Interface vs. boundary in the Dirac operator index

Hidenori Fukaya
(The University of Osaka)

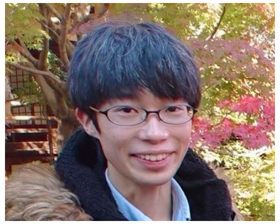
THE UNIVERSITY OF OSAKA

with Shoto Aoki (RIKEN iTHEMS), Hajime Fujita (Japan Women's U.), Mikio Furuta (U. Tokyo), Shinichiroh Matsuo(Nagoya U.), Tetsuya Onogi(U. Osaka), and Satoshi Yamaguchi (U. Osaka),

arXiv:2407.17708, 2503.23921, 2602.12576

Phys-Math collaborators

Physicists



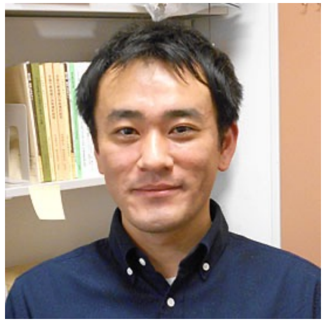
Shoto Aoki
(RIKEN iTHEMS)

Tetsuya Onogi
(U. Osaka)

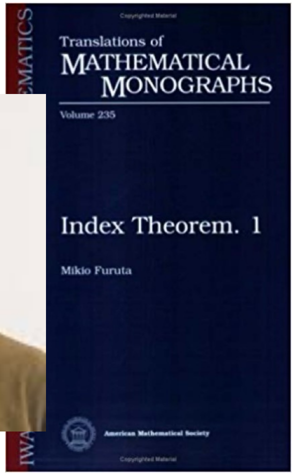
Satoshi Yamaguchi
(U. Osaka)

2nd speaker yesterday

Mathematicians



Hajime Fujita
(Japan Women's U.)



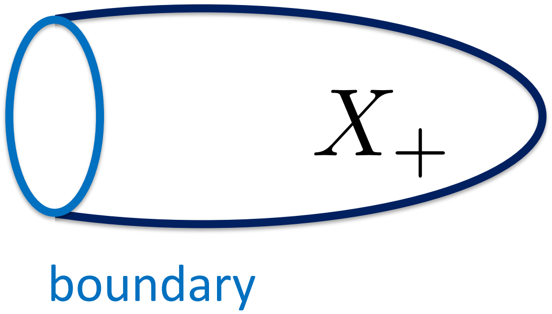
Mikio Furuta
(U. Tokyo)

Shinichiroh Matsuo
(Nagoya U.)

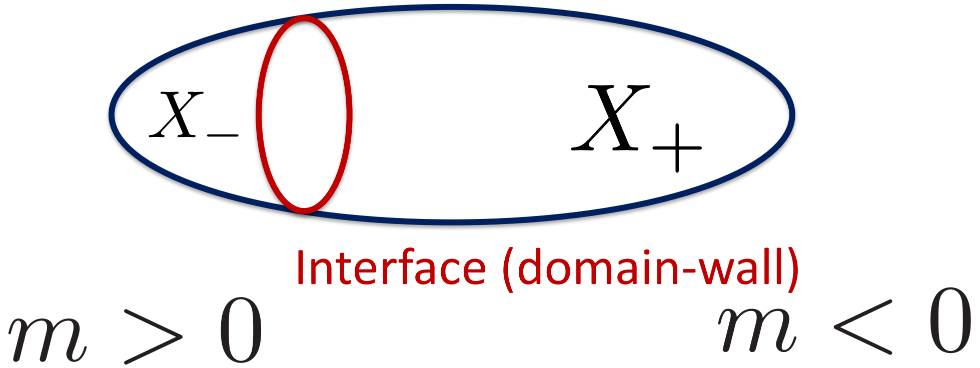
We also had N. Kawai, Y. Matsuki, M. Mori , K. Nakayama and M. Yamashita in the early stage of collaboration.

Mathematical relation between

massless Dirac operators on a manifold with boundary



and massive Dirac operators on a closed manifold with interface.

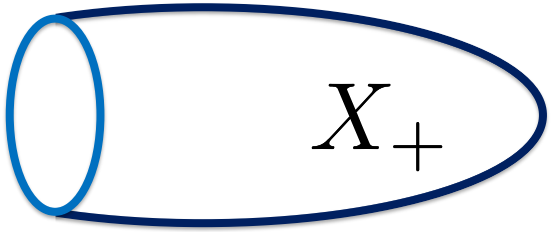


Index (of massless D)= spectral flow (of massive D)

The Atiyah-Patodi-Singer(APS) index of massless Dirac operators on a manifold with boundary, which requires

chiral symmetry

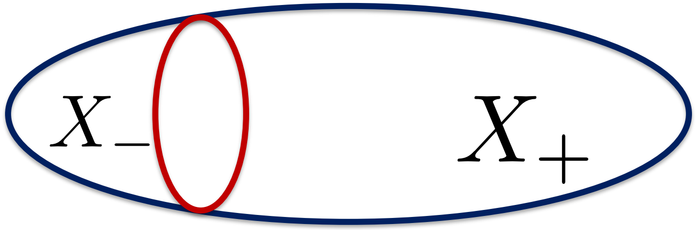
and nonlocal (APS) boundary condition



= the spectral flow of massive Dirac operators on a closed manifold with interface. We do NOT need

chiral symmetry

and non-local boundary condition.



Advantages of the massive expression of Dirac index with (and without) interfaces

- We can avoid nonlocal APS boundary conditions.
- AS and APS indices are unified on a closed manifold: when interface is absent it gives the standard Atiyah-Singer(AS) index.
- Extension to the mod-two version is straightforward.
- Chiral symmetry is NOT necessary: the lattice gauge theory version is straightforward → numerical evaluation is possible.
- This topic offers "interfaces" among math-DG, math-KT, hep-th, hep-lat, cond-mat, etc.

Contents

✓ 1. Introduction

We consider the index in terms of massive Dirac operator with interface.

2. APS index and bulk-boundary correspondence of anomaly (review)

3. Massive Dirac operator without interface and AS index

4. Massive Dirac operator with interface and APS index

5. Lattice formulation of APS index

6. Summary

η invariant in odd dimensions

Let us consider massless Dirac fermion in 3 dimensions with the Pauli-Villars regularization

(the argument does not change in any odd dimensions)

$$Z^{\rm 3D}_{\rm massless} = \lim_{M\to\infty} \det \frac{D^{\rm 3D}}{D^{\rm 3D}+M} = \lim_{M\to\infty} \prod_{\lambda^{\rm 3D}} \frac{i\lambda^{\rm 3D}}{i\lambda^{\rm 3D}+M}$$

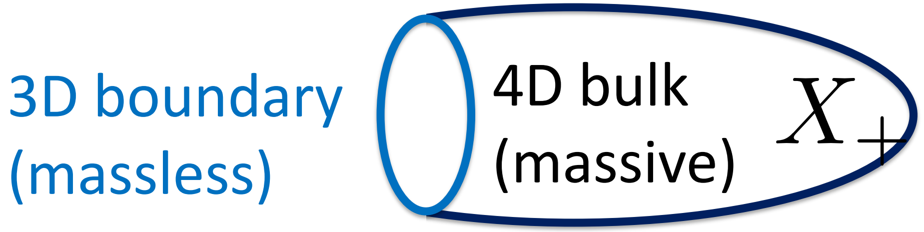
$$\sim \lim_{M\to\infty} \prod_{\lambda^{\rm 3D}} \frac{i\lambda^{\rm 3D}}{M} \propto \exp\left[-\frac{i\pi}{2}\sum_{\lambda^{\rm 3D}} {\rm sgn}(-\lambda^{\rm 3D})\right] = \exp\left[-\frac{i\pi}{2}\eta(iD^{\rm 3D})\right].$$

phase = η invariant.

It is ODD in time-reversal (T) transformation (T anomaly).

Bulk-boundary correspondence of T anomaly

If $Z^{\mathrm{3D}}_{\mathrm{massless}}$ is boundary-localized modes of 4D bulk fermion in a SPT phase, which has a negative mass,



$$Z^{\mathrm{4D}} = \det\left[\frac{D^{\mathrm{4D}} - M'}{D^{\mathrm{4D}} + M'}\right] = \det\left[\frac{D^{\mathrm{4D}} + M'}{D^{\mathrm{4D}} + M'}\right] \exp\left[i\frac{\pi}{16\pi^2}\int FF\right]$$

← Chiral U(1) rotation with $\theta = \pi$

T anomaly cancels in the total system:

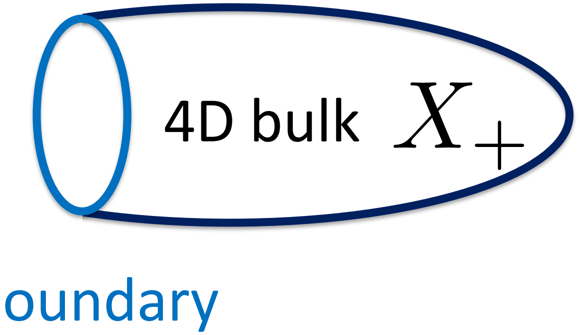
$$Z^{\mathrm{3D}}_{\mathrm{massless}} Z^{\mathrm{4D}} \propto \exp\left[i\pi\left(\frac{1}{16\pi^2}\int FF - \frac{1}{2}\eta(iD^{\mathrm{3D}})\right)\right] = (-1)^{\mathrm{Ind}_{\mathrm{APS}}[D|_{X_+}]}$$

thanks to the APS index theorem. [Witten 2015]

Definition of the Atiyah-Patodi-Singer (APS) index

[Atiyah-Patodi-Singer 1975]

$$\mathrm{Ind}_{\mathrm{APS}}[D|_{X_+}] = \frac{1}{16\pi^2}\int FF - \frac{1}{2}\eta(iD^{3\mathrm{D}})$$

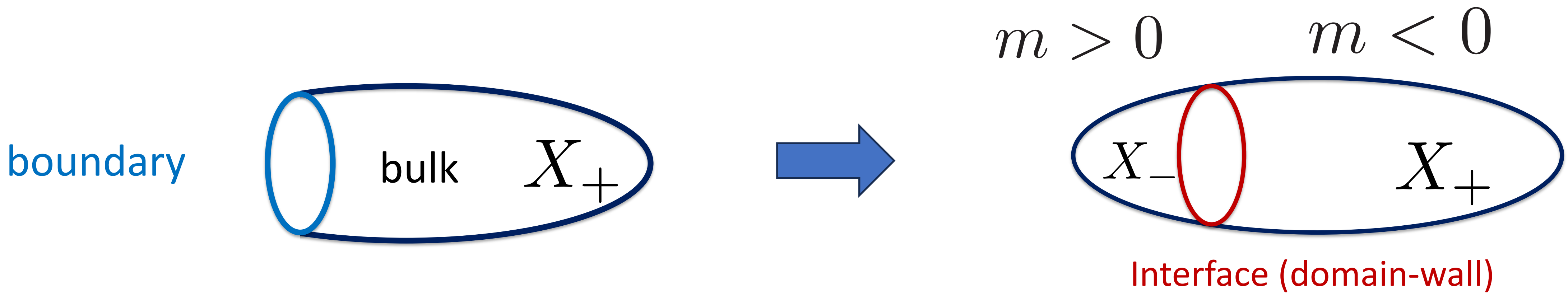


LHS is defined by # of chiral zero modes of a 4D massless Dirac operator on X_+ with a non-local (causality-breaking) boundary condition (APS condition allowing only positive/negative eigenmode components of boundary Dirac operators).

But this operator does NOT exist in the 4D fermion system in the SPT phase, which is essentially massive and local QFT.

Q. Can we describe the APS index with massive fermion without nonlocal APS boundary condition?

We will show the answer=YES. But we replace boundary by interface glued with "outside " to form a closed manifold.



* Surface of topological insulator is not a boundary of a manifold but an interface between topological and normal phases.
(It is nontrivial only when its outside is a normal insulator.)
[Cf. Different explanations by Witten-Yonekura 2019, see also Kobayashi-Yonekura 2021]

Contents

✓ 1. Introduction

We consider the index in terms of massive Dirac operator with interface.

✓ 2. APS index and bulk-boundary correspondence of anomaly (review)

The original definition of the APS index is nonlocal and not natural without “outside”.

3. Massive Dirac operator without interface and AS index (as a warm-up)

4. Massive Dirac operator with interface and APS index

5. Lattice formulation of APS index

6. Summary

Eigenvalues of massive Dirac operator

on a Euclidean 2n-dimensional closed manifold X .

$$H(m) = \gamma_5(D_X + m) \quad D_X := \gamma^\mu(\partial_\mu + iA_\mu)$$

For $D_X\phi = 0,\quad H(m)\phi = \gamma_5 m\phi = \underbrace{\pm}_{\text{chirality}} m\phi.$

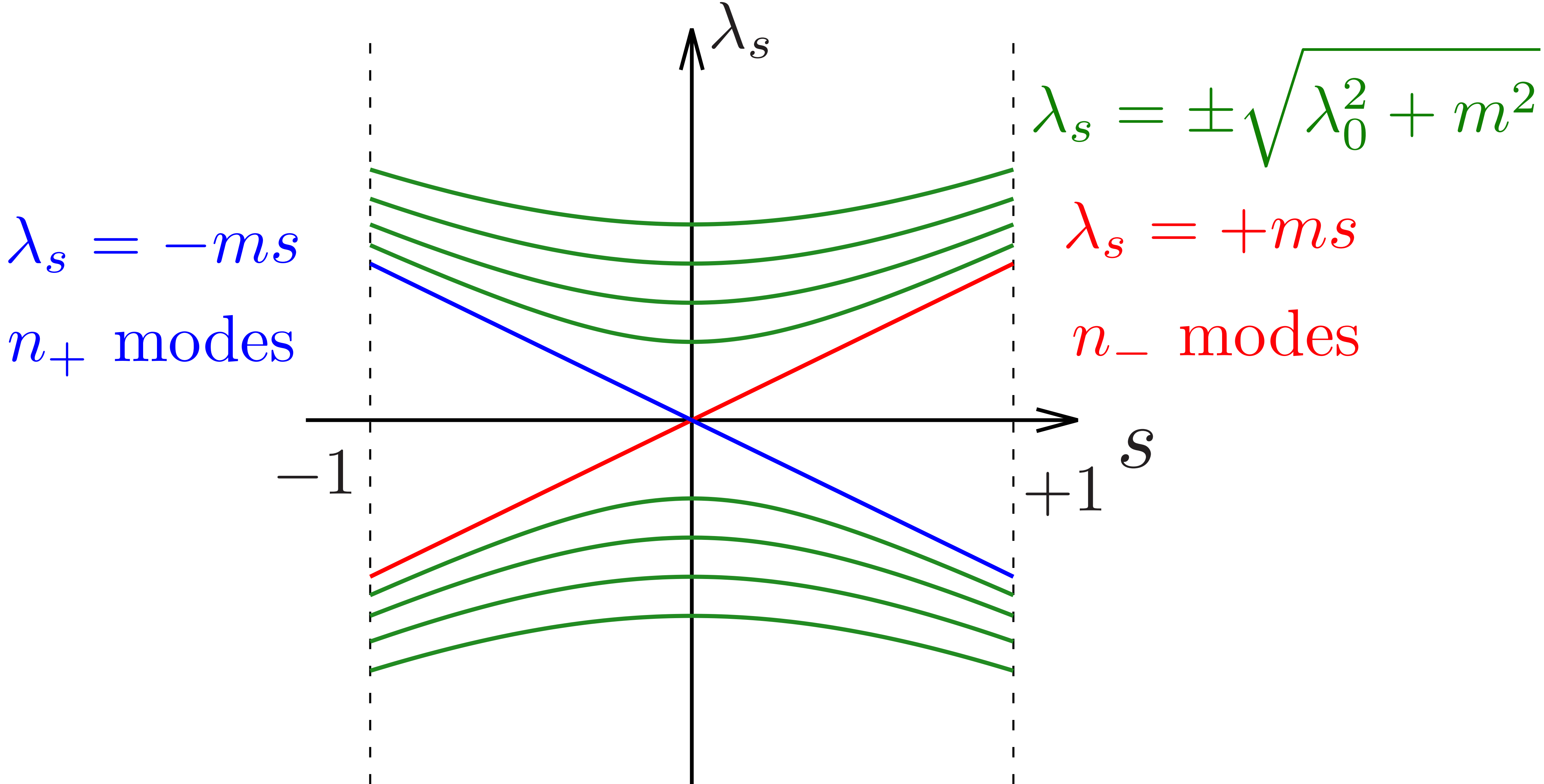
For $D_X\phi \neq 0,\quad \{H(m), D_X\} = 0.$

The eigenvalues are paired: $H(m)\phi_{\lambda_m} = \lambda_m\phi_{\lambda_m}$

$$\Longrightarrow \quad H(m)D_X\phi_{\lambda_m} = -\lambda_m D_X\phi_{\lambda_m}$$

As $H(m)^2 = -D_X^2 + m^2$, we can write them $\lambda_m = \pm\sqrt{\lambda_0^2 + m^2}$

Spectrum of $H_s = \gamma_5(D_X - ms)$ $\quad s \in [-1, 1]$



Spectral flow = Atiyah-Singer index = η invariant

n_+ = # of zero-crossing eigenvalues from + to -

n_- = # of zero-crossing eigenvalues from - to +

$$H_s = \gamma_5(D_X - ms)$$

$$\mathrm{Ind}[D_X] = \quad n_+ - n_-$$ =: spectral flow of H_s $\quad s \in [-1, 1]$

This is also equivalent to the η invariant:

whenever an eigenvalue crosses zero,

$\eta(H_s)$ jumps by two.

$$\Rightarrow \quad n_+ - n_- = -\frac{1}{2}\eta(H_{+1}) + \frac{1}{2}\eta(H_{-1})$$

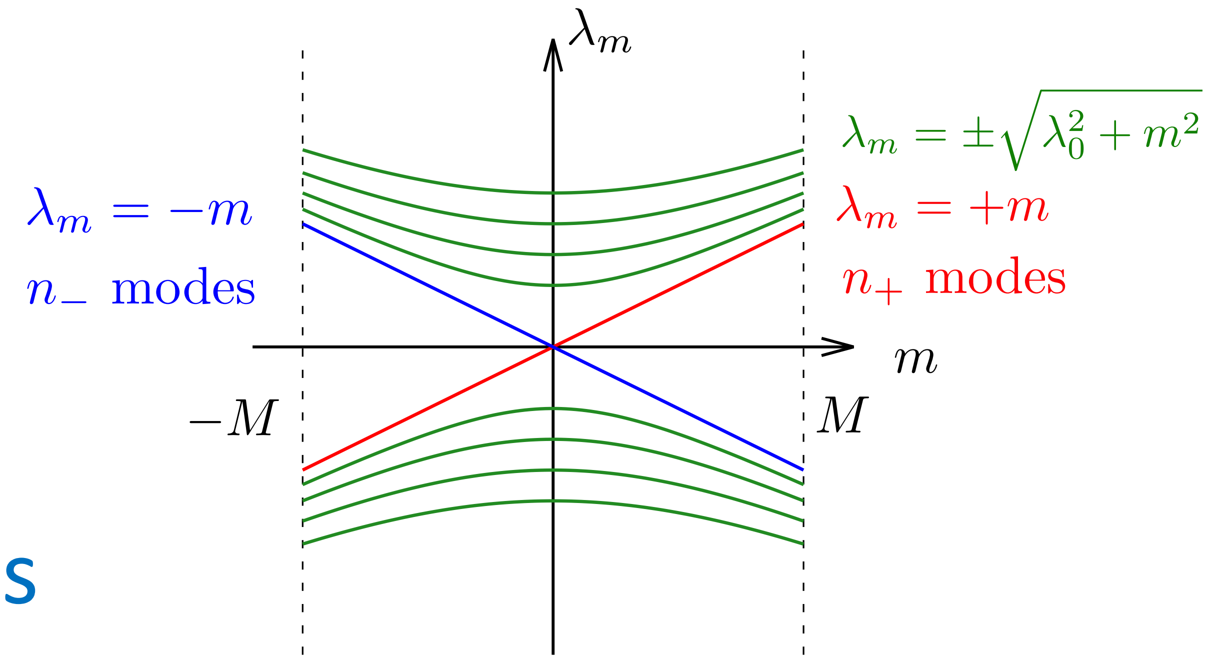
$$\eta(H) = \sum_{\lambda \geq 0}^{reg} - \sum_{\lambda < 0}^{reg}$$

Pauli-Villars subtraction

NOTE: η invariant is an even integer on even-dimensional oriented manifolds.

Suspension isomorphism in K-theory

Massless: counting index by points



Massive: counting index by lines

$$K^0(\text{point}) \cong K^1(I, \partial I)$$

point — with chirality operator

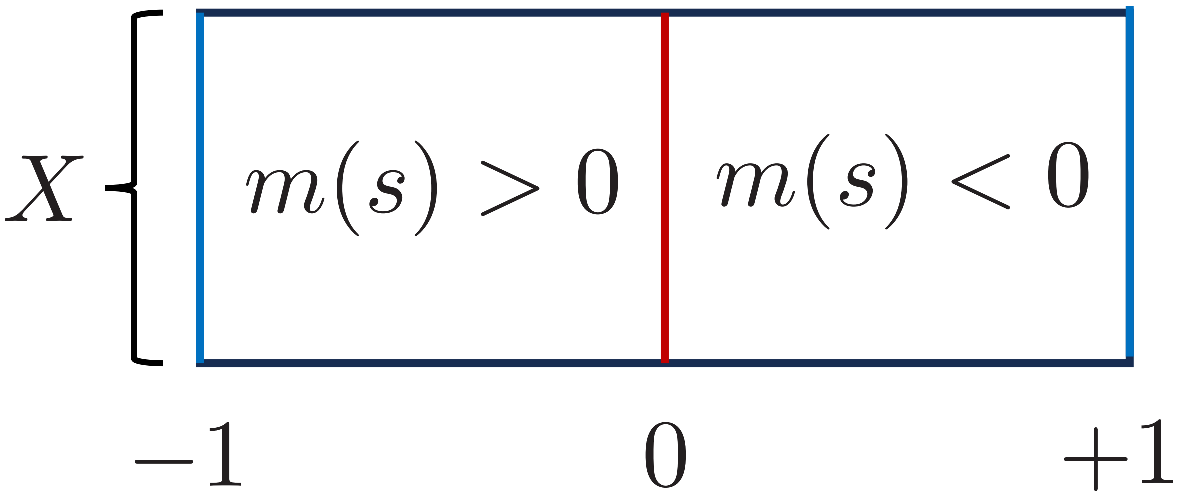
line=interval — without chirality operator

⇨ The two formulations of the index agree.

Proof in 2n+1 dimensions

$$H_s = \gamma_5(D_X - ms)$$

It is interesting to interpret s as an extra dimension:
$X \times [-1, 1]$ has 2 boundaries (s=$\pm$1) and 1 interface (s=0).



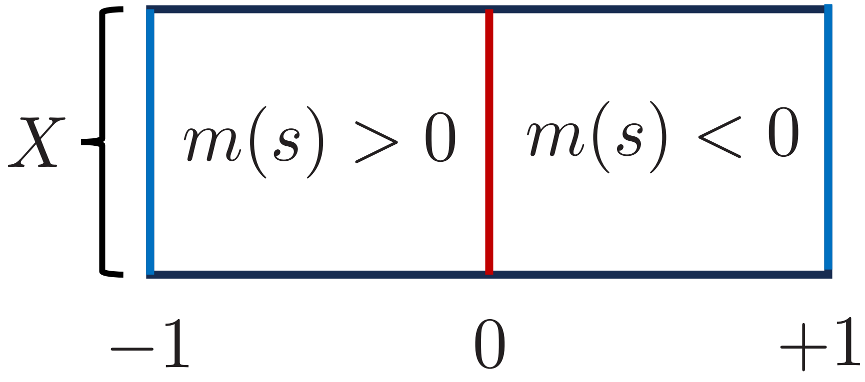
We consider a higher dim.
Dirac operator

$$D_{X\times[-1,1]} = \begin{pmatrix} 0 & \partial_s + H_s \\ -\partial_s + H_s & 0 \end{pmatrix}$$

Proof in 2n+1 dimensions

$$D_{X\times[-1,1]} = \begin{pmatrix} 0 & \partial_s + H_s \\ -\partial_s + H_s & 0 \end{pmatrix}$$

$$H_s = \gamma_5(D_X - ms)$$



Then APS index theorem

(with curvature term=0 in odd dim.) tells

$$\mathrm{Ind}_{\mathrm{APS}}[D_{X\times[-1,1]}] = -\frac{1}{2}\eta(H_{+1}) + \frac{1}{2}\eta(H_{-1}) = \mathrm{sf}[H_s]$$

Also, the zero modes are localized at the interface and essentially determined by D_X : $\mathrm{Ind}_{\mathrm{APS}}[D_{X\times[-1,1]}] = \mathrm{Ind}[D_X]$

These two lead to $\mathrm{Ind}_{\mathrm{APS}}[D_{X\times[-1,1]}] = \mathrm{Ind}[D_X] = \mathrm{sf}[H_s]$

massless massive

Contents

✓ 1. Introduction

We consider the index in terms of massive Dirac operator with interface.

✓ 2. APS index and bulk-boundary correspondence of anomaly (review)

The original definition of the APS index is nonlocal and not natural without “outside”.

✓ 3. Massive Dirac operator without interface and AS index (as a warm-up)

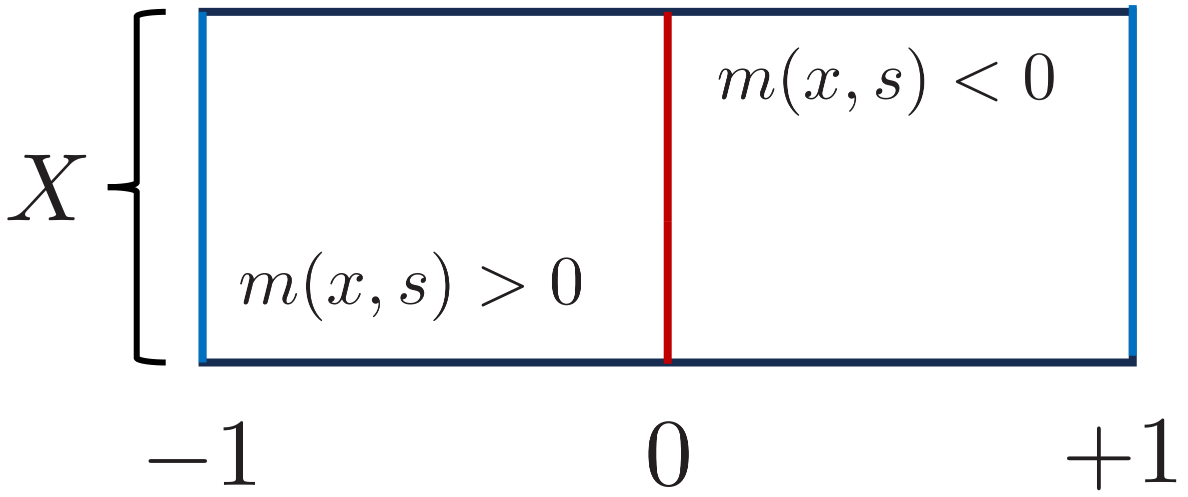
The spectral flow (= η invariant) of massive Dirac operator = AS index.

4. Massive Dirac operator with interface and APS index

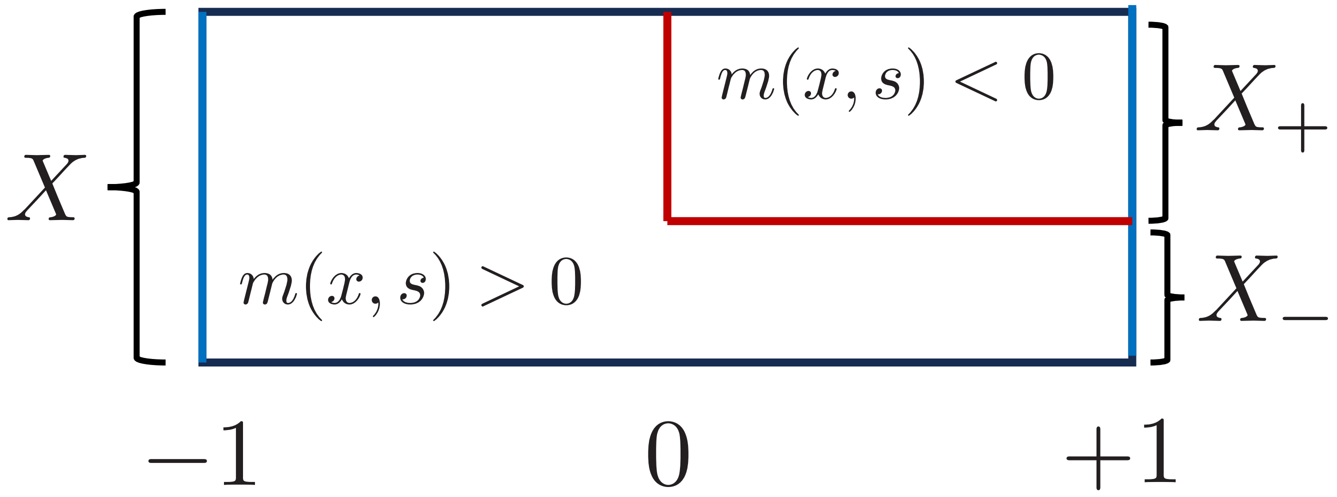
5. Lattice formulation of APS index

6. Summary

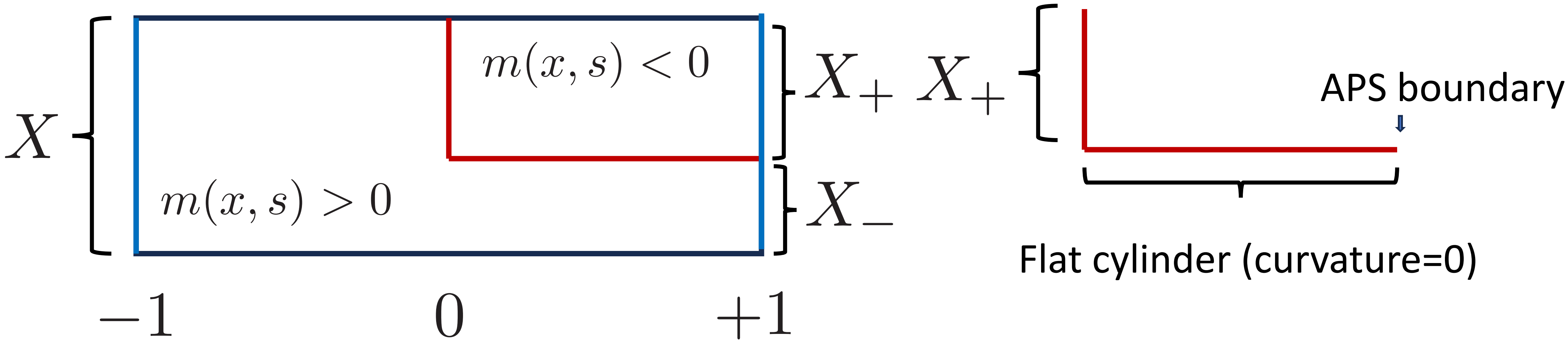
Deforming interface in 2n+1 dimensions



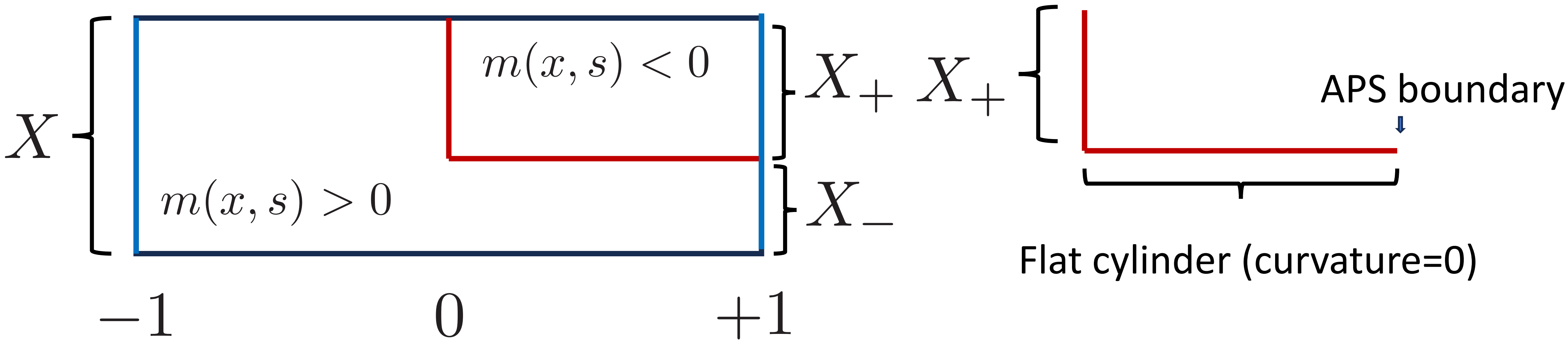
Deforming interface in 2n+1 dimensions



Deforming interface in 2n+1 dimensions



Deforming interface in 2n+1 dimensions



$$\Rightarrow \mathrm{Ind}_{\mathrm{APS}}[D_{X\times[-1,1]}] = \mathrm{Ind}_{\mathrm{APS}}[D_{X_+}] = \mathrm{sf}[H_s]$$

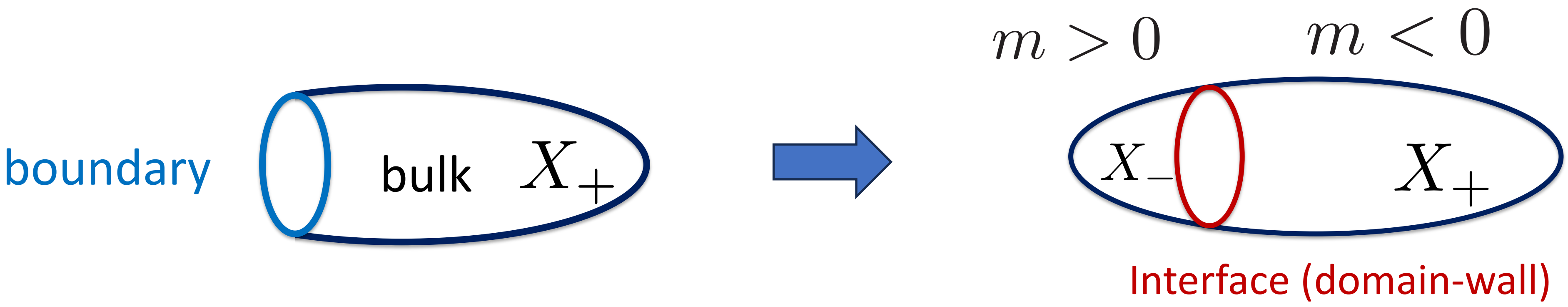
where $H_{+1} = \gamma_5(D_X - m\epsilon(x))$ is a domain-wall Dirac operator with an interface in the mass:

$$\epsilon(x) = \begin{cases} +1 & \text{for } x \in X_+ \\ -1 & \text{for } x \in X_- \end{cases}$$

Theorem 1

[F-Furuta-Matsuo-Onogi-Yamaguchi-Yamashita 2019]

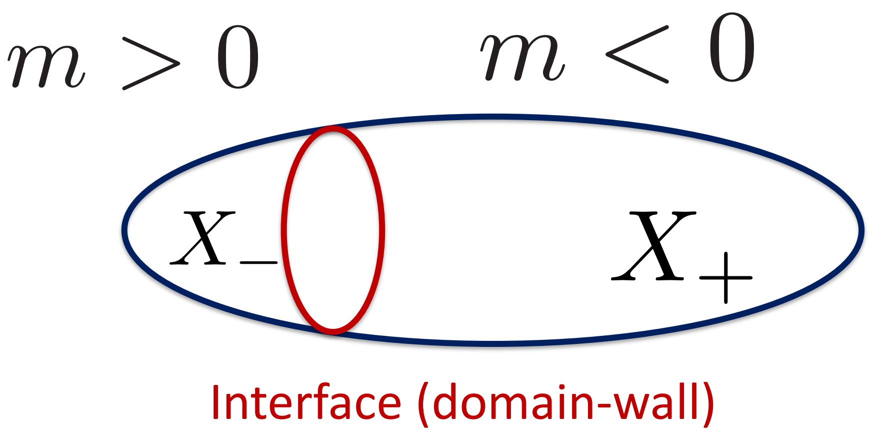
1910.01987



For any APS index of a massless Dirac operator on an even-dimensional Riemannian manifold X_+ with boundary, there exists a massive Dirac operator on a closed manifold $X = X_+ \cup X_-$ with interface on the location of the original boundary and its spectral flow (or η invariant) is equal to the original index.

$$\mathrm{Ind}_{\mathrm{APS}}[D|_{X_+}] = \mathrm{sf}\left[\gamma\left(D_X - \frac{s+1}{2}m\epsilon - \frac{s-1}{2}m\right)\right] = -\frac{1}{2}\eta^{\mathrm{PV}}(\gamma(D_X - m\epsilon))$$

The massive Dirac operator unifies various types of the index formulas.



- AS and APS indices are unified on a closed manifold. We do not need any nonlocal boundary conditions.
- Application to the mod-two version is straightforward (counting # of pairs of zero-crossing eigenmodes).
- Chiral symmetry is NOT necessary: the lattice gauge theory version is straightforward -> Numerical evaluation is possible (NEXT).

Contents

✓ 1. Introduction

We consider the index in terms of massive Dirac operator with interface.

✓ 2. APS index and bulk-boundary correspondence of anomaly (review)

The original definition of the APS index is nonlocal and not natural without "outside".

✓ 3. Massive Dirac operator without interface and AS index (as a warm-up)

The spectral flow (= η invariant) of massive Dirac operator = AS index.

✓ 4. Massive Dirac operator with interface and APS index

The spectral flow of massive Dirac operator with interface = APS index

5. Lattice formulation of APS index

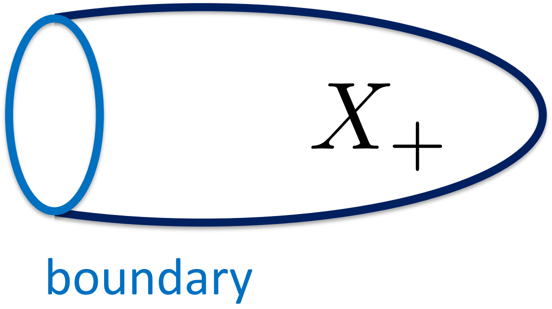
6. Summary

Massive expression is easier on the lattice.

The APS index of massless Dirac operator requires

chiral symmetry

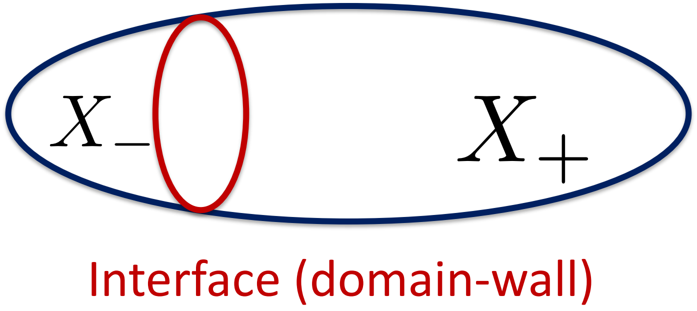
and APS boundary condition.



But the spectral flow of massive Dirac operators on a closed manifold with interface does NOT need

chiral symmetry

and non-local boundary condition.



-> The standard Wilson fermion Dirac operator is good enough.

Wilson Dirac operator on a square lattice

$$D_W = \sum_i \left[\gamma^i \frac{\nabla_i^f + \nabla_i^b}{2} - \frac{a}{2} \nabla_i^f \nabla_i^b \right]$$

Wilson term (is needed to remove doubler modes), which breaks chiral symmetry.

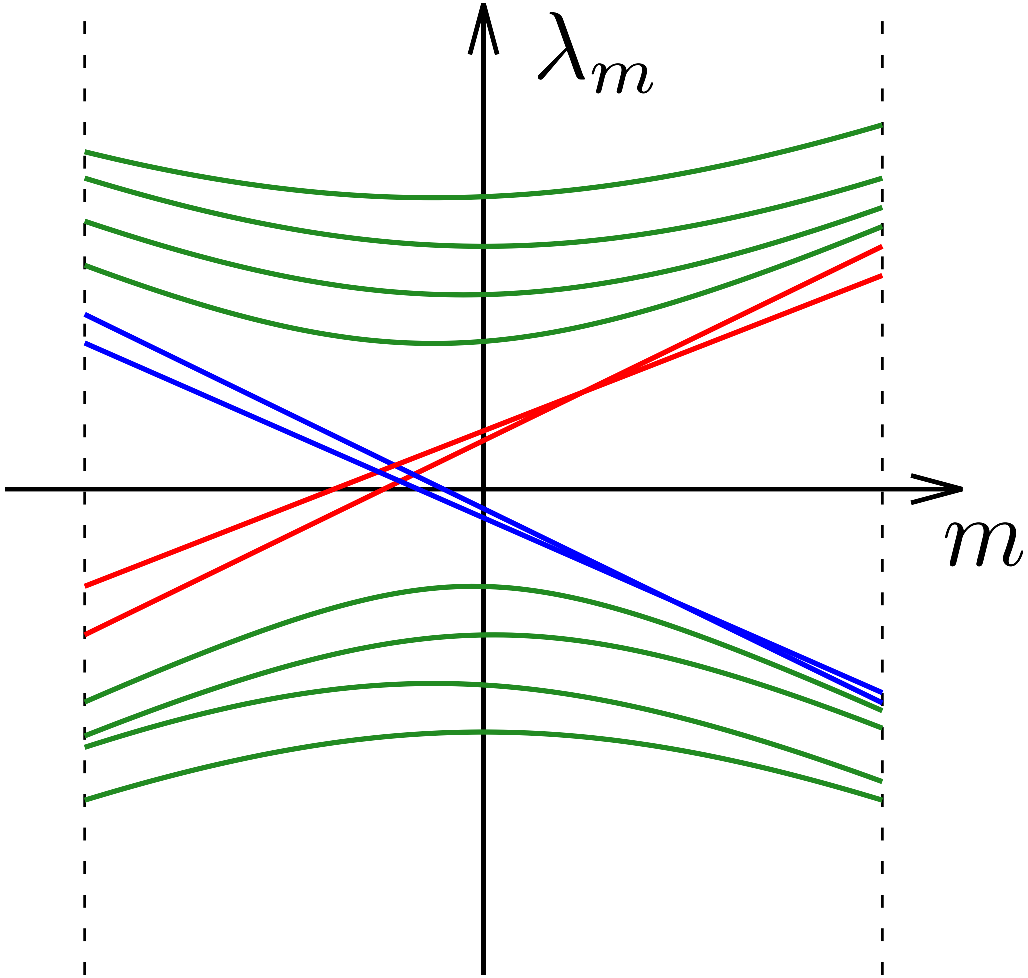
$$a\nabla_i^f \psi(\boldsymbol{x}) = U_i(\boldsymbol{x})\psi(\boldsymbol{x}+\boldsymbol{e}_i) - \psi(\boldsymbol{x})$$

$$a\nabla_i^b \psi(\boldsymbol{x}) = \psi(\boldsymbol{x}) - U_i^\dagger(\boldsymbol{x}-\boldsymbol{e}_i)\psi(\boldsymbol{x}-\boldsymbol{e}_i)$$

* In mathematics, the Wilson term is important in that it guarantees the ellipticity.

Counting lines (massive) is more stable than counting points (massless) against chiral symmetry breaking (on a lattice)

Standard massless definition:
Where is m=0?
What are zero modes?



Spectral flow of massive Dirac:
We can still count the crossing lines.

Note) this fact is known even before overlap Dirac by Itoh-Iwasaki-Yoshie 1982 and other literature, but its mathematical meaning was not discussed. See also Adams, Kikukawa-Yamada, Luescher, Fujikawa, and Suzuki

Theorem 2

[Aoki-Fujita-F-Furuta-Matsuo-Onogi-Yamaguchi 2026]

2602.12576

There exists a finite lattice spacing a_0 such that for any $a < a_0$

$$\mathrm{Ind}_{\mathrm{APS}}[D|_{X_+}] = \mathrm{sf}\left[\gamma\left(D_W - \frac{s+1}{2}m\epsilon - \frac{s-1}{2}m\right)\right] = -\frac{1}{2}\eta(\gamma(D_W - m\epsilon)),$$

continuum — lattice

for sufficiently large m holds, where

$$\epsilon(x) = \begin{cases} +1 & \text{for } x \in X_+ \\ -1 & \text{for } x \in X_- \end{cases}$$

NOTE :

1. The total space is limited to a flat torus: $T^{2n} = X_+ \cup X_-$, although the interface Y can be any 2n-1 dimensional curved submanifold.
2. $ma < 2$ must be satisfied to avoid doublers [Cf. Yamaguchi's talk].
3. η invariant is an even integer on even-dimensional oriented manifolds.

Theorem 3

[Aoki-Fujita-F-Furuta-Matsuo-Onogi-Yamaguchi 2026]

2602.12576

When the Dirac operator is real, there exists a finite lattice spacing a_0 such that for any $a < a_0$

$$\mathrm{Ind}^{\mathrm{mod}-2}_{\mathrm{APS}}[D|_{X_+}] = \mathrm{sf}^{\mathrm{mod}-2}\left[(H_W)_s\right]_{\text{lattice}}$$

continuum

$$(H_W)_s = i\tau_2 \otimes D_W - \tau_1 \otimes \left(\frac{s+1}{2}m\epsilon + \frac{s-1}{2}m\right),$$

for sufficiently large m holds, where

$$\epsilon(x) = \begin{cases} +1 & \text{for } x \in X_+ \\ -1 & \text{for } x \in X_- \end{cases}$$

NOTE : mod-two spectral flow counts # of zero-crossing "pairs".

Sketch of the proof by K-theory

We consider a continuum-lattice combined Dirac operator

$$\hat{D}_{s,t} = \begin{pmatrix} \gamma(D_{\mathrm{cont.}} + m(x,s)) & t f_a \\ t f_a^* & -\gamma(D_W + m(x,s)) \end{pmatrix}$$

and show it is invertible [which is a sufficient condition for Spec.flow=0] by contradiction.

* Then the Wilson Dirac operator gives a well-defined element in K-theory.

⇨ $\gamma(D_{\mathrm{cont.}} + m(x,s)),\ \ \gamma(D_W + m(x,s))$

have the same spectral flow.

Numerical test on a 2D disk

[Cf. Aoki-F, 2022,2023]

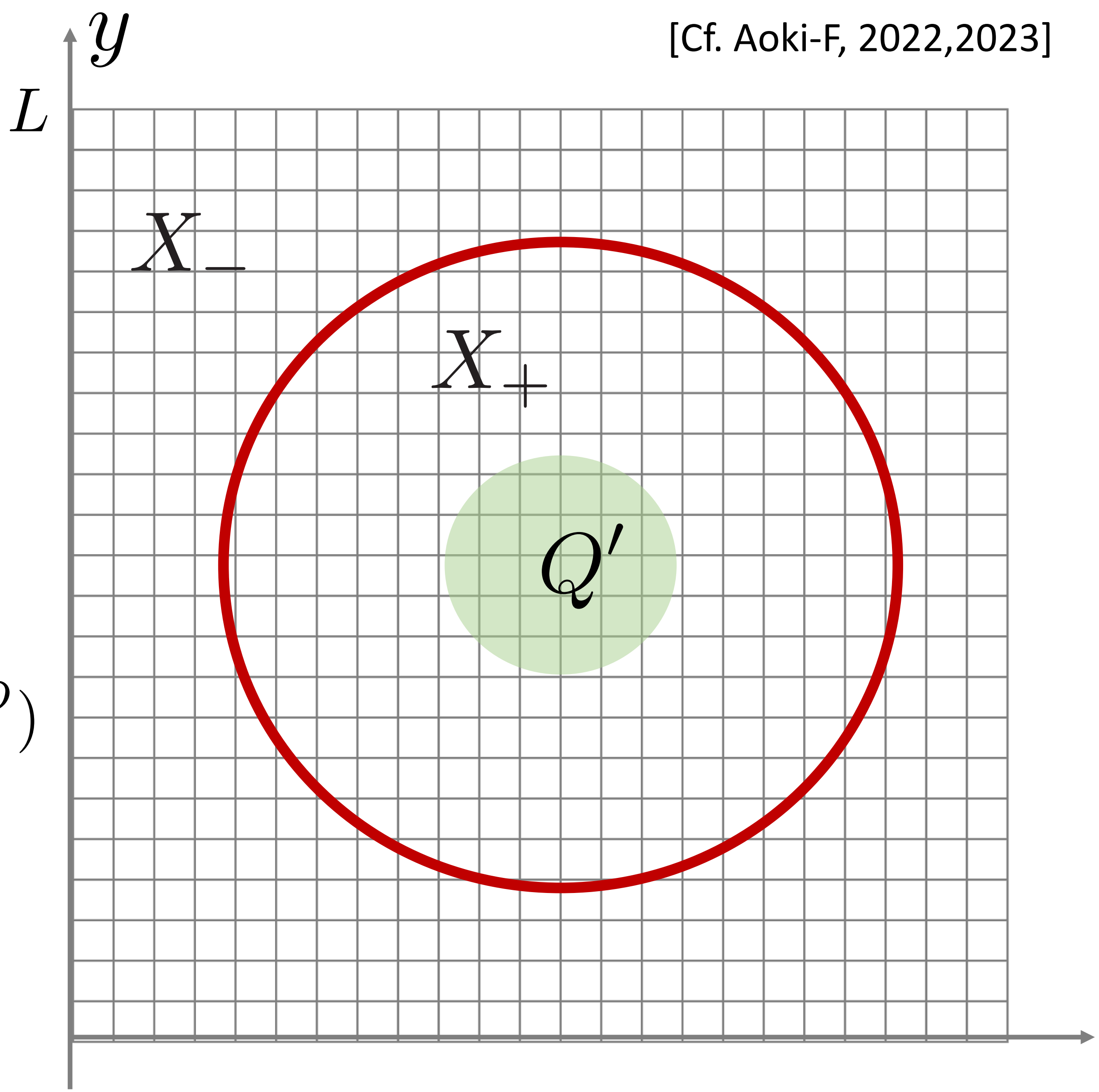
We put a circular curved domain-wall : m=-s/a inside, m=+1/a outside and change s from -1 to 1.

We put U(1) flux Q' and numerically check if the APS index theorem

$$-\frac{1}{2}\eta(\gamma_5 D_{DW}) = \underbrace{\frac{1}{2\pi}\int F}_{=Q'} - \frac{1}{2}\eta(iD^{1D})$$

holds or not.

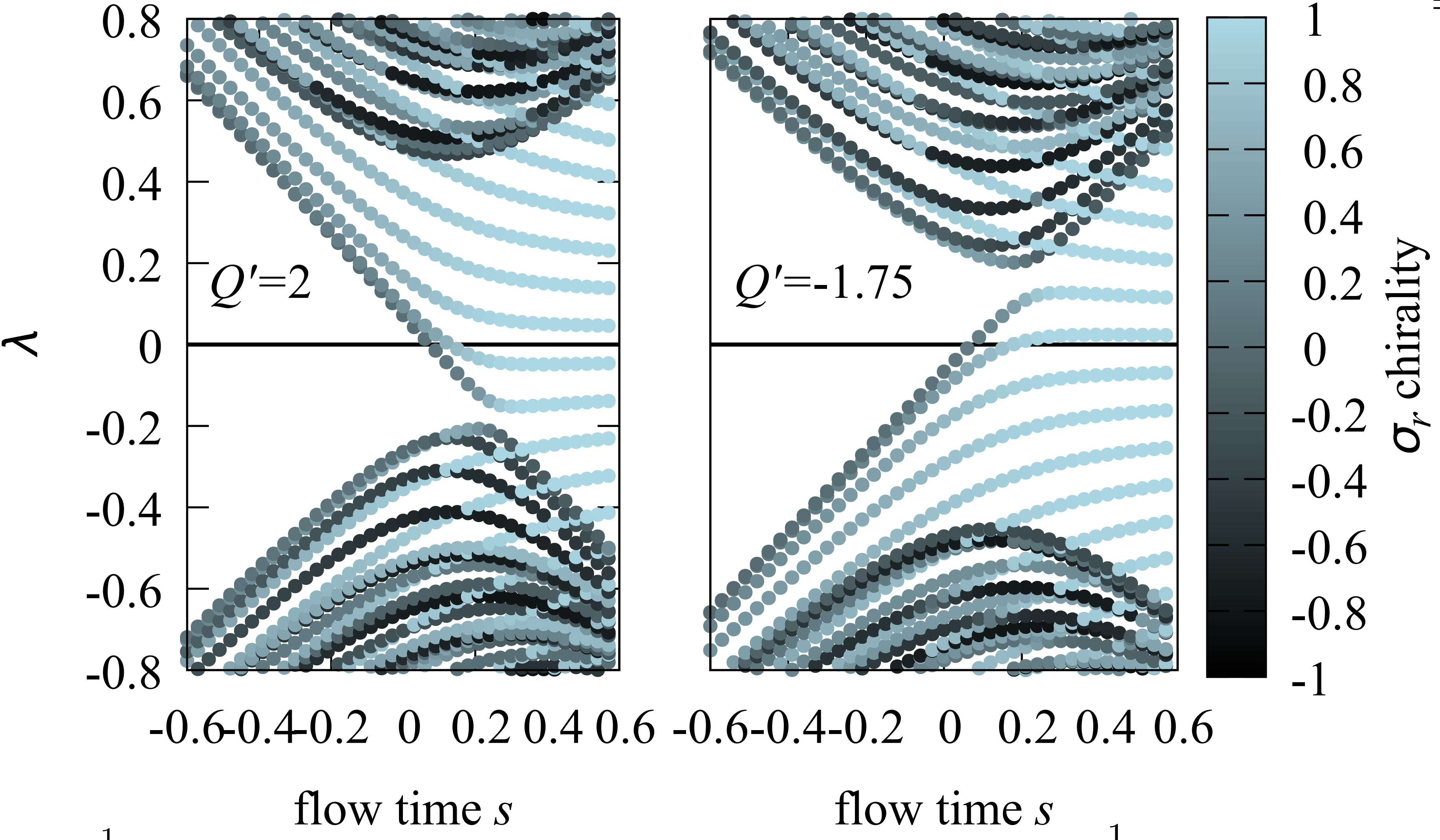
L=33, DW radius=10, flux radius=6.



APS index on a 2D disk

$$-\frac{1}{2}\eta(\gamma_5 D_{DW}) = \underbrace{\frac{1}{2\pi}\int F}_{=Q'} - \frac{1}{2}\eta(iD^{1D})$$

$$\eta(H) = \sum_{\lambda\geq 0}^{reg} - \sum_{\lambda<0}^{reg}$$



$-\frac{1}{2}\eta(iD^{1D}) = 0.$

$-\frac{1}{2}\eta(iD^{1D}) = -0.25$

Edge-localized chiral : $\sigma_r = (\sigma_1 x + \sigma_2 y)/r \sim 1$ modes appear on the 1-dimensional circle domain-wall = the source of boundary eta invariant.

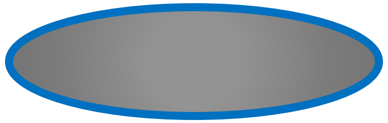
Consistent with the APS Index theorem.

Numerial test for Majorana S^1 domain-wall fermion

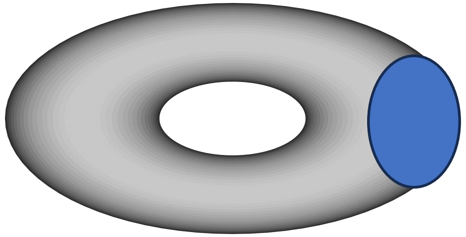
Free Wilson Dirac operator is real:

$$iH_m = \sigma_1 \partial_x + \sigma_3 \partial_y + i\sigma_2 (W + m(x,s))$$

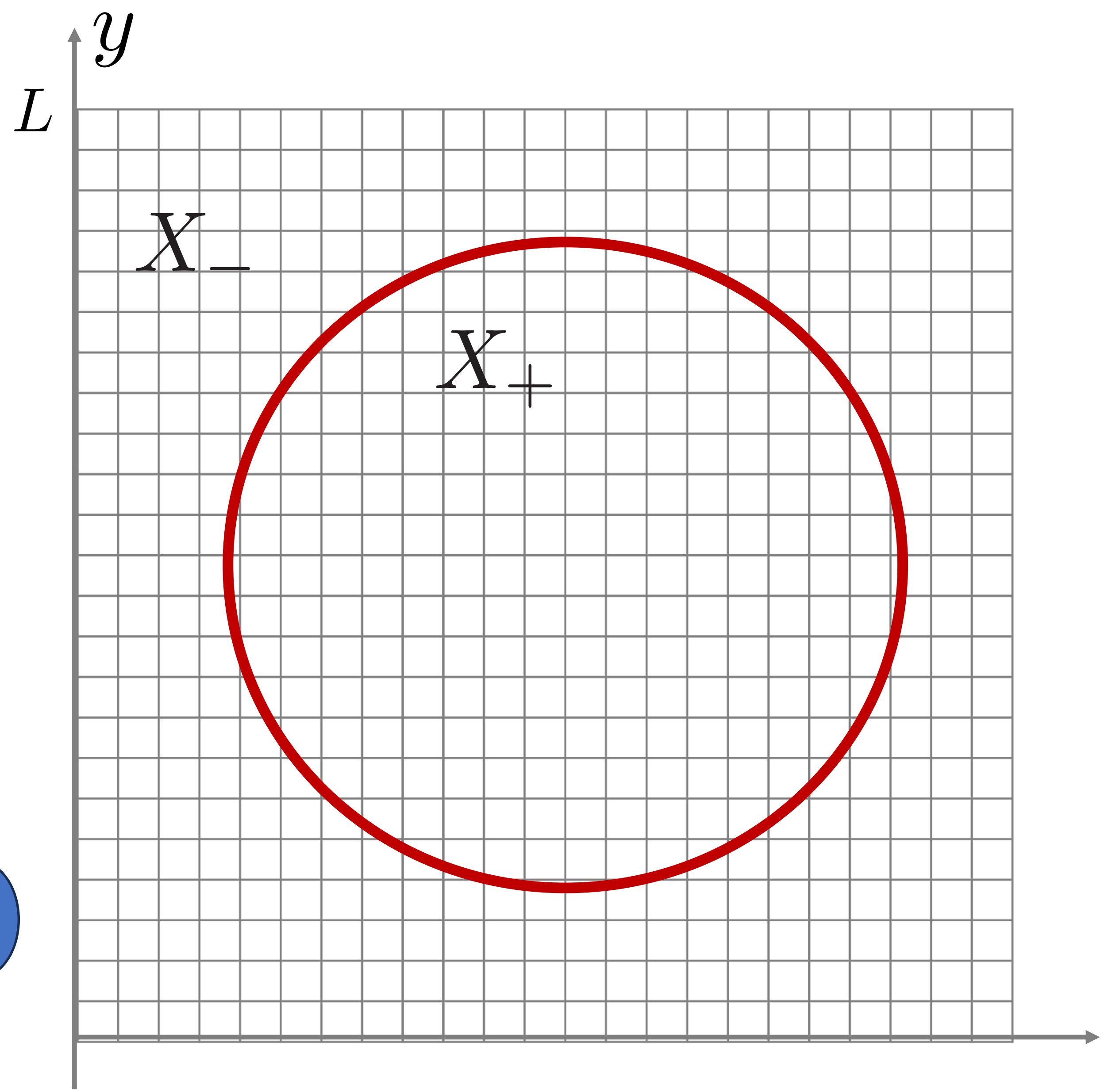
m(x,s)=-s/a inside the domain-wall
=disk



m(x,s)=-s/a outside the domain-wall
= torus with a S^1 hole.



The continuum mod-two APS index = 0 and 1 respectively.

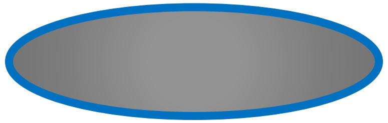


Numerial test for Majorana S^1 domain-wall fermion

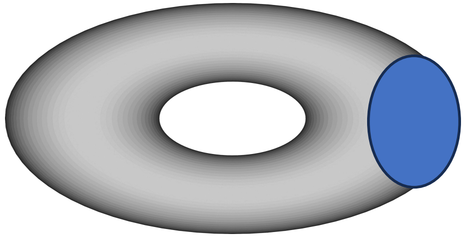
Free Wilson Dirac operator is real:

$$iH_m = \sigma_1\partial_x + \sigma_3\partial_y + i\sigma_2(W + m(x,s))$$

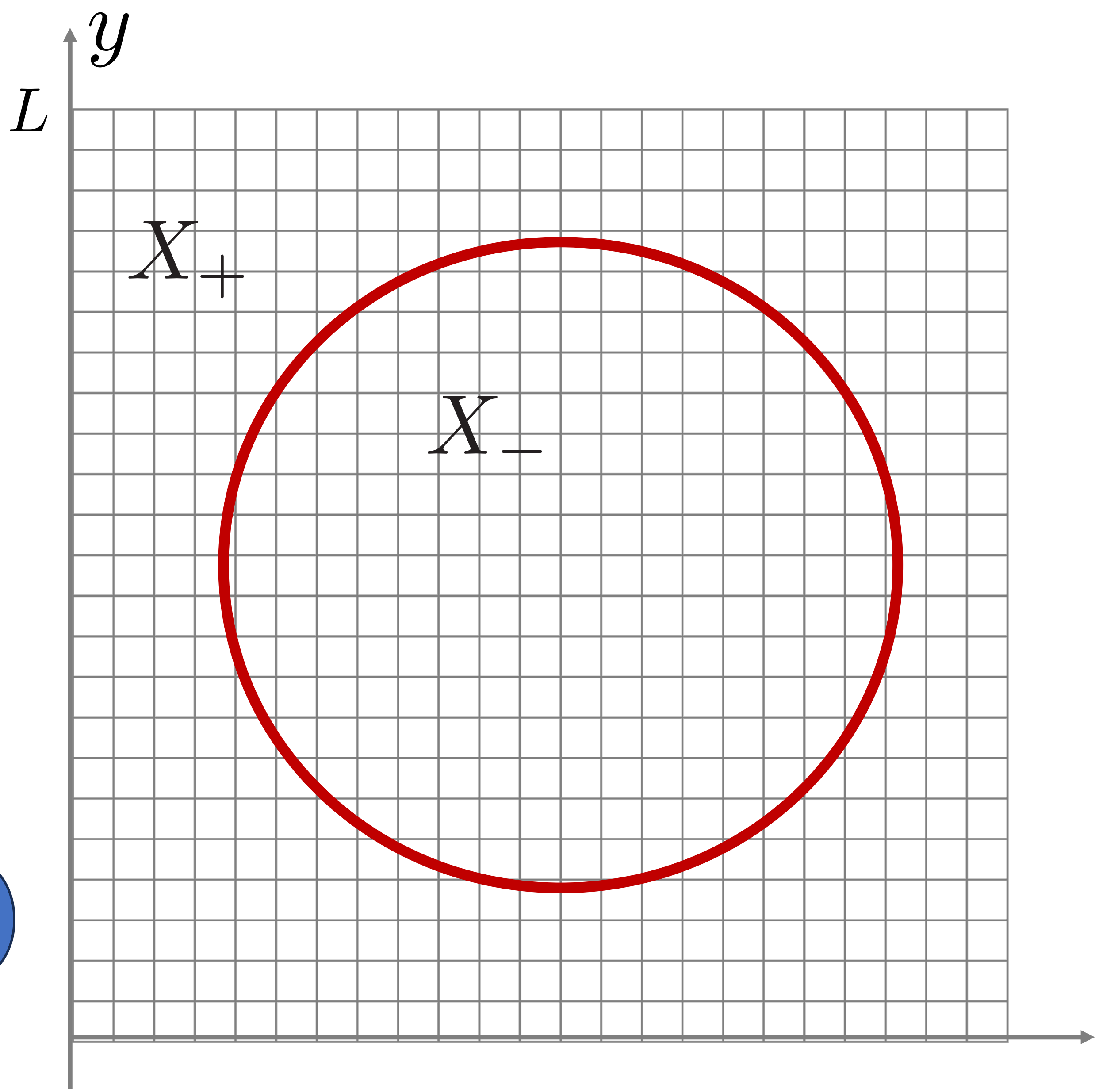
m(x,s)=-s/a inside the domain-wall =disk



m(x,s)=-s/a outside the domain-wall = torus with a S^1 hole.

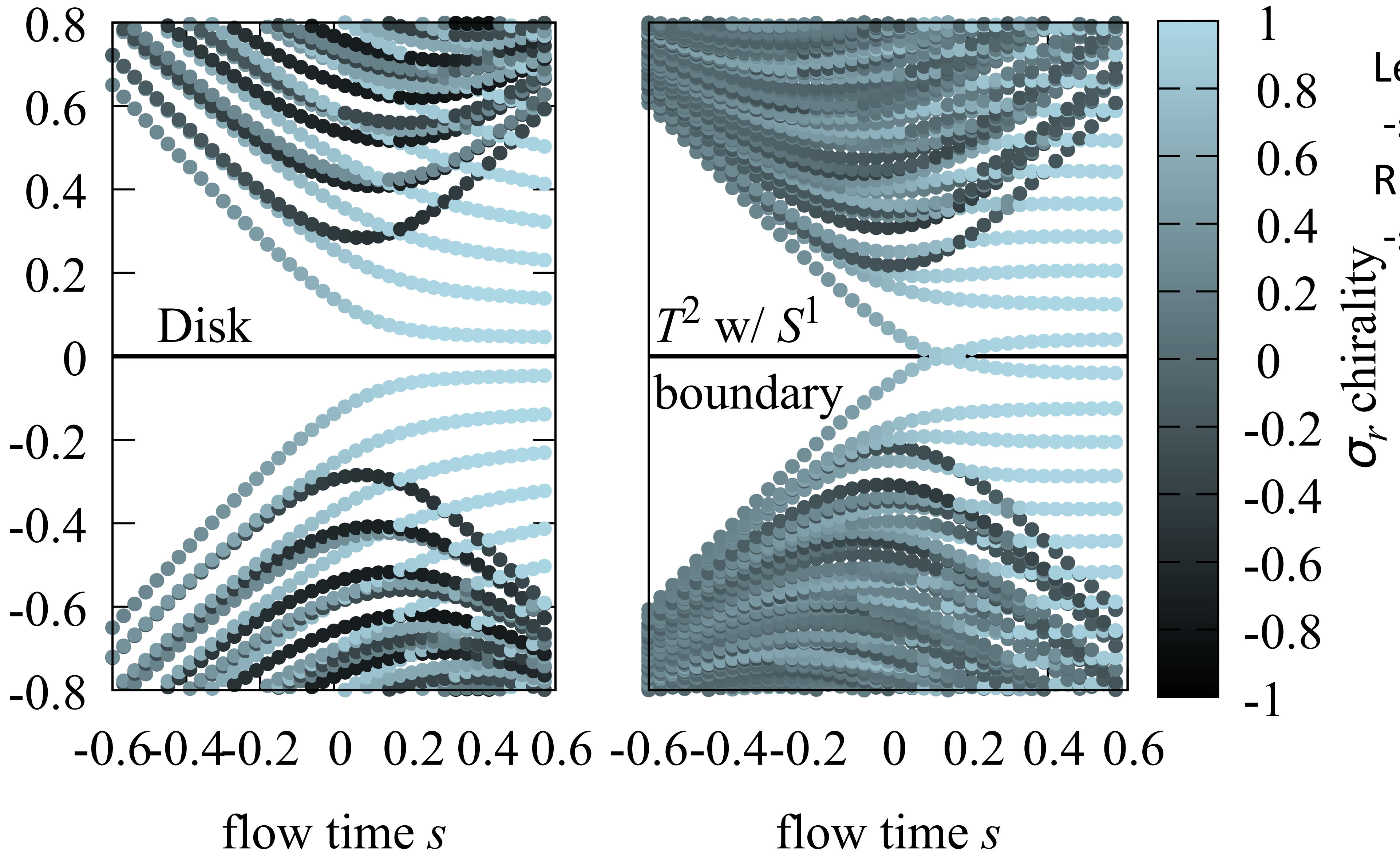


The continuum mod-two APS index = 0 and 1 respectively.



Majorana Dirac spectrum

$$iH_m = \sigma_1 \partial_x + \sigma_3 \partial_y + i\sigma_2 m(s,r)$$



Left panel :
-sM inside S^1 DW
Right :
-sM outside S^1 DW

mod-two spectral flow agrees with the mod-two APS index.

Contents

✓ 1. Introduction
We consider the index in terms of massive Dirac operator with interface.

✓ 2. APS index and bulk-boundary correspondence of anomaly (review)
The original definition of the APS index is nonlocal and not natural without “outside”.

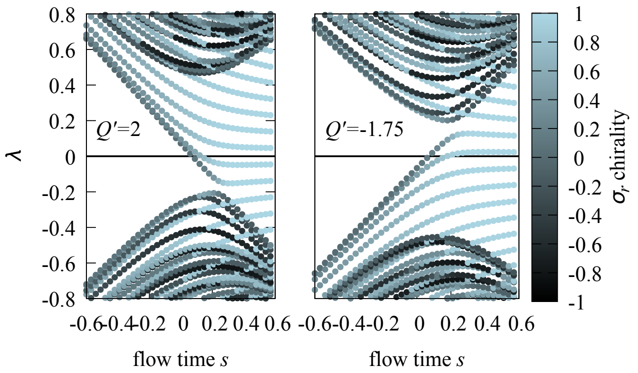
✓ 3. Massive Dirac operator without interface and AS index (as a warm-up)
The spectral flow (= η invariant) of massive Dirac operator = AS index.

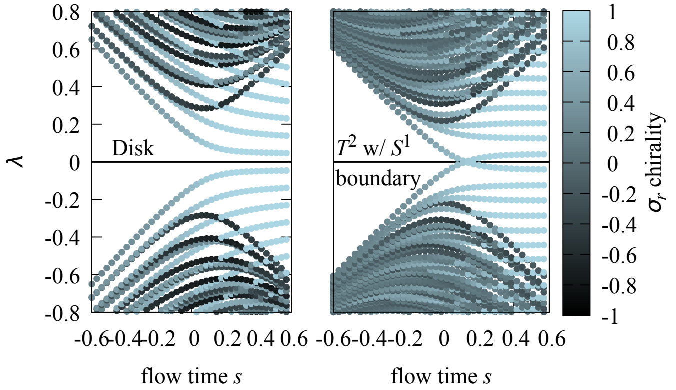
✓ 4. Massive Dirac operator with interface and APS index
The spectral flow of massive Dirac operator with interface = APS index

✓ 5. Lattice formulation of APS index
is given by the spectral flow of the massive Wilson Dirac operator with interfaces

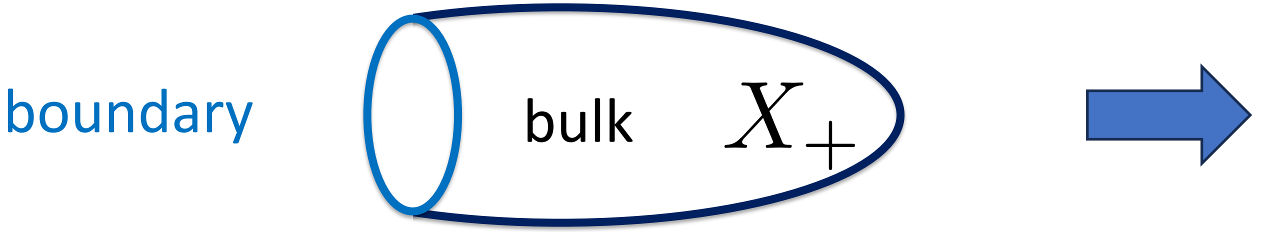
6. Summary

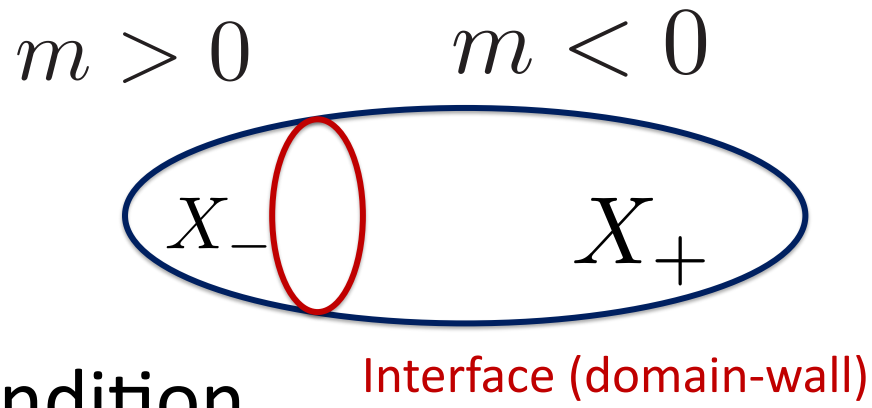
Summary





The massive Dirac operator with/without interfaces unifies various index formulas.





- We can avoid nonlocal APS boundary condition.
- AS and APS indices are unified on a closed manifold.
- The mod-two version is also unified (just counting crossing pairs).
- Chiral symmetry is NOT necessary: the lattice gauge theory version is straightforward → numerical evaluation is possible.

A final remark

As far as we know the anomaly inflow was first discussed by Callan-Harvey 1984 with interfaces (not on a manifold with boundary).

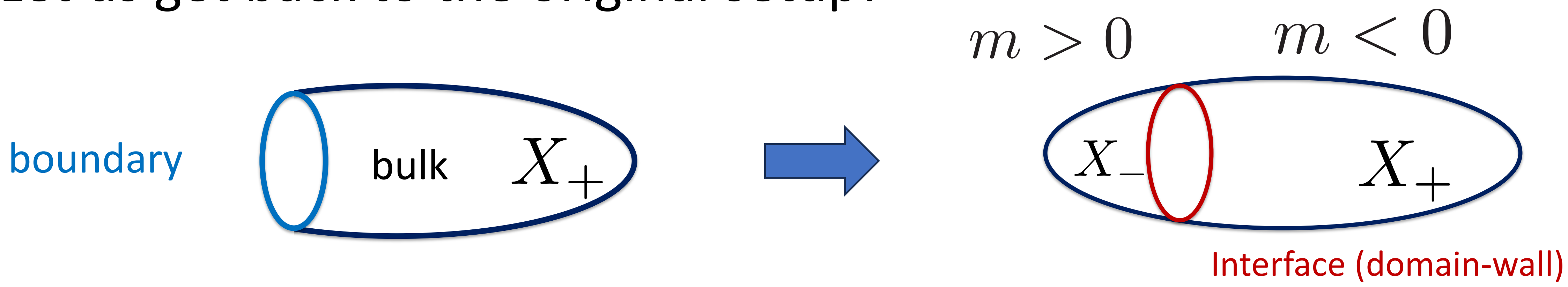
ANOMALIES AND FERMION ZERO MODES ON STRINGS AND DOMAIN WALLS

C.G. CALLAN, Jr.[1] and J. A. HARVEY[2]

Jadwin Hall, Princeton University, Princeton, New Jersey 08544, USA

Received 10 September 1984

Let us get back to the original setup!



Outlook

More nontrivial invariants (Z_8, Z_{16}, η…). [S. Yamaguchi's talk yesterday, based on Araki, F, Onogi, Yamaguchi 2025]. So far VERY GOOD numerical results but rigorous mathematical proof is missing.

Lattice version is limited to the case flat bulk + curved domain-wall.
-> curved bulk and curved domain-wall may be realized by higher co-dimensional junctions of interfaces?