Interfaces for Phases and Phase Gates

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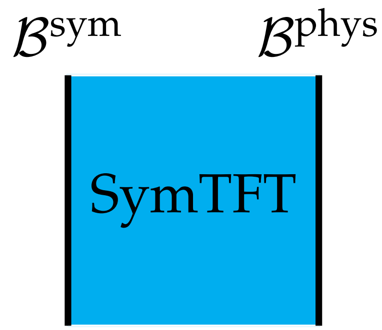
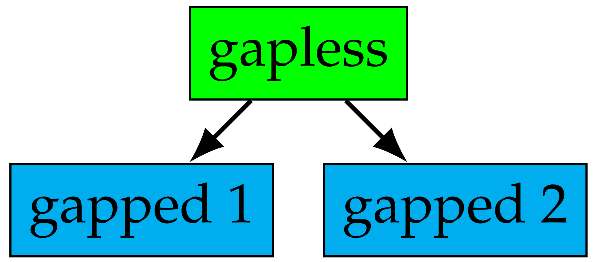
Generalized/Categorical Symmetries are a vast subject. Today is about a surprising "**interface**" between advances in quantum phases with generalized symmetries and quantum information.

A powerful tool to systematically study symmetries is the Symmetry Topological Field Theory (SymTFT) [Adam and Eve...]

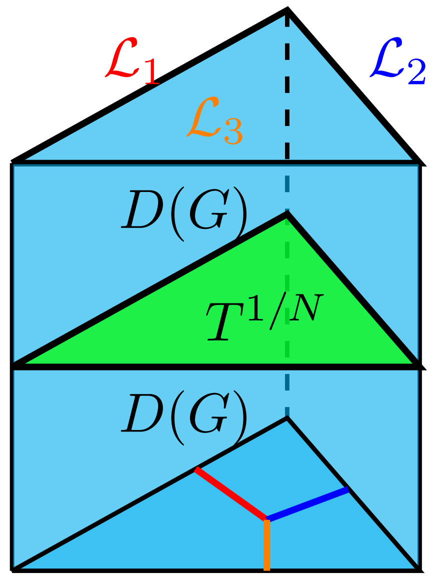


Two applications have a mathematically very close connection:

Classification of quantum phases of matter with generalized symmetries, gapped and gapless [Chatterjee Wen][Bhardwaj, Bottini, Pajer, SSN][Bhardwaj, Pajer, SSN, Warman][Wen, Potter]...



Topological Codes:
Universal gate sets implemented transversally in 2D using non-abelian quantum doubles $D(G)$ [Warman, SSN, 12/2025]



Plan

I. The Many Faces of Interfaces

II. Phases

III. Phase Gates

1. Surface Code and Gates

2. Non-Abelian Surface Code and Non-Clifford Gates

I. The Many Faces of Interfaces

Main lesson from this talk is: topological defects and interfaces between TQFTs have vast, and often surprising, applications.

Setup:

Consider G a finite group and let $D(G)$ be its quantum double.
As a 2+1d TQFT this is a G-gauge theory or Dijkgraaf Witten theory.

Its topological defects are anyons labeled by

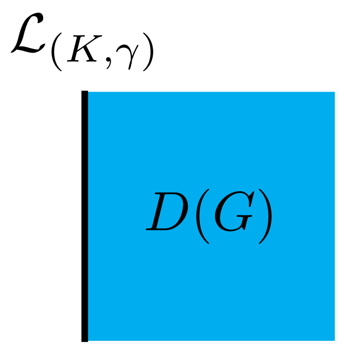
$$([g], \mathbf{R})\,, \quad [g] = \text{conj class}\,,\ \mathbf{R} = \text{irrep of centralizer of } g.$$

Example $G = \mathbb{Z}_2$:
$e = ([1], 1_-)$ charges and $m = ([g], 1)$ fluxes.

Interfaces

First let us consider interfaces of $D(G)$ to the trivial TQFT, i.e. **gapped boundary conditions** (BC). These are classified by subgroups $K < G$ and SPTs

$$(K, \gamma)\,,\ \gamma \in H^2(K, U(1)) \qquad \longleftrightarrow$$



Why?
$D(G)$ has a canonical Dirichlet BC, which ends all pure charges $([1], \mathbf{R})$.
We obtain new BCs from gauging a subgroup K, but stacking before with a K-SPT phase is also admissible

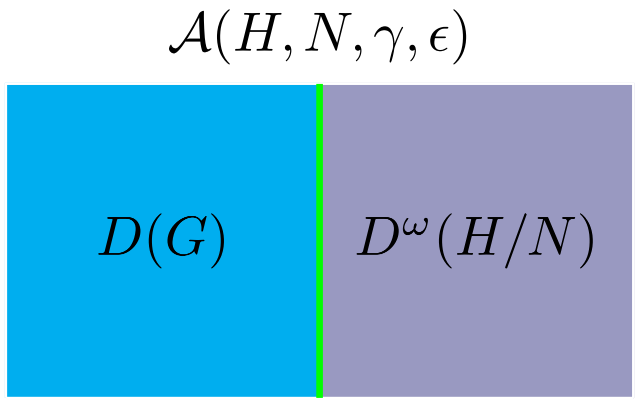
$$\mathcal{L}_{(K,\gamma)} = (\mathcal{L}_{\text{Dirichlet}} \otimes \gamma)/K$$

Such BCs are given in terms of Lagrangian, i.e. maximal, algebras in $D(G)$ and as a vector space we can decompose these into anyons, which "condense" at the boundary:

$$\mathcal{L}_{(K,\gamma)} = \bigoplus_{([g],\mathbf{R})} n^{(K,\gamma)}_{([g],\mathbf{R})} ([g], \mathbf{R})$$

Non-maximal condensations of anyons leads to interfaces with non-trivial topological orders. These are classified by

$(H, N, \gamma, \epsilon) \quad \longleftrightarrow$



where $H < G$, $N < H$ normal, $\gamma \in H^2(N, U(1))$ and $\epsilon : H \times N \to U(1)$ encodes the H-action.

Why?
Fold along the interface and determine the gapped BCs of this folded topological order.

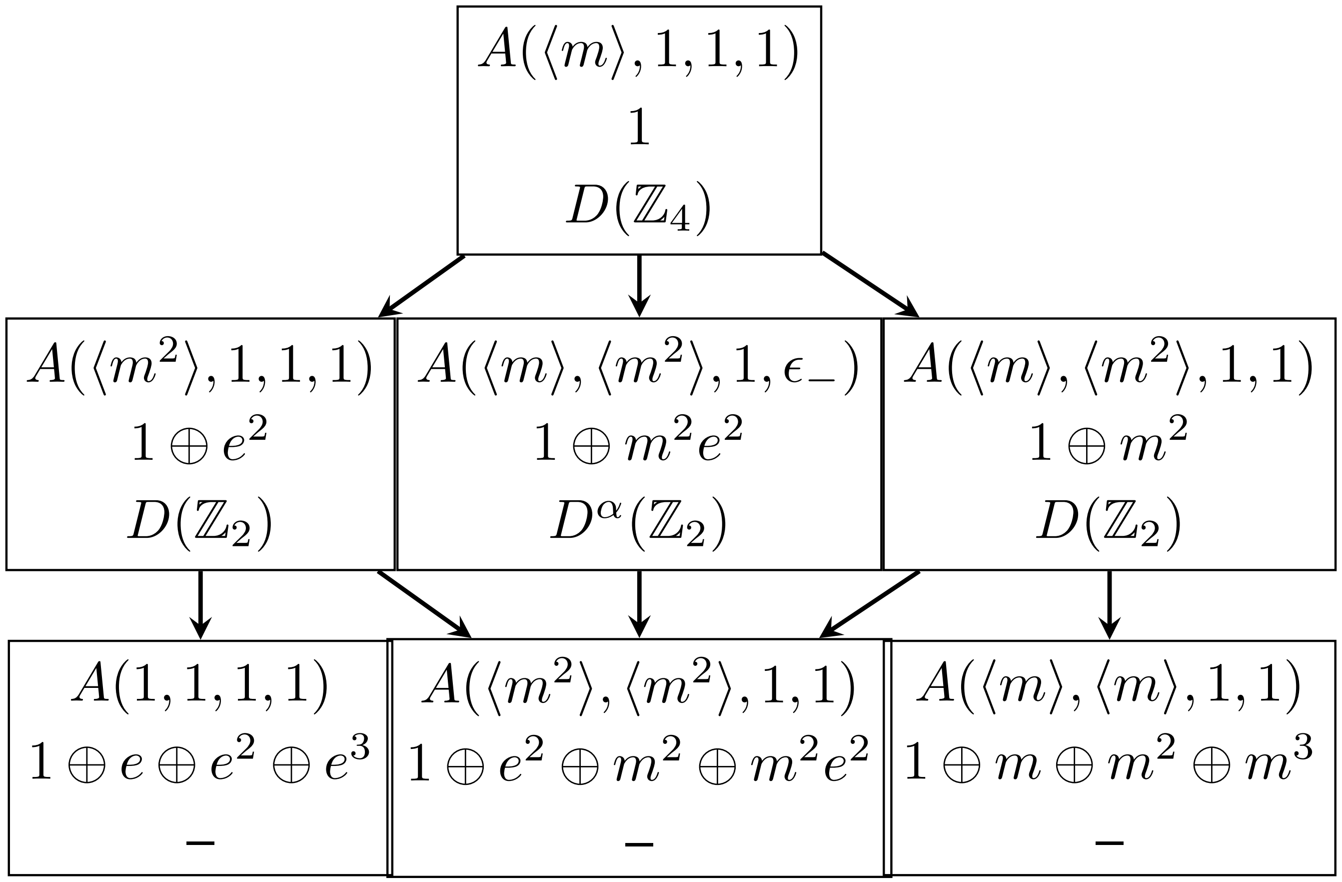
These form non-maximal algebras, which as vector spaces have an anyon decomposition.

$$\mathcal{A} = \bigoplus_{([g], \mathbf{R})} n_{([g], \mathbf{R})} ([g], \mathbf{R})$$

Mathematical Classification: [Davydov, Simmons][Ostrik][Natale], with some updates and useful decomposition into anyons and algebra structure [Gai, SSN, Warman - to appear].

Example: $G = \mathbb{Z}_4 = \langle m; m^4 = 1 \rangle$

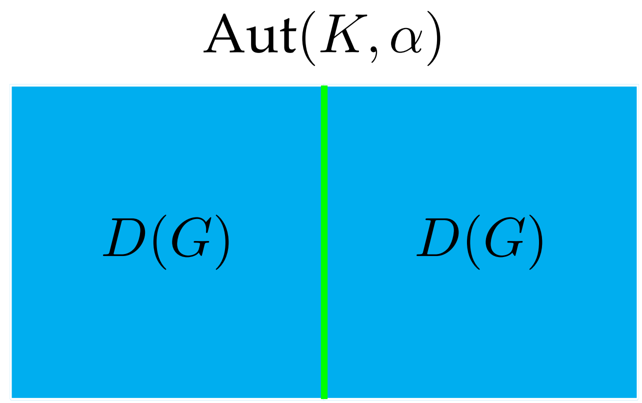
All condensable algebras $\mathcal{A}(H, K, \gamma, \epsilon)$ form a partially ordered set:



where we also list the reduced topological orders and $\alpha \in H^2(\mathbb{Z}_2, U(1)) = \mathbb{Z}_2$.

Automorphism Interfaces

We can also consider interfaces from $D(G)$ to itself:



Such automorphisms were classified by [Davydov]

$$K \subseteq G \times G\,, \quad \text{and} \quad \alpha \in H^2(K, U(1))$$

such that $p_1(K) = p_2(K) = G$, where $p_k : G \times G \to G$ are the projections onto the k-th factor and the cocyle needs to be such that

$$\epsilon(g,h) := \frac{\alpha(g,h)}{\alpha(ghg^{-1},\, g)}$$

on $(K \cap (G \times \{\text{id}\})) \times (K \cap (\{\text{id}\} \times G))$ is non-degenerate.

If $K = G_{\text{diag}}$ this is a purely SPT-automorphism.

This setup has numerous applications.

Same math, very different physical motivation.

1. Quantum Phase Transitions with generalized symmetries
2. Transversal Gates in non-Abelian topological quantum codes.

II. Categorical Phases and Landau

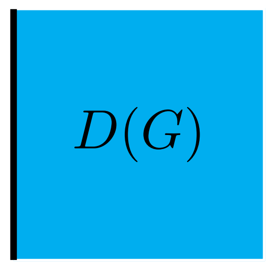
Starting with $D(G)$ is not trivial from the point of view of phases: for non-abelian G there are plenty of non-invertible symmetries that have this as their SymTFT.

Consider a symmetry $\mathcal{S}$ in 1+1d which is gauge related to G. So its SymTFT is

$$\text{SymTFT}(\mathcal{S}) = D(G)$$

and there exists $(K_{\mathcal{S}}, \gamma_{\mathcal{S}})$ such that the topological defects that generate $\mathcal{S}$ can be realized by stacking γ and gauging K.

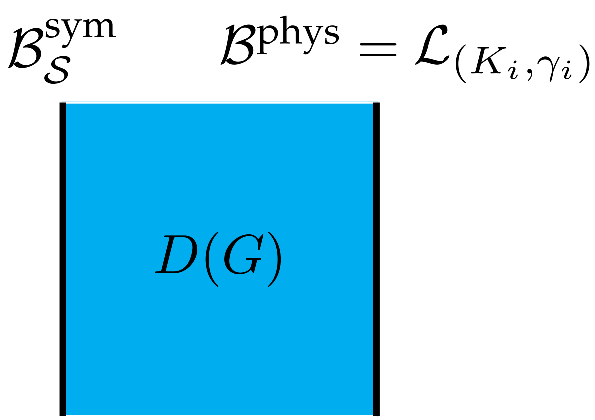
$$\mathcal{L}_{(K_{\mathcal{S}}, \gamma_{\mathcal{S}})} \to \mathcal{S}$$



Example: $\mathcal{S} = \text{Rep}(G)$ for $K = G$.

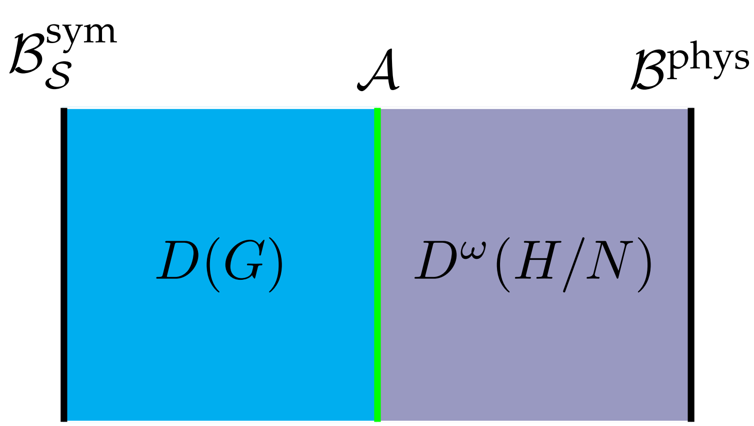
$\mathcal{S}$-symmetric phases from $D(G)$

- $\text{SymTFT}(\mathcal{S}) = D(G)$
- Symmetry boundary $\mathcal{B}^{\text{sym}}_{\mathcal{S}}$: $\mathcal{L}_{(K_{\mathcal{S}},\gamma_{\mathcal{S}})}$
- Physical boundary: classification of gapped phases which are in 1-1 with $\mathcal{L}_{(K_i,\gamma_i)}$.



[Bhardwaj, Bottini, Pajer, SSN][2] [Wen, Potter] [Chatterjee, Wen] [Huang, Cheng] [Bhardwaj, Pajer, SSN, Tiwari, Warman, Wu] [Bhardwaj, SSN, Tiwari, Warman] [Bhardwaj, Bottini, Tiwari, SSN][2] [Bottini, SSN] [Inamura, Bhardwaj, Huang, Tiwari, SSN]...

- Interfaces: Second order phase transitions (gapless) via KT-transformation

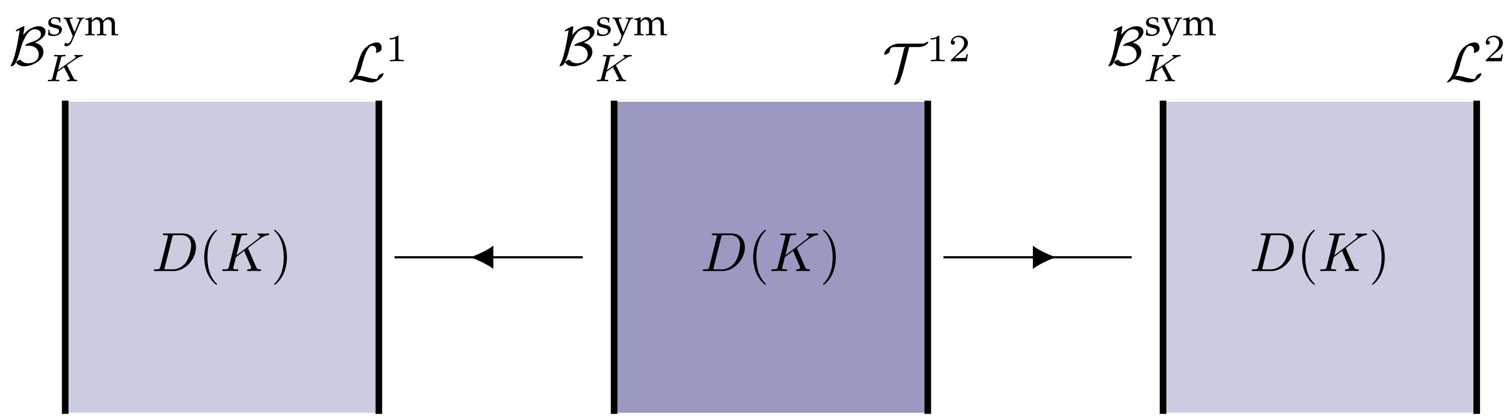


[Chatterjee, Wen] [Bhardwaj, Bottini, Pajer, SSN] [Bhardwaj, Pajer, SSN, Warman] [Bottini, SSN] [Warman, Fan, Tiwari, Pichler, SSN] [Rui Wen][2] [Bhardwaj, Gai, Huang, Inamura, SSN, Tiwari] [Antinucci, Copetti, SSN] ...

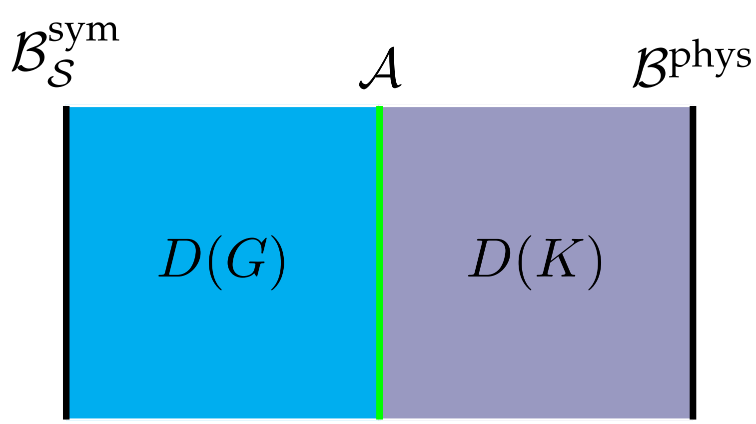
All this extends of course to $\mathcal{S}$ any fusion (higher) category.

Gapless Sandwichology

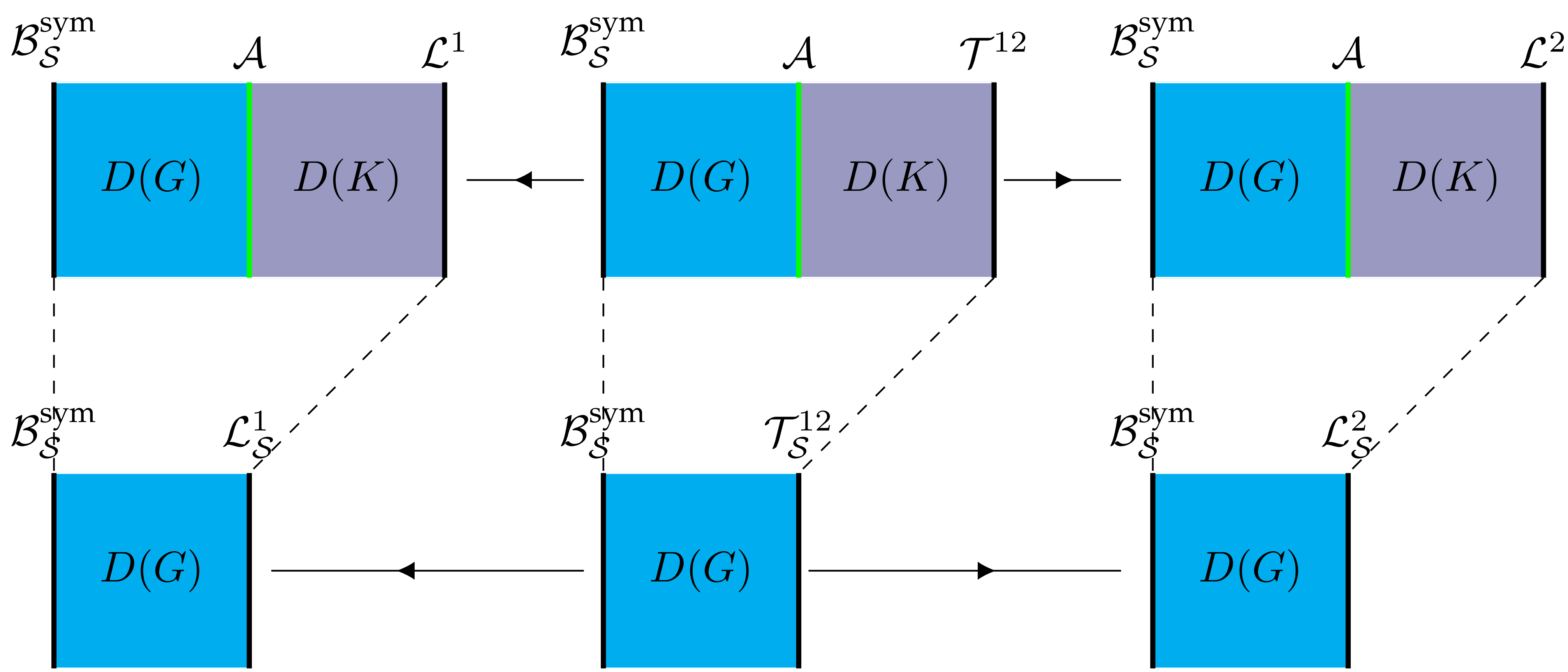
Say you have a second order phase transition for the symmetry group K: from a gapped phase 1 via a gapless phase $\mathcal{T}^{12}$ to gapped phase 2.
Example: $K = \mathbb{Z}_2$, and the two phases are the SSB and trivial phases, transitioning through the Ising CFT.
This can be embedded into a SymTFT in terms of 3 sandwiches:



Consider a larger TO $D(G)$, and an interface $\mathcal{A}$ such that the reduced order is $D(K) = D(H/N)$



We can then obtain three SymTFT configurations for the larger TO:

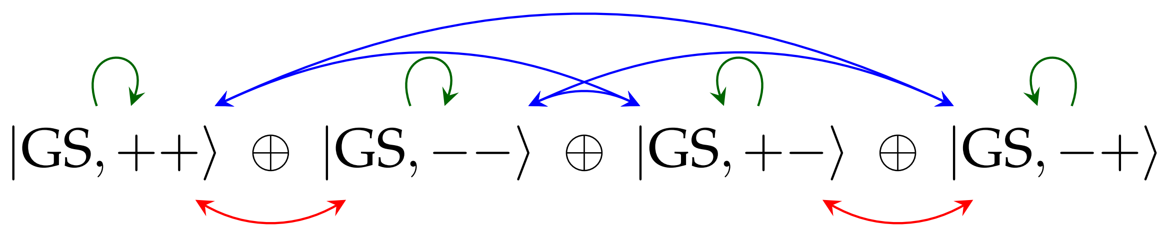


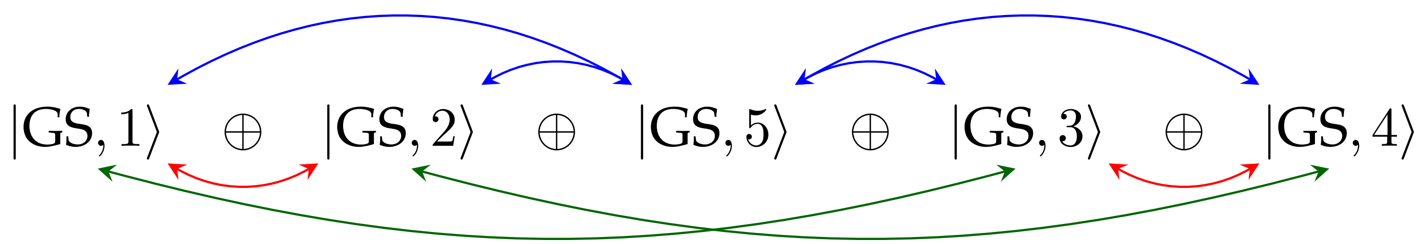
Compatifying the first interval results in an $\mathcal{S}$-symmetric set of theories: that model a transition from a gapped phase via a gapless phase to a gapped phase. This is the SymTFT analog of a "KT transformation".

Note this works of course for any symmetry category – not necessarily group-theoretical.

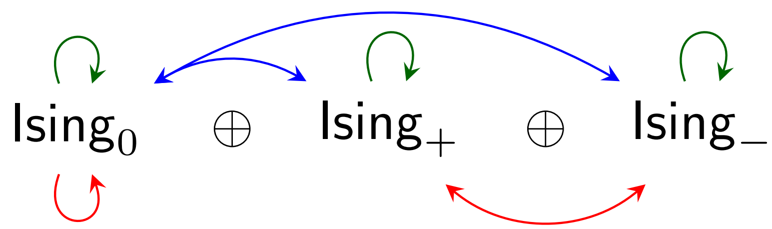
Example: Rep(D_4) Transition: Ising igSSB

Rep(D_4) is a non-invertible symmetry with SymTFT $D(D_4)$. The SymTFT predicts there are among others two gappes phases – $\mathbb{Z}_2 \times \mathbb{Z}_2$ SSB and the Rep(D_4) SSB, where the red/green are 1d irreps and blue 2d irrep:



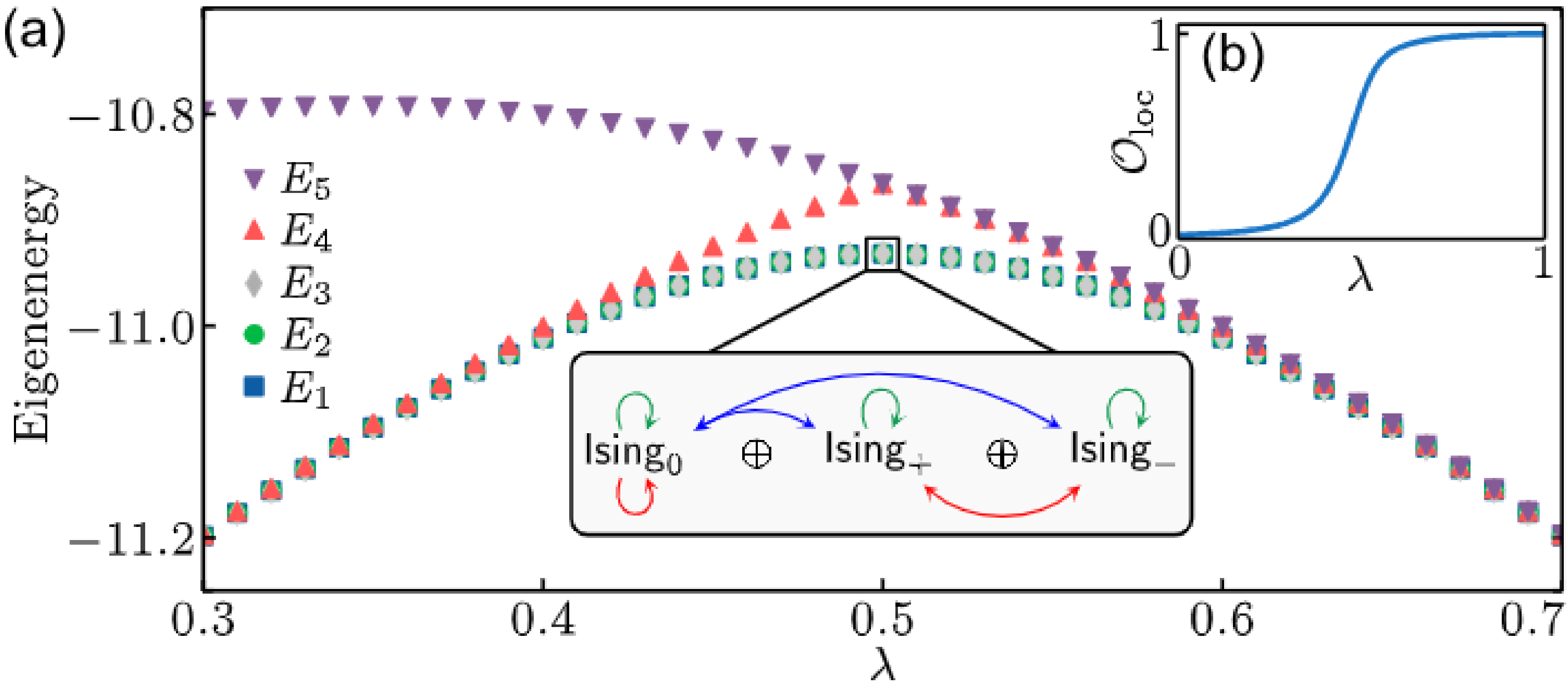


The algebra corresponds to $\mathcal{A}(H, N, \gamma, \epsilon) = \mathcal{A}(D_4, \langle r^2, sr\rangle, 1, 1\rangle)$ with reduced TO $D(\mathbb{Z}_2)$. Inputting the Ising transition the gapless phase is



igSSB: gapless phase with 3 vacua, that can only be gapped by further SSBing.

$H(\lambda)$ is the linear interpolation between a $\mathbb{Z}_2 \times \mathbb{Z}_2$ SSB and the $\text{Rep}(D_4)$ SSB



The critical point is an Ising igSSB:

3 copies of the Ising CFT related by spontaneously broken Ising category

[Alison Warman, Fan Yang, Apoorv Tiwari, Hannes Pichler, SSN]

Twin Algebras

[Gai, SSN, Warman - to appear] The anyon decomposition does not uniquely fix the condensable algebra:

Twin algebras: two algebras $\mathcal{A}(H_i, N_i, \gamma_i, \epsilon_i)$ with the same anyons but inequivalent algebra structures.

1. Twin algebras with non-conjugate subgroups:
   Two $H_i \subset G$, forming a Gassmann triple $|H_1 \cap [g]| = |H_2 \cap [g]|$ for all $[g]$.
   Gives Lagrangian and non-Lagrangian twins.

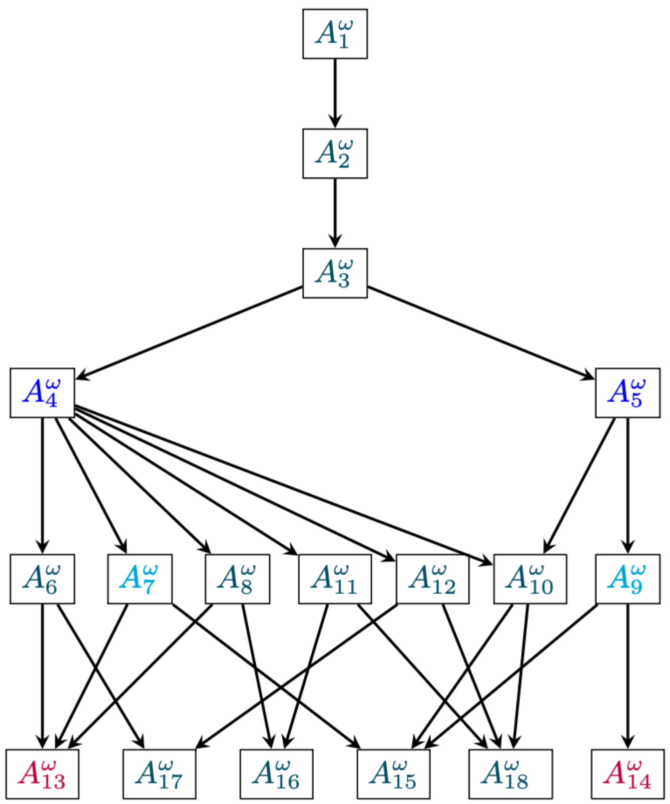
2. Twin algebras with inequivalent SPTs γ:
   Lagrangian: 2-cocycle γ differ by a Bogomolov multiplier
   [Davydov][Pollman, Turner][Kobayashi, Watanabe]
   Non-Lagrangian: different 2-cocycles beyond Bogomolov – for G^ω

3. Twin algebras with inequivalent ϵ:
   Different ϵ same H, N, γ. Non-Lagrangian twins.

Twin Algebras

[Gai, SSN, Warman - to appear]

Smallest rank $G_{32,43} = (\mathbb{Z}_2 \times \mathbb{Z}_2) \ltimes \mathbb{Z}_8$, with Gassmann triple twins.



The twin algebras:

(A_4^ω, A_5^ω), (A_7^ω, A_9^ω) and $(A_{13}^\omega, A_{14}^\omega)$.

Consider the two transitions governed by

$$T_{13,7,15}: \qquad A_{13}^\omega \overset{A_7^\omega}{\longleftrightarrow} A_{15}^\omega$$

$$T_{15,9,14}: \qquad A_{15}^\omega \overset{A_9^\omega}{\longleftrightarrow} A_{14}^\omega .$$

Both are Ising type transitions, but the OPE of order parameters is different. Seems to be a group-symmetry with non-Landau transition – beyond SPT type transitions.

Many other applications

- This approach has been sucessfully applied, as an alternative to other methods like module categories [Thorngren, Wang] to classify phases with **categorical symmetries**.
- **Mixed Phases:** SymTFT for doubled space with left and right symmetry $\mathcal{S}_L \times \mathcal{S}_R$, characterizes strong and weak symmetric phases. [SSN, A. Tiwari, A. Warman, C. Zhang] [R. Luo, Y.-N. Wang, Z. Bi.][M. Qi, R. Sohal, X. Chen, D. T. Stephen, A. Prem]
- Anomalies (minimal anoumalous subcategories) [Antinucci, Copetti, SSN]
- Extension to continuous internal [Brennan, Sun][Antinucci, Benini][Bonetti, del Zotto, Minasian][Apruzzi, Bedogna, Dondi] and spacetime symmetries [Apruzzi, Dondi, Garcia-Extebarria, Lam, SSN]
- ...

What was extremely useful for the following is the technical knowhow of condensable algebras for $D(G)$ type doubles.

III. Transversal Non-Clifford Gates in 2D

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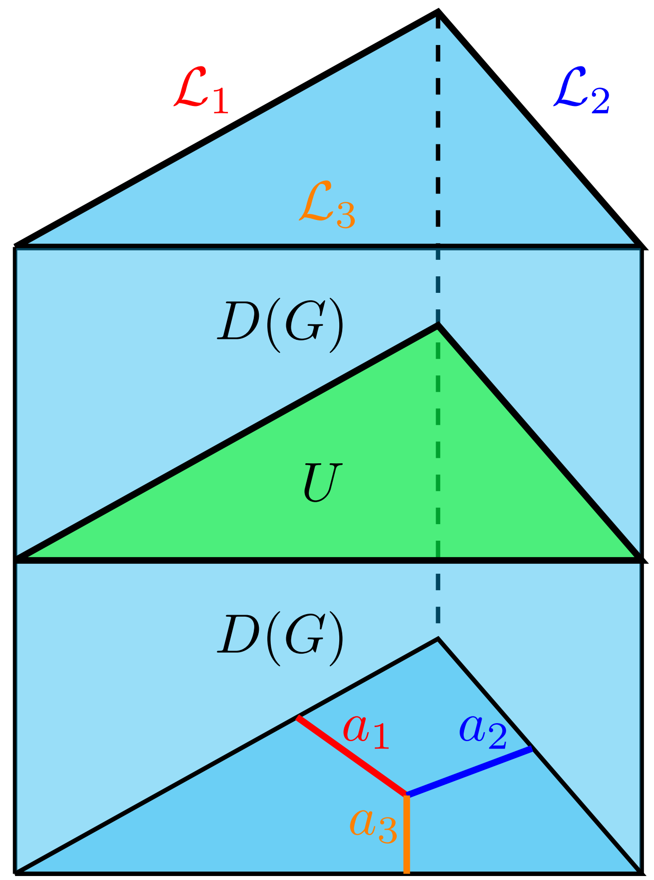
The mathematical tools are essentially the same, but with different interpretation:

- **Physical Hilbert space:**
  The Hilbert space of the surface code associated to $D(G)$. [Kitaev]
  The ground states are the code space.
- **Logical space:**
  This is determined by gapped BCs $\mathcal{L}_i$ specified by (K_i, γ_i)
- **Code-switching:**
  Interfaces to other topological orders.
- **Transversal unitary gates:**
  Automorphism interfaces U

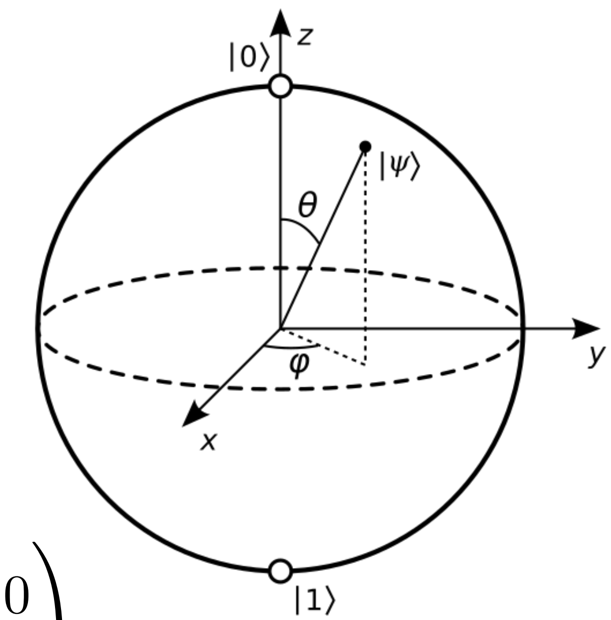


Let's start at the beginning: Qubits

- Quantum computing:
  apply unitary operators (quantum gates) to qubits (or qudits)

- Single qubit

$$\text{basis} = \left\{ |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$



The Pauli group $\mathcal{P}$ is generated by

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad i\mathbb{I} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

satisfying $XZ = -ZX$.

Quantum computing: Clifford hierarchy

The Clifford hierarchy is a sequence of gates $\cdots \mathcal{C}^{(i)} \subset \mathcal{C}^{(i+1)} \cdots$ [Gottesman, Chuang].

- Clifford level 1: Pauli group

$$\mathcal{C}^{(1)} = \mathcal{P} \ \ni \ Z = \mathrm{diag}(1, -1)$$

- Clifford level 2: Clifford gates map Paulis to Paulis

$$\mathcal{C}^{(2)} = \{U \ : \ UPU^\dagger \in \mathcal{P}, \, \forall P \in \mathcal{P}\} \ \ni \ S = \mathrm{diag}(1, i)$$

  can be easily simulated classically [Gottesman, Knill]

- Clifford level $k \geqslant 3$: Non-Clifford gates do not form groups

$$\mathcal{C}^{(k)} = \{U \ : \ UPU^\dagger \in \mathcal{C}^{(k-1)}, \, \forall P \in \mathcal{P}\} \ \ni \ T^{1/N} = \mathrm{diag}(1, e^{i\pi/(4N)})$$

  (here $k = N + 2$)

Universal Gate Sets

Clifford gates alone are not enough to approximate all unitaries. A theorem by [Solovay][Kitaev] states that any unitary can be sufficiently well approximated with Cliffords plus one non-Clifford.

A universal gate-set is e.g.

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \; S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \; T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}, \; \text{CNOT} = |0\rangle\langle 0| \otimes \mathbb{I} + |1\rangle\langle 1| \otimes X \,.$$

For certain application it is better to implement non-Clifford gates ”natively”, without the need to approximate.

QFT (quantum Fourier Transform) e.g. relies heavily on phase gates

$$R_N = T^{1/N} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4N} \end{pmatrix}$$

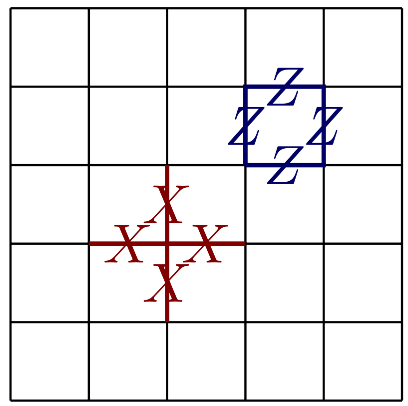
Topological quantum computing

- Idea of using anyons for topological quantum computation [Kitaev, '97]
- Logical states = inequivalent ground states of local Hamiltonian
  $\Rightarrow$ they differ by non-contractible anyon configurations
  $\Rightarrow$ information is stored non-locally and is topologically-protected
- **Transversal gate**: this can be realized by a product of one-qubit rotations on the physical qubits.
- Harmonic oscillator of topological codes:
  Surface code (stabilizer code) (based on $D(\mathbb{Z}_n)$)
  $\Rightarrow$ easy decoding, Clifford gates, lattice surgery etc.
  $\Rightarrow$ However: like the harmonic oscillator it has short-comings...

Surface Code 101

2D lattice with qubit on each edge. The Hamiltonian is given by

$$H = -\sum_v A_v - \sum_p B_p$$



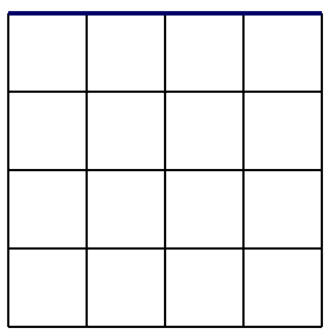
Consider the (abelian) **Pauli stabilizer group** $\mathcal{S} = \{A_v, B_p\}$. The code space is stabilized by $\mathcal{S}$:

$$\mathcal{H}_{\text{code}} = \{|\psi\rangle : S|\psi\rangle = |\psi\rangle\,,\ \forall S \in \mathcal{S}\}$$

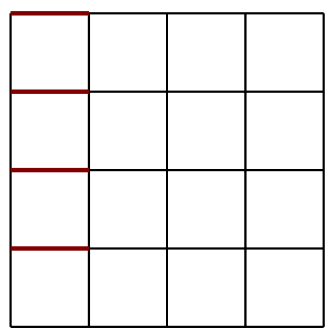
Errors: e-anyon string violates A_v, m-anyon string of Xs violates B_p.

The logical Pauli operators are:

$\overline{Z}$



$\overline{X}$



$|+\rangle$ e.g. gets mapped to $|-\rangle$ by $\overline{Z}$ (inserts e).

How to construct unitary gates in topological codes?

General theorems obstructing non-Clifford gates i.e. universal computation, in 2D surface codes:

- [Eastin, Knill '08] No **exact** universal transversal gate set in a single local error detecting code.
- [Bravyi, König '12] (BK) Topologically-protected gates in 2D Pauli stabilizer codes are necessarily Clifford and kth Clifford hierarchy can be transversally implemented only in k dimensions using Pauli stabilizers.

Usually universal computation can be achieved via magic state distillation or lattice surgery.

Recent advances in magic state preparation: Non-Clifford states/gates and transversal $T = \mathrm{diag}(1, e^{i\pi/4})$ from $D^{\omega}(\mathbb{Z}_2^3) = D(D_4)$, [Huang, Cheng][Davydova eta al][Huang, Warman, SSN, Chen]. The BK bound was improved upon by one dimension in [Kobayashi, Zhu, Hsin].

Main Result: Transversal Non-Clifford Gates in 2D

[Warman, SSN, 12/25]

By relaxing the assumption of Pauli stabilizer codes in Bravyi-Konig, we show that we can realize in 2D non-Clifford gates at any level in the Clifford hierarchy.

More precisely: we construt topologically-protected Non-Clifford phase gates

$$T^{1/N} = \text{diag}\left(1, e^{i\pi/(4N)}\right), \quad N \in \mathbb{N}$$

at any Clifford-hierarchy level from $D(D_{4N})$ surface codes in 2D
For $N = 2^k$ these have realization in terms of qubits only.

$\Rightarrow$ completely bypasses [Bravyi–König]

Main Theorem [Warman, SSN]

Theorem. (Tranversal Gates from SPT-Stacking.) Consider the **quantum double** $D(G)$ of a finite group G on a spatial triangle, with three gapped boundaries labelled by subgroups $K_1, K_2, K_3 \subseteq G$, chosen to encode a **single logical qubit**. Let $\alpha \in H^2(G, U(1))$ be a group 2-cocycle whose restriction on each boundary is trivial in group cohomology, i.e. there exist $\beta^{(i)} : K_i \to U(1)$ such that

$$\alpha|_{K_i} = \delta\beta^{(i)} \qquad \text{for } i \in \{1,2,3\}$$

Define the transversal unitary $U_{\alpha,\beta}$ by stacking the 2D purely spatial SPT circuit for α on a single time-slice, with $\beta^{(i)}$ on each 1D boundary.

Then the induced logical gate $U_{\alpha,\beta}$ is diagonal in the $\overline{Z}$ basis $\{\overline{|0\rangle}, \overline{|1\rangle}\}$, with

$$U_{\alpha,\beta}\overline{|0\rangle} = \overline{|0\rangle}\,,$$

$$U_{\alpha,\beta}\overline{|1\rangle} = \frac{\alpha(g_1,g_2)\beta^{(3)}(g_1g_2)}{\beta^{(1)}(g_1)\,\beta^{(2)}(g_2)}\,\overline{|1\rangle}\,,$$

for any state labeled by $g_1 \in K_1$, $g_2 \in K_2$ representing $\overline{|1\rangle}$.

$U_{\alpha,\beta}$ preserves the logical codespace (it is an automorphism of $D(G)$). **Its lattice realization involves only operators acting on $O(1)$ lattice sites: a local error can thus enlarge its support only by $O(1)$ sites.**

Corollary: Transversal Clifford-Hierarchy Gates from 2D

Corollary(Transversal Non-Clifford Gates from $D(D_{4N})$)
For any integer $N \geq 1$, consider the order-$8N$ dihedral group (of symmetries of a $4N$-gon)

$$G = D_{4N} = \langle r, s \mid r^{4N} = s^2 = \mathrm{id},\ srs = r^{-1} \rangle \,.$$

Consider the Main Theorem applied to $G = D_{4N}$ applied to the three gapped boundaries determined by the subgroups

$$K_1 = \langle rs \rangle \cong \mathbb{Z}_2 \,, \quad K_2 = \langle s \rangle \cong \mathbb{Z}_2 \,, \quad K_3 = \langle r \rangle \cong \mathbb{Z}_{4N}$$

Consider the non-trivial cocycle

$$\alpha_N \in H^2(D_{4N}, U(1)) = \mathbb{Z}_2$$

On the state labeled by $g_1 = rs,\ g_2 = s,\ g_1 g_2 = r$, and $U_{\alpha,\beta}$ evaluates to the unitary

$$U_{\alpha,\beta} = T^{1/N} = \mathrm{diag}\left(1, e^{\frac{i\pi}{4N}}\right)$$

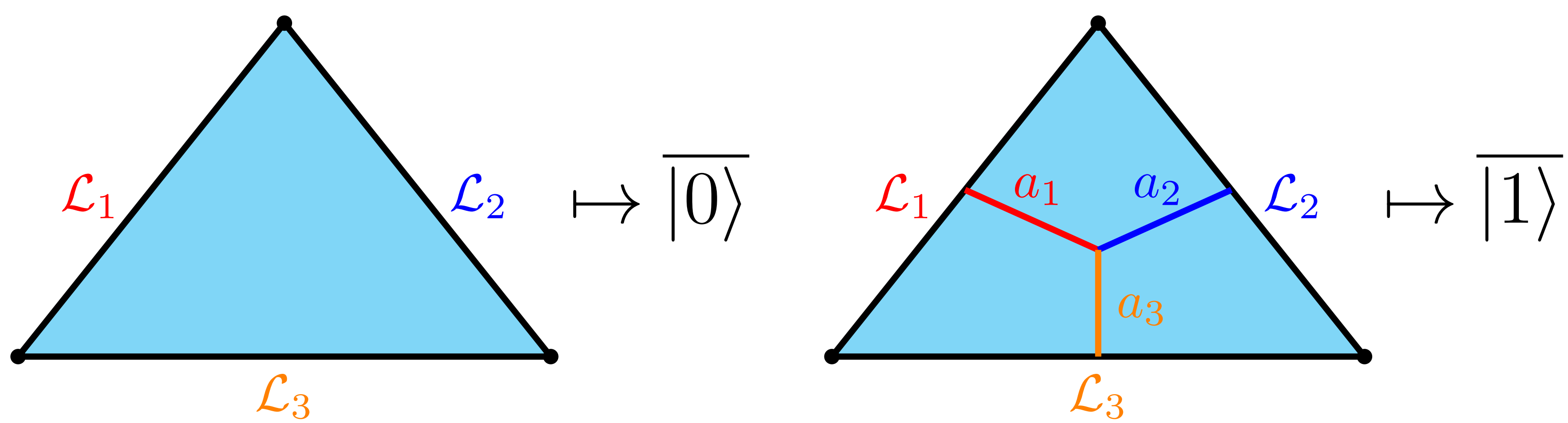
Continuum: Surface Code for one Logical Qubit

- Quantum double $D(G)$
- Spatial configuration:
  2D triangular configuration. Each side is a gapped BC $\mathcal{L}_i$ of $D(G)$ specified by $K_i \subseteq G$
- Logicals:
  Consider two triple-junction of anyons (a_1, a_2, a_3) and (b_1, b_2, b_3) with -1 braiding:

$$(a_1, a_2, a_3) \mapsto \overline{X}$$

$$(b_1, b_2, b_3) \mapsto \overline{Z}$$

- Topological qubit logical basis states



Continuum: Transversal Gates from SPT stacking

Stacking 2D SPTs act as transversal gates: [Warman, SSN]

- SPT given by $\alpha \in H^2(G, U(1))$

$$\alpha(g,h)\alpha(gh,k) = \alpha(g,hk)\alpha(h,k)$$

  This defines an **automorphism interface**.

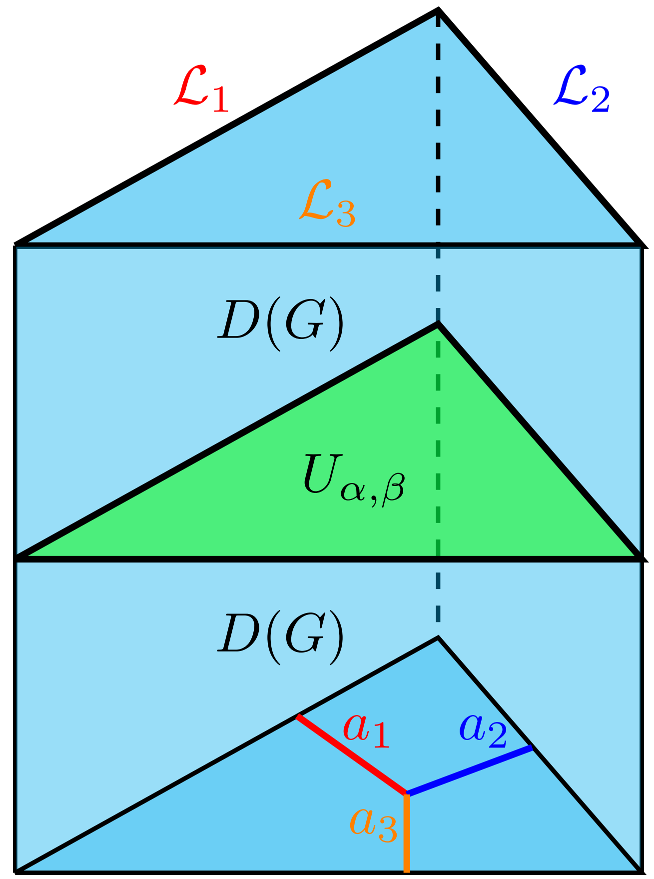
- For α to end on the boundaries we need a boundary 1-cochain $\beta^{(i)}$

$$\alpha|_{K_i}(g,h) = \frac{\beta^{(i)}(g)\beta^{(i)}(h)}{\beta^{(i)}(gh)} = \delta\beta^{(i)}(g,h)$$

- Construct logical gate $U_{\alpha,\beta}$ by stacking α on 2D spatial slice with 1D boundary counter-terms β :

$$U_{\alpha,\beta}\overline{|0\rangle} = \overline{|0\rangle}$$

$$U_{\alpha,\beta}\overline{|1\rangle} = \frac{\alpha(g_1,g_2)\beta^{(3)}(g_1g_2)}{\beta^{(1)}(g_1)\,\beta^{(2)}(g_2)}\,\overline{|1\rangle}$$

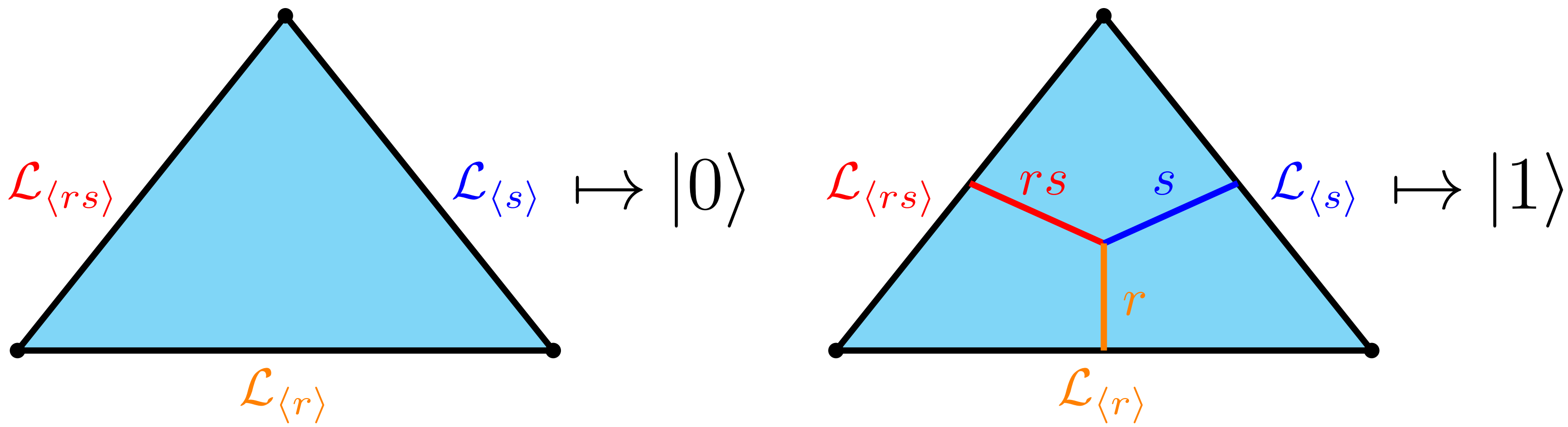


Application to $D(D_{4N})$

- Group of symmetries of $4N$-gon $\quad D_{4N} = \langle r, s \,|\, r^{4N} = s^2 = \mathrm{id}\,,\; srs = r^{-1} \rangle$
- Irreps of dimension 1, generated by 1_r and 1_s:

$$1_r(r) = +1\,, \qquad 1_r(s) = -1\,,$$
$$1_s(r) = -1\,, \qquad 1_s(s) = +1\,,$$

- Topological qubit logical basis states



have eigenvalues $+1$ and -1 under $(1_{rs}, 1_s, 1_r) \Rightarrow$ logical qubit states

Phase Gates from $D(D_{4N})$

- There is one SPT $\alpha \in H^2(D_{4N}, U(1)) = \mathbb{Z}_2$ which evaluates as follows:

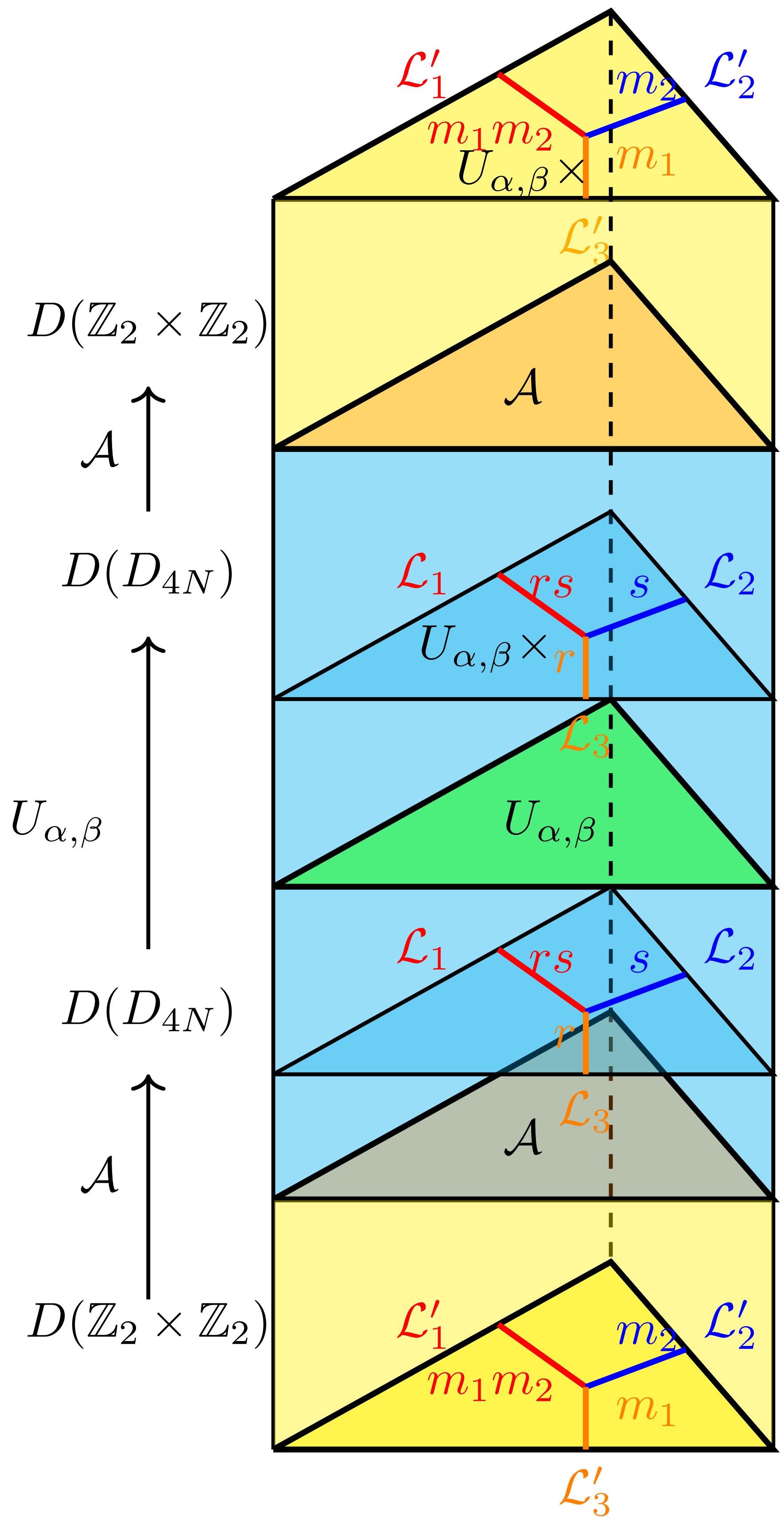
$$\alpha(\mathrm{id}, \mathrm{id}) = 1, \qquad \alpha(rs, s) = 1$$
$$\alpha|_{\langle rs \rangle} \equiv 1, \qquad \alpha|_{\langle rs \rangle} \equiv 1$$
$$\alpha|_{\langle r \rangle} = \delta\beta_N, \qquad \beta_N(r) = e^{i\pi/(4N)}$$

- For the state $|0\rangle$: $g_1 = g_2 = \mathrm{id} \Rightarrow U_{\alpha,\beta}(\mathrm{id}, \mathrm{id}) = 1$
- For the state $|1\rangle$: $g_1 = rs,\ g_2 = s \Rightarrow U_{\alpha,\beta}(rs, s) = \beta_N(r) = e^{i\pi/(4N)}$
- Thus, $U_{\alpha,\beta}$ encodes the topologically-protected Non-Clifford phase gate

$$T^{1/N} = \mathrm{diag}(1, e^{i\pi/(4N)})$$

acting on the logical qubit.

Interfaces: Code Switching $D(G' = \mathbb{Z}_2 \times \mathbb{Z}_2)$ to $D(G = D_{4N})$ and back to $D(G' = \mathbb{Z}_2 \times \mathbb{Z}_2)$: The transversal gate $U_{\alpha,\beta}$ is performed within the non-abelian patch $D(D_{4N})$. The code switching is done by condensation of anyons or gauging $\mathcal{A}$ algebras, that determine the interfaces between the TOs.



Lattice: Non-Abelian Kitaev Quantum Double $D(G)$

Kitaev quantum double for non-abelian G, $D(G)$, has local Hilbert space $\mathcal{H} = \mathbb{C}[G]$ with onb $\{|h\rangle \,:\, h \in G\}$, act with

$$L^g|h\rangle = |gh\rangle\,, \qquad R^g|h\rangle = |hg^{-1}\rangle$$

and diagonal operators for each irrep $\mathbf{R}$ of G and $j,k \in \{1,\cdots,\dim(\mathbf{R})\}$

$$Z^{\mathbf{R}}_{j,k}|g\rangle = M^{\mathbf{R}}_{j,k}(g)|g\rangle$$

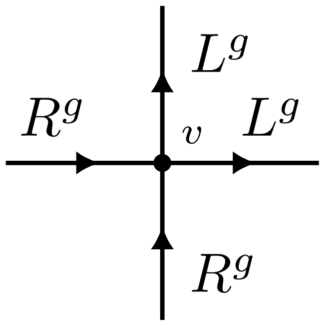
where $M^{\mathbf{R}}(g)$ is the matrix representation of g in the irrep $\mathbf{R}$. The set

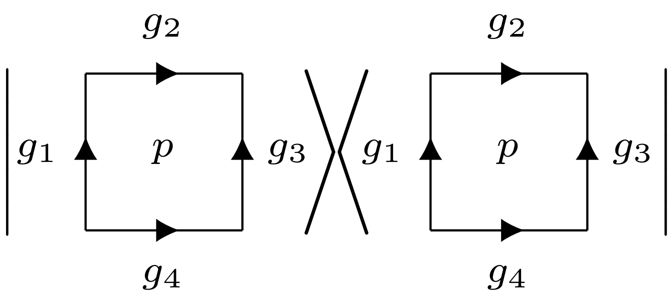
$$\{Z^{\mathbf{R}}_{j,k} \,:\, \mathbf{R} \in \text{Irreps}(G)\,,\, j,k = 1,\cdots,\dim(\mathbf{R})\}$$

is comprised of $\sum_{\mathbf{R}\in\text{Irreps}} \dim(\mathbf{R})^2 = |G|$ operators, forming a basis of diagonal operators on $\mathcal{H}$.

Hamiltonian

The Hamiltonian has vertex and plaquette terms

$A_v^g =$ 

$B_p^g = \sum_{g_1,g_2,g_3,g_4} \delta_{g,\, g_1 g_2 g_3^{-1} g_4^{-1}}$ 

These do not commute:

$$A_s^g A_s^h = A_s^{gh}\,, \qquad (A_s^g)^\dagger = A_s^{g^{-1}}$$

$$B_s^g B_s^h = \delta_{g,h} B_s^h\,, \qquad (B_s^g)^\dagger = B_s^g$$

$$A_s^g B_s^h = B_s^{ghg^{-1}} A_s^g\,,$$

We consider this model on a triangular patch with specific BCs determined by Lagrangian algebras, labeled by $K_i \subset G$

$$H = -\sum_v A_v - \sum_p B_p - \sum_{i=1}^{3} \sum_{s_i} (A_{s_i}^{K_i} + B_{s_i}^{K_i}) \, .$$

Here, the bulk operators are commuting projectors and are defined by

$$A_v := \frac{1}{|G|} \sum_g A_v^g \, , \qquad B_p := B_p^{\mathrm{id}}$$

and the boundary ones as

$$A_{s_i}^{K_i} := \frac{1}{|K_i|} \sum_{k \in K_i} A_{s_i}^k \, , \qquad B_{s_i}^{K_i} := \sum_{k \in K_i} B_{s_i}^k \, ,$$

where $\{s_i\}$ are sites along the boundary i and $A_{s_i}^k, B_{s_i}^k$ are truncated at the boundary.

Non-Abelian Stabilizers

This is a commuting projector Hamiltonian and we identify the ground states as

$$A_v|\psi\rangle = B_p|\psi\rangle = A_{s_i}^{K_i}|\psi\rangle = B_{s_i}^{K_i}|\psi\rangle = |\psi\rangle$$

Let

$$\overline{|\mathrm{id}, \mathrm{id}\rangle} := \prod_v A_v \bigotimes_l |\mathrm{id}\rangle_l$$

For the pure flux anyons

$$a_1 := ([g_1], 1)\,, \quad a_2 := ([g_2], 1)\,, \quad a_3 := ([g_1 g_2], 1)$$

the state corresponding to (a_1, a_2, a_3) is

$$\overline{|g_1, g_2\rangle} := \prod_v A_v \; L_{\xi_1}^{g_1} L_{\xi_2}^{g_2} L_{\xi_3}^{g_1 g_2} \bigotimes_l |\mathrm{id}\rangle_l$$

where L_ξ are the ribbon operators (anyons) that are arranged in the Daimler-star configuration. These can be shown to be stabilized by all the vertex and plaquette terms.

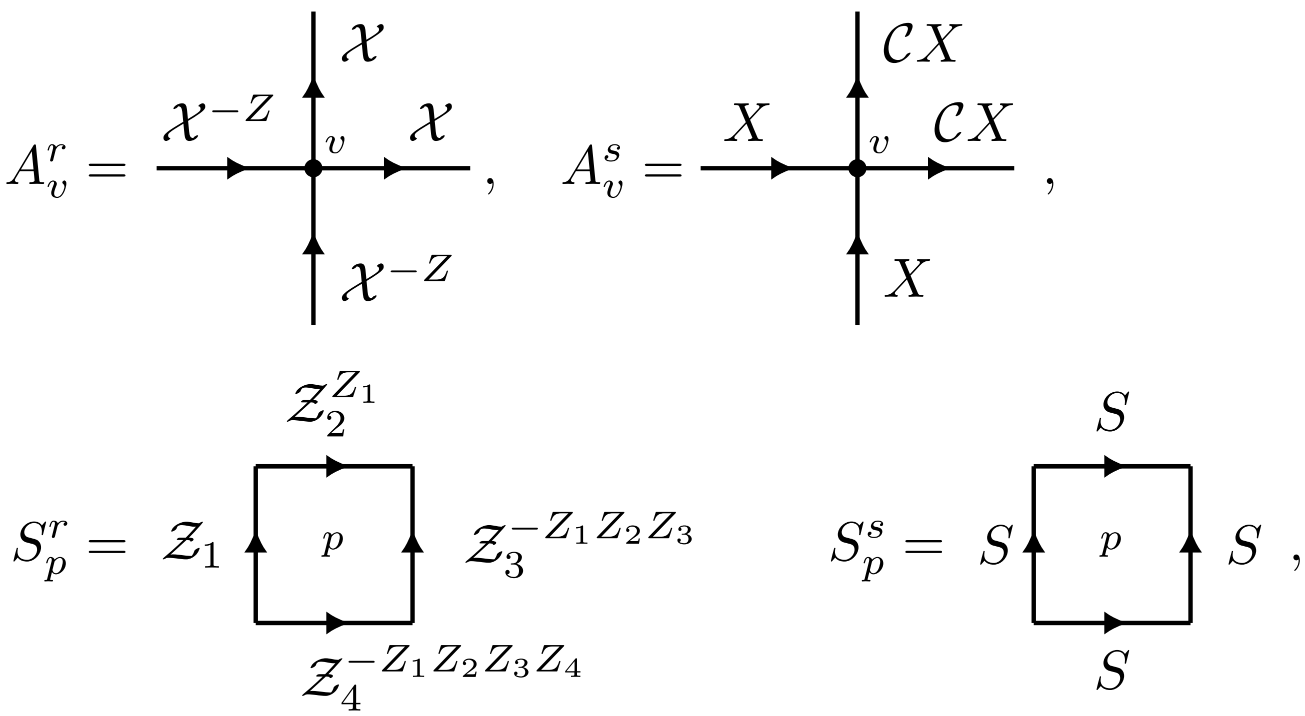
Some note on Non-Abelian Stabilizers and QEC

Oddly the literature contains hardly anything on non-abelian stabilizers. In fact the proper definition and requirements on these stabilizer groups seems to not have been settled. For S_3: [Verresen, Tantivasadakarn,Vishwanath] and D_4 [Huang, Warman, SSN, Chen]. We decided that there is a sensible way to define them: We invented a prescription that seems a reasonable generalization

In [Warman, SSN] we define a stabilizer group for D_{4N}, that stabilizes the code space, while the local errors (elementary excitations) have distinct non-trivial eigenvalues and form a group of unitaries:

$$\mathcal{S}_{D(D_{4N})} = \left\langle A_v^r \,,\, A_v^s \,,\, S_p^r \,,\, S_p^s \right\rangle \,,$$

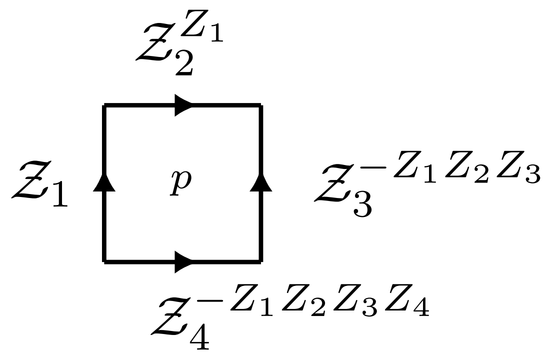
For D_4:



$A_v^r =$
$\mathcal{X}$
$\mathcal{X}^{-Z}$
v
$\mathcal{X}$
$\mathcal{X}^{-Z}$
,
$A_v^s =$
$\mathcal{C}X$
X
v
$\mathcal{C}X$
X
,
$S_p^r =$
$\mathcal{Z}_2^{Z_1}$
$\mathcal{Z}_1$
p
$\mathcal{Z}_3^{-Z_1 Z_2 Z_3}$
$\mathcal{Z}_4^{-Z_1 Z_2 Z_3 Z_4}$
$S_p^s =$
S
S
p
S
S
,

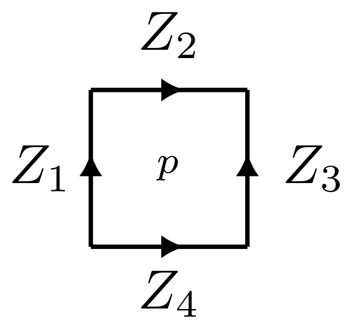
And for D_{4N}

$$S_p^r =$$



$$S_p^s = \zeta_{4N}^{(\mathbb{I} - Z_1 Z_2 Z_3 Z_4)/2}$$
$$= \tfrac{1}{2}(\mathbb{I} + Z_1 Z_2 Z_3 Z_4) + \tfrac{1}{2}(\mathbb{I} - Z_1 Z_2 Z_3 Z_4)\zeta_{4N}$$

$$(S_p^s)^{2N} =$$



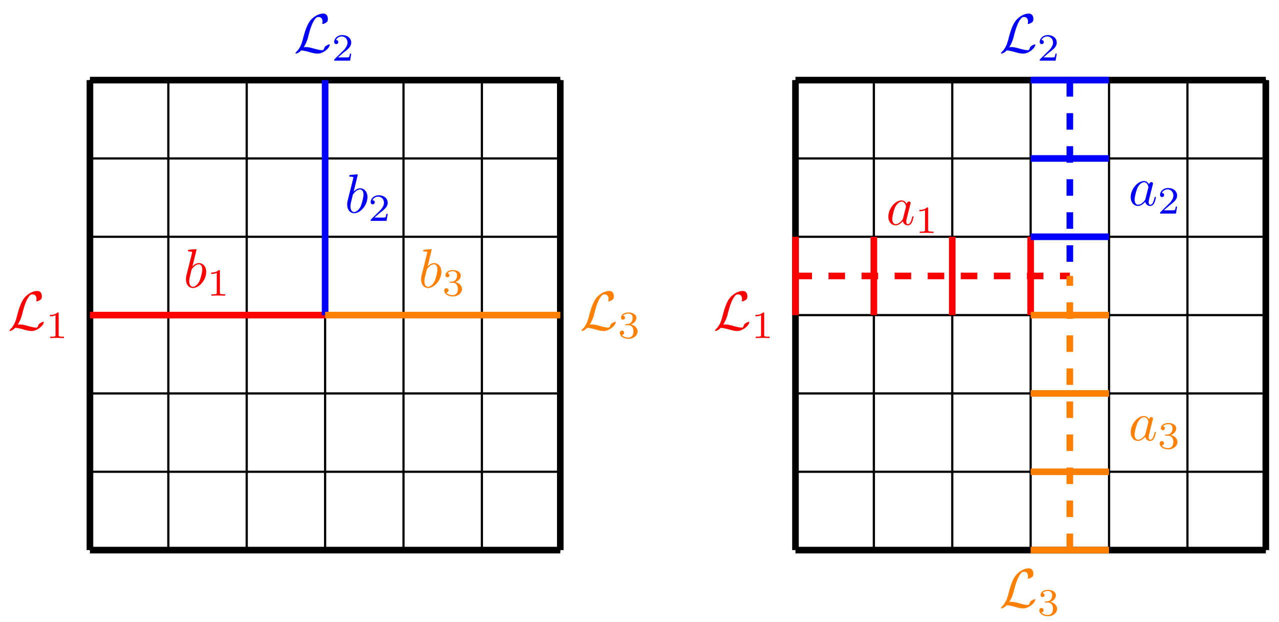
.

Unlike qubit Pauli stabilizers, these are not just the vertex and plaquette terms, but constructed out of these. The physical implementation uses Clifford-hierarchy gates. The stabilizer group $\mathcal{S}_{D(D_{4N})}$ is non-Abelian

$$[A_v^r, A_v^s] = (A_v^r)^{-2}$$
$$\left[A_v^r, S_{p_{NE}}^r\right] = (S_{p_{NE}}^s)^{-2}$$
$$\left[A_v^s, S_{p_{NE}}^r\right] = (S_{p_{NE}}^r)^2 \,.$$

BC in the Lattice Description

Surface codes with gapped boundary conditions were discussed in [Beigi, Shor, Whalen]. The configuration we will implement as as follows – where the three Lagrangians are fixed by the requirement of realizing logical qubits:

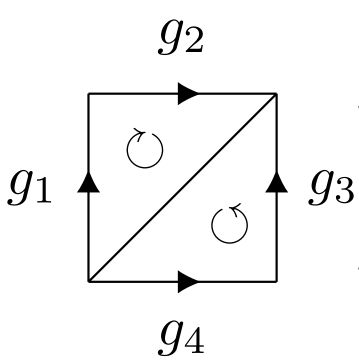


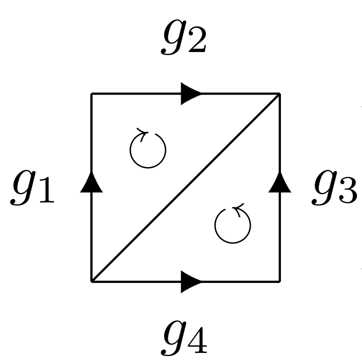
b_i are the electric (purely irreps) anyons

a_i are the magnetic (pure conjugacy class) anyons

Transversal Gates

The 2-cocyle α is evaluated on a triangulated plaquette p by acting with the corresponding operator M_p^α:

$M_p^\alpha \Big|$


$\Big\rangle = \frac{\alpha(g_4,g_3)}{\alpha(g_1,g_2)} \Big|$


$\Big\rangle$.

The boundary 1-cochains $\beta^{(i)}$ for $i \in \{1, 2, 3\}$ are realized as a diagonal gate $M^{\beta^{(i)}}$ acting on each lattice edge e along the boundaries in the basis $\{|g\rangle : g \in G\}$ as follows:

$$M_e^{\beta^{(i)}} |g\rangle_e = \beta^{(i)}(g)|g\rangle_e \,.$$

The transversal gate $U_{\alpha,\beta}$ will therefore be implemented as:

$$U_{\alpha,\beta} = \prod_p M_p^\alpha \prod_{e\in\mathcal{B}_1} \left(M_e^{\beta^{(1)}}\right)^\dagger \prod_{e\in\mathcal{B}_2} \left(M_e^{\beta^{(2)}}\right)^\dagger \prod_{e\in\mathcal{B}_3} M_e^{\beta^{(3)}} \,.$$

Plugging in the data for the $D(D_{4N})$ realizes the R_N phase gates.

Beyond just Math: Qubit Realization

The construction so far requires $4N$-dits and qubits.

For $8N = 2^n$ we can map the local Hilbert space $\mathcal{H} = \{|g\rangle\,,\ g \in D_{2^{n-1}}\} \cong (\mathbb{C}^2)^n$ is the Hilbert space for n qubits

- The first $n-1$ qubits correspond to $r^{2^{n-2}}, r^{2^{n-3}}, \cdots, r$, while the qubit corresponds to $s \Rightarrow$ map: $|r^a s^j\rangle \mapsto |\text{bin}(a)\rangle|j\rangle$
- Stabilizer group with Clifford-hierarchy operators
- This generates non-Clifford gates at every level of the Clifford hierarchy (if $N$is not of this type these are beyond Clifford hierarchy gates!).
- Summary table:

N	n	D_{4N}	Gate	Total phys. qbts
1	3	$D_4 = \mathbb{Z}_4 \rtimes \mathbb{Z}_2$	T	$3 \times N_{\text{edges}}$
2	4	$D_8 = \mathbb{Z}_8 \rtimes \mathbb{Z}_2$	$T^{1/2}$	$4 \times N_{\text{edges}}$
4	5	$D_{16} = \mathbb{Z}_{16} \rtimes \mathbb{Z}_2$	$T^{1/4}$	$5 \times N_{\text{edges}}$
2^{n-3}	n	$D_{2^{n-1}} = \mathbb{Z}_{2^{n-1}} \rtimes \mathbb{Z}_2$	$T^{2^{3-n}}$	$n \times N_{\text{edges}}$

Quantum computing applications: summary

- Non-abelian quantum double $D(G)$ codes are somewhat under-explored (to put it mildly)
- A logical qubit can be encoded in a non-trivial configurations of anyons
- Stack $\alpha \in H^2(G, U(1))$ on spatial slice with boundary counter-terms β $\Rightarrow$ Combined with group automorphisms, this can give rise to many other transversal gates
- From $G = D_{4N} \Rightarrow$ topologically-protected qubit quantum gates

$$U = T^{1/N} = \text{diag}\left(1, e^{i\pi/(4N)}\right), \quad N \in \mathbb{N}$$

  and qubit realization for $8N = 2^n$
- This collapses the Bravyi-Koenig theorem that states one requires k dims for kth level in the Clifford hierarchy to 2D!

Conclusions and Outlook

- Interfaces between TOs have many faces/applications.
- Here I focused in 2+1d TOs that are $D(G)$. In principle one could ask how this generalize for SymTFTs of generic fusion categories – which in the context of phase transitions has been done. What is the QI setup for that?
- Open questions:
  # Define what precisely is required from a non-abelian, Clifford stabilizer group
  # QEC: for non-abelian anyons, due to the non-invertible fusion (anyons can be absorbed) errors need to be corrected time-sensitively ”Just in time decoder”. What are thresholds for D_{4N}?
  # Transversal multi-qubit gates in 2D?
  # Can you extend this to so-called qLDPC codes, and use some non-abelian structure there to get universal gate sets?