr}\right) \\ \text{Outgoing wave} & \text{Incoming wave} \end{cases}$$

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

$$u_{\ell}(r) = \begin{cases} u_{\ell,<}(r) & (r < a) \\ \exp\left(i\delta_{\ell}^{(L)} + i\delta_{\ell}^{(S)}\right) \left[\cos\delta_{\ell}^{(S)}F_{\ell}(r) + \sin\delta_{\ell}^{(S)}G_{\ell}(r)\right] & (r \ge a) \\ \rightarrow \frac{1}{2i} \left(\left(-i\right)^{\ell} \exp\left(2i\delta_{\ell}^{(L)} + 2i\delta_{\ell}^{(S)}\right)e^{ipr} - i^{\ell}e^{-ipr}\right) \\ \text{Outgoing wave} & \text{Incoming wave} \end{cases}$$

$$\frac{u_{\ell}'}{u_{\ell}} \text{ is continuous at } r = a \qquad \qquad \frac{u_{\ell,<}'}{u_{\ell,<}} = \frac{\cos \delta_{\ell}^{(S)} F_{\ell}' + \sin \delta_{\ell}^{(S)} G_{\ell}'}{\cos \delta_{\ell}^{(S)} F_{\ell} + \sin \delta_{\ell}^{(S)} G_{\ell}}$$

[44]

$$u_{\ell}(r) = \begin{cases} u_{\ell,<}(r) & (r < a) \\ \exp\left(i\delta_{\ell}^{(L)} + i\delta_{\ell}^{(S)}\right) \left[\cos\delta_{\ell}^{(S)}F_{\ell}(r) + \sin\delta_{\ell}^{(S)}G_{\ell}(r)\right] & (r \ge a) \\ \rightarrow \frac{1}{2i} \left(\left(-i\right)^{\ell} \exp\left(2i\delta_{\ell}^{(L)} + 2i\delta_{\ell}^{(S)}\right)e^{ipr} - i^{\ell}e^{-ipr}\right) \\ \text{Outgoing wave} & \text{Incoming wave} \end{cases}$$

$$\frac{u_{\ell}'}{u_{\ell}} \text{ is continuous at } r = a \qquad \qquad \frac{u_{\ell,<}'}{u_{\ell,<}} = \frac{F_{\ell}' + \tan \delta_{\ell}^{(S)} G_{\ell}'}{F_{\ell} + \tan \delta_{\ell}^{(S)} G_{\ell}}$$

$$u_{\ell}(r) = \begin{cases} u_{\ell,<}(r) & (r < a) \\ \exp\left(i\delta_{\ell}^{(L)} + i\delta_{\ell}^{(S)}\right) \left[\cos \delta_{\ell}^{(S)}F_{\ell}(r) + \sin \delta_{\ell}^{(S)}G_{\ell}(r)\right] & (r \ge a) \\ \rightarrow \frac{1}{2i} \left(\left(-i\right)^{\ell} \exp\left(2i\delta_{\ell}^{(L)} + 2i\delta_{\ell}^{(S)}\right)e^{ipr} - i^{\ell}e^{-ipr}\right) \\ \text{Outgoing wave} & \text{Incoming wave} \end{cases}$$

$$\frac{u_{\ell}'}{u_{\ell}} \text{ is continuous at } r = a \qquad \qquad \tan \delta_{\ell}^{(S)} = -\frac{F_{\ell}' - F_{\ell}\left(u_{\ell,<}'/u_{\ell,<}\right)}{G_{\ell}' - G_{\ell}\left(u_{\ell,<}'/u_{\ell,<}\right)}$$

$$S_{\ell} = \exp\left(2i\delta_{\ell}^{(L)}\right) \times \exp\left(2i\delta_{\ell}^{(S)}\right)$$

$$\frac{u_{\ell}'}{u_{\ell}} \text{ is continuous at } r = a \qquad \qquad \tan \delta_{\ell}^{(S)} = -\frac{F_{\ell}' - F_{\ell}\left(u_{\ell,<}'/u_{\ell,<}\right)}{G_{\ell}' - G_{\ell}\left(u_{\ell,<}'/u_{\ell,<}\right)}$$

$$S_{\ell} = \exp\left(2i\delta_{\ell}^{(L)}\right) \times \frac{1+i\tan\delta_{\ell}^{(S)}}{1-i\tan\delta_{\ell}^{(S)}}$$

$$\tan \delta_{\ell}^{(S)} = -\frac{F_{\ell}' - F_{\ell} \left(u_{\ell,<}' / u_{\ell,<} \right)}{G_{\ell}' - G_{\ell} \left(u_{\ell,<}' / u_{\ell,<} \right)}$$

$$S_{\ell} = \exp\left(2i\delta_{\ell}^{(L)}\right) \times \frac{1+i\tan\delta_{\ell}^{(S)}}{1-i\tan\delta_{\ell}^{(S)}}$$

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

$$S_{\ell} = \exp\left(2i\delta_{\ell}^{(L)}\right) \times \frac{1+i\tan\delta_{\ell}^{(S)}}{1-i\tan\delta_{\ell}^{(S)}}$$

$$\tan \delta_{\ell}^{(S)} = -\frac{F_{\ell}' - F_{\ell} \left(u_{\ell,<}' / u_{\ell,<} \right)}{G_{\ell}' - G_{\ell} \left(u_{\ell,<}' / u_{\ell,<} \right)}$$

Solutions for $V(r) = V_{long}(r)$

$$F_{\ell}(r) \simeq \frac{C_{\ell}}{(2\ell+1)!!} (pr)^{\ell+1}$$

$$G_{\ell}(r) \simeq \frac{(2\ell-1)\,!!}{C_{\ell}} (pr)^{\ell+1}$$

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

$$S_{\ell} = \exp\left(2i\delta_{\ell}^{(L)}\right) \times \frac{1+i\tan\delta_{\ell}^{(S)}}{1-i\tan\delta_{\ell}^{(S)}}$$

$$\tan \delta_{\ell}^{(S)} = -p^{2\ell+1}C_{\ell}^2 \frac{f_{\ell}' - f_{\ell}\left(u_{\ell,<}'/u_{\ell,<}\right)}{g_{\ell}' - g_{\ell}\left(u_{\ell,<}'/u_{\ell,<}\right)}$$

Solutions for $V(r) = V_{long}(r)$

$$f_{\ell}(r) \equiv \frac{F_{\ell}}{C_{\ell} p^{\ell+1}} \simeq \frac{r^{\ell+1}}{(2\ell+1)!!} \qquad g_{\ell}(r) \equiv \frac{C_{\ell}}{p^{\ell}} G_{\ell} \simeq \frac{(2\ell-1)!!}{r^{\ell}}$$

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

$$S_{\ell} = \exp\left(2i\delta_{\ell}^{(L)}\right) \times \frac{1+i\tan\delta_{\ell}^{(S)}}{1-i\tan\delta_{\ell}^{(S)}}$$

$$\tan \delta_{\ell}^{(S)} = -p^{2\ell+1}C_{\ell}^2 \frac{f_{\ell}' - f_{\ell}\left(u_{\ell,<}'/u_{\ell,<}\right)}{g_{\ell}' - g_{\ell}\left(u_{\ell,<}'/u_{\ell,<}\right)}$$

Solutions for $V(r) = V_{long}(r)$

$$f_{\ell}(r) \equiv \frac{F_{\ell}}{C_{\ell} p^{\ell+1}} \simeq \frac{r^{\ell+1}}{(2\ell+1)!!} \qquad g_{\ell}(r) \equiv \frac{C_{\ell}}{p^{\ell}} G_{\ell} \simeq \frac{(2\ell-1)!!}{r^{\ell}}$$

$$k_{\ell}(p) \equiv -\frac{p^{2\ell+1}C_{\ell}^{2}}{\tan \delta_{\ell}^{(S)}} = \frac{g_{\ell}' - g_{\ell}(u_{\ell,<}'/u_{\ell,<})}{f_{\ell}' - f_{\ell}(u_{\ell,<}'/u_{\ell,<})}$$

 $k_{\ell}(p)$ is *almost* independent on p (will be discussed later)

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

$$S_{\ell} = \exp\left(2i\delta_{\ell}^{(L)}\right) \times \frac{1+i\tan\delta_{\ell}^{(S)}}{1-i\tan\delta_{\ell}^{(S)}}$$

$$\tan \delta_{\ell}^{(S)} = -p^{2\ell+1}C_{\ell}^2 \frac{f_{\ell}' - f_{\ell}\left(u_{\ell,<}'/u_{\ell,<}\right)}{g_{\ell}' - g_{\ell}\left(u_{\ell,<}'/u_{\ell,<}\right)}$$

Solutions for $V(r) = V_{long}(r)$

$$f_{\ell}(r) \equiv \frac{F_{\ell}}{C_{\ell} p^{\ell+1}} \simeq \frac{r^{\ell+1}}{(2\ell+1)!!} \qquad g_{\ell}(r) \equiv \frac{C_{\ell}}{p^{\ell}} G_{\ell} \simeq \frac{(2\ell-1)!!}{r^{\ell}}$$

$$k_{\ell}(p) \equiv -\frac{p^{2\ell+1}C_{\ell}^{2}}{\tan \delta_{\ell}^{(S)}} = \frac{g_{\ell}' - g_{\ell}(u_{\ell,<}'/u_{\ell,<})}{f_{\ell}' - f_{\ell}(u_{\ell,<}'/u_{\ell,<})}$$

$$u_{\ell,<}(r)$$
 : p independent
 $f_{\ell}(r)$: p independent
 $g_{\ell}(r)$: *almost* p independent

$$S_{\ell} = \exp\left(2i\delta_{\ell}^{(L)}\right) \times \frac{k_{\ell}(p) - ip^{2\ell+1}C_{\ell}^{2}}{k_{\ell}(p) + ip^{2\ell+1}C_{\ell}^{2}}$$



$$S_{\ell} = \exp\left(2i\delta_{\ell}^{(L)}\right) \times \frac{k_{\ell}(p) - ip^{2\ell+1}C_{\ell}^{2}}{k_{\ell}(p) + ip^{2\ell+1}C_{\ell}^{2}}$$

$$k_{\ell}(p) = \frac{g_{\ell}' - g_{\ell}(u_{\ell,<}'/u_{\ell,<})}{f_{\ell}' - f_{\ell}(u_{\ell,<}'/u_{\ell,<})}$$

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

$$S_{\ell} = \exp\left(2i\delta_{\ell}^{(L)}\right) \times \frac{k_{\ell}(p) - ip^{2\ell+1}C_{\ell}^{2}}{k_{\ell}(p) + ip^{2\ell+1}C_{\ell}^{2}}$$

Solutions for $V(r) = V_{long}(r)$

$$f_{\ell}(r) \simeq \frac{r^{\ell+1}}{(2\ell+1)!!} + \cdots$$

$$k_{\ell}(p) = \frac{g_{\ell}' - g_{\ell}(u_{\ell,<}'/u_{\ell,<})}{f_{\ell}' - f_{\ell}(u_{\ell,<}'/u_{\ell,<})}$$

$$g_{\ell}(r) \simeq \frac{(2\ell-1)\,!!}{r^{\ell}} + \dots + z_{\ell}(p) \frac{r^{\ell+1}}{(2\ell+1)\,!!} + \dots$$

$$k_{\ell}(p) = \frac{g_{\ell}' - g_{\ell}(u_{\ell,<}'/u_{\ell,<})}{f_{\ell}' - f_{\ell}(u_{\ell,<}'/u_{\ell,<})}$$

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[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

$$k_{\ell}(p) = \frac{g_{\ell}' - g_{\ell}(u_{\ell,<}'/u_{\ell,<})}{f_{\ell}' - f_{\ell}(u_{\ell,<}'/u_{\ell,<})}$$

$$e_{\ell} = \exp\left(2i\delta_{\ell}^{(L)}\right) \times \frac{k_{\ell}(p) - ip^{2\ell+1}C_{\ell}^{2}}{k_{\ell}(p) + ip^{2\ell+1}C_{\ell}^{2}} \qquad k_{\ell}(p)$$
Solutions for $V(r) = V_{long}(r)$

$$f_{\ell}(r) \simeq \frac{r^{\ell+1}}{(2\ell+1)!!} + \dots$$
Leading term
$$g_{\ell}(r) \simeq \frac{(2\ell-1)!!}{r^{\ell}} + \dots + z_{\ell}(p)\frac{r^{\ell+1}}{(2\ell+1)!!} + \dots$$
(basically) leading term
Sizable in some cases

[56]

SE for Spherical-well potential



Z function for Spherical-well potential



SE for finite range Coulomb potential



Z function for finite range Coulomb potential





[60]