Multi-Entropy Measures for Topologically Ordered States

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Interfaces and symmetries,
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Based on:

- "*Multi wavefunction overlap and multi entropy for topological ground states in (2+1) dimensions*",
  Bowei Liu, Junjia Zhang, Shuhei Ohyama, Yuya Kusuki, SR (24)

- "*Revealing the chiral central charge from permutation defects in topological states*",
  Yarden Sheffer, Ruihua Fan, Ady Stern, Erez Berg, and SR (25)

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Outline

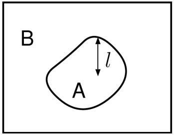
- Introduction; entanglement probes for topologically ordered phases
- Multientropy measures
- Tripartition setup: Multientropy (and reflected entropy)
- Quadpartition setup: Modular commutator and lens space multientropy
- Summary

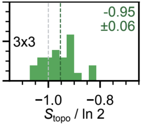
Introduction

- Entanglement has been utilized as a useful probe for many-body quantum systems, both for ground states and for dynamics
- E.g., topologically oredred phases in (2+1)d.
- Topological entanglement entropy [Levin-Wen, Kitaev-Preskill (05)]

$$S_A = const. \times \ell - \ln D$$

The constant negative term (total quantum dimension) encodes an emergent gauge structure.





[Satzinger et al.(21)]

Topological order and anyons

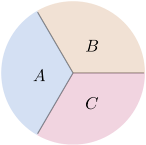
- Quantum dimension d_a; Total quantum dimension $D = \sqrt{\sum_a d_a^2}$
- Topological spin (self-statistical angle): $\theta_a = \exp 2\pi i h_a$.
- Fusion: $a \times b = \sum_c N_{ab}^c c$.
- Other data ...

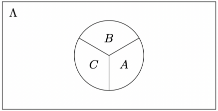
- Total quantum dimension does not fully distinguish different topological orders.

Can we have a finer entanglement probe?
Can we extract and detect all anyon data?

- Various entanglement quantities have been studied; Mutual information, Entanglement spectrum, Entanglement negativity, Computable cross-norm or realignment negativity.

- Go beyond bipartition and divide the total system into three or more subregions. How are these subsytems entangled with each other?





- Three qubits can entangle in a way different from two-qubit entanglement

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle\right], \quad |\mathrm{W}\rangle = \frac{1}{\sqrt{3}}\left[|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle\right]$$

- Multipartite entanglement in many-body ground states?

What can we learn about topological order from multipartite entanglement?

- It is convenient to have specific, computable "measures". Today, I will focus on multientropy measures.

Central charges

- As we will see, multientropy measures can detect central charges.
- Can mostly be understood from the perspective of boundaries.
- *Chiral central charge* $c_- = c_L - c_R$. Related to thermal Hall transport: $\kappa = \frac{\pi k_B^2 T}{6} \times c_-$ [[Kane-Fisher]]
- *Total central charge* $c_{tot} = c_L + c_R$ for ungappable degrees of freedom.

Central charges

- Anyon data and c_- mod 8 are related as:

$$e^{\frac{2\pi i}{8}c_-} = \frac{\sum_a d_a^2\theta_a}{\left|\sum_a d_a^2\theta_a\right|} =: \zeta_1$$

- *Higher central charge* [Ng-Schopieray-Wang(19); Ng-Rowell-Wang-Zhang(22)]

$$\zeta_n := \frac{\sum_a d_a^2\theta_a^n}{\left|\sum_a d_a^2\theta_a^n\right|}, \quad n \in \mathbb{N}$$

Related to gappability of edge [Kaidi-Komargodski-Ohmori-Seifnashri-Shao(22)]

Can we detect central charges from bulk wavefunctions?

Multientropy measures

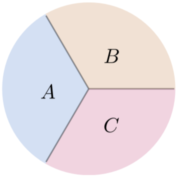
- Multientropy: [Gadde-Krishna-Sharma (22-), Penington-Walter-Witteveen (22), Harper-Takayanagi-Tsuda (24) ...]

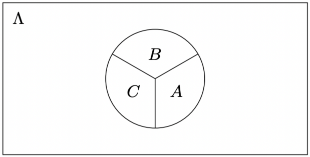
$$\langle\Psi|^{\otimes R}(\pi_A \otimes \pi_B \otimes \pi_C) \mid \Psi\rangle^{\otimes R}$$

where

$R = \#$ of replicas

π_A, π_B, π_C : permutations acting on replicas in the subregion.



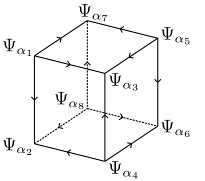


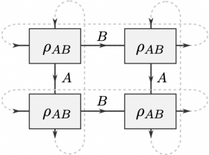
Example

- [Gadde-Krishna-Sharma (22-), Penington-Walter-Witteveen (22), Harper-Takayanagi-Tsuda (24) ...]

$$Z(A\!:\!B\!:\!C) = \langle\Psi|^{\otimes 4}(\pi_A \otimes \pi_B \otimes \pi_C) \mid \Psi\rangle^{\otimes 4},$$
$$\text{with} \quad \pi_A = (12)(34), \quad \pi_B = (13)(24), \quad \pi_C = e,$$

$$G(A\!:\!B\!:\!C) = -(1/2)\ln Z(A\!:\!B\!:\!C)$$





- (More general version)
- Related to *Rényi reflected entropy:*

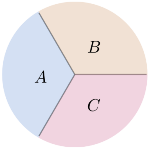
$$G(A\!:\!B\!:\!C) = (1/2)S^R_{2,2}(A\!:\!B) + S_2(AB)$$

where
$S^R_{2,2}(A\!:\!B)$: Rényi reflected entropy for ρ_{AB};
$S_2(A)$: the second Rényi for ρ_A.

More examples applied to topologically ordered states

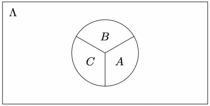
Tripartition setup:

- Reflected entropy [Siva-Zou-Soejima-Mong-Zaletel (21)] [Yuhan Liu-Kusuki-Sohal-Kudler-Flam-SR (21, 23)]
- Multientropy [Bowei Liu-Zhang-Ohyama-Kusuki-SR (24)]



Quadpartition setup:

- Modular commutator [Kim-Shi-Kato-Albert(22);Fan (22); Zou-Shi-Sorce-Lim-Kim (22);Fan-Sahay-Vishwanath(22)]
- Lens-space multientropy and higher-central charge [Sheffer-Stern-Berg (25)]
- Other examples [Sheffer-Fan-Stern-Berg-SR (25)]



Other related works:

- Chiral central charge from twist defects [Sopenko (24-25)];
  Higher central charge from partial rotation [Kobayashi-Wang-Soejima-Mong-SR(23)]
- ...

What do they measure?

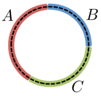
- Reflected entropy can capture tripartite entanglement [Akers-Rath (19)]
  ($h = S^R(A:B) - I(A:B)$)

$$h_{\mathrm{GHZ}} = 0, \quad h_{\mathrm{W}} = 1.49 \ln 2 - 0.92 \ln 2 > 0.$$

- $h = 0$ iff a state is a "sum of triangle states".
  A fixed-point MPS is a SOTS. (1d system is gapped if and only if $h = 0$).
  [Zou-Siva-Soejima-Mong-Zaletel (20)]

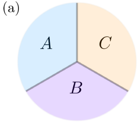
- Ground state of 1d critical spin chain [Zou-Siva-Soejima-Mong-Zaletel (20)]

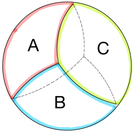
$$h = \frac{c}{3} \ln 2$$



for N_A/N and N_B/N in the limit $N \to \infty$.

Results for tripartition setup





For generic (chiral) topological liquid deep in topological limit

[Liu-Sohal-Kudler-Flam-SR (21), Liu-Kusuki-Kudler-Flam-Sohal-SR (23)] [Bowei Liu-Zhang-Ohyama-Kusuki-SR(24)]

$$h = S^R(A:B) - I(A:B) = \frac{c}{3}\ln 2, \quad c\text{: central charge.}$$

$$\kappa = G(A:B:C) - \frac{1}{2}\left(S_2(A) + S_2(B) + S_2(C)\right) = \frac{c}{8}\ln 2$$

Method: Bulk-boundary correspondence, CFT

Solvable lattice models (Levin-Wen models): $h = 0$.

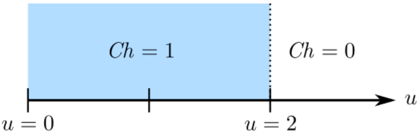
[Siva-Zou-Soejima-Mong-Zaletel (21)]

Method: Isometric tensor network

Method: Free fermion numerics

- Lattice fermion model $f_{\mathbf{r}} = (f_{\uparrow\mathbf{r}}, f_{\downarrow\mathbf{r}})$:

$$H = \frac{-i}{2}\sum_{\mathbf{r}}\sum_{\mu=x,y}\left[f_{\mathbf{r}}^{\dagger}\tau_{\mu}f_{\mathbf{r}+\mathbf{a}_{\mu}} - f_{\mathbf{r}+\mathbf{a}_{\mu}}^{\dagger}\tau_{\mu}f_{\mathbf{r}}\right] + \frac{1}{2}\sum_{\mathbf{r}}\sum_{\mu=x,y}\left[f_{\mathbf{r}}^{\dagger}\tau_{z}f_{\mathbf{r}+\mathbf{a}_{\mu}} + f_{\mathbf{r}+\mathbf{a}_{\mu}}^{\dagger}\tau_{z}f_{\mathbf{r}}\right] + u\sum_{\mathbf{r}} f_{\mathbf{r}}^{\dagger}\tau_{z}f_{\mathbf{r}},$$



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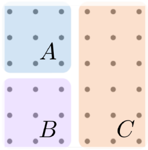
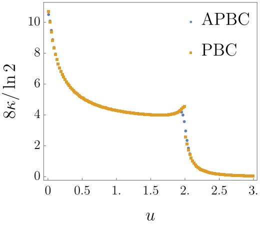
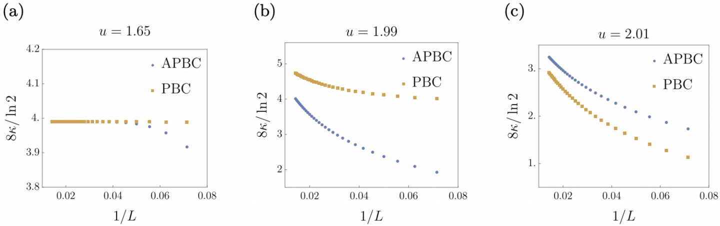


Multientropy numerics

- Analytical result: $\kappa = \frac{c}{8} \ln 2$
- κ is subsystem size independent.
- κ is minimal in the topological phase around $u = 1.34$
- $\kappa \sim (c/8) \ln 2 \times 4$
- Four trijunctions as opposed to two may result in a factor of 2.





- Conjecture:

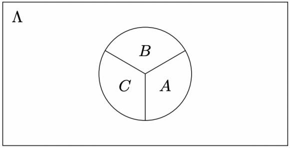
$$\kappa \geq (c_{tot}/8)\ln 2 \quad \text{for multientropy}$$
$$h \geq (c_{tot}/3)\ln 2 \quad \text{for reflected entropy}$$

  where c_{tot} is the central charge of the total ungappable degrees of freedom

- May have an implication on numerics; non-zero h may be an obstruction to have a finite dim PEPS representation
  C.f. diverging correlation length of chiral PEPS [Poilblanc ...] :

Quad partition setup

- We now consider multientropy for "quad partition setup"



- (The phase of) the expectation value

$$\mathcal{M} := \langle \Psi^{\otimes R} | \pi_A \pi_B \pi_C | \Psi^{\otimes R} \rangle$$

Rényi modular commutator J_n

[Sheffer-Fan-Stern-Bern-SR (2025); See also Gass-Levin (2025)]

- Choose: $R = 2n + 1$ and

$$\pi_A = (1, \ldots, 2n+1)$$
$$\pi_B = (1, \ldots, n+1)$$
$$\pi_C = (n+1, \ldots, 2n+1)$$

- The modular commutator

$$J(A, B, C)_\rho := i \, \mathrm{Tr} \left(\rho_{ABC} \left[K_{AB}, K_{BC} \right] \right)$$

where $K_{AB} = -\ln \rho_{AB}$ is the modular operator.

- The modular commutator and chiral central charge: $J = \frac{\pi}{3} c_-$.

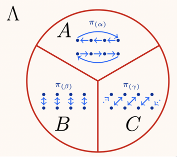
- J is related to J_n in the replica limit:

$$J = \lim_{n \to 0} \frac{i}{n^2} (J_n - \overline{J_n}), \quad J_n = \langle \Psi | \rho_{AC}^n \rho_{AB}^n | \Psi \rangle$$

Lens-space multi entropy Φ_r

- Choose: $R = 2r$ for $r \geq 2$, and

$$\begin{aligned}
&\pi_A(1,t) = (1,t-1); \quad \pi_A(2,t) = (2,t+1),\\
&\pi_B(s,t) = (s+1,t),\\
&\pi_C(1,t) = (2,t-1); \quad \pi_C(2,t) = (1,t+1),
\end{aligned}$$



Indexing the replicas with a tuple (s,t) with $s = 1, 2$, $t = 1, ..., r$; addition is defined mod 2 for s and mod r for t.

- For non-chiral topologically-ordered phases [Sheffer-Stern-Berg (24)]

$$\Phi_r \propto \sum_a d_a^2 \theta_a^r,$$

The quantity on the RHS is proportional to the higher central charge. C.f. higher central charge from partial rotation [Kobayashi-Wang-Soejima-Mong-SR (23)]

Main results for quad partition setup

- n-th Rényi modular commutator:

$$J_n \propto \exp(-\frac{2\pi i c_-}{24}\frac{2n^2}{(2n+1)(n+1)}).$$

- Lens-space multi entropy

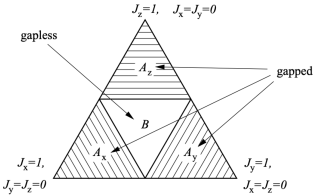
$$\Phi_r \propto \exp(-\frac{2\pi i c_-}{24}(r+\frac{2}{r}))\sum_a d_a^2\theta_a^r$$

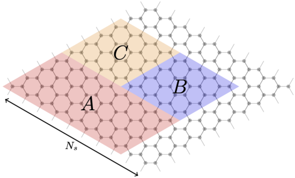
(Up to a real, non-universal proportionality factor)

Kitaev honeycomb model

$$H = J_x \sum_{\langle ij \rangle} X_i X_j + J_y \sum_{\langle ij \rangle} Y_i Y_j + J_z \sum_{\langle ij \rangle} Z_i Z_j + \kappa \sum_{\langle ijk \rangle} X_i Y_j Z_k$$

- Solvable model in terms of Majorana fermion + $\mathbb{Z}_2$ gauge field, exhibiting both $\mathbb{Z}_2$ spin liquid (toric code) and Ising topological order





(a)

n or r	1	2	3	4	5
J_n	-1/144	-1/90	-3/224	-2/135	
Φ_r		0	1/9	1/32	1/5
K_n	-1/144	-1/40	-5/112	-2/135	

(b)

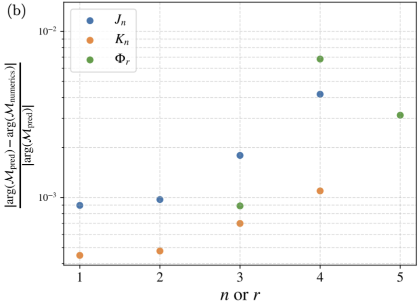
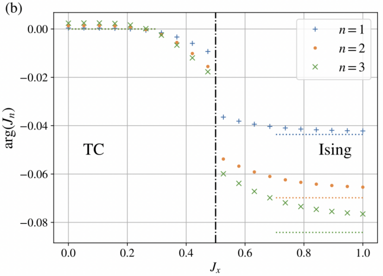
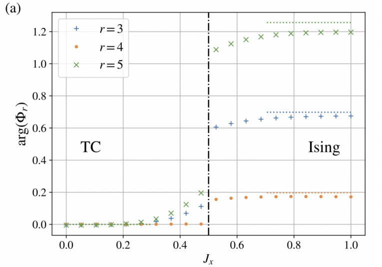


FIG. 10. (a) Analytical predictions on the angles of J_n, Φ_r and K_n. These values are calculated from the known topological data of the Ising topological phase: $c_- = 1/2$, quantum dimensions $d_0 = d_\psi = 1$, $d_\sigma = \sqrt{2}$, and spins $\theta_0 = 1$, $\theta_\psi = -1$, $\theta_\sigma = e^{2\pi i/16}$. (b) The relative error for the three measures J_n, Φ_r, K_n. These are calculated for the Kitaev model on a torus with $n_s \times n_s$ sites, with parameters $n_s = 16$, $J_x = J_y = J_z = 1$, $K = 0.3$. The errors increase with the number of replicas, but are below 0.005 for all examples considered here.

Modular commutator J_n

Lens-space multi entropy Φ_r





Symmetry enrichment

- "Charged modular commutator" [Fan, Sahay, Vishwanath]

$$\mathcal{S} = \mathrm{Tr}\,\rho_{ABC}\left[Q_{AC}^2, K_{AB}\right] = -2i\sigma_{xy}$$

  where Q_{AC} is the total $U(1)$ charge operator in the region AC.
- Rényi version:

$$\mathcal{S} = \lim_{n\to 0,\mu\to 0} \frac{-2}{\mu^2 n}\left(\mathcal{S}_{\mu,n} - \overline{\mathcal{S}_{\mu,n}}\right), \quad \mathcal{S}_{\mu,n} = \langle\Psi|e^{i\mu Q_{AC}}\rho_{AB}^n|\Psi\rangle.$$

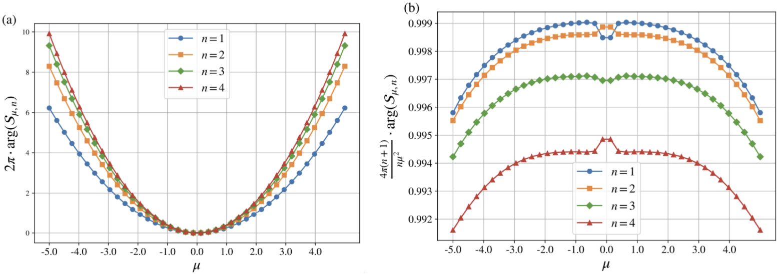
  which can be written as

$$\mathcal{S}_{\mu,n} = \left\langle \Psi^{\otimes n+1}\right| e^{i\mu Q_{AC}}\pi_{AB}\left|\Psi^{\otimes n+1}\right\rangle$$

  where π_{AB} is a cyclic permutation.
- By the simlar calculations as J_n and Φ_r, we can show, for $\mu \ll L/\xi$,

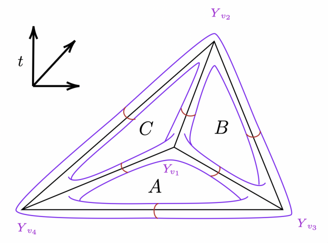
$$\mathcal{S}_{\mu,n} \propto \exp\left(i\sigma_{xy}\frac{n\mu^2}{2(n+1)}\right)$$

Chern insulator



Method of calculations

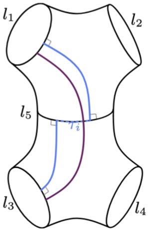
- $\langle \Psi^{\otimes R} | \pi_A \pi_B \pi_C | \Psi^{\otimes R} \rangle$ can be interpreted as a partition function $Z_{3d}(M)$ on 3d manifold M, obtained by gluing replicas in A, B, C through permutation defects.
- Bulk-boundary correspondence: the reduced density matrix ρ_A is mostly supported near the entangling boundary ∂A. We can "reduce" $Z_{3d}(M)$ to the 2d partition function $Z(\Sigma)$ of edge theory (CFT) on 2d surface Σ.
- Σ can be constructed by (i) considering surface Y enclosing the 1d defect lines; gluing and cutting R copies of Y according to the permutations. Genus: $g - 1 = \sum_{v_i} (1/2)(|\pi_{IJ}| + |\pi_{JK}| + |\pi_{IK}|) - o_{IJK}$.



Method of calculations

- Σ can be decomposed into pair-of-pants (spheres with three panctures).
- The moduli of Σ; $3g-3$ length parameters l_i (all go to zero as ξ/L), and $3g-3$ twist parameters τ_i.
- Each τ_i contributes to the phase of $Z(\Sigma)$ as $\langle 0|e^{i\tau_i P}|0\rangle = e^{-i\frac{c_-}{24}\tau_i}$. In total,

$$Z(\Sigma) \propto e^{-i\frac{c_-}{24}\sum_i \tau_i}$$



Summary and discussion

- For the traipartition setup, calculated the multientropy measure κ, and conjectured to be bound from below by the total ungappable central charge.
- For the quadpartition setup, introduced Rényi modular commutator, and its symmetry enrichment. Calculated these quantities and showed that they are given in terms of the chiral central charge and the Hall conductivities.
- Numerical calculations
- Are h (reflected entropy) and κ (multientropy) monotone under the RG?
- Spurious contributions? (c.f. partial rotation)
- Similar (related) idea: *Multiwave function overlap* in (2+1)d. [Shuhei Ohyama and SR (24) and Bowei Liu et al (24)] can detect higher Berry phase and SPT topological invariants $H^3(G, U(1))$.