The Frontier of Particle Physics: Exploring Muons, Quantum Science, and the Cosmos

Quantum computing for stochastic inflation: calculating eigenvalues of differential operators and estimating the decay rate of the perturbation distribution tail

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Based on Phys. Rev. Research 7, 023251 (2025) (with Y. Tada (Rikkyo Univ.))

Self-introduction: Koichi Miyamoto

- My career

  2013: got PhD (cosmology @ Kawasaki-Lab., ICRR, U. Tokyo)

  2013 - 2017: Mitsubishi UFJ Morgan Stanley Securities (investment bank)

  - Developed derivative pricing models

  2018 - 2020: Mizuho-DL Financial Technology (consulting firm)

  - Developed credit scoring models (e.g. decision-making on lending)
  - 2019 - : Research on applications of quantum computing to finance

  2021 - : Current job

- Current research topics

  ➢ applications of quantum algorithms to various fields in industry & science

  ✓ finance, cosmology, bioinformatics, ...

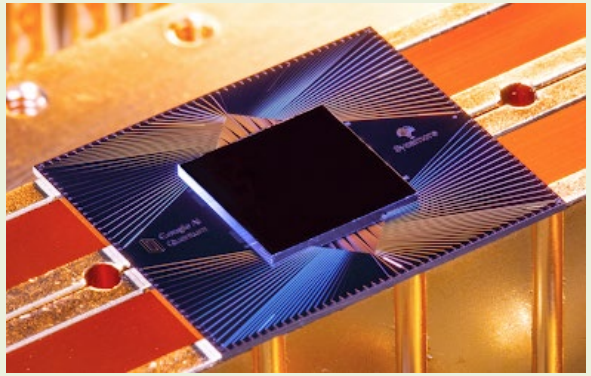
1. Basics of quantum computing

Quantum computing era is coming

- Many news on quantum computers reported recently
  - Tech giants developing quantum computers (Google, IBM, ...)
  - Japanese institutions too (RIKEN, Fujitsu, **UOsaka**†)
- We expect disruptive innovations in various industries via speedup of heavy calculations by quantum computers
  - chemical, finance, pharmaceutical, ...
- Fundamental science might benefit too
  - chemistry, condensed matter physics, ...
  - **high energy physics too!**

† https://qiqb.osaka-u.ac.jp/en/20231220pr/

UOsaka's quantum computer
https://xtech.nikkei.com/atcl/nxt/column/18/00001/08291/



Google's quantum chip
https://ai.googleblog.com/2019/10/quantum-supremacy-using-programmable.html

What is a quantum computer?

- classical computer (we are using in today's daily life)
  - ➢ calculate by manipulating bits (0 or 1)

0 or 1

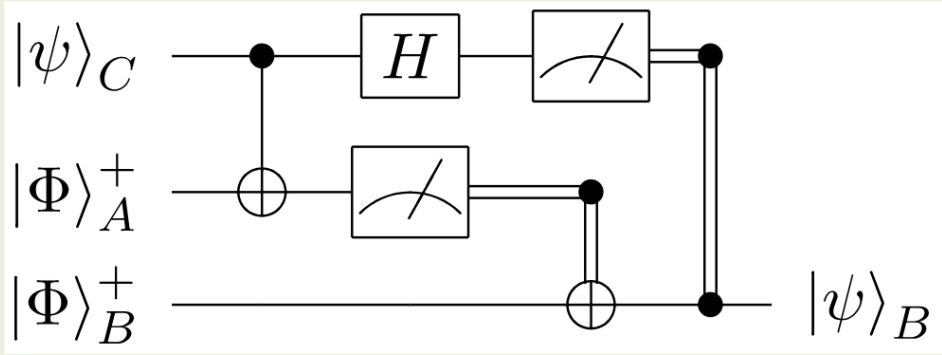
Classical bit

- quantum computer
  - ➢ calculate by manipulating **quantum bits (qubits)** that takes a **superposition** of $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
  - ➢ a qubit: two-level quantum system (2-dim Hilbert space)
    n qubits: its n-th tensor product (2^n-dim Hilbert space)

Quantum bit

- Combining **quantum gates** (elementary operations on qubits), we construct a **quantum circuit** and perform desired computation

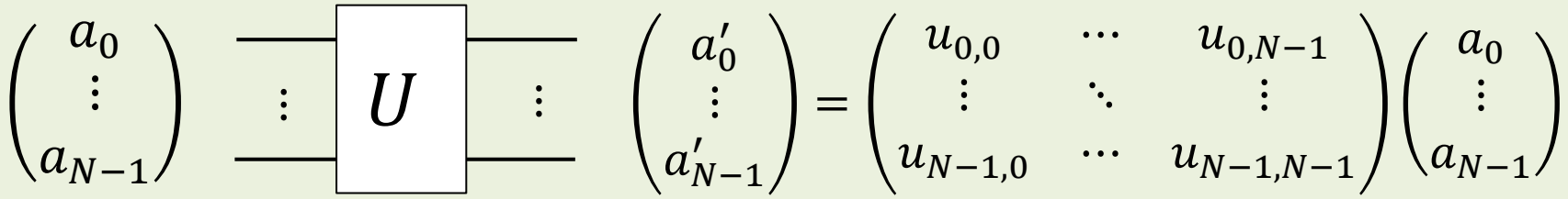
Example of quantum circuits



https://en.wikipedia.org/wiki/Quantum_circuit

Why are quantum computers “efficient”?

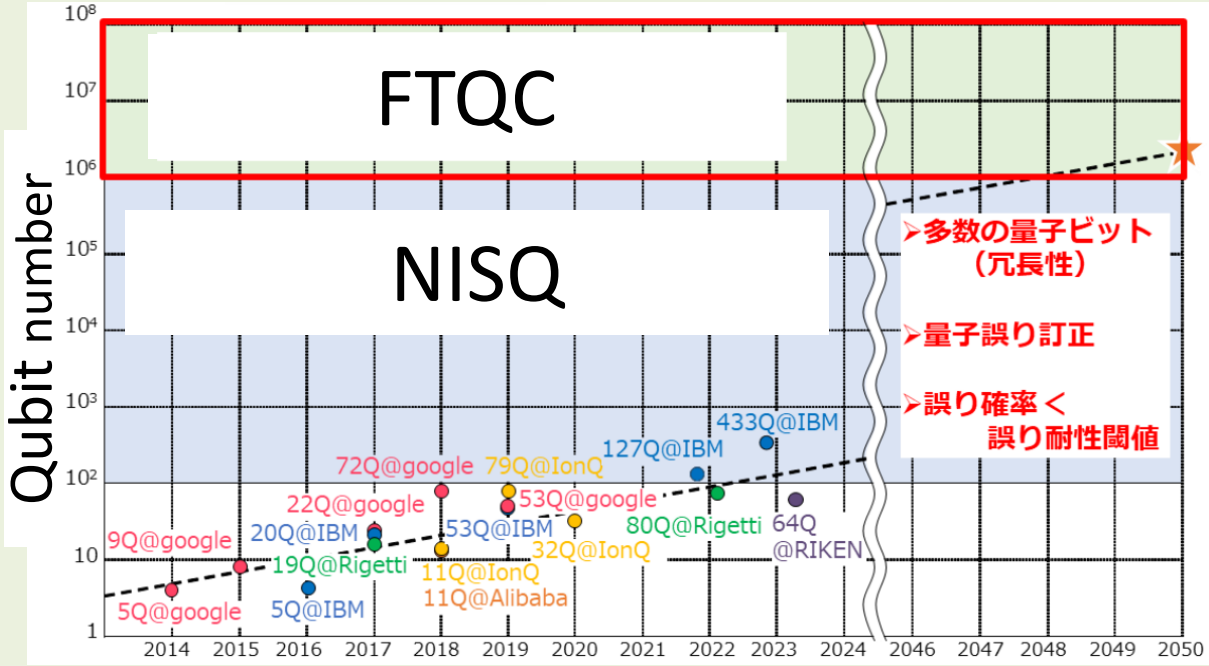
- Using n qubits, we can
  - encode a 2^n-dim vector: $|\Psi\rangle = \sum_{i=0}^{2^n-1} a_i |i\rangle$
  - do $2^n \times 2^n$ matrix operations only by ONE operation of a quantum circuit



- However, this does not immediately speed up all the calculations
  - How to extract the result $a'_0, \ldots, a'_{N-1}$ from the quantum states?
  - Implementing a general $2^n \times 2^n$ unitary is costly: $O(2^n)$ elementary gates
- Nonetheless, for some important problems, **quantum algorithms** output quantities of interest as **numbers** with fewer steps than classical methods
  - Apply q-algos to problems we want to solve on a case-by-case basis

NISQ & FTQC

- Today's quantum computer: **NISQ (Noisy Intermediate-Scale Quantum)** device
  - $O(10-100)$ qubits prone to noise (e.g. $|0\rangle$ flips to $|1\rangle$)
  - many usages have been proposed, but quantum advantage is still unclear
- Development goal: **fault-tolerant quantum computer (FTQC)**
  - error correction
    - ✓protect qubits by redundancy : combine $O(10^4)$ physical qubits to ONE logical qubit
  - can perform long calculations with theoretically guaranteed speed-up
  - $O(10^6)$ phys. qubits ($O(10^2)$ logical) will be needed for practical tasks
  - a few decades to realize



Japanese gov. prospect

https://www.jst.go.jp/moonshot/sympo/20230328/pdf/00_20230328_kitagawa.pdf

Why should we study quantum computing application NOW?

- This talk focuses on FTQC algorithm
- Question:
  Should we NOW search applications of quantum computing, esp. FTQC, even though it will come after decades?
- Yes! Because...
  - Not all calculation task are sped up.
    There are only quantum algorithms for specific problems.
    → To prepare for the quantum computing era,
    we should consider whether we can apply quantum algorithms to problems we are interested in, and how.
  - Finding use cases and estimating the impact promote quantum computer development.

Quantum algorithms useful for high-energy physics

- Quantum computer can speed up large matrix operations, which leads to various practical algorithms

Quantum algorithms	Possible usage
Solve linear equation systems (matrix inversion)	
derive → solve partial differential equaltions (PDEs)	Solve evolution equations (e.g. cosmological perturbation)
derive → machine learning - linear regression - support vector machine - clustering ...etc	Analysis of experimental data
Find matrix eigenvalues & eigenvectors	Find a ground energy & ground state

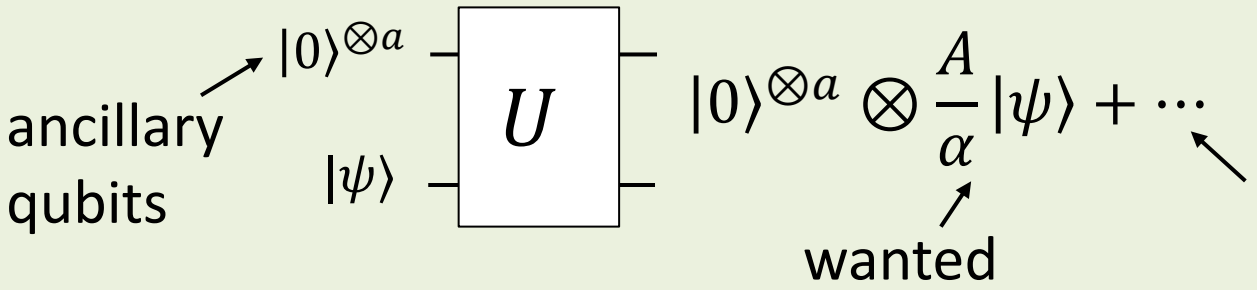
...etc

- This talk focuses on **<u>calculating eigenvalues of large matrices</u>** that appears in **<u>analyzing the primordial perturbation in stochastic inflation</u>**

Building-block technique (1): Block encoding

- Implement a nonunitary matrix as an upper-left block of a unitary
  - nonunitary $A \Rightarrow$ not implementable as a quantum circuit
  - $\frac{1}{\alpha}\begin{pmatrix} A & * \\ * & * \end{pmatrix}$ can be unitary $\Rightarrow$ implementable as a quantum circuit
    (α: factor to suppress norms of rows/cols in A/α smaller than 1)

- Definition: For s-qubit operator A, if $(s+a)$-qubit unitary U_A satisfies $\left\| A - \alpha(\langle 0|^{\otimes a} \otimes I) U (|0\rangle^{\otimes a} \otimes I) \right\| \leq \epsilon$, we say U_A is a (α, a, ϵ)-block-encoding of A



- If A is sparse, we can construct its block-encoding efficiently†
  - If we have the circuit $O_{\text{ent}}^{A}: |i\rangle|j\rangle|0\rangle \to |i\rangle|j\rangle|A_{ij}\rangle$ to compute entries in A, U_A is constructed by $O(1)$ uses of O_{ent}^{A}

† Gilyen et al., STOC 2019, pp. 193-204

Building-block technique (2): Quantum singular value transformation

- Singular value
  - Any matrix can be decomposed as $A = V \begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \end{pmatrix} W^\dagger$ $(\sigma_i \geq 0,$ V, W: unitary)
  - If A is Hermite and positive-definite, singular value = eigenvalue
- Quantum singular value transformation (QSVT)†
  - Given a block-enc. of A, construct a block-enc. of $g_S(A)$: $U_{g_S(A)} = \begin{pmatrix} g_S(A) & * \\ * & * \end{pmatrix}$

    Here, $g_S(A)$ is transformation of A's singular values by a function g:

$$A = V \begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \end{pmatrix} W^\dagger \rightarrow g_S(A) = V \begin{pmatrix} g(\sigma_1) & & \\ & g(\sigma_2) & \\ & & \ddots \end{pmatrix} W^\dagger$$

  - enables various operations related to A
    - ✓e.g.) inversion: $g(x) = \frac{1}{x} \Rightarrow U_{g_S(A)} = \begin{pmatrix} A^{-1} & * \\ * & * \end{pmatrix}$

† Gilyén et al., STOC 2019 pp. 193-204

2. Improved quantum algorithm for calculating eigenvalues of differential operators

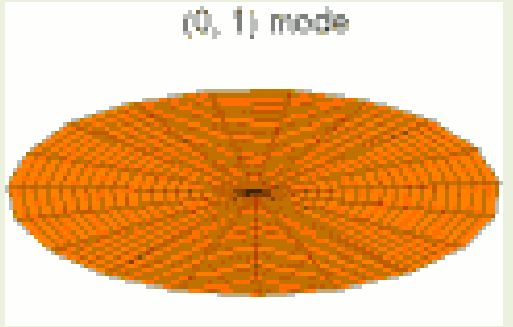
Eigenvalues of differential operators

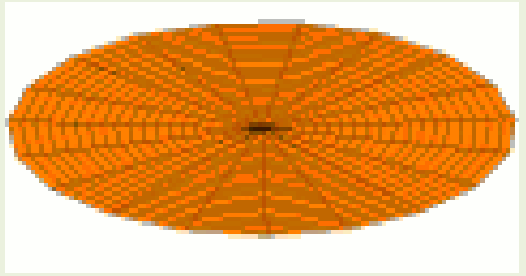
- **Partial differential equations** are widely used to describe physical phenomena
  - e.g.) Wave equation: $\frac{\partial^2}{\partial t^2} f(t, \mathbf{x}) = c^2 \Delta f(t, \mathbf{x}),\ \Delta \coloneqq \sum_i \frac{\partial^2}{\partial x_i^2}$
- **Eigenvalues of the differential operator** $\mathcal{L}$ are often of interest
  - $\mathcal{L} f(\mathbf{x}) = \lambda f(\mathbf{x}), \lambda \in \mathbb{C}$
    $\Rightarrow (\lambda, f)$: eigenvalue & eigenfunction
  - Important quantities that characterize the behavior of the solution
    - ✓e.g.) vibration of a membrane

$$\mathcal{L} = c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

$\Rightarrow \lambda$: eigenfrequency, f: eigenmode

https://commons.wikimedia.org/wiki/Category:Drum_vibration_animations

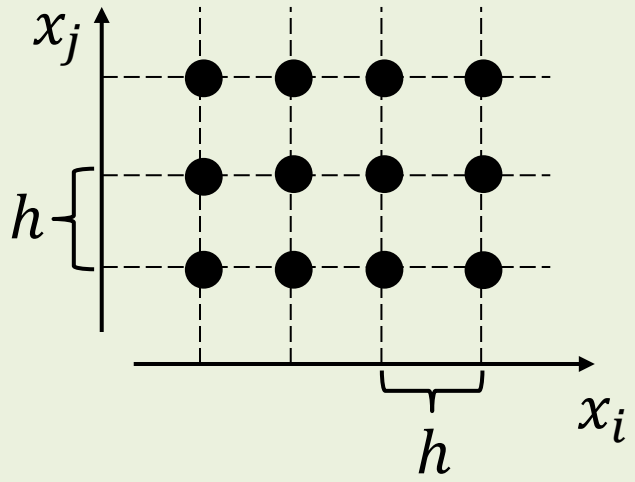




A common way: finite difference method

- Set grid points in the space and approximate derivatives by the **finite difference method** (FDM)
  - e.g., central diff. $\frac{\partial}{\partial x_i} f(\mathbf{x}) \simeq \frac{1}{2h}\left(f(\mathbf{x}+h\mathbf{e}_i) - f(\mathbf{x}-h\mathbf{e}_i)\right)$

    ($\mathbf{e}_i$: the unit vector in the ith direction)



- This converts the differential op. $\mathcal{L}$ into a matrix L, then we apply some method for matrix eigenvalue problem to L

$$\text{e.g., } \mathcal{L} = \frac{\partial^2}{\partial x^2} \rightarrow L = \begin{pmatrix} -2/h^2 & 1/h^2 & & & \\ 1/h^2 & -2/h^2 & 1/h^2 & & \\ & \ddots & \ddots & \ddots & \\ & & 1/h^2 & -2/h^2 & 1/h^2 \\ & & & 1/h^2 & -2/h^2 \end{pmatrix}$$

- But FDM suffers from **the curse of dimensionality**
  - In d-dim cases, if we set n_{gr} grid points in each direction, L is $n_{\mathrm{gr}}^d \times n_{\mathrm{gr}}^d$

    → for large d, intractable in classical computing!

Previous works

- Quantum algorithms can perform exponentially large matrix calculations
  - e.g.) HHL for matrix inversion (pioneering algorithm in 2000s):
    $O(\mathrm{poly}\log N)$ computational complexity for $N \times N$ matrices
- In fact, some works in the 2000s† proposed quantum algorithms for calculating differential operator eigenvalues, based on that for matrix eigenvalues‡
  - But, not consider multi-dimensional cases or rigorously evaluate the dependence of computational complexity on d
  - No paper since then, so recent progress in quantum algorithms has not been incorporated
- Let's improve the quantum algorithm using state-of-the-art techniques such as **block encoding** & **quantum singular value transformation**!

† Szkopek et al., PRA 72, 062318 (2005); Papageorgiou et al., Quantum Inf. Process. 4, 87 (2005); Bessen, J. Complex. 22, 660 (2006)
‡ Abrams & Lloyd, PRL 83, 5162 (1999)

Our quantum algorithm: problem setting

- Consider operators of the Sturm–Liouville type

$$\mathcal{L} = -\sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left(a_i(\mathbf{x})\frac{\partial}{\partial x_i}\right) + a_0(\mathbf{x}) \quad (a_0, a_1, \dots, a_d : \overline{\mathcal{D}} \to \mathbb{R}_+)$$

on $\mathcal{D} \coloneqq (U, L) \times \cdots (U, L) \subset \mathbb{R}^d$

  - ➢ includes Laplacian Δ, (a part of) Fokker-Planck,
    the problem in stochastic inflation considered later, and so on
  - ➢ We impose the Dirichlet boundary condition ($f = 0$ on $\partial\mathcal{D}$)
  - ➢ All the eigenvalues are positive

Our quantum algorithm: finite-difference approx.

- Set n_{gr} points at equal intervals of h in each direction and approximate $\mathcal{L}$ as†

$$\mathcal{L}f(\mathbf{x}_{\mathbf{j}}^{\mathrm{gr}}) \approx \sum_{i=1}^{d} \frac{-1}{h^2}\Big[a_i(\mathbf{x}_{\mathbf{j}}^{\mathrm{gr}} + \frac{h}{2}\mathbf{e}_i)f(\mathbf{x}_{\mathbf{j}}^{\mathrm{gr}} + h\boldsymbol{e}_i) - \Big(a_i(\mathbf{x}_{\mathbf{j}}^{\mathrm{gr}} + \frac{h}{2}\boldsymbol{e}_i) + a_i\,(\mathbf{x}_{\mathbf{j}}^{\mathrm{gr}} - \frac{h}{2}\boldsymbol{e}_i)\Big)f(\mathbf{x}_{\mathbf{j}}^{\mathrm{gr}}) + a_i(\mathbf{x}_{\mathbf{j}}^{\mathrm{gr}} - \frac{h}{2}\boldsymbol{e}_i)f(\mathbf{x}_{\mathbf{j}}^{\mathrm{gr}} - h\boldsymbol{e}_i)\Big] + a_0(\mathbf{x}_{\mathbf{j}}^{\mathrm{gr}})$$

($\mathbf{x}_{\mathbf{j}}^{\mathrm{gr}}$: grid point in $\mathcal{D}$ labeled by $\mathbf{j} = (j_1, \dots, j_d) \in \{1, \dots, n_{\mathrm{gr}}\}^{\times d}$)

- By this, $\mathcal{L}$ is converted into a Hermitian matrix $L \in \mathbb{R}^{N_{\mathrm{gr}} \times N_{\mathrm{gr}}}$
  ($N_{\mathrm{gr}} = n_{\mathrm{gr}}^d$: total # of grid points)

- When $n_{\mathrm{gr}} \to \infty$, L's eigenvalues λ_k^L converge to $\mathcal{L}$'s eigenvalues λ_k‡

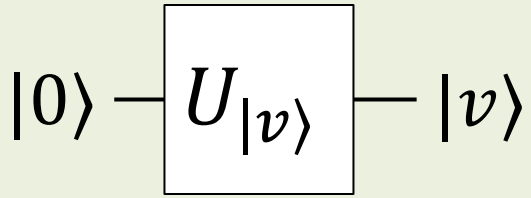
$$\left|\lambda_k^L - \lambda_k\right| = O\left(\frac{1}{n_{\mathrm{gr}}^2}\right)$$

† Larsson and Thomée, "Partial differential equations with numerical method" (2003) ‡ Kuttler, SIAM J. Numer. Anal., 7, 206 (1970)

Quantum algorithm to find the first eigenvalue of a matrix

- We are often interested in the first (=smallest) eigenvalue of $\mathcal{L}$

- We use a QSVT-based algorithm to find the first eigenvalue of a Hermitian matrix L ¶
  - Suppose that we have
    - block-encoding U_L of L
    - quantum circuit $U_{|v\rangle}$ to prepare a state $|v\rangle$ that **overlaps** the first eigenvector $|\psi_1\rangle$ of L well: $|\langle\psi_1|v\rangle| = O(1)$



  - Then, we can find an estimate λ_1' of L's first eigenvalue λ_1 with accuracy ϵ ($|\lambda_1' - \lambda_1| \leq \epsilon$), with $\tilde{O}(\|L\|/\epsilon)$ queries‡ to U_H & $U_{|v\rangle}$

Not dependent on L's size

¶ Lin and Tong, Quantum 4, 372 (2020) ‡ $\tilde{O}$: big-O with log-factors omitted

Our quantum algorithm: outline

- Assumptions: we have the following quantum circuits
  a. O_{a_i} to calculate $a_i(x)$: $O_{a_i}|\mathbf{x}\rangle|0\rangle = |\mathbf{x}\rangle|a_i(\mathbf{x})\rangle$
     (Recall: $\mathcal{L} = -\sum_{i=1}^{d} \partial_{x_i}\left(a_i(\mathbf{x})\partial_{x_i}\right) + a_0(\mathbf{x})$)
  b. $O_{\tilde{f}_1}$ to generate $\left|\tilde{f}_1\right\rangle = \sum_{\mathbf{j}} \tilde{f}_1(\mathbf{x}_{\mathbf{j}}^{\mathrm{gr}})\,|\mathbf{j}\rangle$ ‡ that encodes a trial function $\tilde{f}_1$ overlapping the first eigenfunction f_1 well ($\left|\int_{\mathcal{D}} f_1(\mathbf{x})\tilde{f}_1(\mathbf{x})\mathrm{d}\mathbf{x}\right| = O(1)$)

- Procedure
  1. Discretize the differential operator $\mathcal{L}$ to the matrix L
  2. Implement a block-encoding of L with O_{a_i}'s
  3. Estimate L's first eigenvalue by the aforementioned quantum algorithm for matrix eigenvalue finding

‡ The normalization factor is omitted.

Our quantum algorithm: computational complexity

- We find an estimate of $\mathcal{L}$'s first eigenvalue λ_1 with accuracy ϵ, with $\tilde{O}(d^3/\epsilon^2)$ queries to O_{a_i}'s and $O_{\tilde{f}_1}$.
  - Polynomial w.r.t d

- Regarding the dependency on ϵ, compared to Szkopek et al. (2005) ($\tilde{O}(1/\epsilon^3)$), our algorithm makes an improvement.

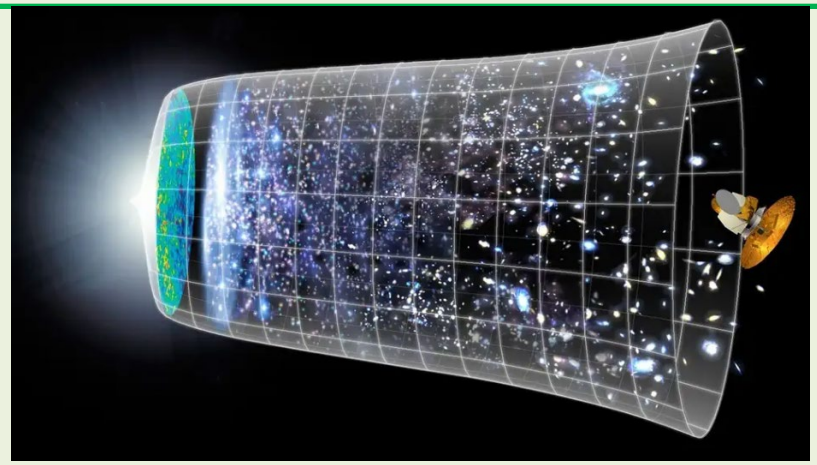
Loading a function to a quantum state

- How to generate $|f\rangle = \sum_{\mathbf{j}} f(\mathbf{x}_{\mathbf{j}}^{\mathrm{gr}}) \, |\mathbf{j}\rangle$ that encodes the function f?
  - For simple f which is written by elementary functions, some methods for encoding have been proposed (e.g. Grover & Rudolph, arXiv:quant-ph/0208112)
  - We assume that a trial function $\tilde{f}_1$ can be set to such a simple one (which is the case in the stochastic inflation example)

3. Application to estimating the decay rate of the perturbation distribution tail in stochastic inflation

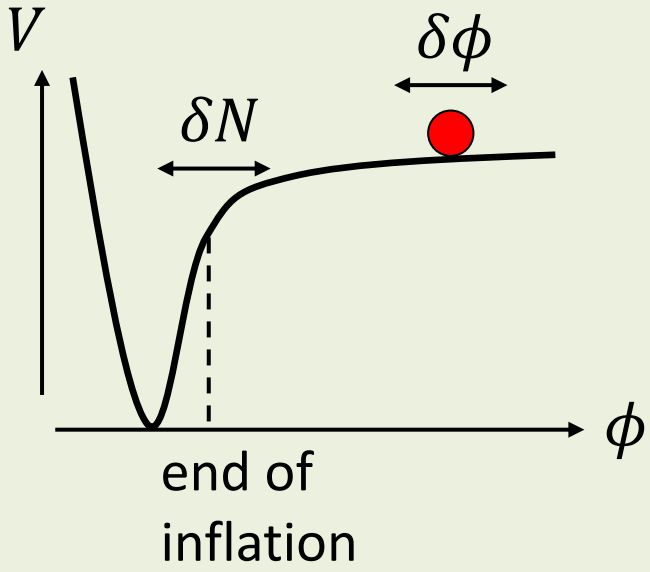
Primordial perturbation generated by inflation

- Inflation
  - ➢accelerated space expansion in the early universe
  - ➢total e-fold: $N_{tot} > 50 - 60$
    - ✓e-fold: $N(t) = \ln a(t)/a_0$ (a: scale factor)
      (Space expanded $e^{N(t)}$ times)
  - ➢caused by the potential of scalar fields (*inflatons*) slowly rolling to the potential minimum



https://www.newscientist.com/definition/cosmic-inflation/

- In inflation, the primordial curvature perturbation is generated by inflaton's quantum fluctuation



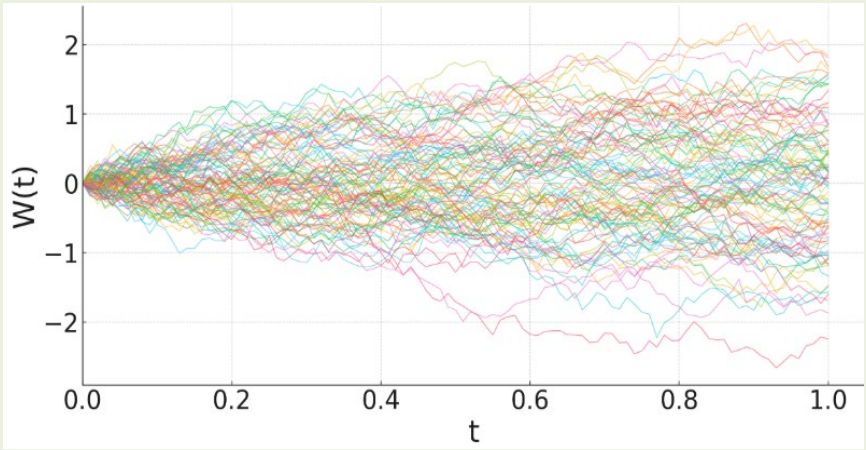
- δN-formalism
  - ➢curvature pert. ζ = fluctuation of the total e-fold δN ‡

‡ in some gauge

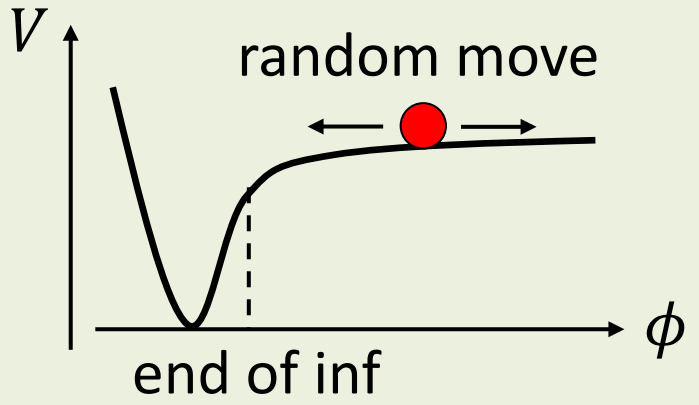
Stochastic inflation

- Probabilistic framework to analyze inflationary perturbations†
  - Dynamics of inflatons $\boldsymbol{\phi} = (\phi_1, \dots, \phi_d)$ (coarse-grained on a large scale) is described by the following stochastic differential eq. (Langevin eq.) ¶

$$\mathrm{d}\phi_i = -\frac{1}{v(\boldsymbol{\phi})}\partial_{\phi_i} v(\boldsymbol{\phi})\mathrm{d}N + \sqrt{2v(\boldsymbol{\phi})}\mathrm{d}W_i$$



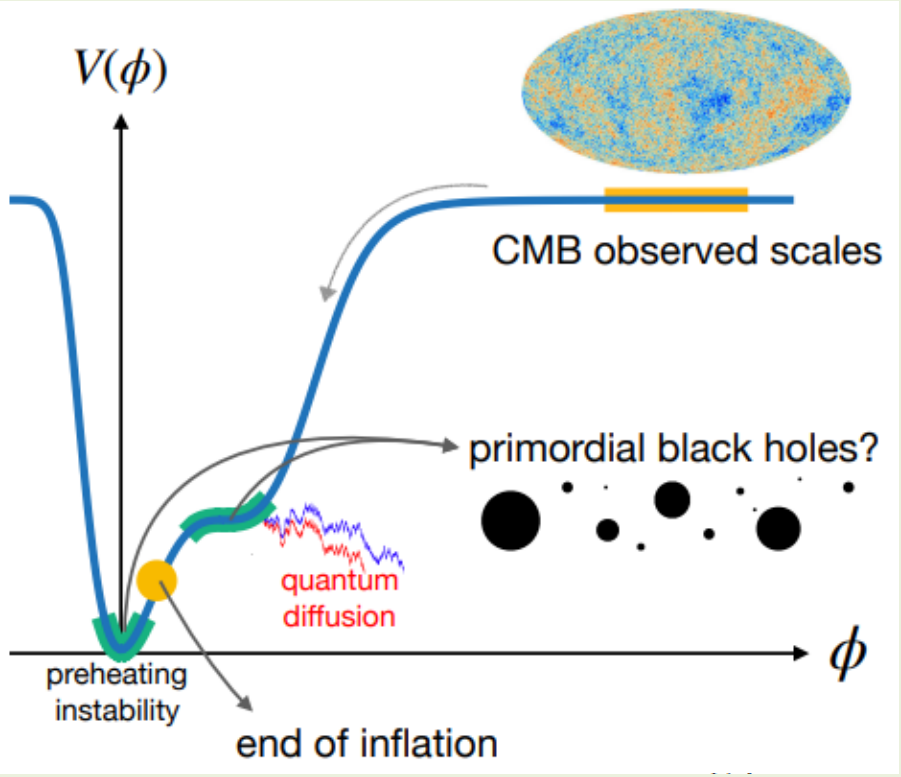
    - ✓ $v = V/24\pi^2$, V: inflatons' potential
    - ✓ W_i: Wiener process (Brownian motion)
      - $\mathrm{d}W_i(t) = W_i(t + \mathrm{d}t) - W_i(t)$ obeys normal dist. with mean=0 & variance= $\mathrm{d}t$
    - ✓ e-fold N is used as a variable that represents time
  - δN = fluctuation of the time when $\boldsymbol{\phi}$ reaches the end-of-inflation point



† For a review, see Cruces, Universe 8, 334 (2022) ¶ We take a unit that the reduced Planck mass is 1.

Eigenvalue problem in stochastic inflation

- If inflatons go through a very flat region (e.g., inflection point), random movement dominates slow-roll
  - → **Fat tail** in the probability distribution of density perturbations
  - → primordial black holes



Vennin, arXiv:2009.08715

- Conditioned that inflatons are at $\boldsymbol{\phi}$ at some time, the probability density of $\mathcal{N}$, e-fold to the end of inflation, obeys the adjoint Fokker-Plack eq.

$$\partial_{\mathcal{N}} P(\mathcal{N}|\boldsymbol{\phi}) = \mathcal{L}_{\mathrm{FP}}^{\dagger} P(\mathcal{N}|\boldsymbol{\phi}),\ \mathcal{L}_{\mathrm{FP}}^{\dagger} = \sum_{i=1}^{d} \left(-\frac{\partial_{\phi_i} v}{v} \partial_{\phi_i} + v \partial_{\phi_i}^2 \right)$$

- Eigenvalues of $\mathcal{L}_{\mathrm{FP}}^{\dagger}$ = decay rate of $P(\mathcal{N}|\boldsymbol{\phi})$ w.r.t. $\mathcal{N}$

  ⇒ If $\mathcal{L}_{\mathrm{FP}}^{\dagger}$ has small eigenvalues, $P(\mathcal{N}|\boldsymbol{\phi})$ may have a fat tail!

Applying our quantum algo to find the eigenvalue of $\mathcal{L}_{\mathrm{FP}}^{\dagger}$

- d may be large (multifield inflation) → classically intractable
- Our quantum algorithm can be applied
  - ➢ $\mathcal{L}_{\mathrm{FP}}^{\dagger}$ is not of the Sturm-Liouville type, but can be transformed to $\widetilde{\mathcal{L}_{\mathrm{FP}}^{\dagger}}$ of that type with the same eigenvalues
- Issue: Can we choose a trial function $\tilde{f}_1$ overlapping the first eigenfunc f_1 well?
  - ➢ $\left|\int_{\mathcal{D}} f_1(\mathbf{x})\tilde{f}_1(\mathbf{x})\mathrm{d}\mathbf{x}\right|$ should be as large as possible
  - ➢ But we do not know f_1...
- Idea: f_1 is expected to have a simple shape (no node, single bump,...)
  → **How about a Gaussian?**
  - ➢ Let's confirm through a test case!

Test case: Hybrid inflation with an inflection point

- 2-field model with the following potential

$$V(\phi,\psi) = V_\phi(\phi) + V_0\left[\left(1-\left(\frac{\psi}{M}\right)^2\right)^2 + 2\left(\frac{\phi\psi}{\phi_c M}\right)^2\right]$$

- For ϕ's potential, we take an inflection-type one

$$V_\phi(\phi) = V_0\beta(\phi-\phi_c)^3$$

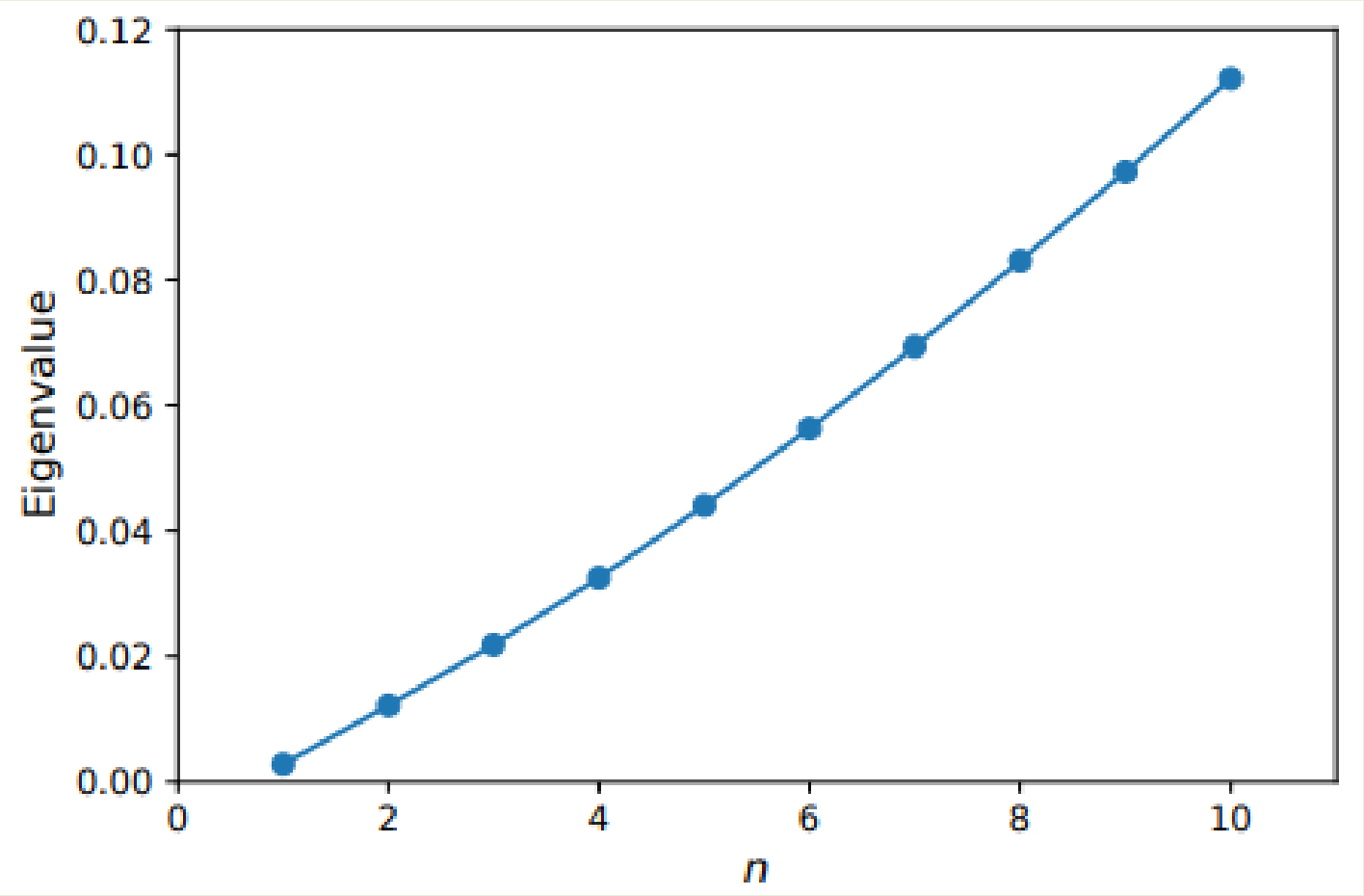
- In 2-dim cases, calculating eigenvalues by FDM can be tractable by classical computers, so we have performed it and seen the overlap between f_1 and the Gaussian trial function $\tilde{f}_1$.

- Tested parameters:

$$V_0 = 10^{-15}, M = 10^{16}\mathrm{GeV}, \phi_c = \sqrt{2}M, \beta = 10^4$$

Test case: Hybrid inflation with an inflection point

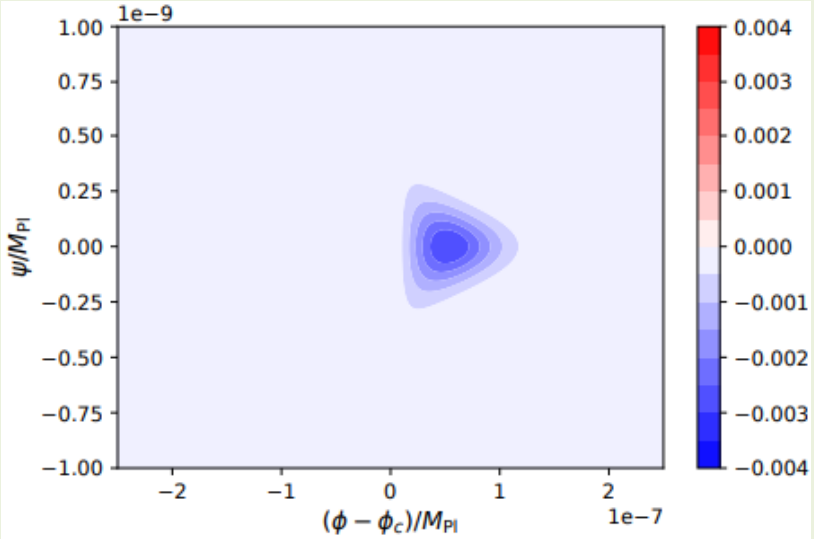
- Lowest eigenvalues



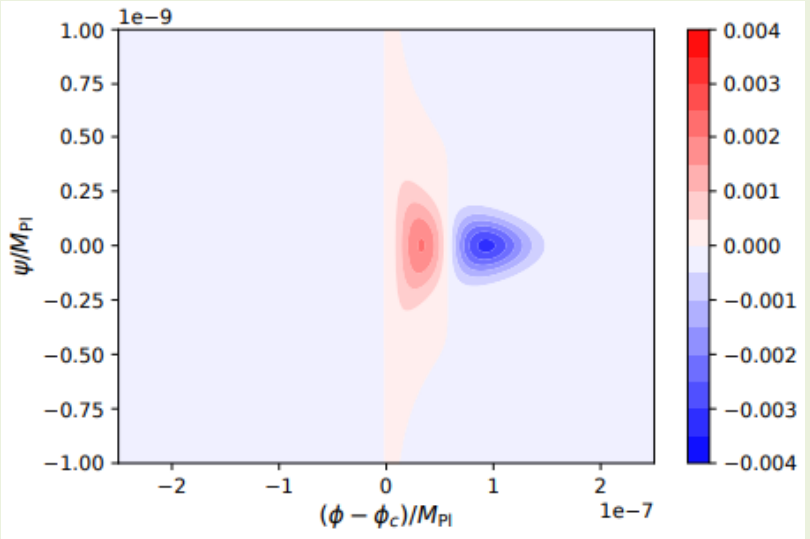
➢There are small eigenvalues

Test case: Hybrid inflation with an inflection point

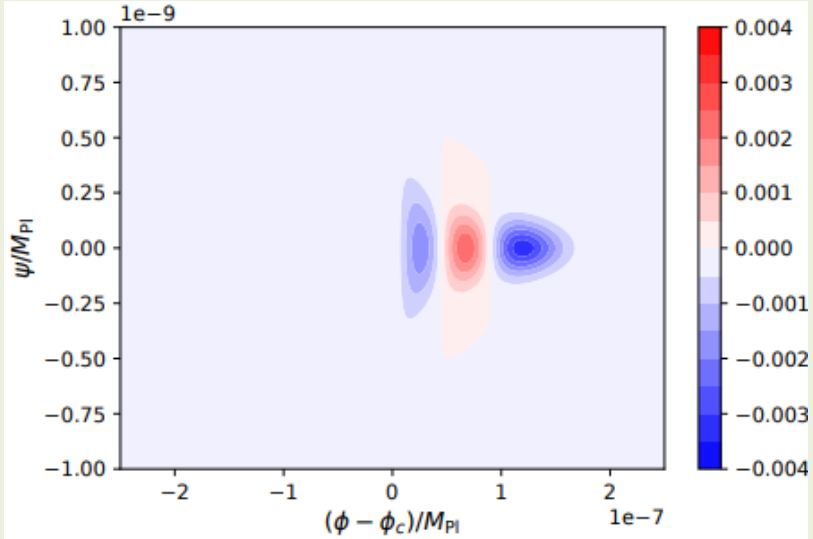
- Lowest eigenfunctions



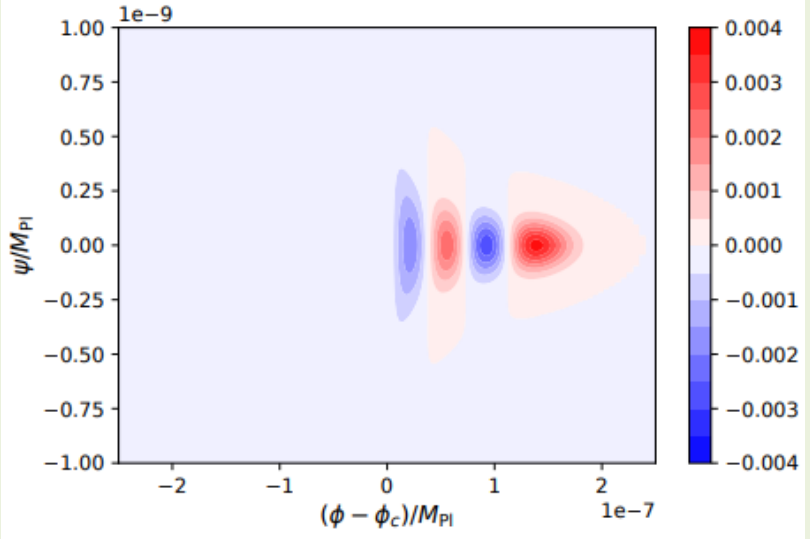
(a) $n = 1$



(b) $n = 2$



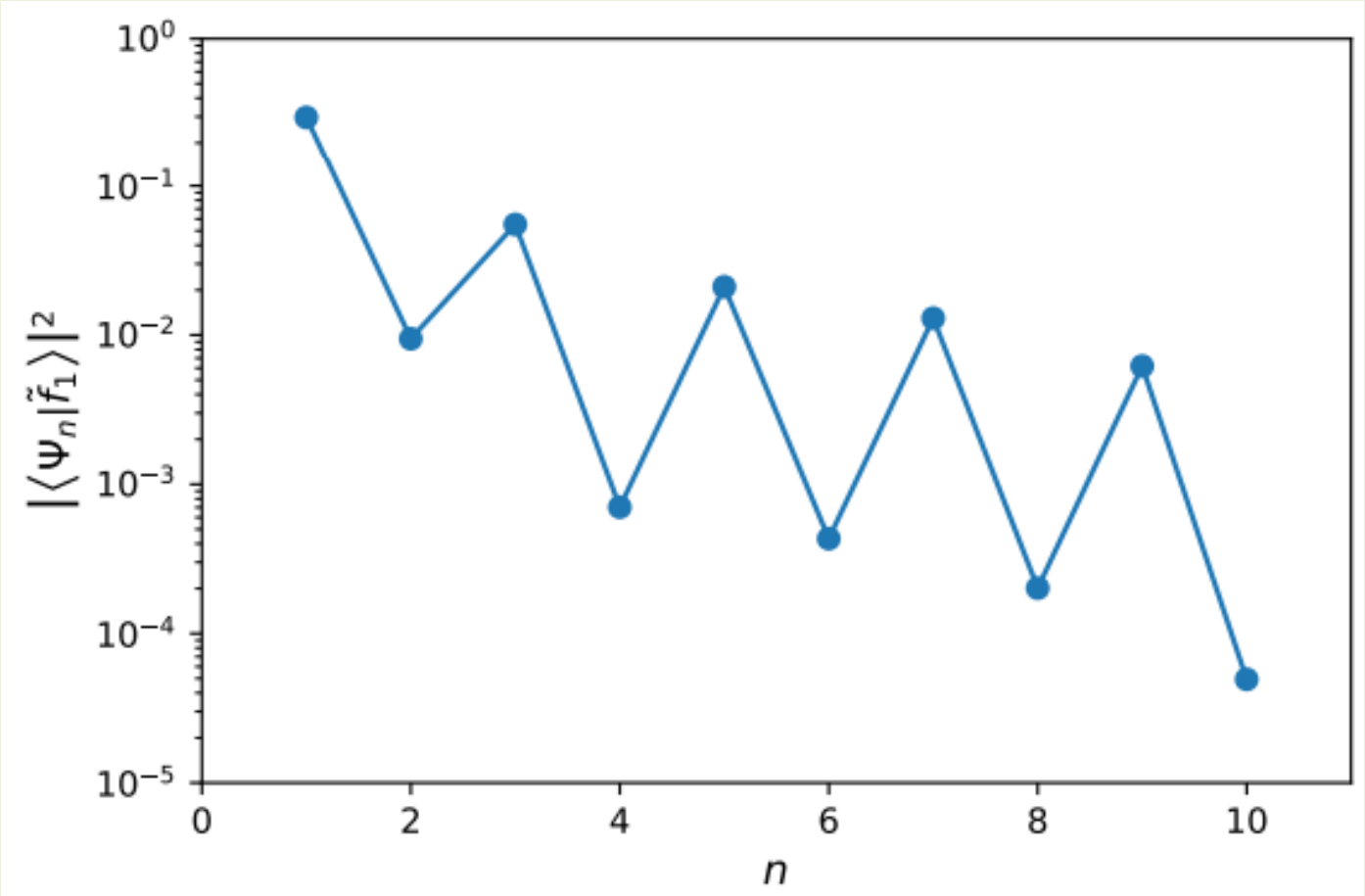
(c) $n = 3$



(d) $n = 4$

Test case: Hybrid inflation with an inflection point

- Overlap between the trial function and the lowest eigenfunctions



➢ Overlap with the first eigenfuction is about 0.3

→ Our quantum algorithm is expected to work!

4. Summary

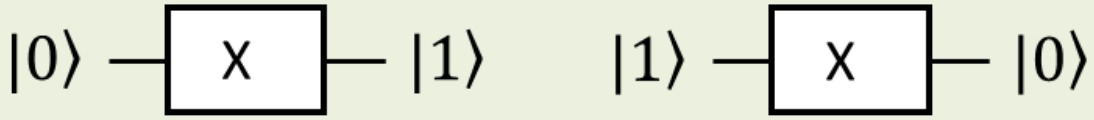
Summary

- Quantum computing may speed up various numerical problems, especially large matrix operations, including HEP-related ones.
- Calculating eigenvalues of differential operators is an important task, but the FDM approach suffers from the curse of dimensionality.
- We proposed an improved quantum algorithm for this task based on QSVT
  - computational complexity: $\tilde{O}(d^3/\epsilon^2)$ (d: dimension, ϵ: accuracy)
- HEP use-case: stochastic inflation
  - small eigenvalues of the adjoint Fokker-Planck op.
    → fat tail in the probability distribution of the density perturbation
    → PBH
  - Demonstrated the FDM for hybrid inflation with an inflection-type potential
    → Gaussian trial function works

Supplement

Quantum gate & quantum circuit

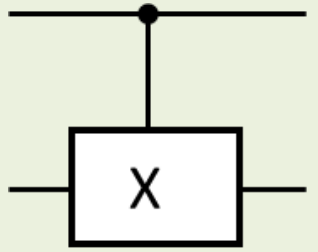
- quantum gate: operation on qubits (unitary operation on the state vector)
  - example of 1-qubit gates
    - ✓ NOT : flip 0 ⇔ 1 $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$



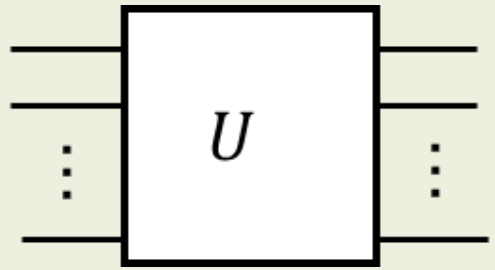
    - ✓ Hadamard: $H = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$

  - example of 2-qubit gates
    - ✓CNOT (controlled NOT):
      flip target qubit if control qubit is 1

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



- Combining quantum gates, we construct a **<u>quantum circuit</u>** and perform desired computation
  - arbitrary unitary transformation
    by combining several types of elementary gates



Quantum algorithm to find the first eigenvalue of a matrix

- Now, L is Hermitian and positive-definite, so L's eigenvalue = L's singular value

- We can estimate the smallest eigenvalue λ_1 as follows
  - By QSVT, we can apply a step-function $\Theta_\lambda(x) = \begin{cases} 1; x < \lambda \\ 0; x > \lambda \end{cases}$ to eigenvalues, which transforms a quantum state as

    $|0\rangle|\psi\rangle = |0\rangle \otimes \sum_i a_i|v_i\rangle$

    $\rightarrow U_{\Theta_\lambda}|0\rangle|\psi\rangle = |0\rangle \otimes \sum_{\lambda_i<\lambda} a_i|v_i\rangle + |1\rangle \otimes \sum_{\lambda_i>\lambda} a_i|v_i\rangle$

    ($|v_i\rangle$: i-th eigenvector of L)
  - For $U_{\Theta_\lambda}|0\rangle|\psi\rangle$, $\lambda < \lambda_1 \Rightarrow \Pr(\text{ancilla} = |0\rangle) = 0$

    $\lambda > \lambda_1 \Rightarrow \Pr(\text{ancilla} = |0\rangle) > 0$
  - By binary search on λ, we find the threshold that separates $\Pr(\text{ancilla} = |0\rangle) = 0$ and $\Pr(\text{ancilla} = |0\rangle) > 0$

    $\rightarrow$ It is an estimate of λ_1