Haagerup Symmetry in $(E_8)_1$?

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[arXiv:2512.08225] Jan Albert, **YH**, Justin Kaidi, Yunqin Zheng
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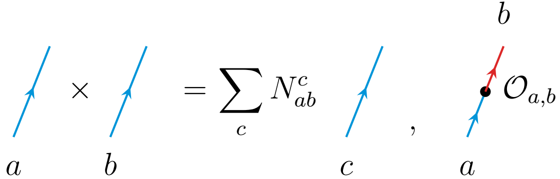
Why Do We Study Haagerup CFT?

- **Symmetries play a central role in theoretical physics.**
- **In two-dimensional rational conformal field theories (RCFTs), Verlinde lines have long been studied.** [Verlinde '88]
- **Generalized global symmetries:**

  symmetries = topological defects

  [Fröhlich, Fuchs, Runkel, Schweigert '09; Kapustin, Seiberg '14; Gaiotto, Kapustin, Seiberg, Willett '14; ...]
- **In 2d theories, symmetries are described by "fusion categories."**

  [Moore, Seiberg '89; Bhardwaj, Tachikawa '17; Chang, Lin, Shao, Wang, Yin '18; ...]



Why Do We Study Haagerup CFT?

- **It is widely believed that every fusion category can be realized as the symmetry of a unitary CFT with a single vacuum.**
- **For many fusion categories, the CFT realizing them is unknown.**
- **Haagerup fusion category is a prime example.**
  [Haagerup '93; Grossman, Snyder '11]
- **There are 3 distinct Haagerup categories, $\mathcal{H}_1,\ \mathcal{H}_2,\ \mathcal{H}_3$, which are related by gauging. So, we will focus on $\mathcal{H}_3$.**
- **$\mathcal{H}_3$ has 6 simple lines $\{\mathbf{1},\ \alpha,\ \alpha^2,\ \rho,\ \alpha\rho,\ \alpha^2\rho\}$ satisfying the fusion rules:**

$$\alpha^3 = \mathbf{1},\ \alpha\rho = \rho\alpha^2,\ \rho^2 = \mathbf{1} + \rho + \alpha\rho + \alpha^2\rho$$

Why Do We Study Haagerup CFT?

- **Several studies have investigated Haagerup CFTs:**
  - Unitary Haagerup CFT with multiple vacua [Bottini, Schafer-Nameki '24]
  - Non-unitary Haagerup CFT with a single vacuum [Gang, Kim, Lee '23]
  - Haagerup topological quantum field theory (TQFT) [Huang, Lin '21]
- **Numerical studies of unitary, single-vacuum Haagerup CFTs via anyon chains:**
  - $c \approx 2$ [Huang, Lin, Ohmori, Tachikawa, Tezuka '21]
  - $c \approx 2$ [Vanhove, Lootens, Damme, Wolf, Osborne, Haegeman, Verstraete '21]
  - $c \approx 2.1$ [Liu, Zou, Ryu '22]
  - $c \approx 1.5$ [Corcoran, Leeuw '24]
  - $c \approx 1.3, 1.8, 2.5$ [Hung, Ji, Shen, Wan, Zhao '25]
- **Therefore, it is important to push the study of Haagerup CFT further, both analytically and numerically.**

Outline

- **Part I: Bootstrapping Haagerup CFTs**
- **Part II: Haagerup Symmetry in Non-diagonal RCFTs**

Part I

Part I: Bootstrapping Haagerup CFTs

Modular Bootstrap

- **Torus partition function of a compact unitary CFT is given by**

$$Z(\tau,\bar{\tau}) = \mathrm{Tr}_{\mathcal{H}}\, q^{L_0-\frac{c}{24}}\,\bar{q}^{\bar{L}_0-\frac{c}{24}} = \sum_{h,\bar{h}} n_{h,\bar{h}}\chi_h(\tau)\bar{\chi}_{\bar{h}}(\bar{\tau})$$

- **Modular transformation:**

$$\mathcal{S}:\tau\to-\frac{1}{\tau},\quad \mathcal{T}:\tau\to\tau+1$$

- **Modular invariance:**

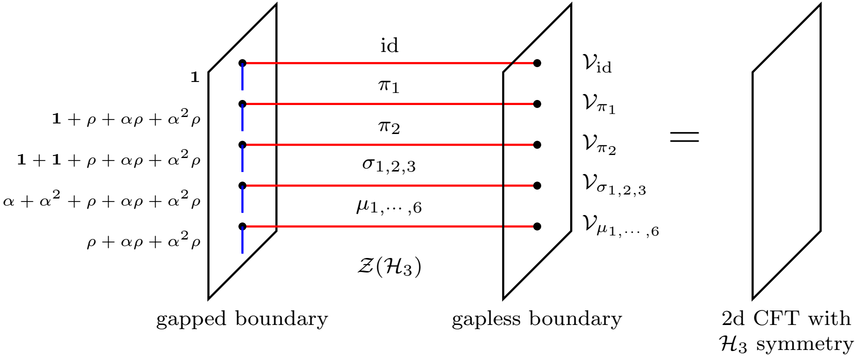
$$Z(\tau,\bar{\tau}) = Z(-\frac{1}{\tau},-\frac{1}{\bar{\tau}}) = Z(\tau+1,\bar{\tau}+1)$$

- **The modular $\mathcal{S}$ invariance imposes strong constraints on the spectrum of the theory $n_{h,\bar{h}}$.**
- **By incorporating the symmetry data into the modular bootstrap, we aim to constrain the spectrum of Haagerup CFTs.**

Symmetry Enhanced Modular Bootstrap

[Lin, Shao '23]

- **SymTFT allows us to see this clearly:**

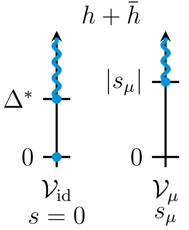


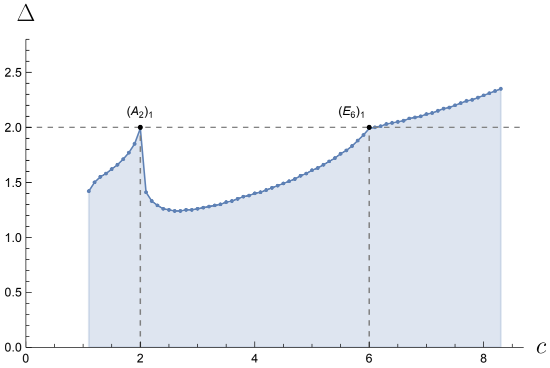
- **The modular crossing equation for $\mathcal{H}_3$ symmetry:**

$$Z_\mu(-\tfrac{1}{\tau},-\tfrac{1}{\bar\tau})=\sum_{\nu\in\mathcal{Z}(\mathcal{H}_3)}S_{\mu\nu}Z_\nu(\tau,\bar\tau),\quad Z_\mu(\tau,\bar\tau)=\sum_{(h,\bar h)\in\mathcal{V}_\mu}n_{\mu;h,\bar h}\chi_h(\tau)\bar\chi_{\bar h}(\bar\tau)$$

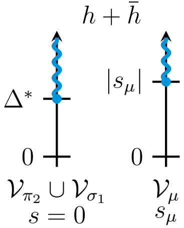
Bootstrapping the Neutral Sector

- **Impose a gap $\Delta \geq \Delta^*$ above the vacuum on the scalar sector ($h - \bar{h} = 0$) in $\mathcal{V}_{\text{id}}$, and assume only the unitarity bound $\Delta \geq |s_\mu|$ on other sectors $\mathcal{V}_\mu$. This plot suggests Haagerup CFTs at $c \approx 2, 6$. Both theories have $(h, \bar{h}) \approx (1, 1)$ state as the lightest neutral scalar.**

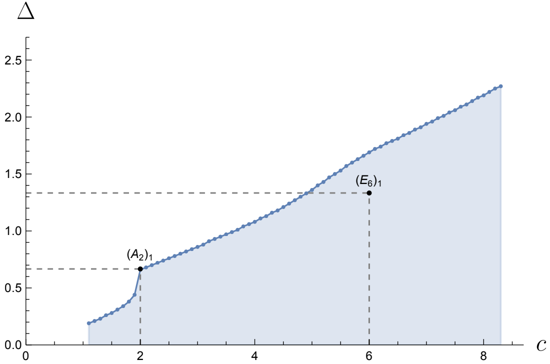




Bootstrapping the $\mathbb{Z}_3$ Charged Sector



- **Next, impose a gap $\Delta \geq \Delta^*$ on the scalar sector in $\mathcal{V}_{\pi_2}\cup\mathcal{V}_{\sigma_1}$, and assume only the unitarity bound $\Delta \geq |s_\mu|$ on other sectors $\mathcal{V}_\mu$. This plot suggests a Haagerup CFT at $c \approx 2$, which has a $(h,\bar{h}) \approx (\frac{1}{3},\frac{1}{3})$ state as the lightest $\mathbb{Z}_3$ charged scalar.**



Part II

Part II: Haagerup Symmetry in Non-diagonal RCFTs

Symmetries in Non-diagonal RCFTs

- **So far, we have seen the importance of symmetries. In particular, symmetries provide strong constraints on the spectrum.**
- **Conversely, how can we identify the symmetries (topological defect lines, TDLs) of a known CFT?**
- **In diagonal RCFTs, a well-known class of TDLs are the Verlinde lines generating $\mathcal{C} \cong \mathrm{Rep}\mathcal{V}$, which commute with the chiral algebra $\mathcal{V}$.**
- **On the other hand, there are many other TDLs that commute with at least the Virasoro algebra. Particularly, how can we determine the TDLs in a non-diagonal RCFT?**

Symmetries in Non-diagonal RCFTs

- **The TDLs of a non-diagonal theory are described by a bimodule category ${}_{\mathcal{A}}\mathcal{C}_{\mathcal{A}}$. These can be calculated using Ocneanu graphs** [Ocneanu '00] **(see also** [Petkova, Zuber '00]**) or α-induction** [Longo, Rehren '95; Bockenhauer, Evans, Kawahigashi '00]**.**
- **I would like to compute them in a more "physicist-friendly" way. This is essentially physical interpretation of α-induction.** [Komargodski, Ohmori, Roumpedakis, Seifnashri '20]

$\mathcal{Z}(\mathcal{H}_3)$ Theory

- **We will focus on non-diagonal modular invariants of a diagonal RCFT $T_{\mathcal{Z}(\mathcal{H}_3)}$, where $\mathrm{Rep}(\mathcal{V}) \cong \mathcal{Z}(\mathcal{H}_3)$.**
- **However, it is still unknown whether the theory $T_{\mathcal{Z}(\mathcal{H}_3)}$ actually exists. At least, it can exist only at $c = 8k$.**
- **As a major development toward solving this mystery,** [Evans, Gannon '10] **conjectured "chiral" $\mathcal{Z}(\mathcal{H}_3)$ theories at $c = 8$, with two possible character vectors.**
- **Importantly, the following discussion holds for any $c = 8k$ where $T_{\mathcal{Z}(\mathcal{H}_3)}$ actually exists.**

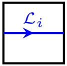
$(E_8)_1$ Has $\mathcal{H}_3 \times \mathcal{H}_3^{\text{op}}$ Symmetry?

- **Consider a theory T_1 at $c = 8k$:**

$$Z^{T_1} = \chi^{T_{\mathcal{Z}(\mathcal{H}_3)}} \mathcal{M}_1 \bar{\chi}^{T_{\mathcal{Z}(\mathcal{H}_3)}}, \quad \mathcal{M}_1 = \begin{pmatrix} \begin{smallmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{smallmatrix} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \end{pmatrix}$$

- $T_{\mathcal{Z}(\mathcal{H}_3)}$ **has 12 Verlinde lines** $\{\mathrm{id}, \pi_1, \pi_2, \sigma_{1,2,3}, \mu_{1,\cdots,6}\}$ **forming** $\mathcal{Z}(\mathcal{H}_3)$**. When we insert** $\mathcal{L}_i$ **along the spatial direction,**

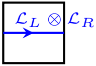
$$Z^{T_{\mathcal{Z}(\mathcal{H}_3)}}\big|^{\mathcal{L}_i} =$$



$$= \chi^{T_{\mathcal{Z}(\mathcal{H}_3)}} D_{\mathcal{L}_i} \bar{\chi}^{T_{\mathcal{Z}(\mathcal{H}_3)}}, \quad (D_{\mathcal{L}_i})_{jj} = \frac{S_{ij}}{S_{1j}}$$

- **Define chiral/anti-chiral symmetries in** T_1**:**

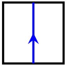
$$Z^{T_1}\big|^{\mathcal{L}_L \otimes \mathcal{L}_R} =$$



$$= \chi^{T_{\mathcal{Z}(\mathcal{H}_3)}} D_{\mathcal{L}_L} \mathcal{M}_1 D_{\mathcal{L}_R} \bar{\chi}^{T_{\mathcal{Z}(\mathcal{H}_3)}}$$

- **The multiplicities in the twisted sector are consistently non-negative integers:**

$$Z^{T_1}\big|_{\mathcal{L}_L \otimes \mathcal{L}_R} =$$



$$= \chi^{T_{\mathcal{Z}(\mathcal{H}_3)}} N_{\mathcal{L}_L} \mathcal{M}_1 N_{\mathcal{L}_R} \bar{\chi}^{T_{\mathcal{Z}(\mathcal{H}_3)}}$$

$(E_8)_1$ Has $\mathcal{H}_3 \times \mathcal{H}_3^{\mathrm{op}}$ Symmetry?

- **What are the TDLs surviving in T_1? We should consider the equivalence classes $[\mathcal{L}_i]$:**

$$\mathcal{L}_{L1} \sim \mathcal{L}_{L2} \sim \mathcal{L}_{R1} \quad \textbf{if } D_{\mathcal{L}_{L1}}\mathcal{M}_1 = D_{\mathcal{L}_{L2}}\mathcal{M}_1 = \mathcal{M}_1 D_{\mathcal{L}_{R1}}$$

- **Focus on the chiral symmetry. From** $[\pi_1] = [\mathrm{id}] + [\mu_i],\ [\pi_2] = 2[\mathrm{id}] + [\mu_i],\ [\sigma_i] = [\sigma_j],\ [\mu_i] = [\mu_j]$**, the only independent lines are** $[\mathrm{id}], [\sigma], [\mu]$**, whose fusion rules are**

$$[\sigma]^2 = 5[\mathrm{id}] + 4[\sigma] + 9[\mu],\ [\mu]^2 = 3[\mathrm{id}] + 3[\sigma] + 6[\mu],\ [\sigma][\mu] = 3[\mathrm{id}] + 3[\sigma] + 8[\mu]$$

- **Looking at** $([\sigma] - [\mu])^2 = 2[\mathrm{id}] + ([\sigma] - [\mu])$**, we find** $[\sigma] - [\mu] = [\alpha] + [\alpha^2]$**.**
- **The unique decomposition satisfying** $[\mu]^2 = 3[\mathrm{id}] + 3[\alpha] + 3[\alpha^2] + 9[\mu]$ **is** $[\mu] = [\rho] + [\alpha\rho] + [\alpha^2\rho]$**.**
- **As a result, the chiral symmetry is identified as** $\mathcal{H}_3$**.**

$(E_8)_1$ Has $\mathcal{H}_3 \times \mathcal{H}_3^{\mathrm{op}}$ Symmetry?

- **Applying the same procedure to the anti-chiral part, we find T_1 has $\mathcal{H}_3 \times \mathcal{H}_3^{\mathrm{op}}$ symmetry.**
- **This is consistent with the number of simple lines, $\mathrm{Tr}[\mathcal{M}_1 \mathcal{M}_1^{\mathrm{T}}] = 36$ and the total quantum dimension of $\mathcal{Z}(\mathcal{H}_3)$, $\sum_i \langle \mathcal{L}_i \rangle^2 = \frac{117}{2}(11 + 3\sqrt{13})$.**
- **If we assume the existence of a non-chiral analog of the conjectured chiral $\mathcal{Z}(\mathcal{H}_3)$ theories at $c = 8$, the corresponding T_1 indeed coincides with $(E_8)_1$. This implies $(E_8)_1$ would have $\mathcal{H}_3$ symmetry!**
  [Möller, Rayhaun '24]
- **In fact, we can directly confirm that gauging $\mathcal{H}_3 \times \mathcal{H}_3^{\mathrm{op}}$ symmetry in T_1 takes us back to $T_{\mathcal{Z}(\mathcal{H}_3)}$.**

T_2 Has a Large Symmetry

- **Next, consider another theory T_2 at $c = 8k$:**

$$Z^{T_2} = \chi^{T_{\mathcal{Z}(\mathcal{H}_3)}} \mathcal{M}_2 \bar{\chi}^{T_{\mathcal{Z}(\mathcal{H}_3)}}, \quad \mathcal{M}_2 = \begin{pmatrix} 1 & 1 & & & & & \mathbf{0} \\ 1 & 1 & & & & & \\ & & 2 & & & & \\ & & & 2 & & & \\ & & & & 2 & & \\ & & & & & 2 & \\ \mathbf{0} & & & & & & \ddots \end{pmatrix}$$

- **Following the same procedure, it turns out T_2 has 20 simple lines:**

$$\{[g]\}, \ \{[g\lambda]\}, \ [\mu_L], \ [\mu_R], \quad g \in \mathbb{Z}_3^2$$

- **They satisfy the commutative fusion rules:**

$$[\mu_i]^2 = 9[\mu_i] + \sum [g], \ [\mu_i][g] = [\mu_i], \ [\mu_L][\mu_R] = \sum [g\lambda],$$
$$[\mu_i][\lambda] = [\mu_L] + [\mu_R], \ [\lambda]^2 = [\mathbf{1}] + 9[\lambda] + [\mu_L] + [\mu_R]$$

- $\{[g]\}, \ [\mu_i]$ **form the near-group category** $\mathbb{Z}_3^2 + 9$**. This is consistent with** $\mathrm{Tr}[\mathcal{M}_2 \mathcal{M}_2^{\mathrm{T}}] = 20$ **and** $\sum_i \langle \mathcal{L}_i \rangle^2 = \frac{117}{2}(11 + 3\sqrt{13})$**.**
[Evans, Gannon '12; Izumi '15]

- **At $c = 8$, the corresponding T_2 would be $(A_2)_1 \times (E_6)_1$.**

Summary

- **The numerical modular bootstrap for $\mathcal{H}_3$ symmetry suggests Haagerup CFTs at $c \approx 2, 6$. Both theories have an $(h, \bar{h}) \approx (1, 1)$ state, and only $c \approx 2$ theory has an additional $(h, \bar{h}) \approx (\frac{1}{3}, \frac{1}{3})$ state.**
- **The theory T_1 at $c = 8k$ realizes $\mathcal{H}_3 \times \mathcal{H}_3^{\mathrm{op}}$ symmetry. If we assume the conjectured theory $T_{\mathcal{Z}(\mathcal{H}_3)}$ at $c = 8$ exists, $(E_8)_1$ would have $\mathcal{H}_3 \times \mathcal{H}_3^{\mathrm{op}}$ symmetry.**
- **The theory T_2 has a large symmetry including the near-group category $\mathbb{Z}_3^2 + 9$. Can this Abelian symmetry guide us to an analytic construction of the $c = 2$ Haagerup CFT?**

$$\begin{tikzcd}
& T_{\mathcal{Z}(\mathcal{H}_3)} \arrow[dl, shift right, "\mathbf{1}+\pi_1+2\pi_2"'] \arrow[dr, shift left, "\mathbf{1}+\pi_1"] & \\
T_1 \arrow[ur, shift right, "\mathcal{A}^{\mathbf{diag}}"'] \arrow[rr, shift left, "\mathbb{Z}_{3\mathbf{B}}"] & & T_2 \arrow[ul, shift left, "\mathcal{A}_{[\lambda]}"'] \arrow[ll, shift left, "\mathbb{Z}_3^{\mathbf{diag}}"] \\
((E_8)_1) & & ((A_2)_1 \times (E_6)_1)
\end{tikzcd}$$

Thank you for your attention!