Impurities with a Cusp:
From Casimir Forces to Sudakov Radiation

Gabriel Cuomo

YUKAWA INSTITUTE FOR THEORETICAL PHYSICS

Interfaces and symmetry, Kyoto, 4th March 2026

ma per seguir virtute e canoscenza

SISSA

Based on:

- *2406.10186* with ***Yin-Chen He*** and ***Zohar Komargodski***
- *work in progress* with ***Simone Giombi*** and ***Luigi Tizzano***

Setup and motivation

Line defects in physics

Line defects appear in many physical systems. An incomplete list includes:

- 2d: boundaries and interfaces in critical systems (e.g. Kondo problem), …

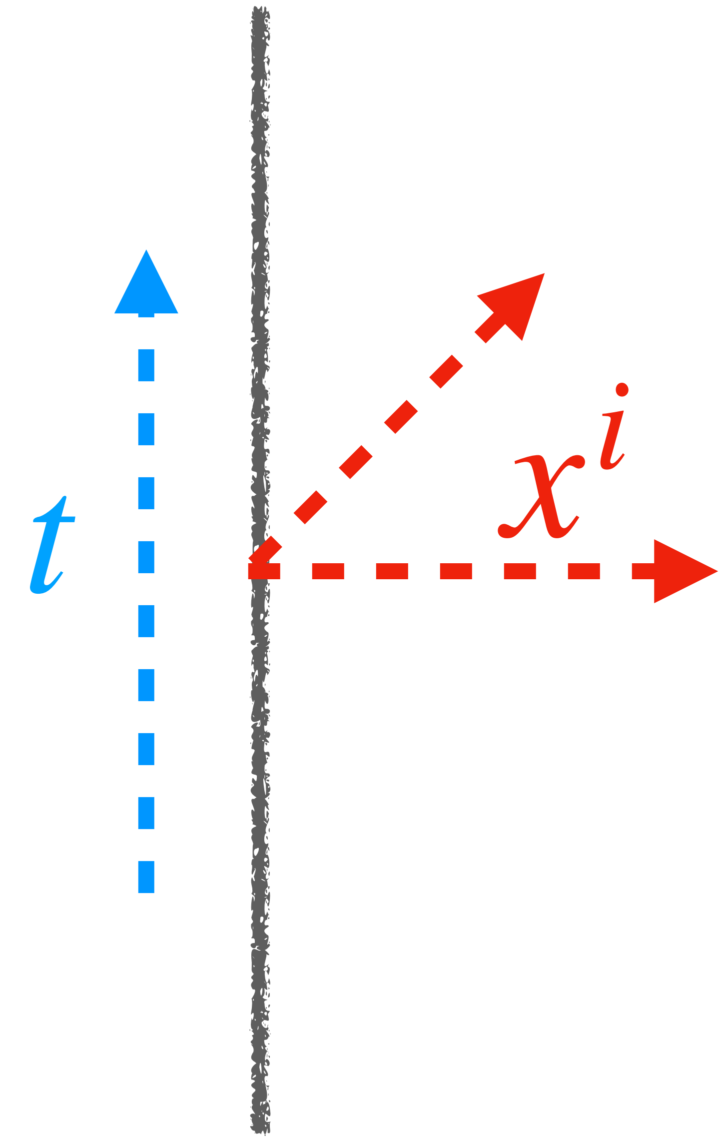
  Wilson 1975, Tsevelick Wiegmann 1985, Cardy 1989, Affleck 1995…

- 3d: impurities, symmetry defects, anyons, …

  Sachdev Buragohain Vojta 1999, Billó Caselle Gaiotto Gliozzi Meineri Pellegrini 2013…

- 4d: Wilson and 't Hooft lines, …

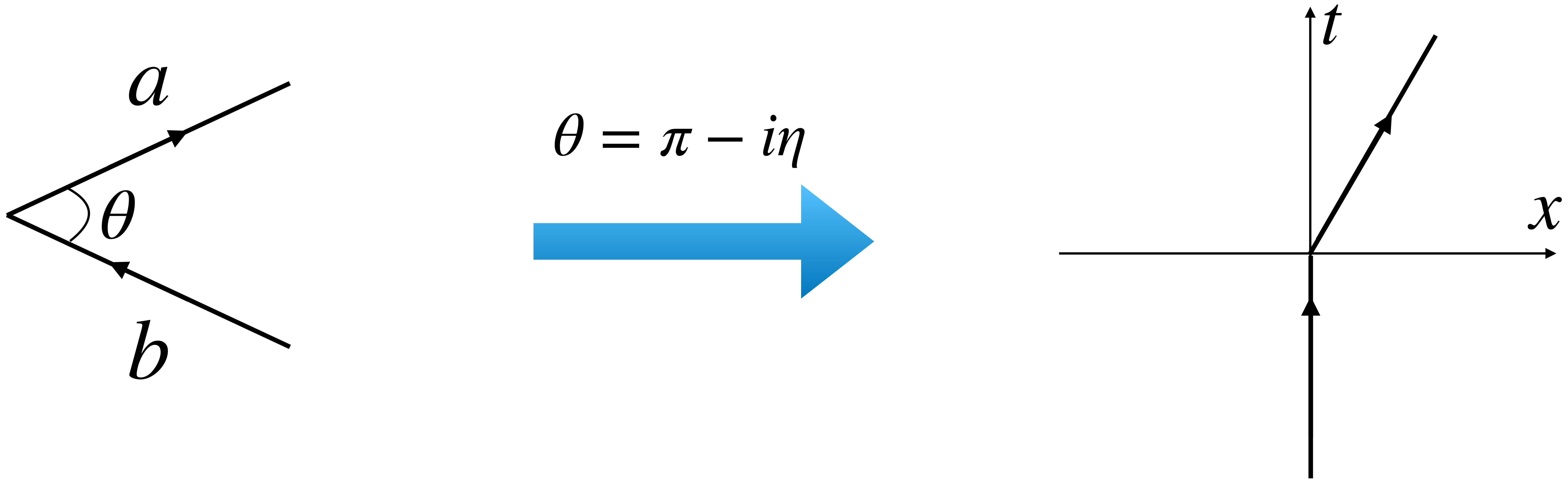
  Wilson 1974, 't Hooft 1978,…



Many interesting examples can be studied within the framework of Quantum Field Theory (QFT) (especially in AdS) and Renormalization Group (RG).

Cusped line defects

A natural configuration involves line defects a and b meeting at a cusp angle

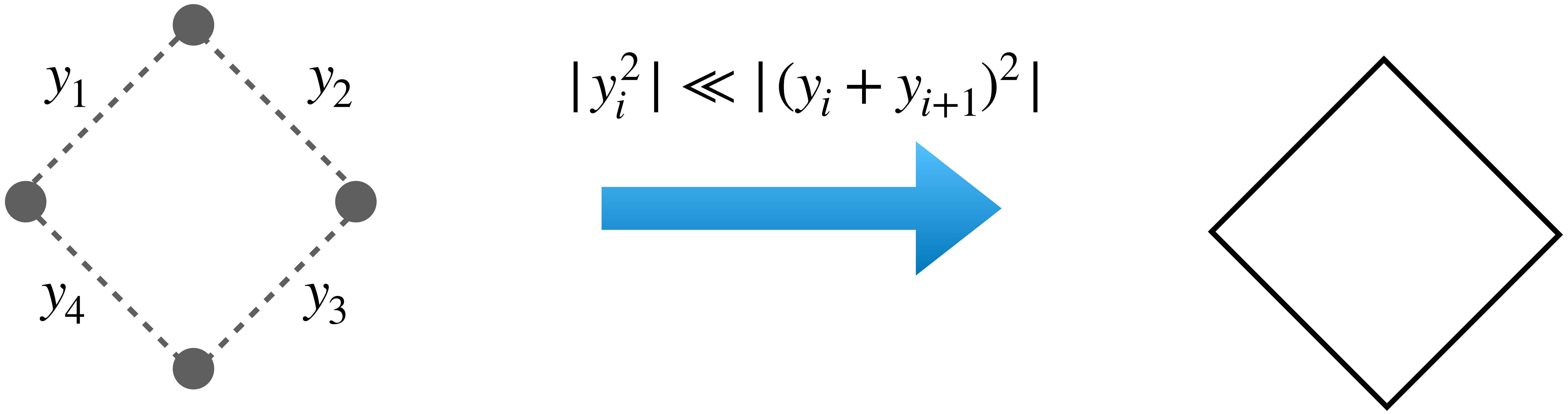


Analytic continuation to complex θ desribes a boost in real time.

Cusped line defects have several applications:

- Connection with interesting Euclidean observables: Casimir energy, fusion, etc.
- Applications to entanglement and Renyi entropies
- Local quench in real time
- Related to amplitudes and correlators in (perturbative?) gauge theories

$$\langle J(x_1)J(x_2)J(x_3)J(x_4)\rangle \xrightarrow{x^2_{ii+1} \ll x^2_{ii+2}} \langle W_{adj}(x_1,x_2)W_{adj}(x_2,x_3)W_{adj}(x_3,x_4)W_{adj}(x_4,x_1)\rangle$$



Alday Eden Korchemsky Maldacena Sokatchev 2010, …, Chen Monni Pang Vita Zhu 2025

Goal: identify some general properties of the CFT partition function in the presence of cusped lines (cusp anomalous dimension)

$$\log Z_{ab}(\theta) = -\Gamma_{ab}(\theta) \log \frac{L_{IR}}{a_{UV}} + \text{finite}$$

Polyakov 1980

Key results

- Small θ limit describes fusion of (nontopological) impurities
- $\Gamma_{a\bar{a}}(\theta)$ is concave $\Longrightarrow$ negative Casimir energy, i.e. "opposites attract"
- Large boost angle (generically): *quasi-free* Effective Field Theory (EFT)

Plan of the talk

1. RG flows on line defects
   A. Review of defect RG flows
   B. Example: pinning field defect
2. The Cusp in Euclidean signature
   A. General properties
   B. Results in the Ising model
3. Lorentzian Cusp
   A. Analytic continuation
   B. Large boost angle limit
4. Summary and outlook

RG flows on line defects

Extended operators as defect QFTs

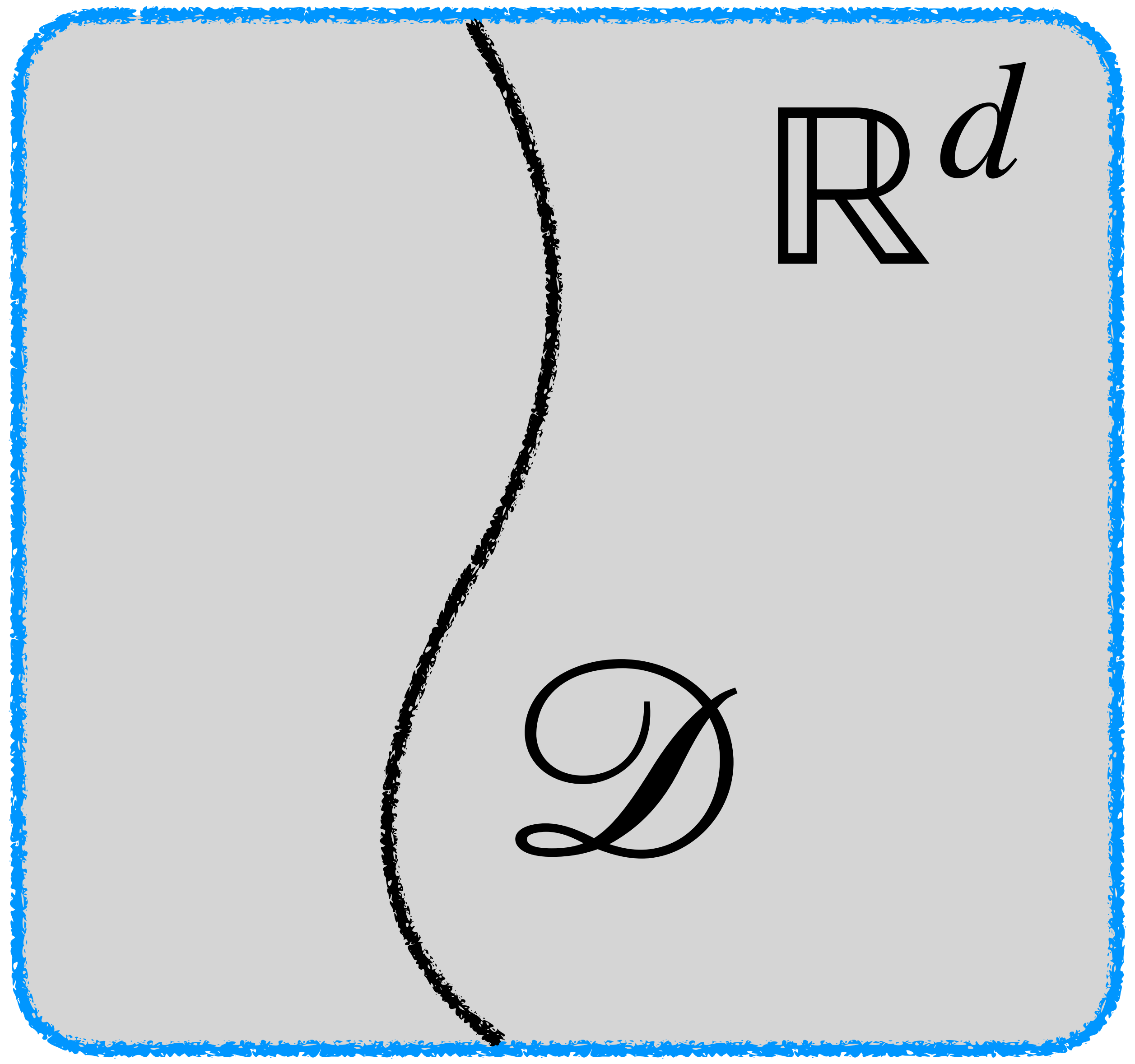
2 equivalent viewpoints on defects in Quantum Field Theory (QFT):

- Defects are extended operators of the bulk QFT :

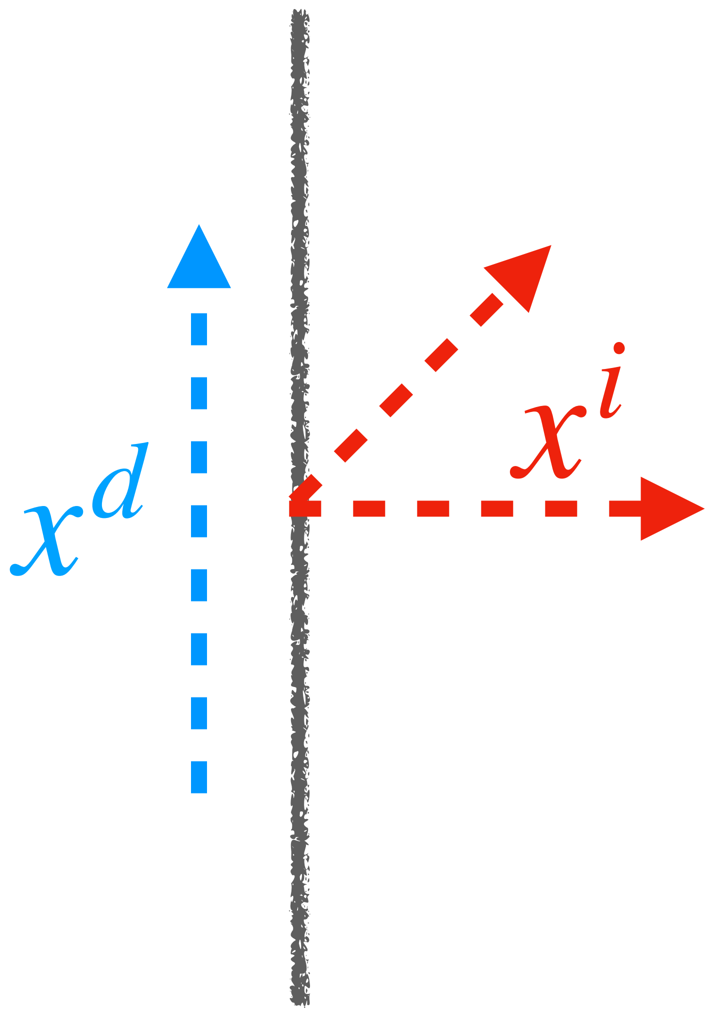
$$\mathcal{D} = e^{i\int_\Sigma \mathcal{L}(\phi)} \quad \leftrightarrow \quad \langle \mathcal{D} \ldots \rangle = \int_{\phi(\Sigma)=\phi_0} D\phi e^{iS[\phi]}$$

- Bulk+defect define a *Defect QFT* (DQFT):

$$\mathcal{L} = \mathcal{L}_{bulk} + \delta^{d-1}_{\mathcal{D}} \mathcal{L}_{defect}$$

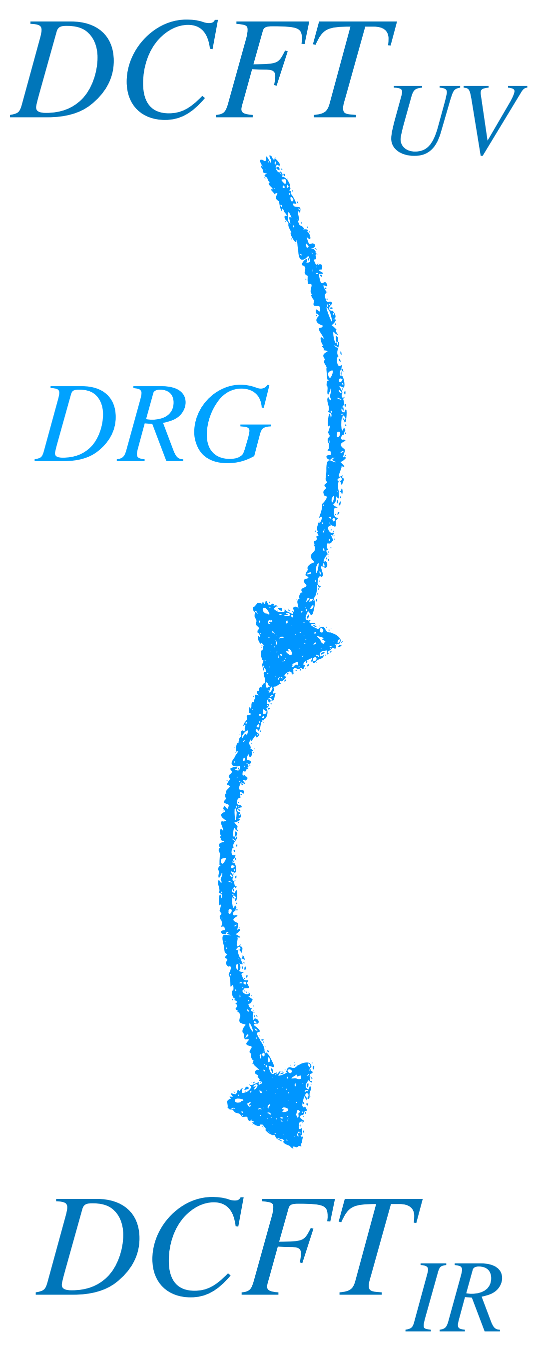


Defects in conformal field theories (CFTs)



Defect CFTs (DCFTs) = DQFTs preserving the p-dimensional conformal group when placed on straight or circular lines

$$SO(d+1,1) \supset \overbrace{SL(2,\mathbb{R})}^{\text{Conformal along } x^d} \times \overbrace{SO(d-1)}^{\text{Rotations of } x^i}$$



Massive DQFTs flow under the *defect renormalization group* (DRG) between UV and IR DCFT fixed points.

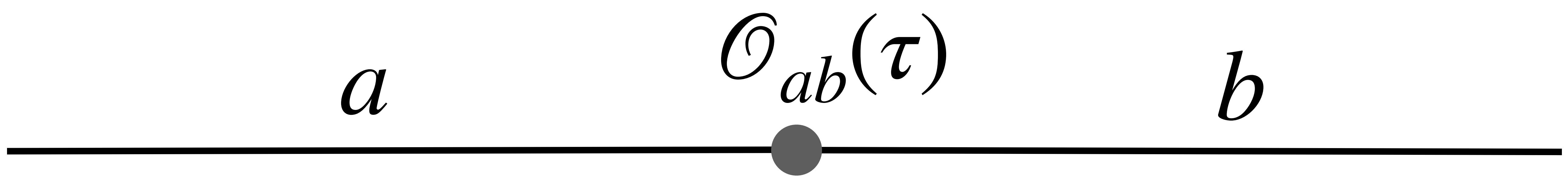
Irreversibility theorems guarantee $DCFT_{UV} \neq DCFT_{IR}$

Properties of DCFTs

Defect operators classified according to their $SL(2,\mathbb{R}) \times SO(d-1)$ quantum numbers

$$\langle \hat{\mathcal{O}}(\tau)\hat{\mathcal{O}}(0)\rangle = \frac{1}{\tau^{2\Delta(\hat{\mathcal{O}})}} \qquad \text{(straight defect)}$$

Defect operators can interpolate between different defects



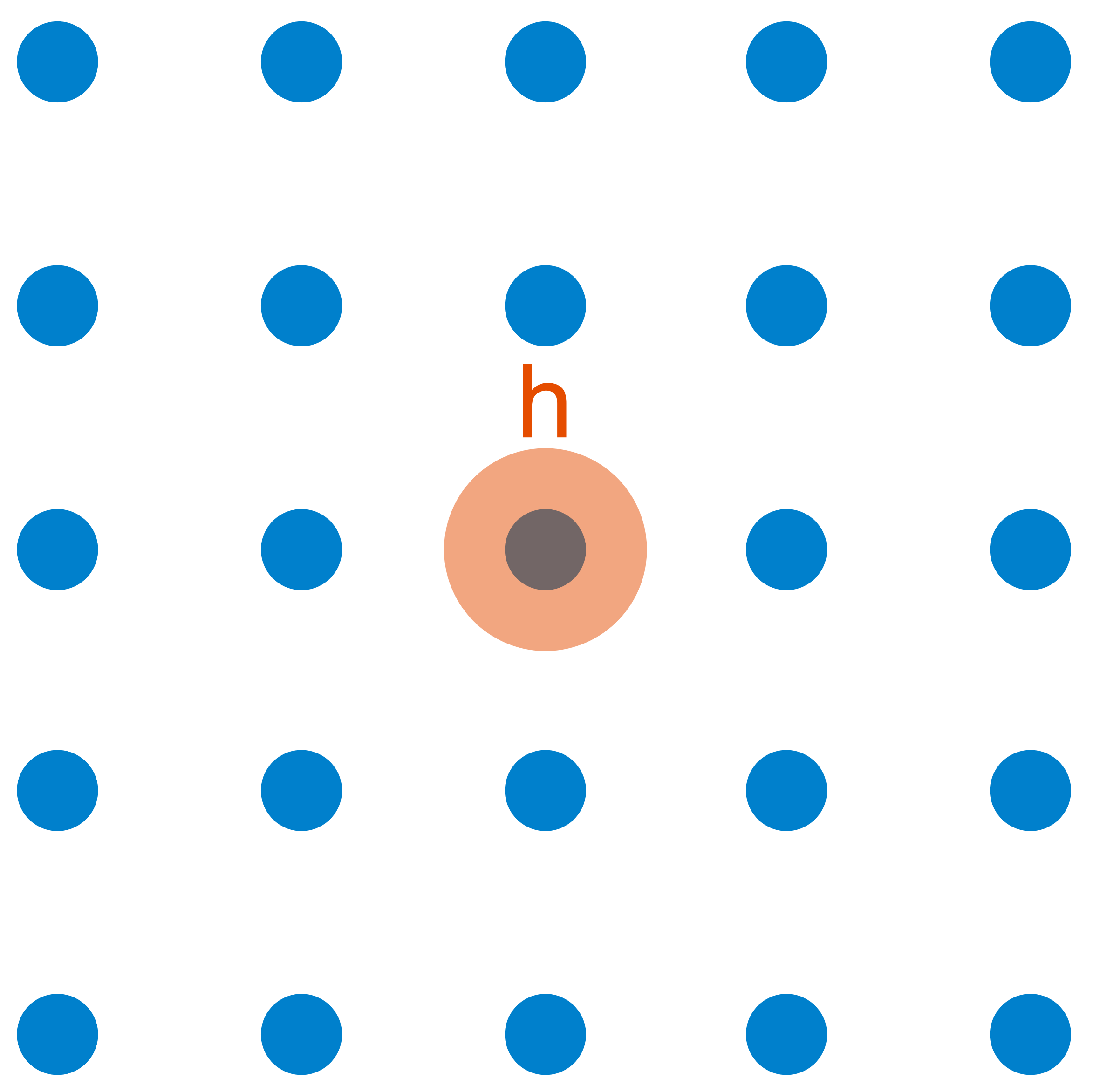
Bulk-to-defect OPE relates bulk and defect operators:

$$\mathcal{O}(\tau, x_\perp) \sim \sum_{\hat{\mathcal{O}}_k} a_k x_\perp^{\Delta(\hat{\mathcal{O}}_k)-\Delta_{\mathcal{O}}} \hat{\mathcal{O}}_k(\tau)$$

Example: localized magnetic (pinning) field

Consider the critical $O(N)$ model in $d < 4$

$$S_{O(N)} \sim \int d^d x \frac{1}{2}(\partial\phi_a)^2 + \frac{\lambda}{4!}(\phi_a^2)^2$$



$$H_{imp} = \delta_{\mathcal{D}}^{d-1} h\phi_1$$

Ex.: Localized external magnetic field

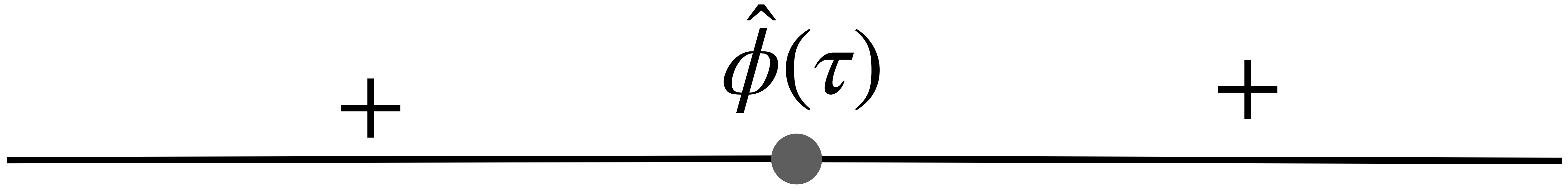
$$S = S_{O(N)} + h\int_{x_\perp = 0} d\tau\, \phi_1(x(\tau))$$

- $\Delta_\phi < 1$: flows to nontrivial DCFT
- Breaks $O(N) \to O(N-1)$ ($\mathbb{Z}_2 \to 1$ for $N = 1$)

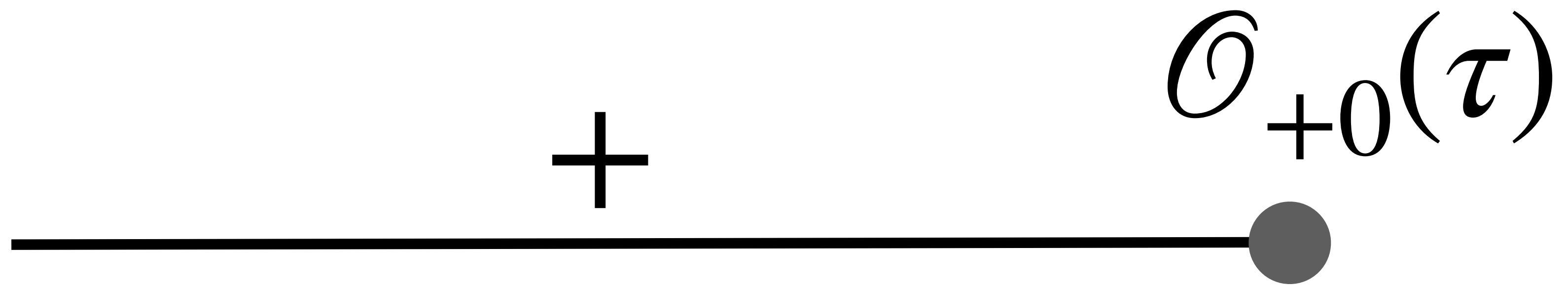
Allais Sachdev 2014

Nonperturbative pinning field data in the $3d$ Ising model

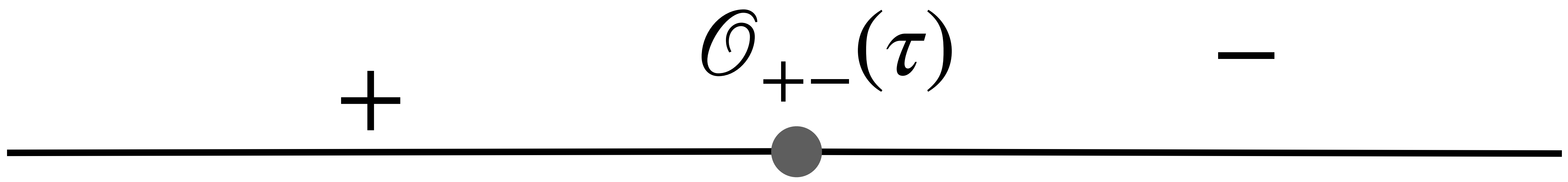
- $\Delta(\hat{\phi}) \simeq 1.6$: isolated defect is RG stable



- $\Delta_{+0} \simeq 0.11$: $1 \oplus +$ is unstable



- $\Delta_{+-} \simeq 0.83$: $+ \oplus -$ is unstable (no SSB)

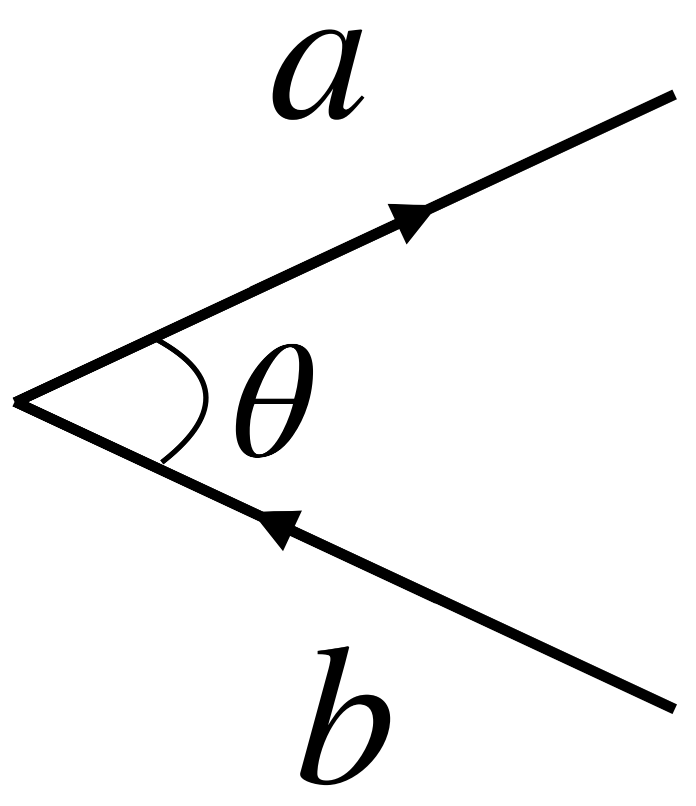


Toldin Assaad Wessel 2016, Hu He Zhou 2023, Zhou Gaiotto He Zou 2024, Lanzetta Liu Metlitski 2025

Cusp anomalous dimension
at Euclidean angle

The cusp anomalous dimension

Cusp breaks $SO(d+1,1) \to \mathbb{R}_D$

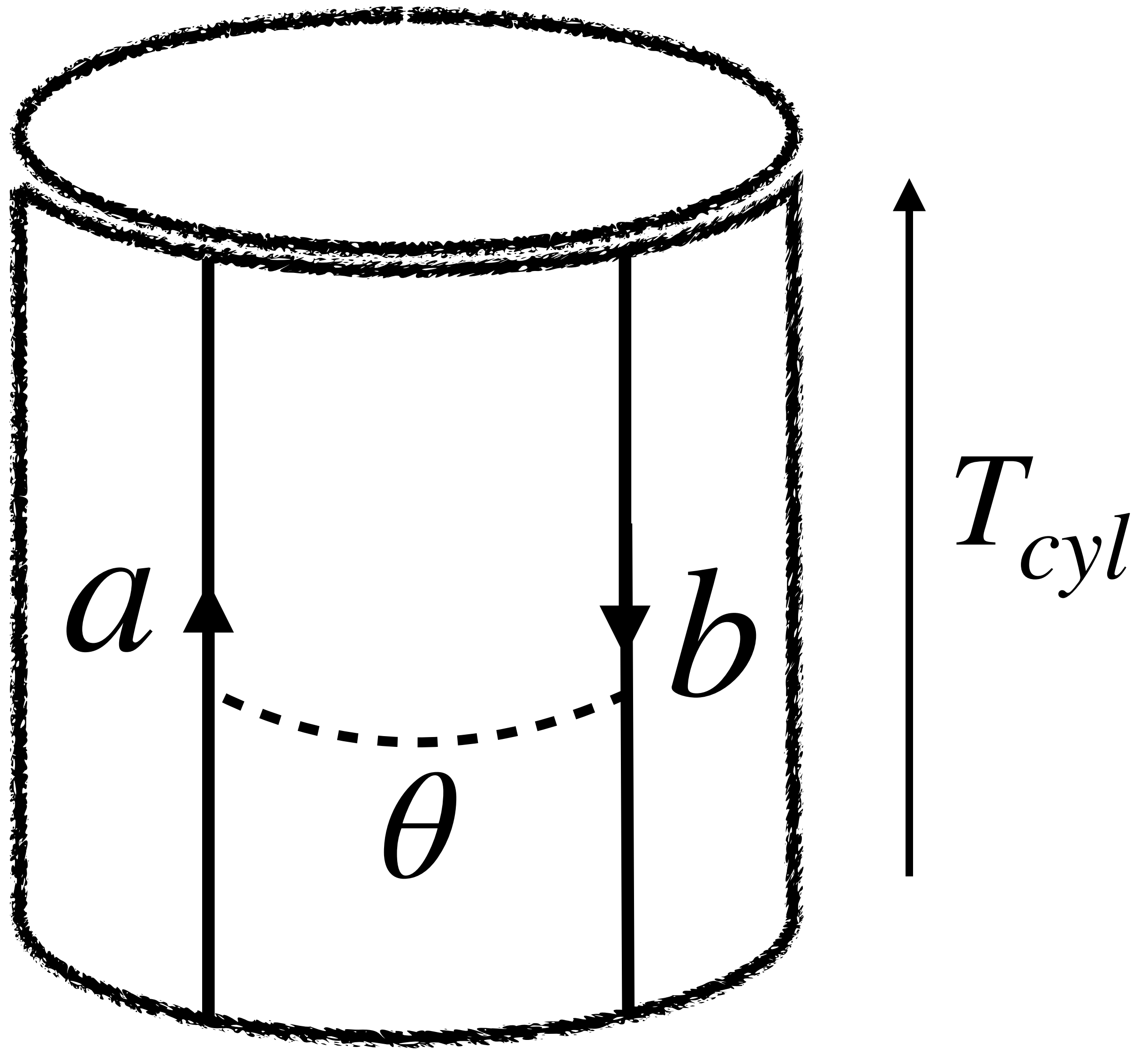


Cusp anomalous dimension

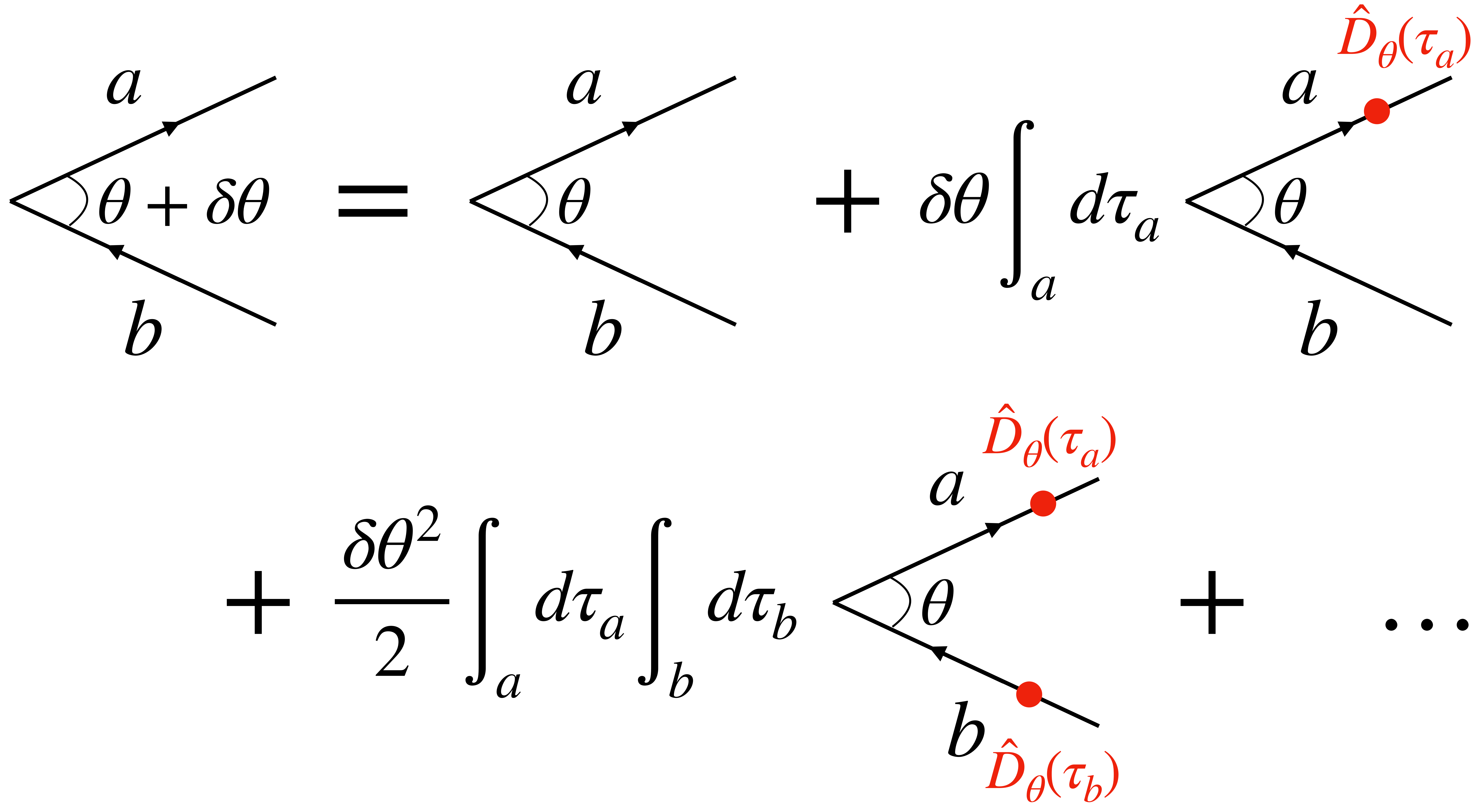
$$\log \frac{Z_{ab}(\theta)}{Z_{CFT}} = -\Gamma_{ab}(\theta) \log \frac{L_{IR}}{a_{UV}}$$

Equivalently, ground-state energy on $\mathbb{R} \times S^{d-1}$ with two defects at distance θ

$$\log \frac{Z_{ab}(\theta)}{\sqrt{Z_{aa}(\pi) Z_{bb}(\pi)}}\Bigg|_{\text{cyl}} = -\Gamma_{ab}(\theta) T_{\text{cyl}}$$

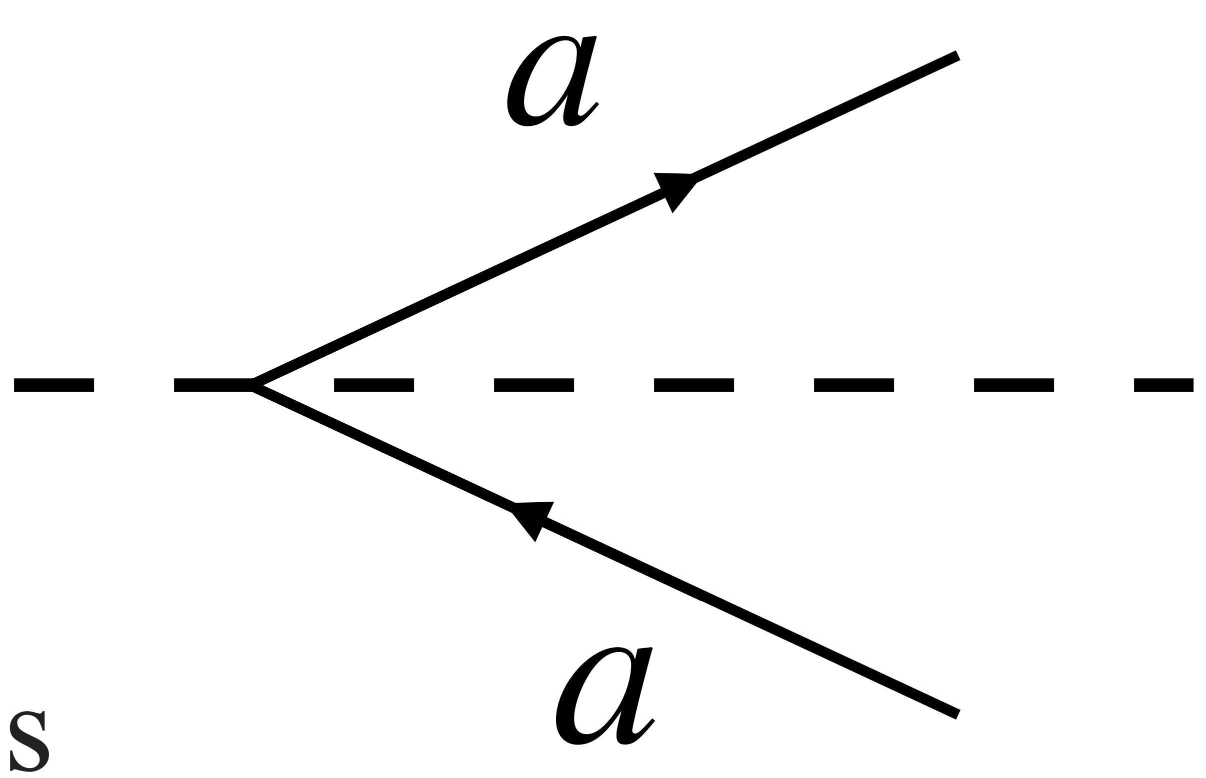


Small deformations controlled by the *Displacement operator*



$$\frac{d^2\Gamma_{ab}(\theta)}{d\theta^2} = -\int dx\, x^{-1} F_{ab}(x)\,,$$

$$F_{ab}\left(\frac{\tau_a}{\tau_b}\right) = \langle D_n(\tau_a) D_n(\tau_b)\rangle_c \tau_a^2 \tau_b^2$$

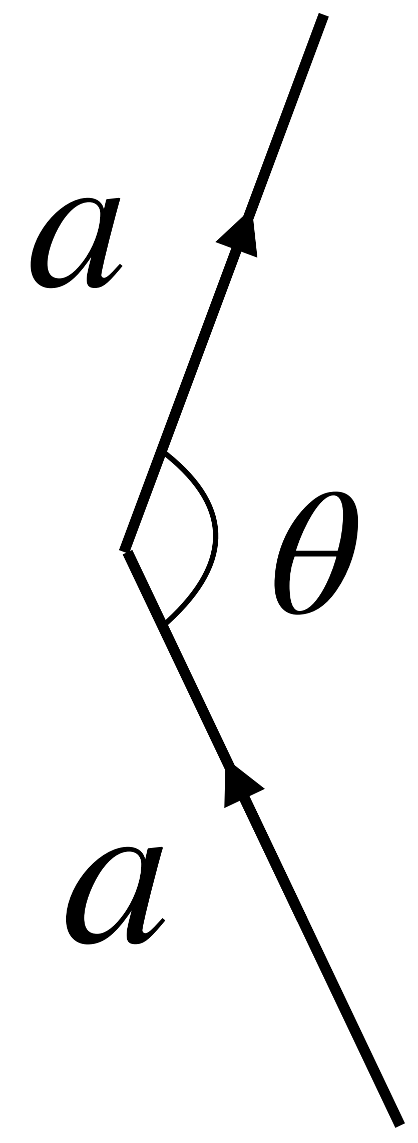


- Reflection positivity: $\int dx\, x^{-1} F_{a\bar{a}}(x) > 0$
- OPE $F(x) \overset{x\to 0}{\sim} x^{\Delta_{min}}$ and $F(x) \overset{x\to\infty}{\sim} x^{-\Delta_{min}}$: integral converges

$$\Gamma''_{a\bar{a}}(\theta) < 0$$

$$\boxed{\Gamma''_{a\bar{a}}(\theta) < 0}$$

Near $\theta \simeq \pi$ cusp fixed by Displacement 2pt function



$$\langle D_i(\tau) D_j(0) \rangle = \frac{C_D \delta_{ij}}{|\tau|^4}$$

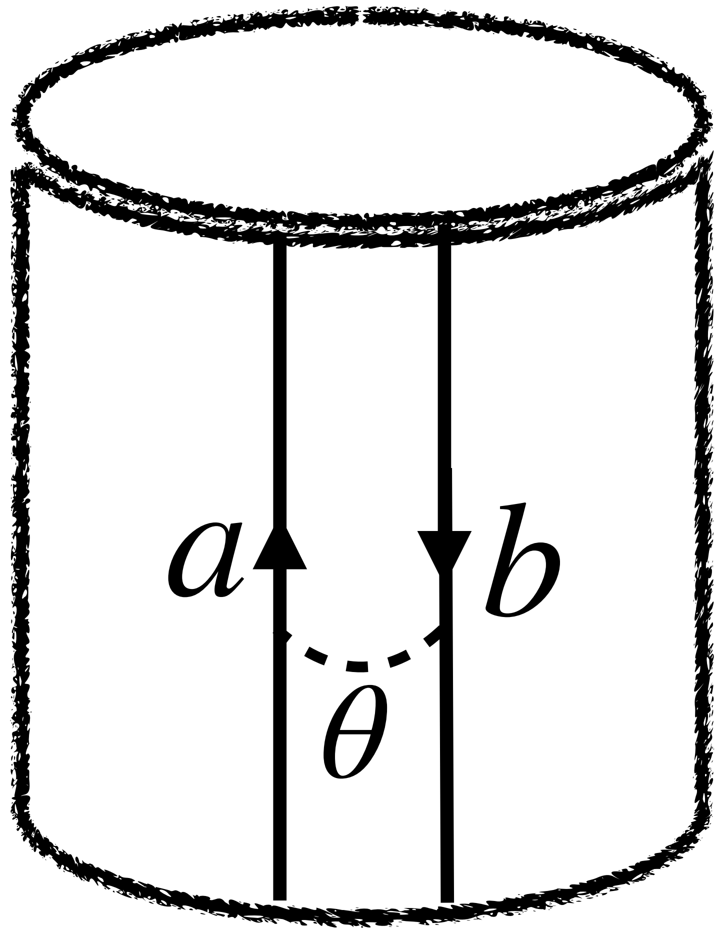
$$\Gamma''_{a\bar{a}}(\theta)|_{\theta=\pi} = -\frac{C_D}{6}$$

Correa Henn Maldacena Sever 2012

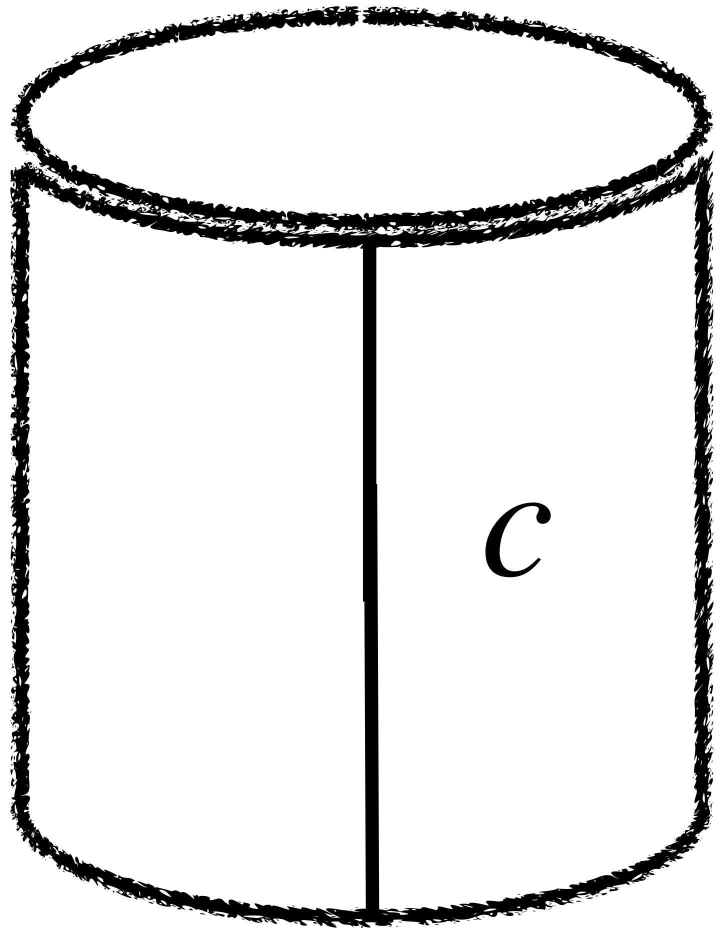
Since $\Gamma_{a\bar{a}}(\pi) = \Gamma'_{a\bar{a}}(\theta)\,|_{\theta=\pi} = 0$ it follows

$$\Gamma_{a\bar{a}}(\theta) < 0\,, \quad \Gamma'_{a\bar{a}}(\theta) < 0$$

Near $\theta \simeq 0$: fusion EFT (generalized multipole expansion)



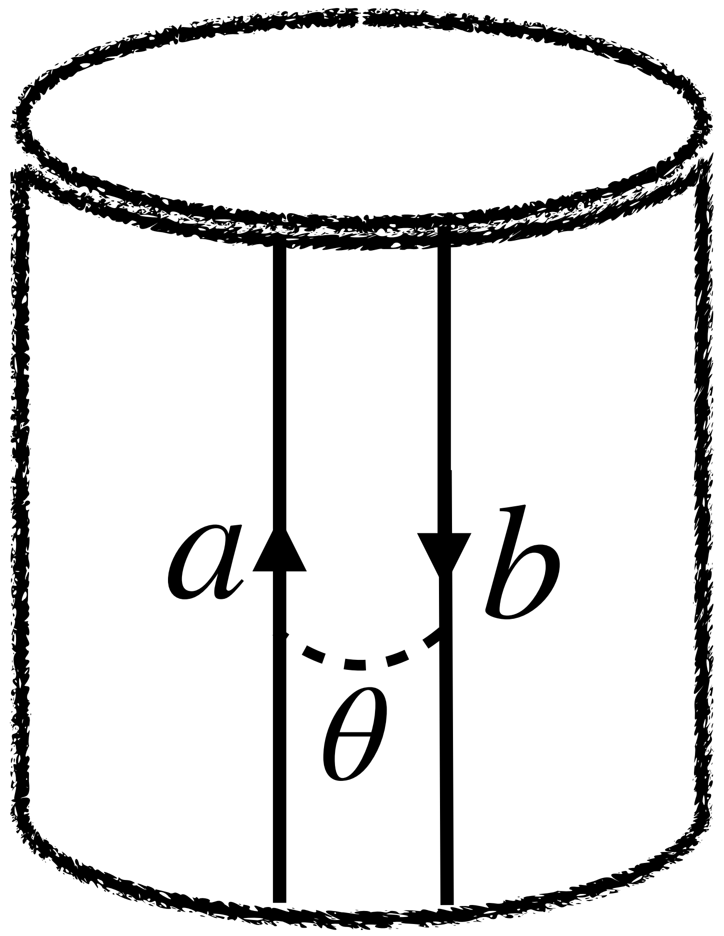
$$\simeq \sum_c e^{-\int dt \frac{C_{abc}}{\theta} + \dots}$$



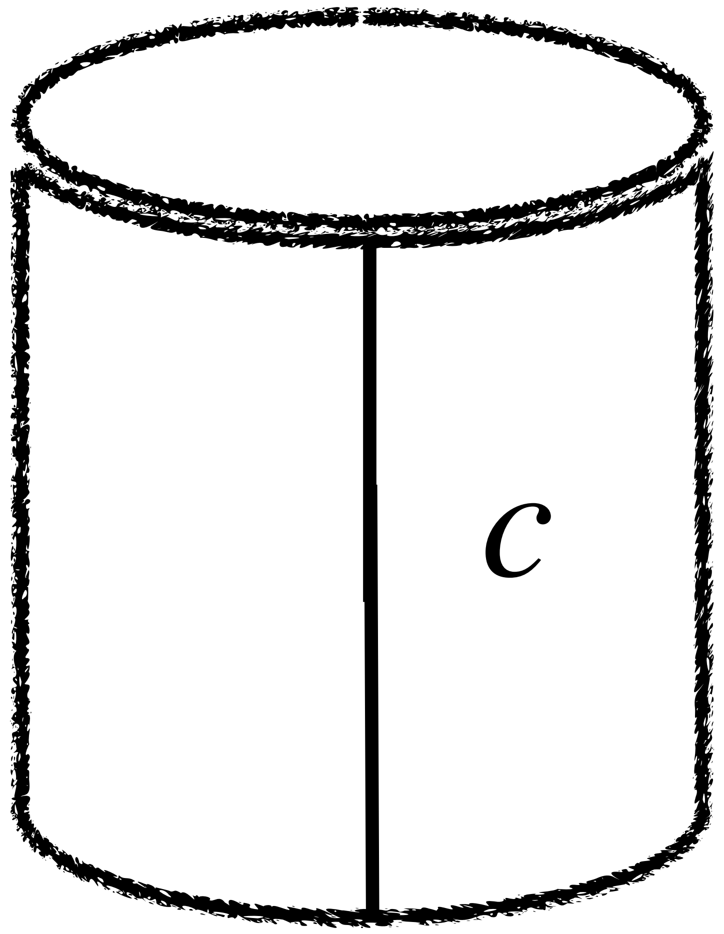
All defect operators may appear (including interpolating ones), thus:

- Expect RHS to include only stable defects
- Line with with the smallest Casimir energy C_{abc} dominates in the IR
- Irrelevant operators $\sim \theta^{\Delta_{irr}-1}\mathcal{O}$ induce power corrections

See also *Dyatlik Khanchandani Popov Wang 2024, Kravchuk Radcliffe Sinha 2024*



$$\simeq \quad e^{-\int dt \frac{C_{abc}}{\theta} + \dots}$$



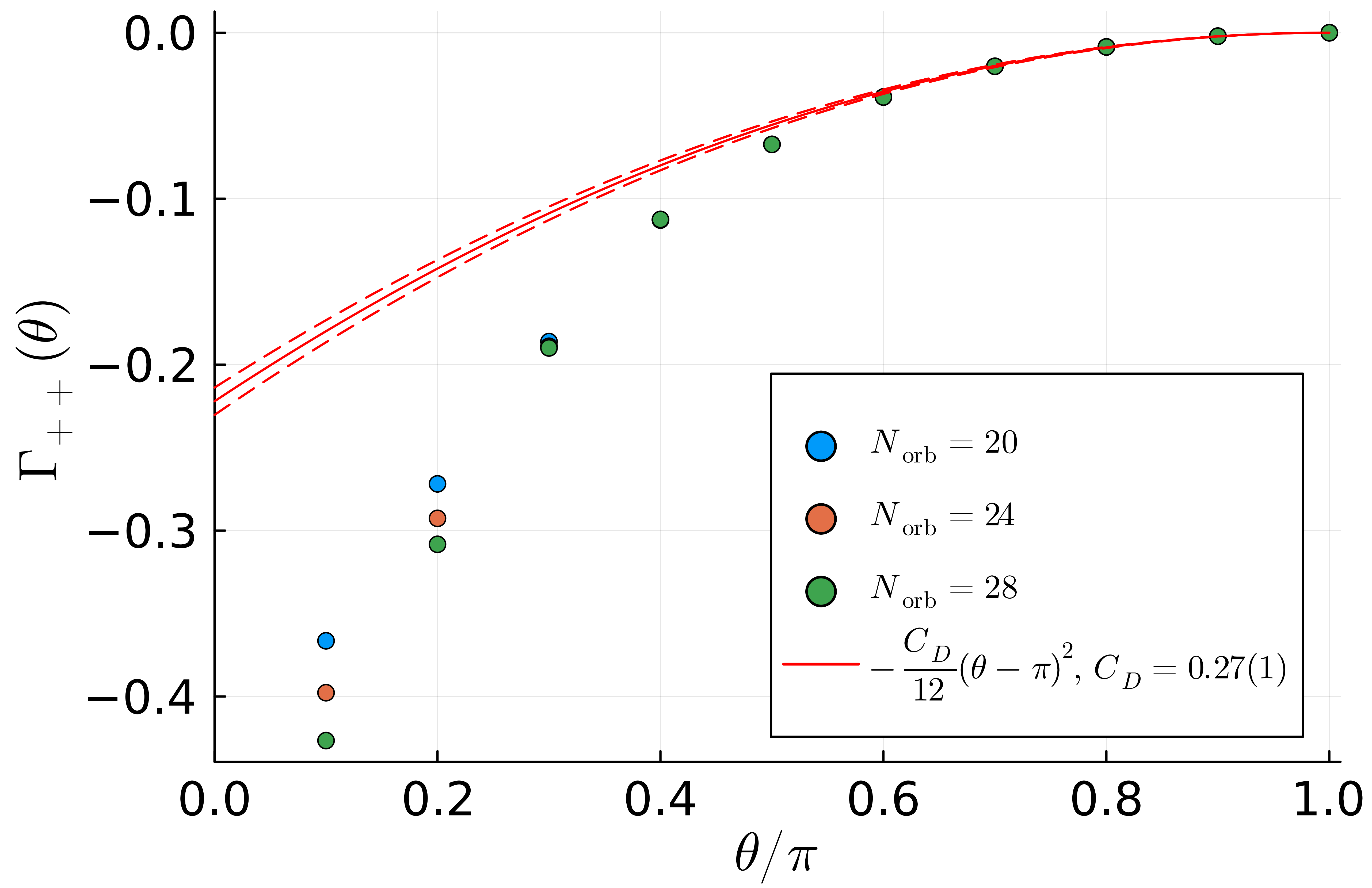
$$\Gamma_{ab}(\theta) \simeq \frac{C_{abc}}{\theta} + \Delta_{c0} + \#\theta^{\Delta_{irr}-1} + \dots$$

Defect creation operator dimension

$$\Gamma''_{a\bar{a}}(\theta) < 0 \quad \Longrightarrow \quad C_{a\bar{a}c} < 0$$

See also *Bachas 2006*

Application: cusp in $3d$ Ising model



Casimir energy obtained fitting $\theta^2\Gamma'_{++}(\theta) = -C_{+++} + a\,\theta^{\Delta_{\hat{\phi}}-1}$

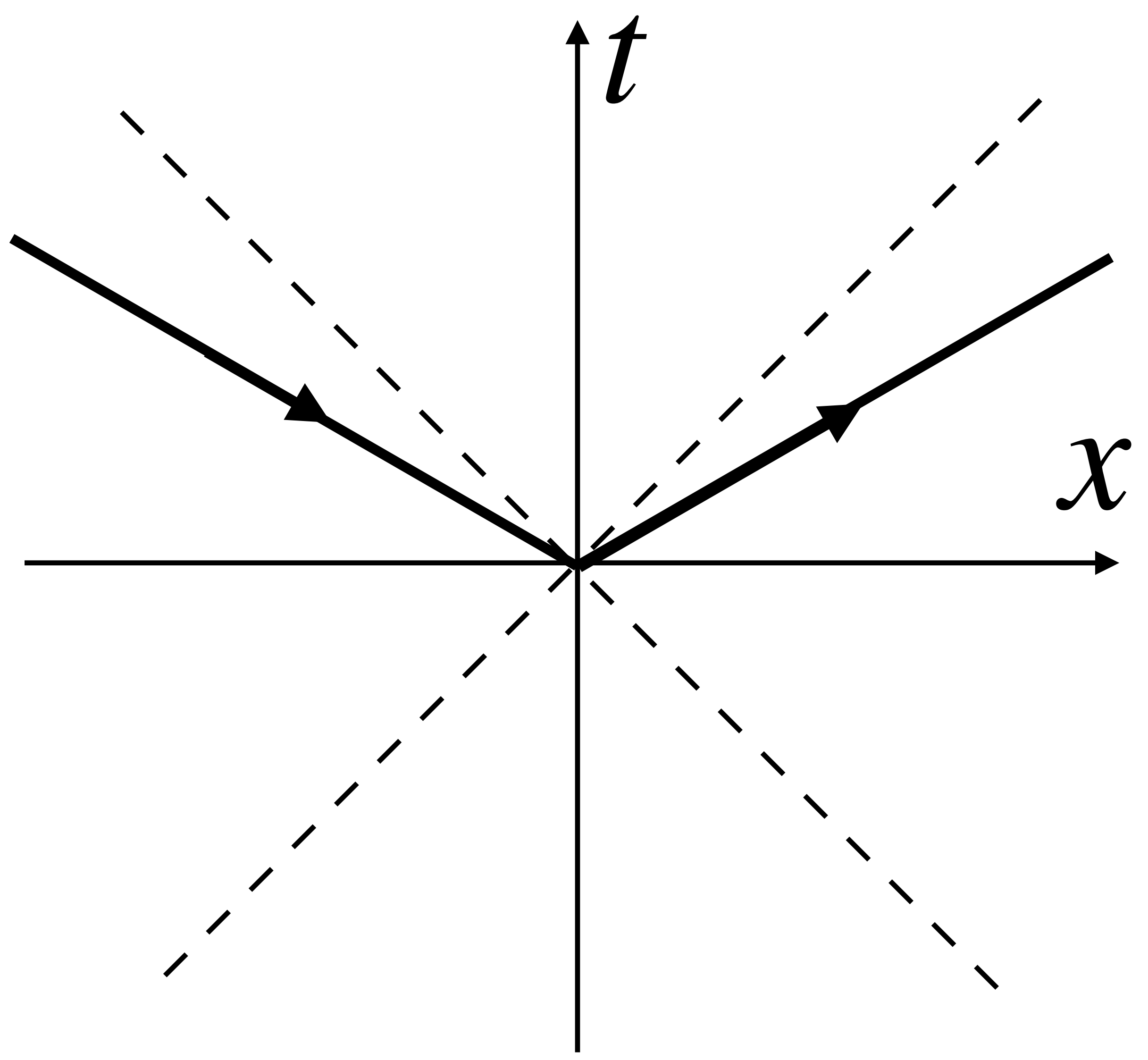
$$C_{+++} = -0.29(2)$$

Cusp anomalous dimension at nonzero boost

Analytic continuation

Analytic continuationt to rapidity $\eta > 0$

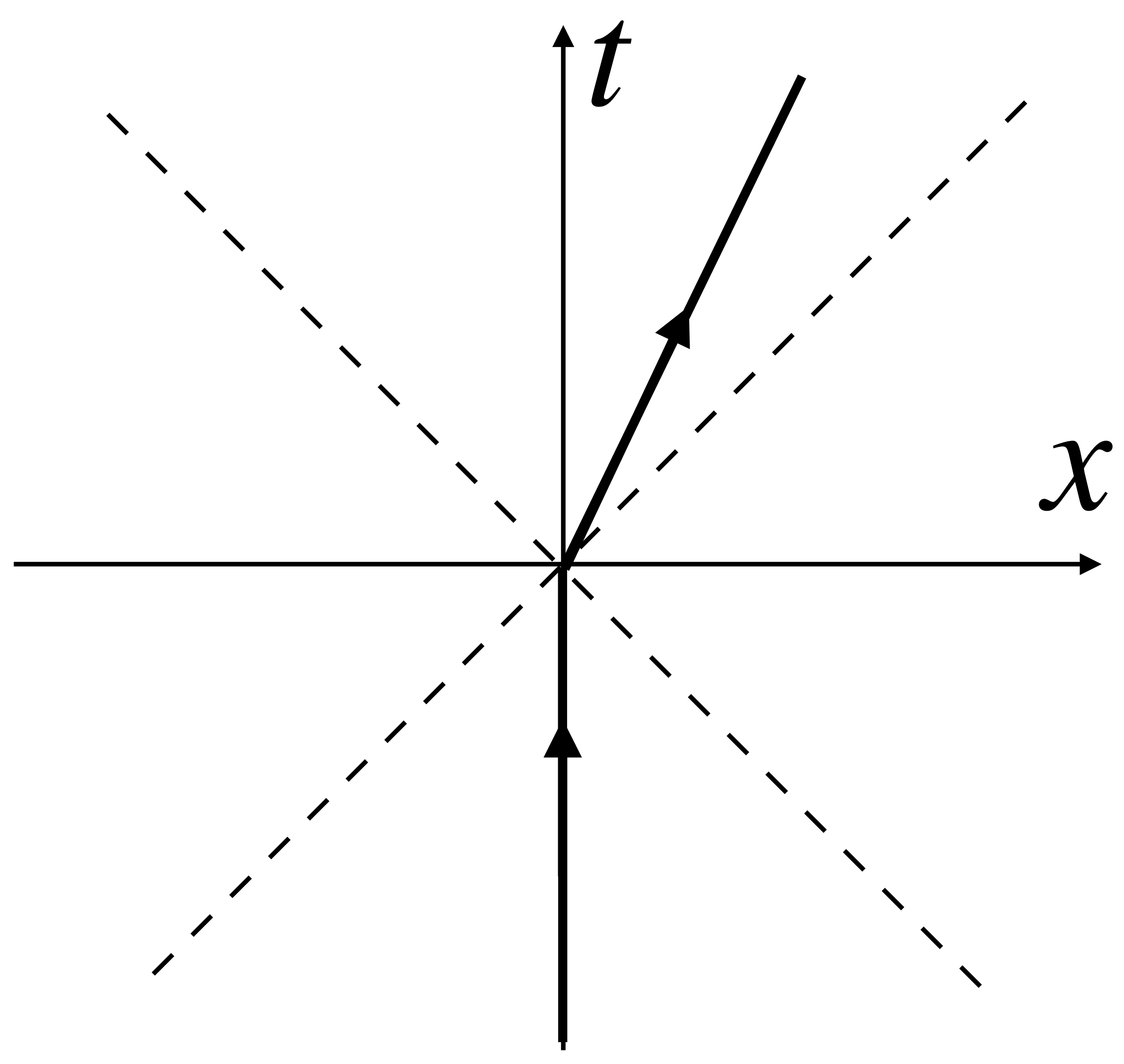
$$\frac{\dot{x}_a \cdot \dot{x}_b}{|\dot{x}_a||\dot{x}_b|} = \cos\theta \xrightarrow{\theta=\pi-i\eta} = -\cosh\eta$$



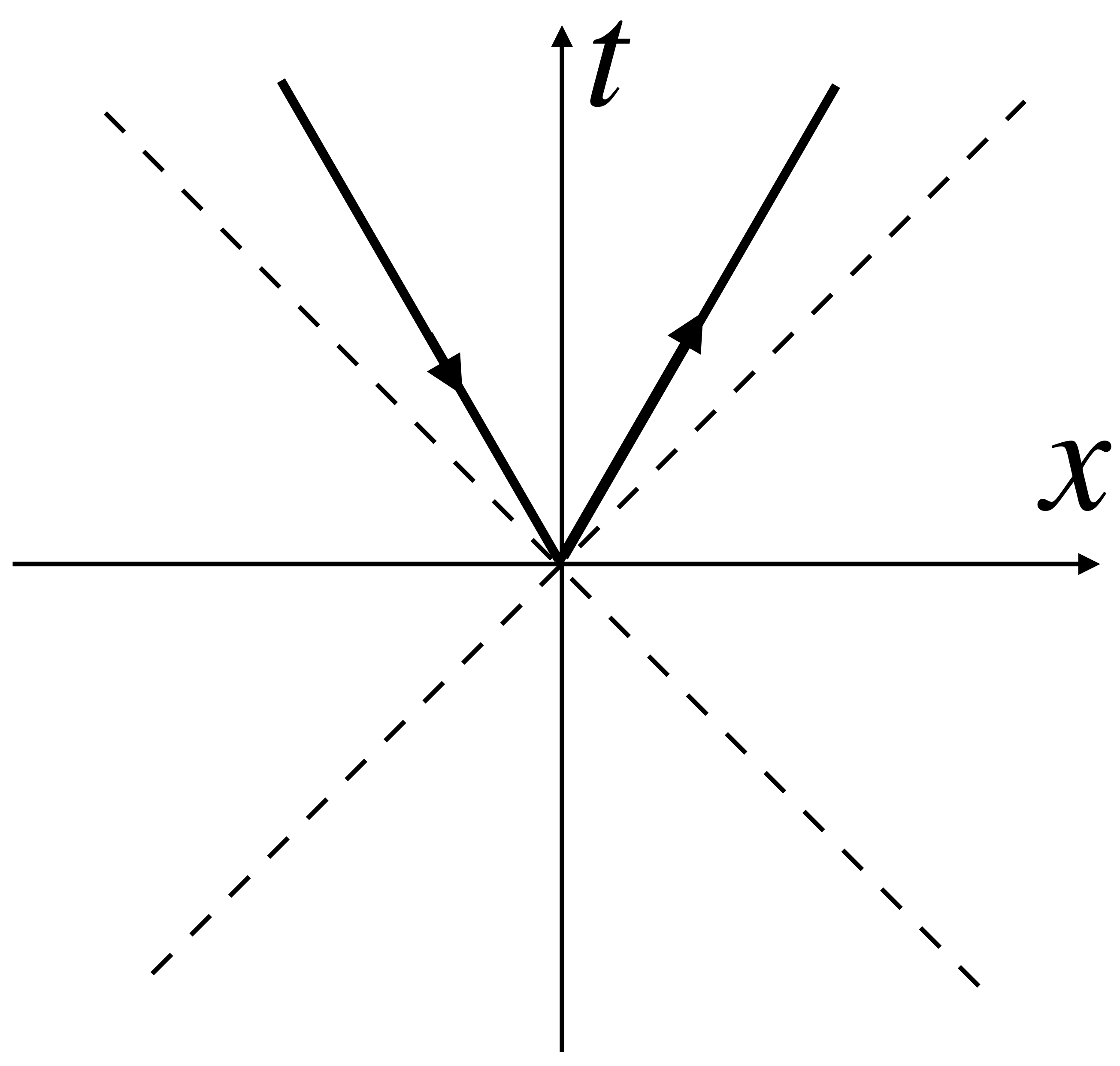
$$\Gamma^{L}_{ab}(\eta) = \Gamma_{ab}(\pi - i\eta)$$

- Lines everywhere spacelike separated: cusp remains real

All other configurations obtained from this, e.g.



$$\Gamma_{ab}^{\text{quench}}(\eta) = \Gamma_{ab}^{L}(\eta)$$



$$\Gamma_{ab}^{\text{Sudakov}}(\eta) = \Gamma_{ab}^{L}(\eta - i\pi + i\epsilon)$$

Large boost angle

$$\Gamma^L_{ab}(\eta) \xrightarrow{\eta\to\infty} ?$$

- Encodes suppression of exclusive production process due to soft radiation upon replacing $e^{2\eta} \to Q^2/m^2$ and $L^2_{IR}/a^2_{UV} \to Q^2/\mu^2_{IR}$
- "Double-log" enhancement for Wilson lines in perturbative gauge theories

$$\Gamma^L_{WW}(\eta) = \gamma_{\rm cusp}\eta + O(1)\,,$$

$$\gamma_{\rm cusp} = \frac{g^2}{(4\pi)^2} + \ldots$$

Consider a different frame:

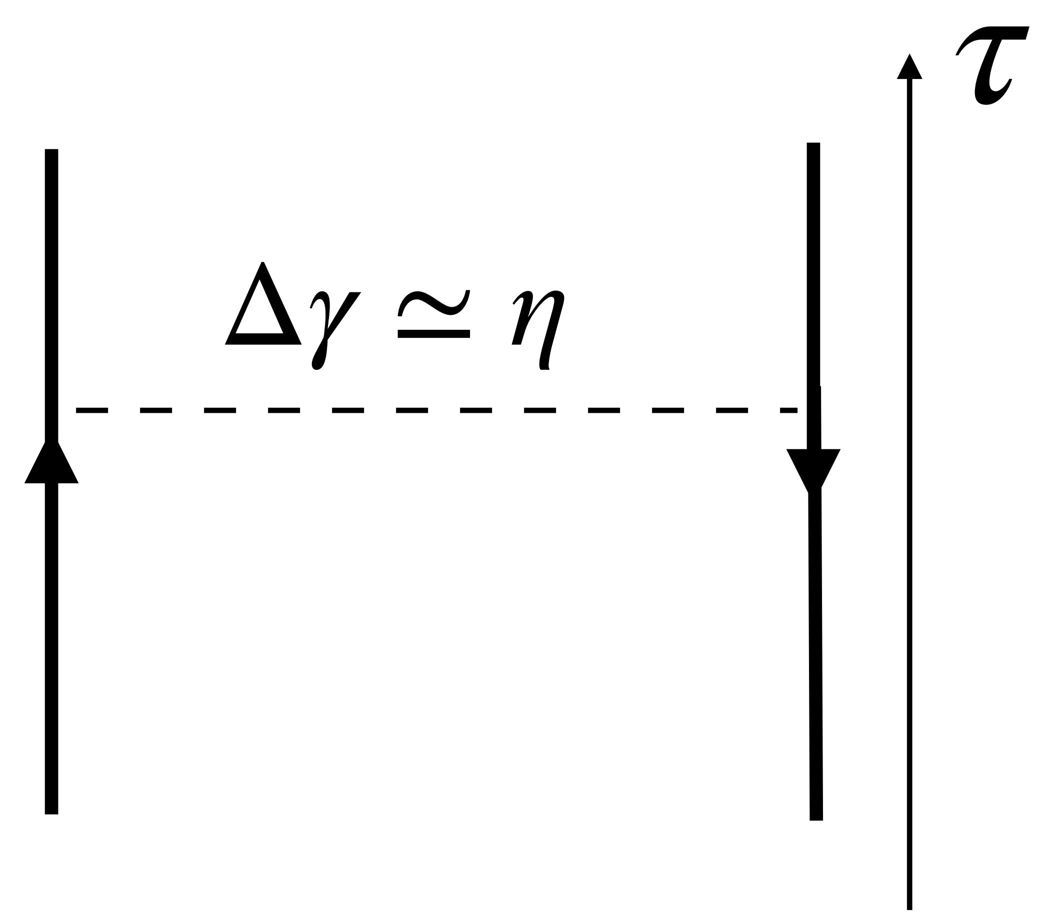
- Weyl rescale $\mathbb{R}^{d-1,1} \to \mathrm{AdS}_3 \times S^{d-3}$ with metric

$$ds^2_{\mathrm{AdS}_3} = -d\alpha^2 + d\gamma^2 \sin^2\alpha + d\tau^2 \cos^2\alpha$$

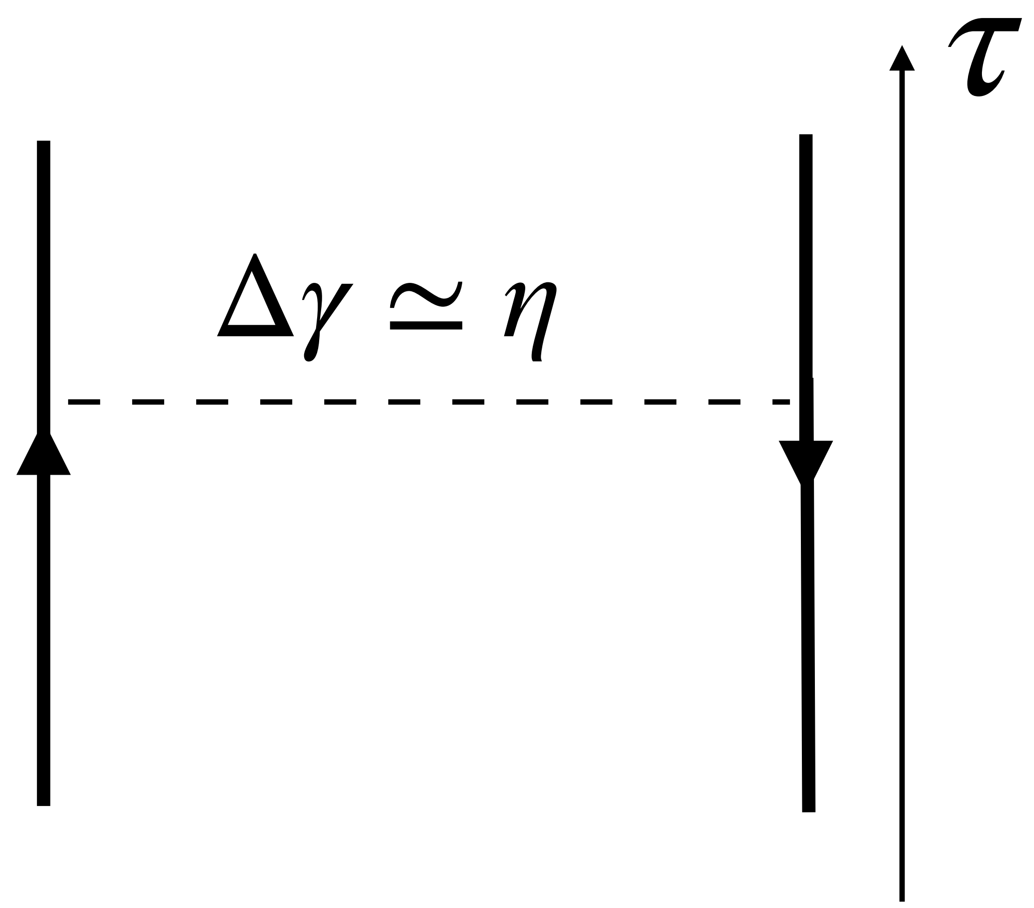
$$i\partial_\tau = \hat{D}\,, \quad i\partial_\gamma = \hat{J}_{+-}$$

- Weyl+Lorentz map lines to large separation

$$\gamma_1 = -\gamma_2 \simeq \eta/2\,, \quad \alpha_1 = \alpha_2 = \frac{\pi}{4}$$



Large separation implies weak interaction:



$$\Gamma^L_{ab}(\eta) = \Delta_{a0} + \Delta_{b0} - V_{\rm int}(\eta) \, ,$$

Defect creation operators' dimension

Vanishes for $\eta \to \infty$

- Interaction term controlled by the lowest twist $\tau = \Delta - J$ operator $\mathcal{O}_{\rm min}$

$$\mathcal{L}_{int} \sim \int d\tau_1 d\tau_2 \langle \mathcal{O}_{\rm min}(x_1) \mathcal{O}_{\rm min}(x_2) \rangle$$

$$V_{\rm int}(\eta) \simeq \lambda_a \lambda_b e^{-\tau_{\rm min}\eta} \times \begin{cases} \eta + \# & \text{if } J(\mathcal{O}_{\rm min}) = 0 \\ 1 & \text{if } J(\mathcal{O}_{\rm min}) > 0 \end{cases}$$

$$\Gamma^L_{ab}(\eta) = \Delta_{a0} + \Delta_{b0} - V_{\text{int}}(\eta)\,,$$

$$V_{\text{int}}(\eta) \simeq \lambda_a \lambda_b e^{-\tau_{\min}\eta} \times \begin{cases} \eta + \# & \text{if } J(\mathcal{O}_{\min}) = 0 \\ 1 & \text{if } J(\mathcal{O}_{\min}) > 0 \end{cases}$$

- Does not hold for defects that cannot end on local operators, such as the fundamental Wilson line in $\mathcal{N} = 4$ super Yang-Mills or monodromy defects

- Wilson lines charged under 1-form symmetry see a string due to the twist-0 gauge field

$$\Gamma_L(\eta) \simeq T_{string} \Delta\gamma = T_{string} \log \eta$$

Alday Maldacena 2007

Application: heavy particle in Ising bath

Consider Ising CFT + heavy Φ

$$\mathcal{L} = \mathcal{L}_{Ising} + |\partial\Phi|^2 - m^2|\Phi|^2 - g\, m^{2-\Delta_\sigma}\sigma|\Phi|^2$$

$$\langle\Phi(p)\Phi^*(p)\rangle \overset{g=0}{=} \frac{1}{p^2 - m^2}\,, \quad \langle 0||\Phi|^2|p_1, p_2\rangle_{\text{amp.}} \overset{g=0}{=} 1$$

Even for $g \ll 1$ coupling to CFT modifies near on-shell behaviour:

$$\langle\Phi(p)\Phi^*(p)\rangle \simeq \int d^d x e^{ipx} G_{free}(x)\mathcal{D}(0,x) \overset{p^2\to m^2}{\sim} \frac{\#}{m^{2\Delta_{+0}}(p^2 - m^2)^{1-\Delta_{+0}}}$$

$$\langle 0||\Phi|^2|p_1, p_2\rangle_{\text{amp.}} \simeq e^{-\left[\Gamma_L\left(\log\frac{Q}{m}\right) - 2\Delta_{+0}\right]\log\frac{Q}{E_{IR}}}\,, \quad Q = \sqrt{-(p_1+p_2)^2}$$

Relevant interactions depends on the virtuality Q^2:

$$\text{coupling} \sim g \frac{m^{2-\Delta_\sigma}}{Q^{2-\Delta_\sigma}}$$

For $0 < g \ll 1$ only soft $Q^2 \ll m^2$ quanta contribute: use heavy particle EFT

$$\Phi = e^{-im\,v\cdot x}\left[\Phi_{near} + \Phi_{off-shell}\right]$$

$$\mathcal{L}_{eff} \simeq \mathcal{L}_{Ising} + 2imv^\mu \Phi^*_{near}\partial_\mu \Phi_{near} - gm^{2-\Delta_\sigma}\sigma|\Phi_{near}|^2$$

Decoupling transformation:

$$\Phi_{near}(x) \to \mathcal{D}(x,0)\Phi_{near}(x)\,, \quad \mathcal{D}(x,0) = e^{-\frac{i}{2}gm^{1-\Delta_\sigma}\int_0^x d|x'|\,\sigma(x')}$$

$$\langle 0||\Phi|^2|p_1, p_2\rangle_{\text{amp.}} \simeq e^{-\left[\Gamma_L\left(\log\frac{Q}{m}\right)-2\Delta_{+0}\right]\log\frac{Q}{E_{IR}}}\,, \qquad Q = \sqrt{-(p_1+p_2)^2}$$

Predictions for small and large η:

$$\Gamma^L(\eta) \simeq \begin{cases} \frac{1}{12}C_D\eta^2 & |\eta| \ll 1 \\ 2\Delta_{+0} - \lambda_{int}^2 e^{-\Delta_\sigma\eta}\left[\eta - \psi^{(0)}(\Delta_\sigma) - \gamma\right] & |\eta| \gg 1 \end{cases}$$

- $\Delta_\sigma \simeq 0.518$, $\Delta_{+0} \simeq 0.11$, $C_D \simeq 0.27$ known nonperturbatively from numerics

- λ_{int}^2 estimate from ϵ-expansion:

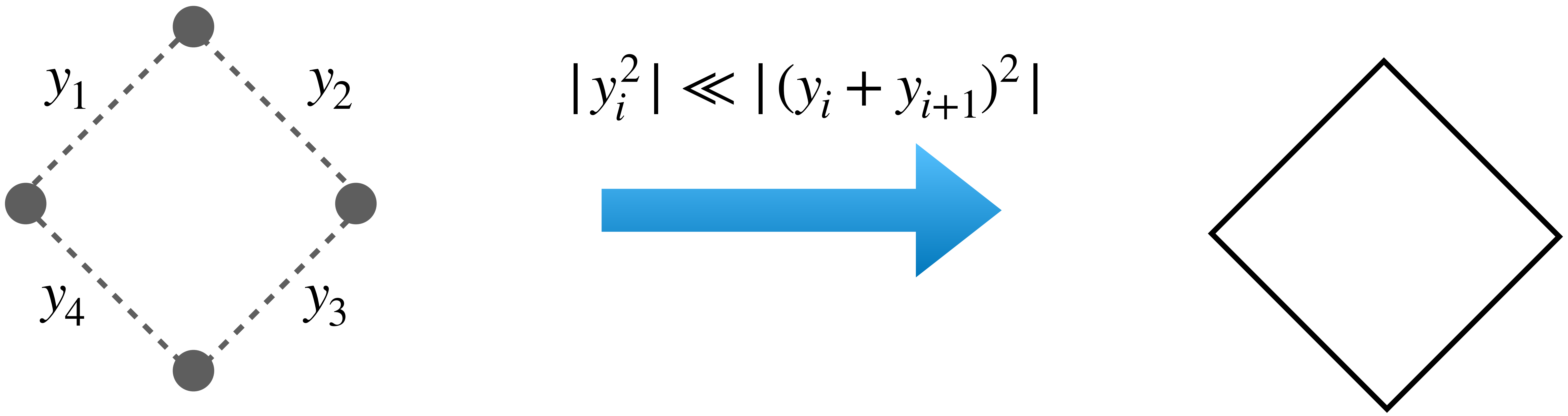
$$\lambda_{int}^2 = \frac{9}{2\pi^2} + \epsilon\left(\frac{27\zeta(3)}{\pi^4} - \frac{2}{3\pi^2}\right) + O\left(\epsilon^2\right)\,.$$

Speculation: transition in adjoint Wilson line

Perturbative factorization theorem in conformal gauge theories ($d = 4$):

- Leading lightlike behaviour of correlation function captured replacing tree-level propagators with **adjoint** Wilson lines

$$\langle J(x_1)J(x_2)J(x_3)J(x_4)\rangle \xrightarrow{x_{ii+1}^2 \ll x_{ii+2}^2} \langle W_{adj}(x_1,x_2)W_{adj}(x_2,x_3)W_{adj}(x_3,x_4)W_{adj}(x_4,x_1)\rangle$$



Alday Eden Korchemsky Maldacena Sokatchev 2010, …, Chen Monni Pang Vita Zhu 2025

$$\langle J(x_1)J(x_2)J(x_3)J(x_4)\rangle \xrightarrow{x_{ii+1}^2 \ll x_{ii+2}^2} \langle W_{adj}(x_1,x_2)W_{adj}(x_2,x_3)W_{adj}(x_3,x_4)W_{adj}(x_4,x_1)\rangle$$

- Exponential doubly logarithmic singularity from perturbative results

$$\Gamma^L_{W_{adj}W_{adj}}(\eta) \sim g^2\eta + O(1)\,,$$

- EFT suggests string breaking transition nonperturbatively:

$$\Gamma^L_{W_{adj}W_{adj}}(\eta) \simeq \text{const.}\,, \quad \eta \gtrsim 1/g^2$$

- Implies that factorization theorem has to be reconsidered nonperturbatively (presumably related to dominance exchange between multi-trace and single-trace operators in OPE)
- Similar considerations hold for EEC in back-to-back limit

Summary and outlook

Summary

RG flows on line defects

- ☑ Review of defect RG
- ☑ Results for the localized magnetic field in the Ising model

Cusped lines

- ☑ General results on the Euclidean cusp and Casimir energy
- ☑ Universal behaviour at large boost angle

Outlook

More on defect RG:

- ❑ Wilson lines and vortex loops at deconfined critical points? Dualities?

  Aharony GC Komargodski Mezei Raviv-Moshe 2023, Komargodski Popov Rayhaun 2025

- ❑ Non-perturbative contraints, e.g. from anomalies?

  Thorngren Wang 2020, Antinucci Copetti Galati Rizi 2025,…

Cusped defects:

- ❑ Lightcone bootstrap for the Lorentzian cusp? Additional symmetries?

  Erramilli Kulp Popov 2025

- ❑ Correlation functions/Wilson loop duality beyond perturbation theory?

- ❑ Lorentzian cusp/quench from fuzzy sphere?

THANK YOU!