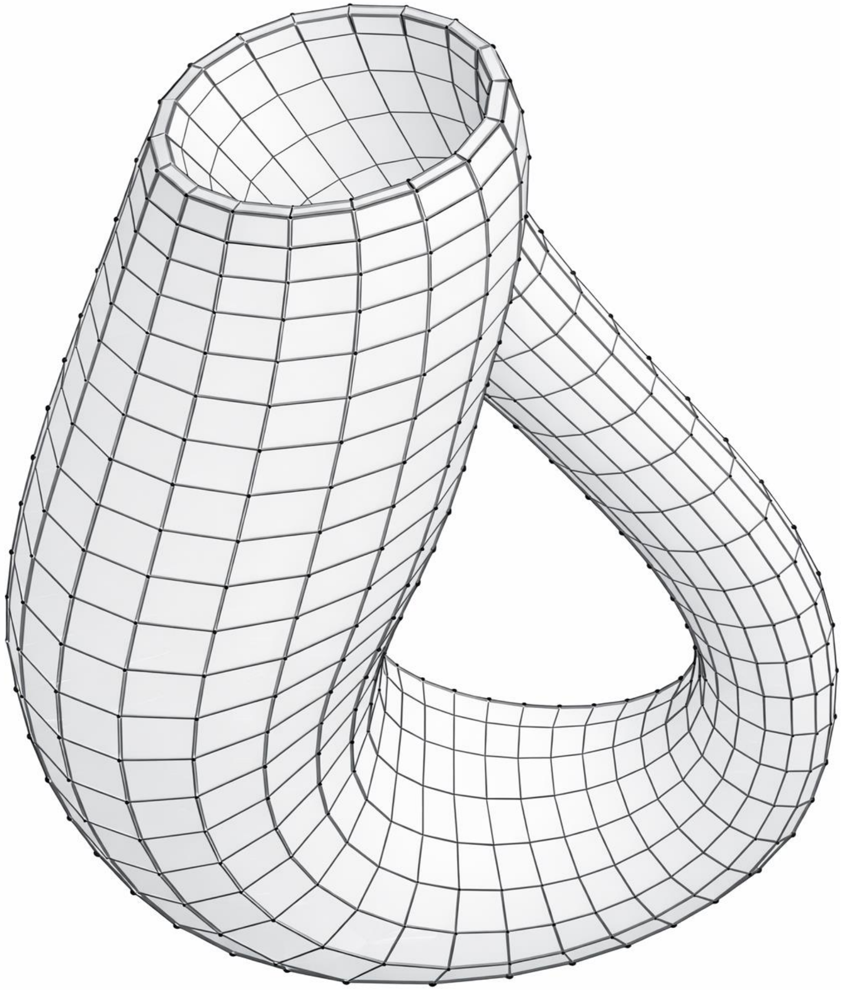
Arf–Brown–Kervaire Invariant on the Lattice



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Based on:

Sho Araki, Hidenori Fukaya, Tetsuya Onogi, SY, 2512.11424

Introduction

Lattice field theory

Finite!

Lattice field theory is a non-perturbative regularization of QFT.

Path-integral becomes mathematically well-defined finite dimensional integral.

↓ Taking the lattice spacing $a \to 0$ appropriately (renormalization)

continuous **QFT**

A way to define QFT.

η invariant

[Atiyah, Patodi, Singer 75]

H : Hermitian operator (e.g. $H = -i\gamma^{\mu} D_{\mu} = -i\not{D}$)

$$\eta(H) = \sum_i \text{sign}\, \lambda_i$$

eigenvalues of H

zeta function regularization

If H is finite-even-dimensional, $\frac{1}{2}\eta(H) \in \mathbb{Z}$

If H is infinite-dimensional, $\frac{1}{2}\eta(H)$ may be fractional

η invariant

[Kapustin, Thorngren, Turzillo, Wang 14], [Witten 15],...
[Freed, Hopkins 16], ...

(symmetry protected topological)

Fermionic SPT phase can be diagnosed by

$$Z_{massive} \sim \exp\left(2\pi i \frac{\eta(-iD)}{2}\right)$$

η invariant on the Lattice

$-iD$ on the lattice, if it exists, is finite dimensional, and thus $\frac{1}{2}\eta(-iD) \in \mathbb{Z}$

$$\Rightarrow \quad \exp\left(2\pi i \frac{\eta(-iD)}{2}\right) = 1$$

Impossible to describe SPT phase on the lattice ?

At least naive treatment should not work.

In this talk:

(a double cover of O(2))

2 dim Majorana fermion with $Pin^-(2)$ symmetry

SPT phase of "Kitaev chain" [Fidkowski, Kitaev 11]

- Construct the lattice path-integral formalism on unoriented surfaces.
- η invariant of this theory (ABK invariant) can be obtained from this theory.
  (Arf-Brown-Kervaire)

cf Hamiltonian formalism [Shapourian, Shiozaki, Ryu 16], [Shiozaki, Shapourian, Gomi, Ryu 17]...

ABK invariant

Two-dimensional Majorana Fermion

Euclidean formulation

$\psi(x)$: single Majorana fermion (2 components)

$\psi^\dagger$ does not appear in this formulation

γ^1, γ^2 2 dim gamma matrices.

C : unitary matrix that satisfies $C\gamma^a C^{-1} = -\gamma^{aT}$, $C^T = -C$

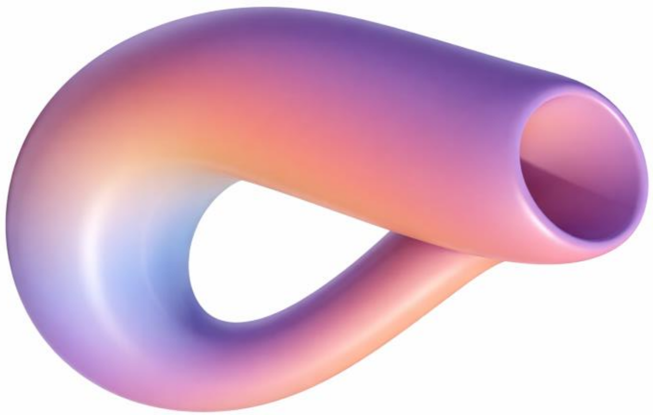
$\gamma_3 := i\gamma^1\gamma^2$

Local reflection in x^1 direction:

$$\psi(x) \xrightarrow{R_1} \gamma^1\gamma_3\psi(x)$$

$\Rightarrow R_1^2 = -1$ (Form Pin⁻(2) combined with local rotation)

Action



Σ : closed 2 dim surface (may be **unoriented**) with Pin$^-$ str

$$S = \int_\Sigma d^2x \sqrt{g} i \psi^T C (D + m) \psi$$

$$D := \gamma^\mu D_\mu$$

The mass term is also invariant under reflection

Partition function

$$S = \int_\Sigma d^2x \sqrt{g} i\psi^T C(D+m)\psi$$

$$Z(m) = \int D\psi e^{-S} = \mathrm{Pf}\big(C(D+m)\big)_{\mathrm{reg}}$$

$$= \prod_i (i\lambda_i + m)_{reg}$$

$i\lambda_i$: eigenvalue of D

eigenvectors appear in pairs

$Du_i = i\lambda_i u_i \Rightarrow v_i := C^{-1}u_i^*$ satisfy $Dv_i = i\lambda_i v_i$

a factor for each pair (u_i, v_i)

Phase of the partition function → η invariant

Heuristic argument

$$Z(m) = \prod_i (i\lambda_i + m)_{reg}$$

$M > 0$

$$\frac{Z(-M)}{Z(M)} = \frac{\prod_i (i\lambda_i - M)}{\prod_i (i\lambda_i + M)} = \frac{\prod_i i\lambda_i}{\prod_i (i\lambda_i + M)} \frac{\prod_i (i\lambda_i - M)}{\prod_i i\lambda_i}$$

$$\xrightarrow{M \to \infty} \prod_i \frac{i\lambda_i}{M} = \prod_i \left|\frac{\lambda_i}{M}\right| \exp \frac{\pi}{2} i \,\mathrm{sign}\lambda_i$$

$$\frac{Z(-M)}{Z(M)} \xrightarrow{M \to \infty} \exp\left(2\pi i \frac{\eta(-iD)}{2}\right)$$

Phase of the partition function → η invariant

Heuristic argument

$$\frac{Z(-M)}{Z(M)} \xrightarrow{M \to \infty} \exp\left(2\pi i \frac{\eta(-iD)}{2}\right)$$

$$\eta(-iD) = \sum_i \operatorname{sign} \lambda_i$$ zeta function regularization

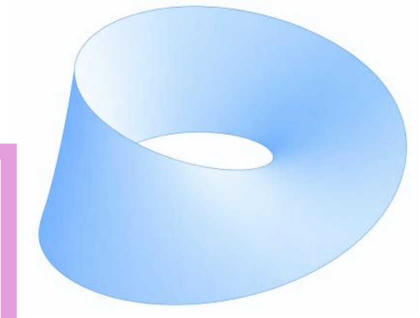
Fact: $4\eta(-iD) \in \mathbb{Z}$ See Eg. [Kaidi, Parra-Martinez, Tachikawa 1911.11780]

$\beta \coloneqq 4\,\eta \mod 8$ is a bordism invariant of Σ. "ABK invariant"

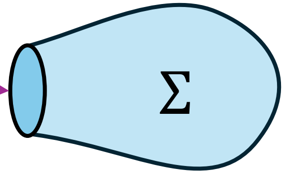
Eg. for RP^2, $\beta = \pm 1$ phase $= \exp(\pm \pi i/4)$

Surface with boundary

eg. Möbius strip

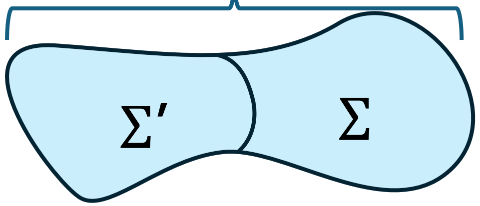


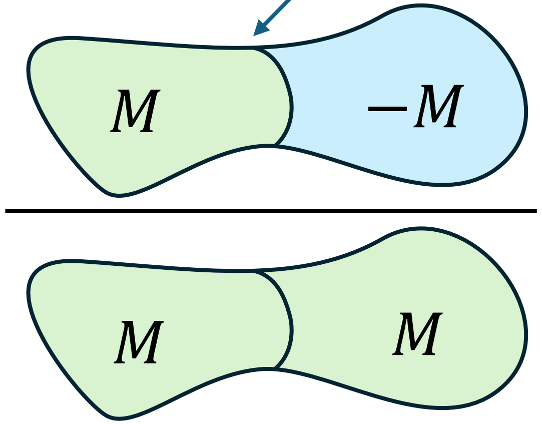
APS boundary condition (non-local)
⇨ D is anti-hermitian ⇨ $\eta(-iD)$ is defined.



This η is related to "domain-wall fermion"

(interface)





$$\xrightarrow{M \to \infty} \; |\cdot| \exp\left(2\pi i \frac{\eta(\Sigma)}{2}\right)$$

[Fukaya, Onogi, SY 17], [Fukaya, Furuta, Matsuo, Onogi, Yamashita, SY 19,20] for APS index using domain-wall. (oriented case)

[Witten, Yonekura 19] for generic η invariants using local boundary.

(Fukaya's talk tomorrow)

ABK invariants (η invariants) on the lattice ?

$$\frac{Z(-M)}{Z(M)} \xrightarrow{M \to \infty} \exp\left(2\pi i \frac{\eta(-iD)}{2}\right)$$

We will argue this relation on the lattice.

Naively, we can only obtain trivial values since finite dimensions.

⇓

Naive argument does not work!

Lattice formulation

Naive discretization

Derivative is replaced by difference

$$\frac{\partial}{\partial x}\psi(x) \longrightarrow \psi(x+1)-\psi(x)$$ Not anti-hermitian

$$\frac{\partial}{\partial x}\psi(x) \longrightarrow \frac{\psi(x+1)-\psi(x-1)}{2} =: \nabla_x\psi(x)$$

Naive Dirac operator $$D_{\text{naive}} \coloneqq \gamma^\mu \nabla_\mu$$

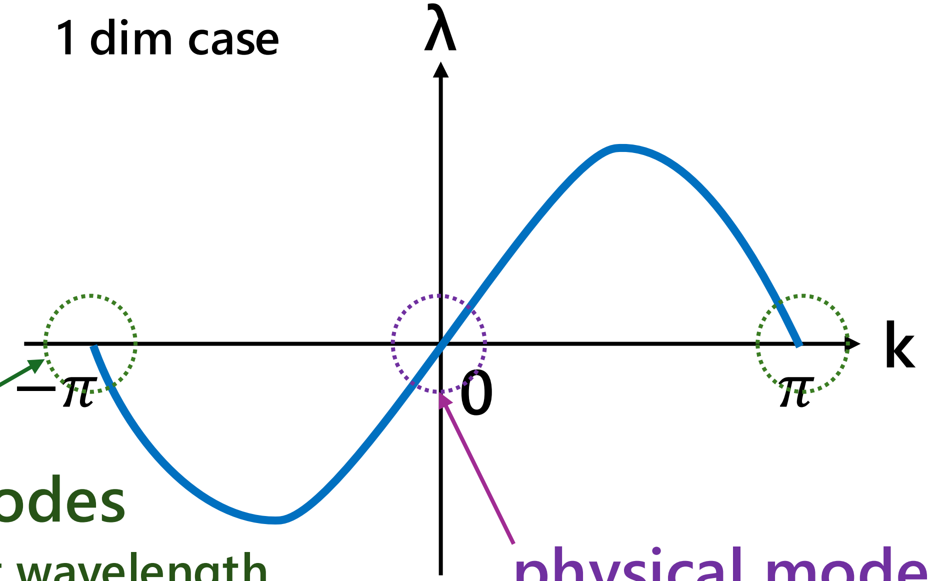
But $$\frac{1}{2}\eta(-iD_{\text{naive}}) \in \mathbb{Z}$$

It must be wrong. But how?

Doublers

Fourier transformation

1 dim case



Doubler modes

Extra light modes with short wavelength

physical modes

Single Majorana fermion with large wavelength

This system does not realize the single Majorana fermion in the continuum limit due to doublers.

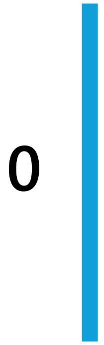
Wilson term

In order to kill doublers but retain physical modes, introduce

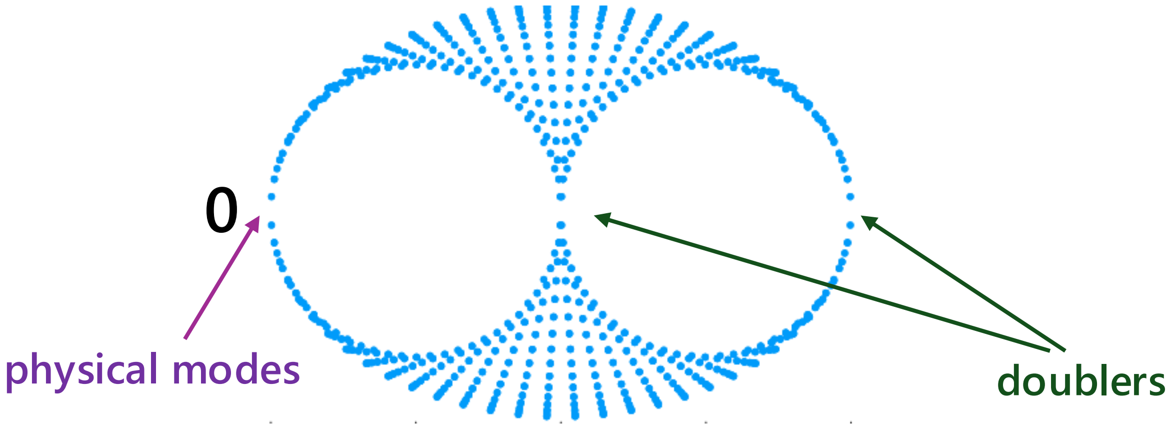
Wilson term ("wavenumber dependent mass term")

$$D_W := D_{\mathrm{naive}} + W \qquad W = \sum_{\mu} W_{\mu} \qquad W_x \psi(x) = 2\psi(x) - \psi(x+1) - \psi(x-1)$$

Eigenvalues of D_{naive}



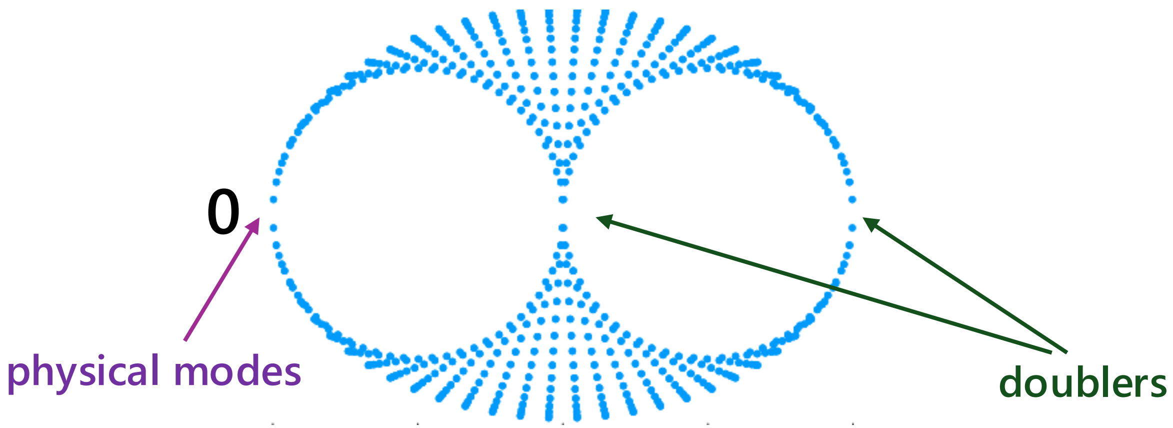
Eigenvalues of D_W



Wilson term

$$D_W := D_{\text{naive}} + W$$

Eigenvalues of D_W



Good news! Invariant under reflection ➡ put on an unorientable surface

Bad news? D_W is not anti-hermitian ➡ $\eta(-iD_W)$ does not make sense.

But this is perfectly consistent.

Phase of the partition function

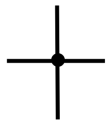
$$Z(m) = \mathrm{Pf}\Big(C(D_W + m)\Big)$$

Pfaffian of a finite dimensional matrix

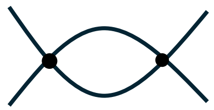
No conceptual difficulty

Construct surfaces (technical issue)

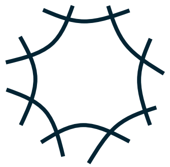
We need four links connected to each site to define difference operator and Wilson term.



The curvature should be localized on plaquettes



positive curvature



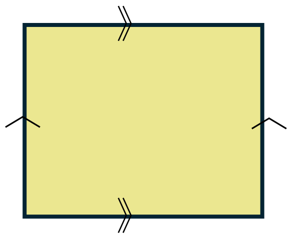
negative curvature

Construct surfaces (technical issue)

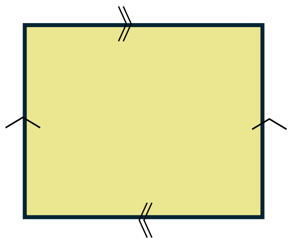
Various surfaces are constructed by cutting and gluing square lattice.

Pin⁻ structures are implemented

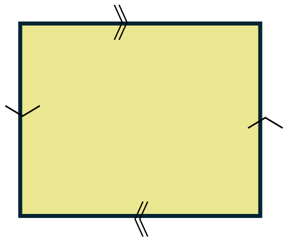
T^2



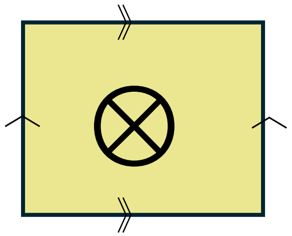
Klein bottle



RP^2

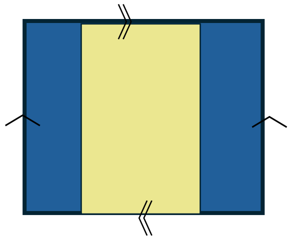


$T^2 \# RP^2$



Surfaces with boundary can be formulated by domain-wall.

Möbius strip



Numerical result

$$\frac{Z(-M)}{Z(M)} = |\bullet| \exp\left(\frac{2\pi i}{8}\beta\right)$$

32 x 32 lattice

$M = 1$ in the lattice unit

Klein bottle

0.0000000000
0.0000000000
6.0000000006
1.9999999994

RP^2

0.9999999994
7.0000000006

$T^2 \# RP^2$

0.9999999997
7.0000000003
5.0000000003
2.9999999997
…

Möbius strip

6.9998749247
1.0001250753

- Close to integers.
- Close to known exact values.

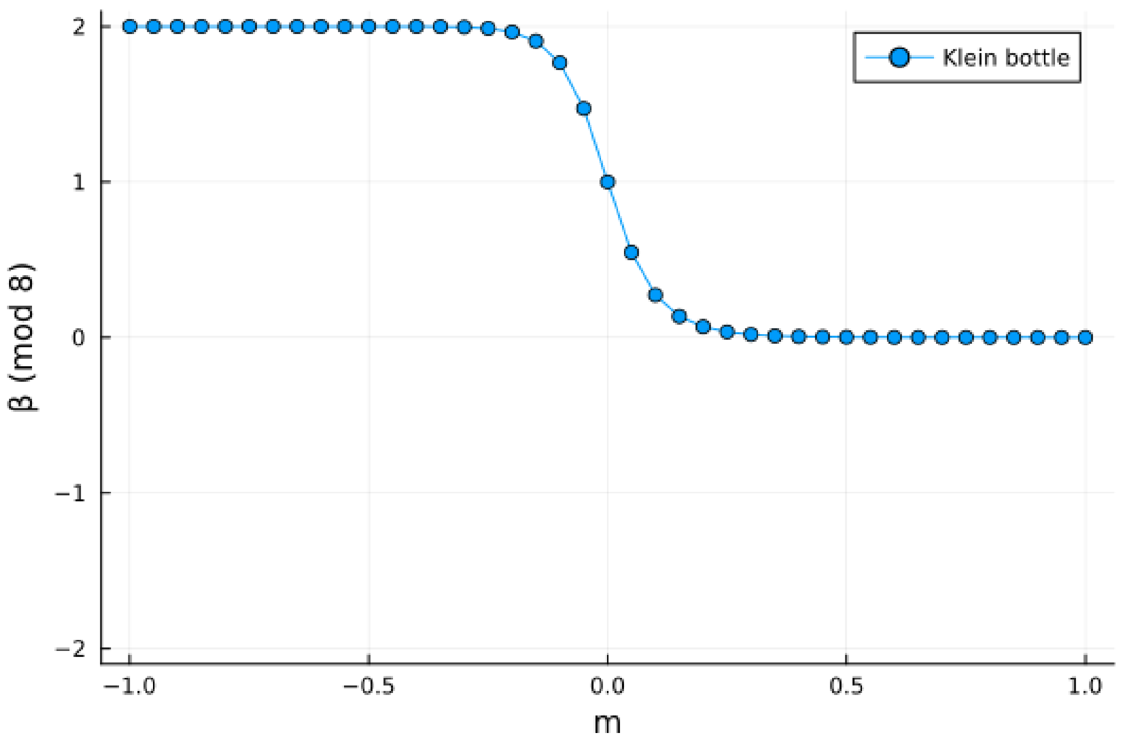
⬇

This lattice formulation is good enough to describe ABK invariants.

Numerical result : mass dependence

$$Z(m) = |\bullet| \ \exp\left(\frac{2\pi i}{8}\beta\right)$$

16 x 16 lattice

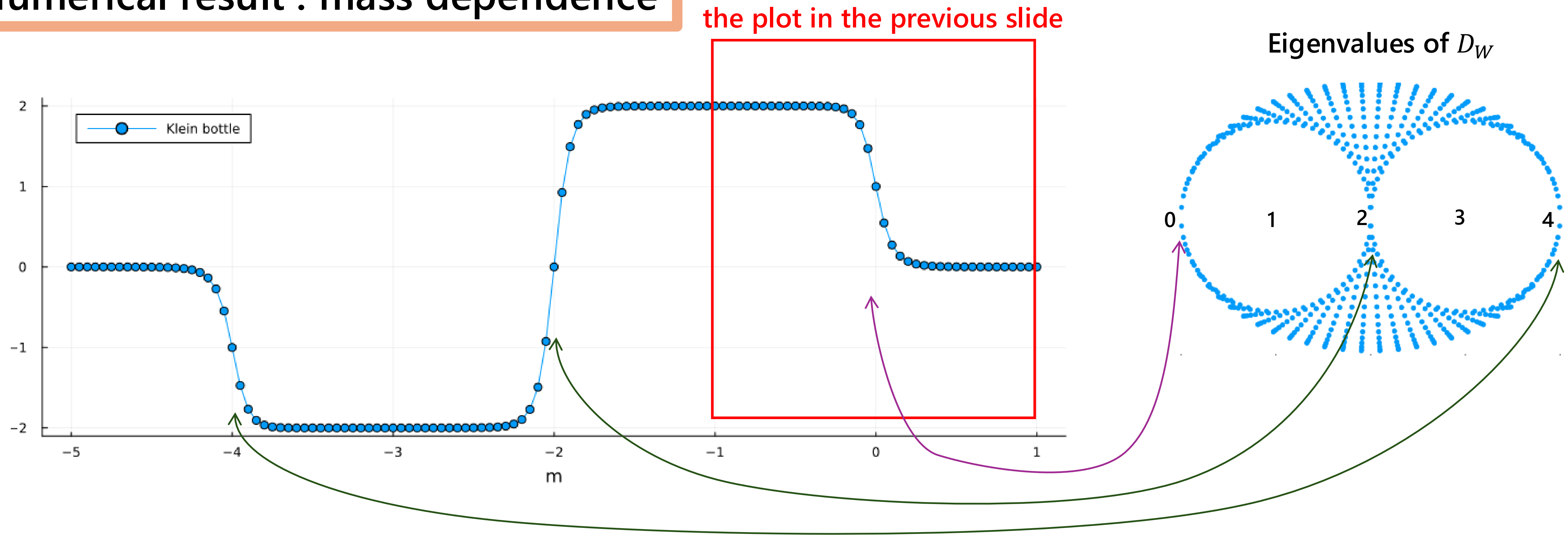


Looks perfect.
But how does this relate to the following argument?

$$\frac{Z(-M)}{Z(M)} \xrightarrow{M \to \infty} \exp\left(2\pi i \frac{\eta(-iD)}{2}\right) = 1$$

for finite dimensions.

Numerical result : mass dependence



Large negative mass may cancel the Wilson term ⇨ Doublers contribute

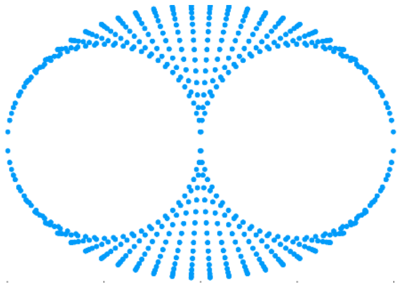
Not a single Majorana fermion anymore!

$$\frac{Z(-M)}{Z(M)} \xrightarrow[M \to \infty]{\text{fixed lattice}} 1$$ **is correct.**

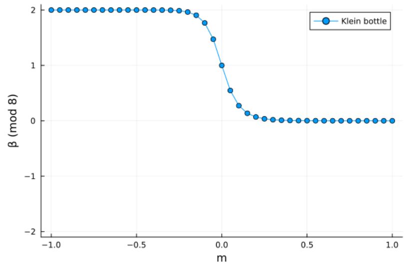
Summary
&
Discussion

Summary 2 dim Majorana fermion with Pin$^-$(2) symmetry

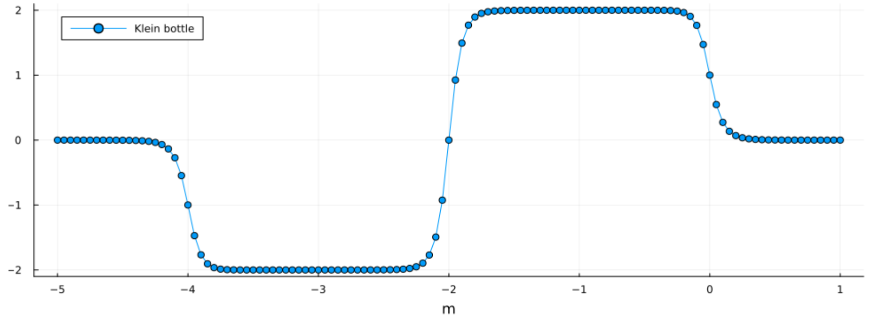
- Construct the lattice path-integral formalism on unoriented surfaces.
  key concepts: doublers and the Wilson term



- This formulation successfully reproduces the ABK invariant.



- Consistent with the heuristic arguments.



Discussion

Exact values are obtained only in the limit $a \to 0$, with $M = 1/a$

In the case of index on the lattice, exact values are obtained from a finite lattice.

[Neuberger 97], [Hasenfratz, Laliena, Niedermayer 98],...
[Fukaya's talk]

Is there a formulation where the exact ABK invariant can be obtained on a finite lattice?

More systematic understanding why this formulation works.

Eg. cutting and gluing properties. Domain-wall (interface) formulation could be useful.