Defect Anomalies and Scattering Amplitudes

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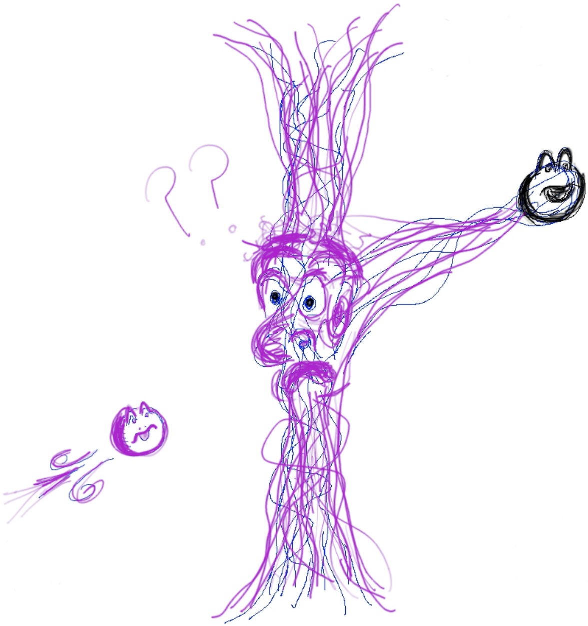
Interfaces & Symmetry
YITP Kyoto



Based on work to appear with **A. Antinucci**, **G. Galati** and **G. Rizi**.

⚠ Today we work in (1+1)-dimensions!

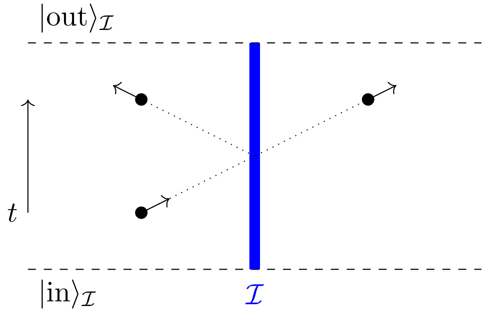
How can we use **Symmetry** to constrain the physics of **Interfaces** ?

Many natural questions: what is a symmetric interface? How can symmetry be realized on defects? What are the correct Ward identities? How to anomalies affect defects and their symmetry-breaking properties?

See e.g. [...; Padayasi et. al. '21; Cuomo, Mezei, Raviv-Moshe '21; Drukker, Kong, Sakkas '22; Herzog, Schaub '23; Cuomo, Zhang '23; Choi, Rayhaun, Sanghavi, Shao '23; Debray et. al. '23; Bhardwaj, CC, Schafer-Nameki '24; Choi, Rayhaun, Zheng '24; Kusuki, Murciano, Ooguri, Pal '24; Antinucci, CC, Galati, Rizi '24 '25; CC '24 '25; Barkeshli et. al. '25; Gabai, Sever, Zhong '25; Popov, Wang '25; Choi, Ha, Kim, Kusuki, Ryu '25; Komargodski, Popov, Rayhaun '25; Girault, Paulos, van Vliet '25; Drukker, Kong, Kravchuk '25; Shao, Zhong '25; Arbalestrier, Argurio, Galati, Paznokas '25; Prembabu, Shao, Verresen '26; ...] for a (growing and incomplete) list of recent works on the subject!

In many cases, a natural observable is the S-matrix in the presence of an interface:

$$S_{\mathcal{I}}(E) = {}_{\mathcal{I}}\langle \text{out}; E | \text{in}; E \rangle_{\mathcal{I}} \,.$$



Proxy in which bulk particles are used to probe the interface physics. This perspective has already led to interesting insights [CC, Cordova L., Komatsu '24; Cordova C., Garcia-Sepulveda, Holfester, Ohmori '24, Berean-Duchter, Derda, Parra-Martinez '25, ...] .

Motivation

Fermion-Monopole Scattering. The (confusing) Callan-Rubakov solution for the scattering of charged fermions onto magnetic monopoles [Callan '82; Rubakov '82]

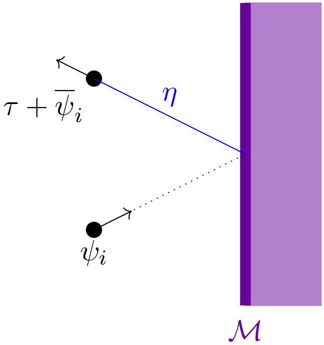
$$e_R \longrightarrow \frac{1}{2}\left(\overline{u}_R^1 + \overline{u}_R^2 + \overline{q}_L^3 + l_L\right)$$

Has been recently rephrased by saying that the outgoing radiation pertains to a different (twisted) Hilbert space $\mathcal{H}_\eta$ [Van Beest et. al. '23 '24] , see also [Hamada, Kitahara, Sato '22] .

The process (for an $SU(N)$ fundamental fermion ψ_i) reads:

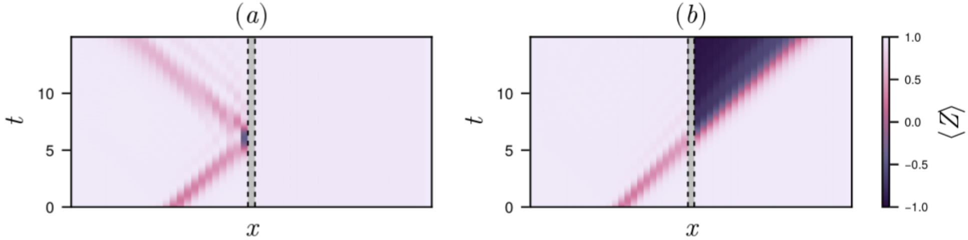
$$\psi_i \longrightarrow \tau + \overline{\psi}_i \,.$$

With τ being a disorder operator.



A "UV" model for this phenomenon [Polchinski '84] has been recently analyzed in [Hamada, Kitahara, Sato '22; Loladze, Okui '24; Loladze, Okui, Tong '25] confirming these predictions. (More of this later!)

Similarly [Ueda et.al. '25] , a numerical study of the scattering of charged wavepackets on lattice interfaces, has shown (perhaps surprisingly) identical features:



This curious scattering phenomenon is thus quite ubiquitous. For lack of a better name, I will call it:

Categorical Scattering

We focus on two questions:

I) What does Categorical Scattering tell us about the asymptotic Hilbert space $\mathcal{H}_{\mathcal{I}}^{(\text{in})}$ in the presence of an interface?

II) Is there a common physical mechanism underlying Categorical Scattering? Can we find/generate new examples?

The answer is deeply entangled to the realization of bulk symmetries on the defect, and to their 't Hooft anomalies.

Roadmap

I) Symmetric Interfaces, Interface Anomalies.

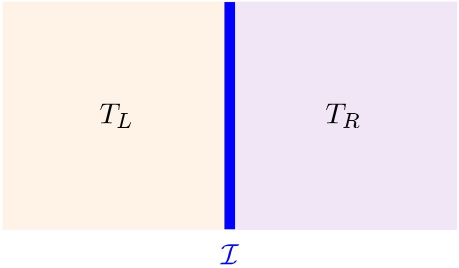
II) Categorical Scattering from Interface Anomalies.

III) Asymptotic Interface Hilbert space $\mathcal{H}_{\mathcal{I}}^{(\mathrm{in})}$.

IV) Examples/Applications.

Symmetric Interfaces

Consider an interface $\mathcal{I}$ between theories T_L and T_R:

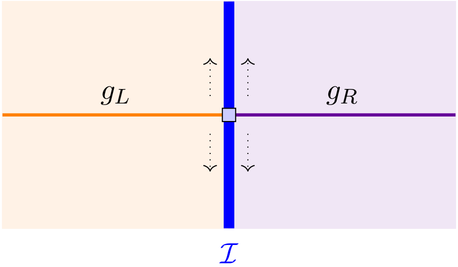


And let G_L and G_R be the symmetries on the two sides. In this talk we will focus on *groups*.

Several of the following remarks can be upgraded to fusion categories e.g. [Choi, Rayhaun, Shao '23; Choi, Rayhaun, Zheng '24; Bartsch, Gai, Schafer-Nameki '26] .

Consider a symmetry defect $g_{\mathcal{I}} = (g_L, g_R)$.

We say that $g_{\mathcal{I}}$ is preserved by the interface if the $g_{\mathcal{I}}$ defect remains topological in the presence of $\mathcal{I}$:

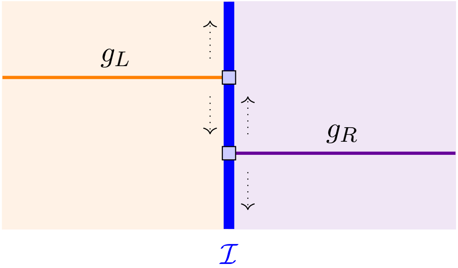


Or:

$$g_L \, \mathcal{I} \, g_R^{-1} = \mathcal{I} \, .$$

We denote by $G_{\mathcal{I}} \subseteq G_L \times G_R$ the symmetry preserved by the interface.

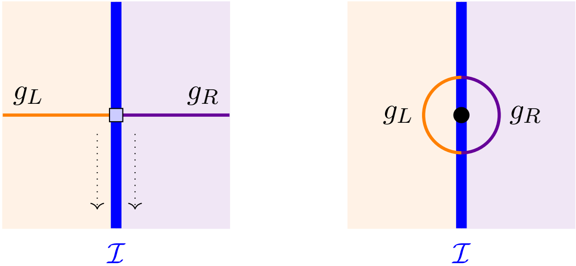
Particularly important for us will be so-called **symmetry reflecting** interfaces [Antinucci, CC, Galati, Rizi '24], which are invariant under one-sided elements $(g_L, 1)$ and $(g_R, 1)$:



(mod anomalies)

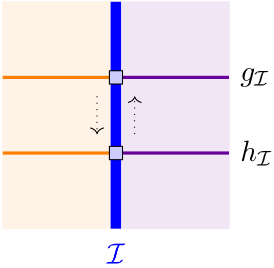
$$g_L \, \mathcal{I} = \mathcal{I} \, g_R = \mathcal{I} \, .$$

A symmetric interface admits an action of $G_{\mathcal{I}}$ on the defect Hilbert space $\mathcal{H}_{\mathcal{I}}$ (i.e. on states) and on operators:

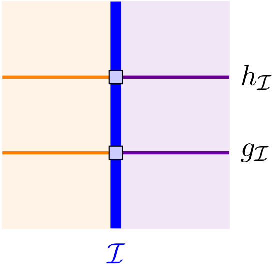


This action is akin to the symmetry action on a 1d quantum mechanical system localized on the interface.

This allows us to define the concept of **Interface Anomalies** [Debray et.al. '23; Antinucci, CC, Galati, Rizi '24; Komargodski, Popov, Rayhaun '25] :



$$= \chi(g_{\mathcal{I}}, h_{\mathcal{I}})$$



with $\chi(g_{\mathcal{I}}, h_{\mathcal{I}}) = \frac{\omega(g_{\mathcal{I}}, h_{\mathcal{I}})}{\omega(h_{\mathcal{I}}, g_{\mathcal{I}})}$ and:

$$\omega \in H^2(G_{\mathcal{I}}, U(1)) \,.$$

Notice that, as the symmetry action is projective, $\mathcal{H}_{\mathcal{I}}$ must be degenerate! Thus defect anomalies cannot be matched by "trivial" defects.

Examples

Continuum Example A standard manner to obtain continuum examples is to couple a $G_{\mathcal{I}}$-symmetric QM system with the correct anomaly to the bulk in a $G_{\mathcal{I}}$-preserving manner.

Consider a bulk O(3) model. We couple the bulk O(3) field ϕ^a to a spin s defect [Komargodski, Popov, Rayhaun '25] via:

$$S = S_{bulk} + \int d\tau z_i^\dagger \dot{z}^i + \gamma \int d\tau S_a \phi^a \,, \quad z^\dagger z = 2s \,, \quad S^a = z^\dagger \Sigma^a z \,.$$

The SO(3) symmetry is preserved by the coupling, but is realized projective on $\mathcal{H}_{\mathcal{I}}$ when s is odd.

The defect anomaly is:

$$\omega = 2s \int w_2(SO(3)) \,.$$

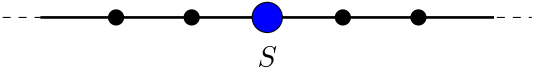
Lattice Example Consider e.g. the spin 1/2 Heisenberg chain:

$$H = g \sum_{j=1}^{2n} \vec{X}_j \cdot \vec{X}_{j+1}\,, \qquad \vec{X}_{2n+1} = \vec{X}_1\,.$$

This is symmetric under $G = SO(3)$, with generator $Q_{\hat{n}} = \sum_j \hat{n} \cdot \vec{X}_j$. In particular there are two $\mathbb{Z}_2$ generators $U = \prod_j Z_j$, $V = \prod_j X_j$.

We insert a spin S impurity at site J:

$$H_S = \gamma \left(\vec{X}_J \cdot \vec{\Sigma}^{(S)} + \vec{\Sigma}^{(S)} \cdot \vec{X}_J \right)\,, \qquad H_{\mathcal{I}} = H + H_S\,.$$



The $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry is preserved, provided we modify the generator in the presence of the impurity:

$$U_{\mathcal{I}} = U\, e^{i\pi \mathbf{Z}^{(S)}}\,, \qquad V_{\mathcal{I}} = V\, e^{i\pi \mathbf{X}^{(S)}}\,.$$

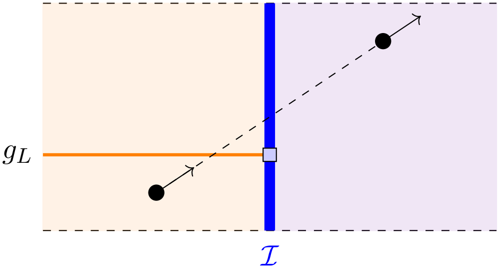
However, depending on S, the system now has a defect anomaly:

$$U_{\mathcal{I}}\, V_{\mathcal{I}} = (-)^{2S} V_{\mathcal{I}}\, U_{\mathcal{I}}\,, \qquad \omega_S = 2\pi i S \int A_U \cup A_V\,.$$

The same observation can be lifted to the full SO(3), the defect anomaly being $2S \int \omega_2(SO(3))$.

Is "Symmetry Reflecting" Reflective?

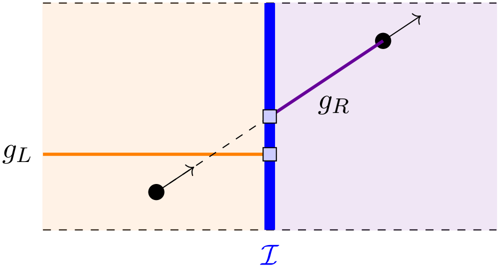
Consider a symmetry-reflecting interface $\mathcal{I}$ under G_L. Ward identities seem to imply that the G_L charge on the left of $\mathcal{I}$ remains conserved.



An incoming particle charged under G_L cannot be naively transmitted through $\mathcal{I}$: the left G_L charge is not conserved and:

$$g_L|\text{in}\rangle_{\mathcal{I}} = q_L|\text{in}\rangle_{\mathcal{I}}\,, \qquad g_L|\text{out}\rangle_{\mathcal{I}} = 1|\text{out}\rangle_{\mathcal{I}}$$

However, in the presence of a defect anomaly, the “missing” charge can be matched by the anomaly, if the outgoing particle is in the correct twisted Hilbert space:



We now have:

$$g_L|\text{in}\rangle_{\mathcal{I}} = q_L|\text{in}\rangle_{\mathcal{I}}\,, \qquad g_L|\text{out}\rangle_{\mathcal{I}} = \chi(g_L, g_R)|\text{out}\rangle_{\mathcal{I}}$$

A defect anomaly allows charge to be stored on topological junctions.
This is the mechanism underlying the existence of Categorical Scattering.

Asymptotic States for Symmetric Defects?

This seems to be at odds with unitarity of the S-matrix as a map between asymptotic states:

$$S : \mathcal{H}_0^{(in)} \longrightarrow \mathcal{H}_0^{(out)} .$$

However unitarity is a sacred concept. Thus we should upgrade our picture of scattering amplitudes:

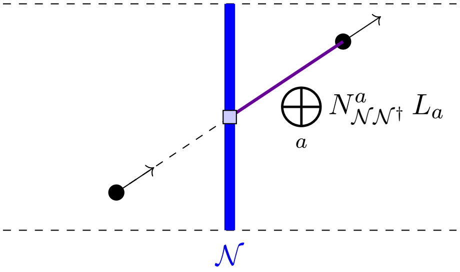
$$S : \mathcal{H}_{\mathcal{I}}^{(in)} \equiv \bigoplus_{g \in G_{\mathcal{I}}} \mathcal{H}_{g\mathcal{I}}^{(in)} \longrightarrow \mathcal{H}_{\mathcal{I}}^{(out)} = \bigoplus_{g \in G_{\mathcal{I}}} \mathcal{H}_{g\mathcal{I}}^{(out)} .$$

Where the Hilbert spaces $\mathcal{H}_{g\mathcal{I}}^{(in)}$ are obtained by performing an LSZ reduction (if available) on the states of:

$$\mathcal{H}_{g_L} \otimes \mathcal{H}_{g_R} .$$

This identification resonates with several other contexts:

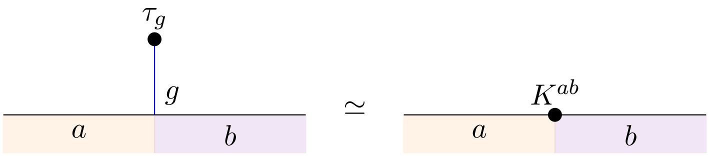
I) Consider a topological (non-invertible defect) $\mathcal{N}$. The tube algebra action of $\mathcal{N}$ on a charged particle has an image in the twisted Hilbert space.



This action [Bartsch, Gai, Schafer-Nameki '26] can be interpreted as a (energy-independent) unitary S-matrix for $\mathcal{N}$.

II) In the context of integrable systems, massive soliton scattering has long been studied [Zamolodchikov, Zamolodchikov, Faddeev, ...] .

On SSB vacua, solitons are the IR avatar of disorder operators $\tau_g \in \mathcal{H}_g$, where g is a spontaneously-broken symmetry [CC, Cordova L., Komatsu '24; Cordova C., Garcia-Sepulveda, Holfester, Ohmori '24; ...] .



The kink Hilbert space in an SSB scenario has standard one-particle states, which can be used to construct an S-matrix.

Applications

Fermion-Rotor Scattering inside a monopole core can be described by the fermion-rotor model:

$$S = \int dx dt \sum_{i=1}^{N} \overline{\chi}_i \left(i\not{D} + \frac{\alpha}{2\pi} f'(x) \gamma^1 \right) \chi^i + \int dt \frac{1}{2M} \dot{\alpha}^2 + i \frac{\theta}{2\pi} \int dt \dot{\alpha} \,,$$

where χ_i is a Weyl fermion and $\alpha \sim \alpha + 2\pi$ a compact rotor.

The profile function $f(x)$ is a regulator $f'(x) \sim \delta(x)$. Furthermore we choose $f(-x) = -f(x)$.

The vector $U(1)$ symmetry is broken to $\mathbb{Z}_N$ by the ABJ anomaly due to the defect coupling:

$$\partial_\mu J^\mu = \frac{N}{2\pi}\dot{\alpha} f'(x)\,.$$

Furthermore, the equations of motion for α, $\frac{J^\perp}{2\pi} = M^{-1}\ddot{\alpha}$ imply that the $\mathbb{Z}_N$ symmetry generator trivializes on the interface:

$$U^k = \exp\left(2\pi i \frac{k}{N}\int dt J^\perp\right) = \exp\left(2\pi i M^{-1}\int dt\ddot{\alpha}\right) = 1\,,$$

which is thus symmetry-reflecting under $\mathbb{Z}_N$.

Furthermore, the system has a further symmetry $U(1)_T$, generated by:

$$\chi_i \to e^{i\beta f(x)} \chi_i \,, \quad \alpha \to \alpha + \beta \,.$$

This symmetry is anomaly-free, but gives opposite charges ($\pm 1/2$ in this normalization) to $\psi_i(x > 0)$ and $\psi_i(x < 0)$. This leads to the Callan Rubakov puzzle:

$$|\text{in}\rangle = |\psi_i, E\rangle \longrightarrow |\text{out}\rangle = \textcolor{red}{?}$$

as there seems to be no local operator with the correct $U(1)_T$ charge on the right (and the incoming fermion cannot be reflected as it is chiral).

The solution follows from analyzing the interplay between $\mathbb{Z}_N$ and $U(1)_T$. In particular, a $\mathbb{Z}_N$ transformation leads to a 2π increase in the interface theta angle:

$$U^k \cdot Z_{\mathcal{I}}[A_T, \theta] = \exp\left(ik \int A_T\right) Z_{\mathcal{I}}[A_T, \theta] \,. \tag{0.1}$$

For a $\mathbb{Z}_N \subset U(1)_T$ this reduces to the defect anomaly†:

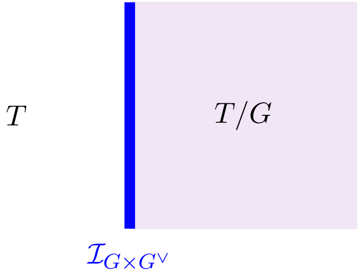
$$\omega(A_T, B) = \frac{2\pi i}{N} \int A_T \cup B \,.$$

The mixed anomaly immediately gives us the solution: the $U(1)_T$ charge is matched by passing to a twisted sector:

$$e^{i\alpha}\, \psi_i(x > 0)\,, \quad e^{i\alpha} = \tau \,. \tag{0.2}$$

Gauging interfaces The observations of [Ueda et.al. '25] can also be reconciled in this framework.

Consider a (topological) interface $\mathcal{I}_{G\times G^\vee}$ between a theory T and T/G, with G an abelian group.



Due to the Dirichlet boundary conditions on the G gauge fields, one can see that $\mathcal{I}$ is symmetric under $G \times G^\vee$, with a defect anomaly:

$$\omega(A, A^\vee) = 2\pi i \int A \cup A^\vee \,.$$

Thus, a charged particle σ in T can naturally scatter into a disorder operator μ in T/G.

Notice that this channel is still allowed if $\mathcal{I}_{G\times G^\vee}$ is non-topological (i.e. we turn on a $G \times G^\vee$ symmetric coupling on the interface).

A lattice example In [Ueda et.al. '25] the authors consider the following interface on the Ising chain:

$$
\begin{aligned}
H_L &= -g_L \sum_{i=-\infty}^{-2} X_{i+1}X_i - \sum_{i=-\infty}^{-1} Z_i \\
H_R &= -\sum_{i=1}^{\infty} Z_{i+1}Z_i - g_R \sum_{i=1}^{\infty} X_i\,, \\
H_{\mathcal{I}} &= -\gamma_L X_{-1}X_0 - \gamma_R Z_0 Z_1\,, \qquad H = H_L + H_R + H_{\mathcal{I}}\,.
\end{aligned}
$$

The Ising $\mathbb{Z}_2$ symmetries are:

$$
\eta_L = \prod_i Z_i\,, \qquad \eta_R = \prod_i X_i\,.
$$

Both of these can be truncated on the defect: $\eta_L^{\mathcal{I}} = \prod_{i=-\infty}^{0} Z_i$, $\eta_R^{\mathcal{I}} = \prod_{i=0}^{\infty} X_i$.

Clearly we also have:

$$[H_{\mathcal{I}}, \eta_L] = [H_{\mathcal{I}}, \eta_R] = 0\,,$$

i.e. the defect is symmetry-reflecting. However:

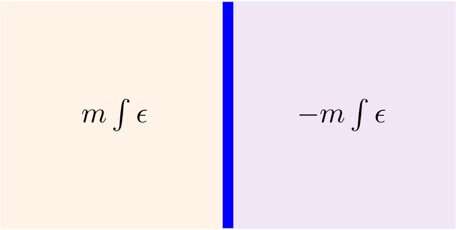
$$\eta_L \eta_R (\eta_R \eta_L)^{-1} = Z_0 X_0 Z_0 X_0 = -1\,.$$

Encoding the nontrivial defect anomaly for the whole family.

For $g_L = g_R = g = \gamma_R$, $\gamma_L = 1$, the interface is topological and describes the results of [Ueda et.al. '25] due to the defect anomaly.

Integrable Categorical Scattering? Another promising way to obtain interesting examples are integrable defects.

Consider the Ising model in the presence of a KW duality defect $\mathcal{N}$. On the two sides we turn on an (integrable) energy perturbation:



$$\mathcal{N}[\lambda] \equiv \mathcal{N} + \lambda \int dt \, \epsilon$$

This is essentially a continuum version of the former lattice model where we further add a marginal ϵ perturbation of the defect.

The $\mathcal{N}$ has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ defect anomaly, but is no-longer topological. The displacement operator is:

$$D \propto \partial_\perp \epsilon \,.$$

However both ϵ and $\mathcal{N}$ preserve integrability. Thus scattering on $\mathcal{N}[\lambda]$ should be integrable.

No-go theorem of [Castro-Alvaredo, Fring, Gohmann '02] is evaded due to defect anomaly implying degeneracy in $\mathcal{H}_{\mathcal{N}[\lambda]}$ (and, in this case, bulk $S = \pm 1$).

Not included/ To Explore

- Higher-dimensional generalizations. Novel integrables examples in (1+1)d.
- "True" **Categorical scattering?** (including non-invertible symmetries) Perhaps building on the results of [Bartsch, Gai, Schafer-Nameki '26] .
- More Lattice examples? How to extract defect anomaly from Lattice Hamiltonian?
- Generating examples by discrete gauging. (Not included for lack of time)
- Applications to (1+1)d gauge theories? Wilson lines in QCD_2?

Thank you!