Conserved charges in deformed 2d CFTs

Federico Ambrosino [Perimeter Institute]

Interfaces and Symmetry, March 5th 2026

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Based on [2504.05277, 2511.02007]
with Runkel & Watts,
+ to appear [FA, Runkel, Watts, Konechny]^2
+ partially [2501.07511] & [2601.18667]

Deformed d dimensional CFT

Goal: understand RG flows

$$\mathcal{T}_{\mathrm{UV}} = \mathrm{CFT}_{\mathrm{UV}} + g \int \varphi \,, \qquad (\Delta_\varphi \lesssim d)$$

Gapped | Gapless

$$\mathcal{T}_{\mathrm{IR}} = \quad \mathrm{TQFT} \quad \mathrm{CFT}_{\mathrm{IR}}$$

Already very hard to solve $\mathrm{CFT}_{\mathrm{d}}$

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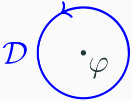
Hopeful to solve $\mathrm{CFT}_{\mathrm{d=2}}$ (Virasoro algebra)

Symmetries strike back

Topological operator = Symmetry

[Gaiotto, Kapustin, Seiberg, Willett]

Today: QFT_2



(Codimension 1 😃)

More defects ⇒ More symmetries ⇒ More data

$$\mathrm{CFT}_{\mathrm{UV}} + g \int \varphi \rightsquigarrow \,?$$

Deformed 2 dimensional CFT

Let us consider 2 dimensions

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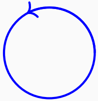
Often we know topological defects $\subset \mathrm{CFT}_{\mathrm{UV}}$ $\mathcal{D}$ 

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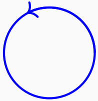
Often we know topological defects $\subset \mathrm{CFT}_{\mathrm{UV}}$ $\mathcal{D}$ 

What can we say about defects along RG?

Invariants under RG flows

$$\mathrm{CFT}_{\mathrm{UV}} + g\int \varphi \rightsquigarrow \textbf{?}$$

When: $[\mathcal{D}, \varphi] = 0 \Leftrightarrow$ 

$=$ 

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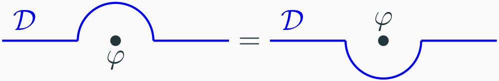

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Many constraints from UV data: [Chang, Lin, Shao, Wang, Yin]

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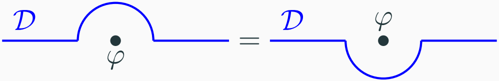
When: $[\mathcal{D},\varphi]=0 \Leftrightarrow$ 

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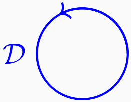
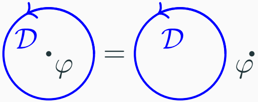
Q Dim $d_{\mathcal{D}} =$ 

RG invariant! 

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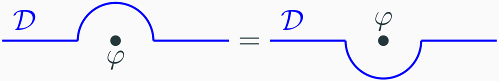

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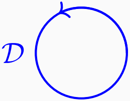

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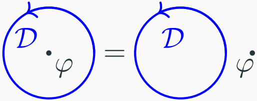
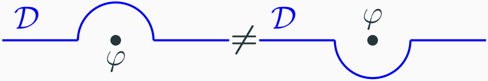


Fixes IR fixed points from RCFTs

[Tanaka, Nakayama][FA, Negro][FA, Prochazka]

Punchline of the talk

Generically:



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New class: **translational invariant defects** [FA, Runkel, Watts]

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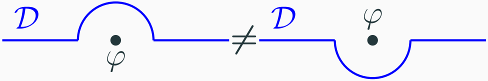

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- Extend class of topological symmetries:
  $\mathcal{F} \simeq \mathcal{D}_{\mathrm{top}} \subset \mathcal{D}_{transl.inv.} \simeq$ DY-module(Rep[$\mathcal{F}$])
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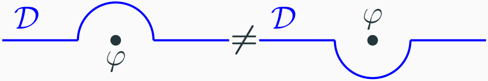

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Beyond topological framework!

Warm up: top. defects in RCFTs
To ($\mathcal{W}$) infinity and beyond!

A primer on Virasoro Minimal models

$\mathcal{M}(p,q)$: Rational 2D CFT, A-series

- Central charge: $c = 1 - \frac{6(p-q)^2}{pq}$ Unitary iff $q = p + 1$
- $\frac{(p-1)(q-1)}{2}$ primaries $\phi_{(r,s)} = \phi_{(p-r,q-s)}$
- weigths $h_{(r,s)}$ and C^t_{rs} all known
- Fusion Category: $\phi_\rho \otimes \phi_\sigma = \sum_\delta \mathcal{N}^\delta_{\rho\sigma}\, \phi_\delta$

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Topological lines $\subseteq$ Verlinde Lines $\{\mathcal{L}_\rho\} \overset{1 \text{ to } 1}{\Longleftrightarrow} \{\phi_\rho\}$

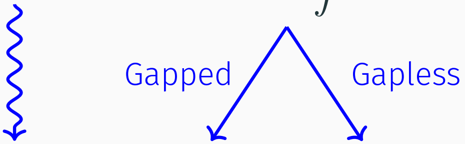
Non-invertible symmetry: $\mathcal{L}_\rho \times \mathcal{L}_\sigma = \sum_\delta \mathcal{N}^\delta_{\rho\sigma}\, \mathcal{L}_\delta$

Ward identity: $\mathcal{L}_\rho$

ϕ_σ $= \left[\frac{S_{\rho\sigma}}{S_{0\sigma}}\right] \cdot \phi_\sigma$

Relevant deformation of minimal models

$$\mathcal{T}_{\mathrm{UV}} = \mathcal{M}(p,q) + g_\rho \int \phi_\sigma \,, \qquad (h_\rho < 1)$$



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Derfomation by primary field

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TODAY

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c-theorem with: $c_{\rm eff} = 1 - \frac{6}{pq}$ $(\mathcal{PT}$-sym) [Ravanini]

We know all* topological defects!

Impose anomaly matching

Strategy: Fix $\mathcal{T}_{UV} = \mathcal{M}(p, q)$

1. For any relevant def. compute all RG invariants
2. Generate all $\mathcal{M}(p', q')$ with $c_{\text{eff}}^{\text{IR}} < c_{\text{eff}}^{\text{UV}}$
3. Exclude $\mathcal{T}_{\text{IR}}$ not fulfilling constraints

Produces putative flows:

$$\mathcal{M}(p, q) \xrightarrow{\phi_{(r,s)}} \mathcal{M}(p', q')$$

New flows

Result: $\mathcal{M}(p,q) \xrightarrow{\phi_{(1,2k+1)}} \mathcal{M}(p,kp-q)$

[Nakayama, Tanaka][FA, Negro]

Only known flows:

- $k=1$: $\phi_{(1,3)}$ [Fendley, Saleur, Al. Zamolodchikov][Al. Zamolodchikov]
- $k=1/2$: $\phi_{(1,2)}, \phi_{(1,5)}$ [Dorey, Dunning, Tateo]
- $k=3$: $\phi_{(1,7)}$ $\quad \mathcal{M}(3,10) \to \mathcal{M}(3,8)$ [Narovlansky, Sun, Tarnopolsky]

Preserve $\mathrm{Rep}[\mathrm{SU}(2)_{q-2}] \supset \mathbb{Z}_2$ symmetry*:

$$\{\mathbf{1}, \mathcal{L}_{(2,1)}, \cdots, \mathcal{L}_{(q-1,1)}\}, \qquad \mathcal{L}_{(q-1,1)} \times \mathcal{L}_{(q-1,1)} = \mathbf{1}$$

$$[\mathcal{L}_{(n,1)}, \phi_{(1,2m+1)}] = 0, \qquad m = 1, \cdots, k$$

* Only $\mathrm{Rep}[\mathrm{PSU}(2)_{q-2}]$ for $k \in \mathbb{Z}/2$

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Suggest a vast generalization [FA, Prochazka]

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$\mathcal{W}_N$ algebra

Virasoro algebra: generated by modes of spin 2 field $T(z)$

$\mathcal{W}_N$ algebra: additional currents of spin $s = 2, \cdots, N$

$$\{T(z), W^{(3)}, \cdots, W^{(N)}\}$$

[Zamolodchikov[2], Fateev, Lukyanov. . .]

$$T(z)W^{(s)}(u) \sim \frac{sW^{(s)}(u)}{(z-u)^2} + \frac{\partial W^{(s)}(u)}{z-u} + \cdots .$$

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$$T(z)W^{(s)}(u) \sim \frac{sW^{(s)}(u)}{(z-u)^2} + \frac{\partial W^{(s)}(u)}{z-u} + \cdots .$$

E.g. $\mathcal{W}^{(3)}$:

$$W(z)W(u) \sim \frac{c/3}{(z-u)^6} + \frac{2T(u)}{(z-u)^4} + \frac{\partial T(u)}{(z-u)^3} + \frac{1}{(z-u)^2}\left(\frac{3}{10}\partial^2 T(u) + \frac{32}{22+5c}\Lambda(u)\right) + \cdots$$

$\mathcal{W}_N$ Minimal models

$\mathcal{W}_N$ rational truncations: $\mathcal{W}_2$ minimal models $\mathcal{W}_2(p,q)$

- Rational CFT: $c^{(2)}_{p,q} = (2-1)\left[1 - \frac{2(2+1)(p-q)^2}{pq}\right]$
- Unitary iff $|p-q| = 1$
- Coset construction $\frac{\widehat{\mathfrak{su}}(2)_\kappa \times \widehat{\mathfrak{su}}(2)_1}{\widehat{\mathfrak{su}}(2)_{\kappa+1}}$ $\kappa{=}p/(q{-}p){-}2$

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[FA, Procházka]

- *Adjoint*-category of deformation $\varphi \in PSU(N)_{q-N}$
- Preserves large symmetry $\mathbf{Rep}[SU(N)_{p-N}] \supset \mathbb{Z}_N$ (Bootstrap?)
- Uniform in rank! $\Leftarrow$Truncations of $\mathcal{W}_\infty$ (Ask me!)
- **Incredible power of topological lines!**

New questions

We found new flows between Virasoro and $\mathcal{W}_N$ models.
Focus on Virasoro

What is special about flows with φ?

Remember: $\phi_{(1,r)}$, $r = 2, 3, 5$ are integrable

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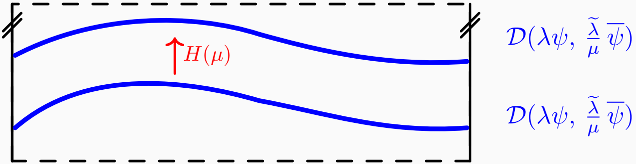
We found:[Ambrosino, Runkel, Watts][2]

Non-top. defects = Non-local conserved charges

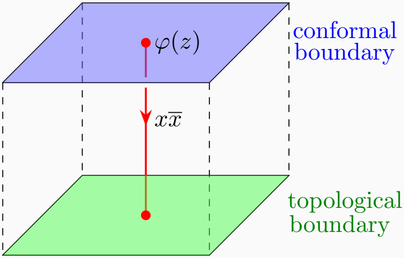
New class of defects!

Translational invariant defects



3d TFT for 2d RCFT

$\mathrm{TFT}_{d=3}$ encoding the $\mathrm{QFT}_{d=2}$

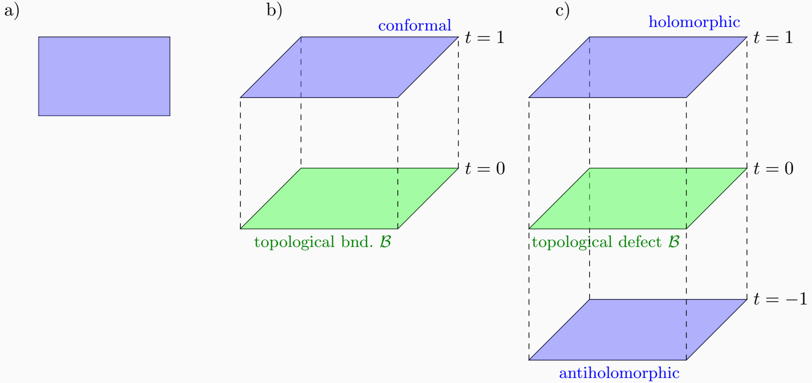


- Topological ops in QFT: Neumann @ top boundary
- Non topological: Dirichlet boundary conditions

All lines = topological anyons of 3D TFT

Chiral TFT

In 2d CFT I like chiral TFT:



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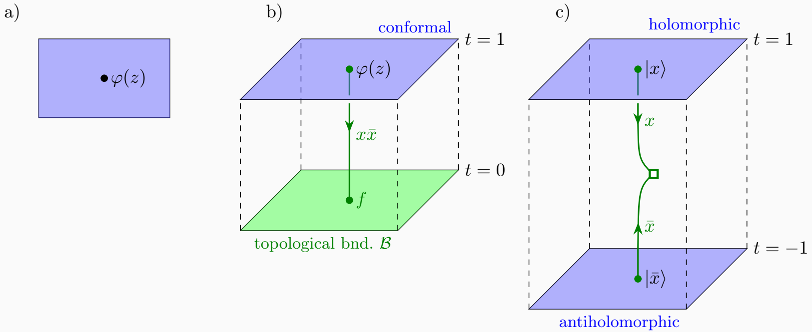
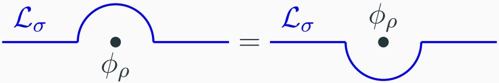
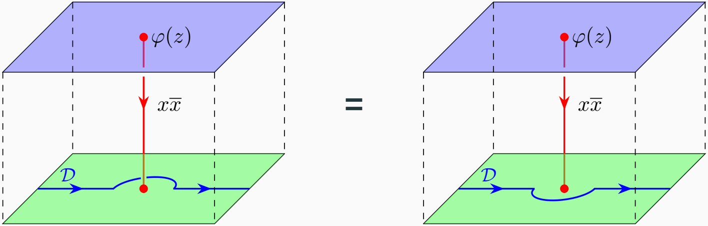


Example: commutation condition from before





Commuting defects and where to find them

Hamiltonian of perturbed CFT: $H(\mu) = H_0 + H_{\text{pert}}(\mu)$

$$H_0 = \frac{2\pi}{L}\left(L_0 + \bar{L}_0 - \frac{c}{12}\right) \quad , \quad H_{\text{pert}}(\mu) = 2i\mu \int_0^L \varphi(s) ds \; .$$

$\mathcal{D}$ topological in H_0:

$$\underbrace{[H_0, \mathcal{D}] = 0}_{\text{Top. in CFT}}, \qquad \& \qquad \underbrace{[\varphi, \mathcal{D}] = 0}_{\text{Comm. with def.}}$$

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Stronger than conserved!

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$\mathcal{L}$ conserved in $H(\mu)$:

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But: $[H_0, \mathcal{L}] \neq 0, \quad [\varphi, \mathcal{L}] \neq 0$

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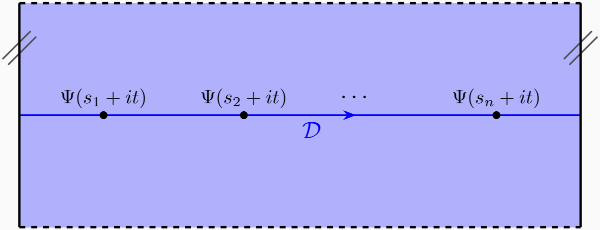
Weaker than topological! $[\mathbb{T}, \mathcal{L}] \neq 0$ What kind of animals?

Perturbed defect operators

Perturb a topological defect $\mathcal{D}$ by (chiral) defect operators:

$$\underbrace{\psi(z)}_{\text{hol}}, \qquad \underbrace{\overline{\psi}(\bar{z})}_{\text{antihol.}}, \qquad \psi, \overline{\psi} \in \mathcal{D}, \qquad \varphi \sim \overline{\psi}\psi$$



$$\mathcal{D}(\lambda\psi + \widetilde{\lambda}\overline{\psi}) = \exp\left(\int_0^L (\lambda\psi(s+it) + \tilde{\lambda}\bar{\psi}(s+it))\, ds\right),$$

s coordinate along defect, t position of defect

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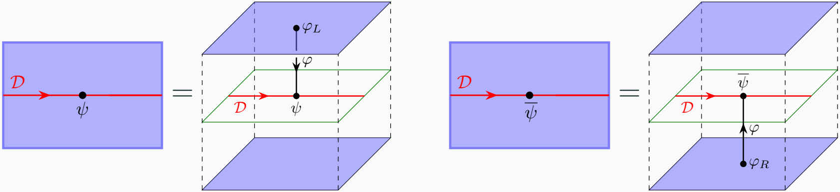

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s coordinate along defect, t position of defect

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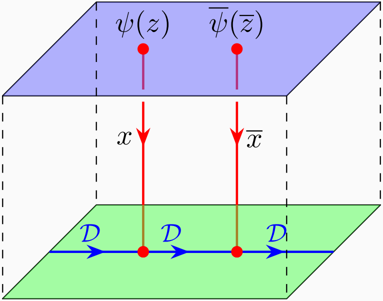

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Bulk commutation condition

Conservation law in **QFT**: (*Complicated:*)

$$\left[\, H(\mu)\,,\, \mathcal{D}(\lambda\psi + \widetilde{\lambda}\overline{\psi})\,\right]_{\color{blue}{\mathrm{QFT}}} \;=\; 0$$

Equivalent to condition in CFT at any order in $\lambda, \widetilde{\lambda}$:

$$\boxed{\mu\,[\mathcal{D},\varphi]_{\color{red}{\mathrm{CFT}}} \;=\; \lambda\widetilde{\lambda}\,[\psi,\overline{\psi}]_{\color{red}{\mathrm{CFT}}}}$$

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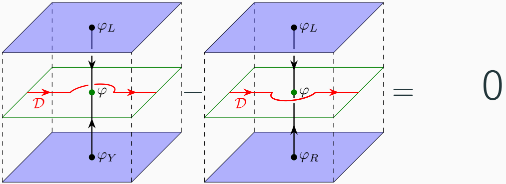
Remember: Topological $\longleftrightarrow [\mathcal{L},\varphi] = 0$

- $\rightarrow$ "Adiabatically" deform Line & Bulk simulatenoeusly
- $\rightarrow$ Rigidly translational invariant
- $\rightarrow$ Often not renormalized (E.g. $h < \frac{1}{2}$ in $\mathcal{M}(p,q)$)

Condition for translation-invariant

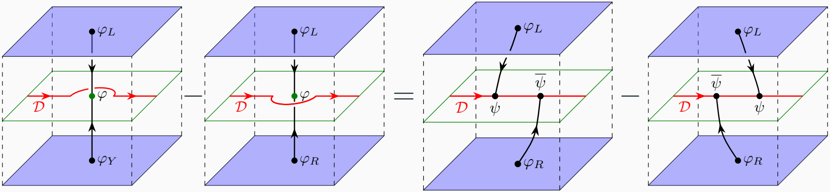
$$\boxed{\text{Topological:} \qquad [\mathcal{D}, \varphi] = 0}$$



Rigid: deformed bulk, undeformed line

Condition for translation-invariant

$$\text{Transl. invariant:} \qquad \mu\,[\mathcal{D},\varphi] \;=\; \lambda\widetilde{\lambda}\,[\psi,\overline{\psi}]$$

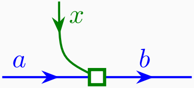


Natural: deformed bulk & line accordingly

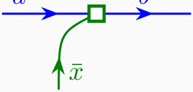
$\mathcal{D}(\lambda\psi + \frac{\widetilde{\lambda}}{\mu}\,\overline{\psi})$ is (non-local) conserved charge

Conditions

Basis for defect fields:

$$\psi = \sum_{a,b\in\mathcal{D}} \kappa_{ab}$$ 

,

$$\overline{\psi} = \sum_{a,b\in\mathcal{D}} \widetilde{\kappa}_{ab}$$ 

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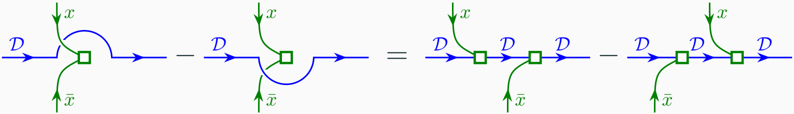
Bulk commutation condition:



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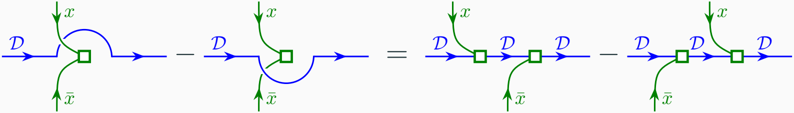

Using standard TQFT rules:

$$\delta_{ac}\left(\mathrm{R}^{(xa)b}-\frac{1}{\mathrm{R}^{(ax)b}}\right)\mathrm{F}_{1b}^{(ax\bar{x})a} = \delta_{b\in\mathcal{D}}\,\kappa_{ab}\widetilde{\kappa}_{bc}-\sum_{d\in\mathcal{D}}\mathrm{F}_{db}^{(xa\bar{x})c}\,\widetilde{\kappa}_{ad}\kappa_{dc}$$

$$\forall a,c\in\mathcal{D},\qquad \forall b\in a\otimes x$$

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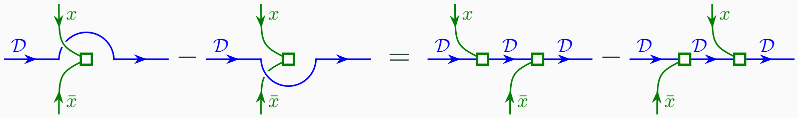

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When solution exists ⇒ Defect is conserved

Solutions in Minimal models

Minimal model $M(p,q)$	perturbing bulk field φ	weights $h=\bar{h}$ of φ	topological defect solving (4)
$q \geq 3$	$(1,2)$	$h_{1,2} = \frac{3}{4}t - \frac{1}{2}$	$(1,1) \oplus (1,2)$
$q \geq 4$	$(1,3)$	$h_{1,3} = 2t - 1$	$(1,2)$
$q \geq 6$	$(1,5)$	$h_{1,5} = 6t - 2$	$(1,3)$
$q = 9, 10, 18$	$(1,7)$	$h_{1,7} = 12t - 3$	$(1,5)$

And many others... E.g. (1,9) on (1,5), (1,6) (1,7), etc...

$$\left[\mathcal{D}(\lambda, \mu/\lambda), \mathcal{D}(\lambda', \mu/\lambda')\right] = 0, \qquad \lambda \neq \lambda'$$

Infinitely many non-local conserved charges!

(Complicated condition)

$(1,7)$ deformation has no local conserved charges [BLZ]

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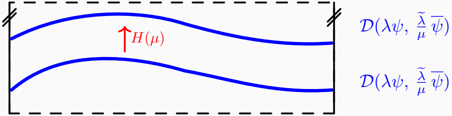
Imply Y-system/TBA equations!! **Exact solvability?**

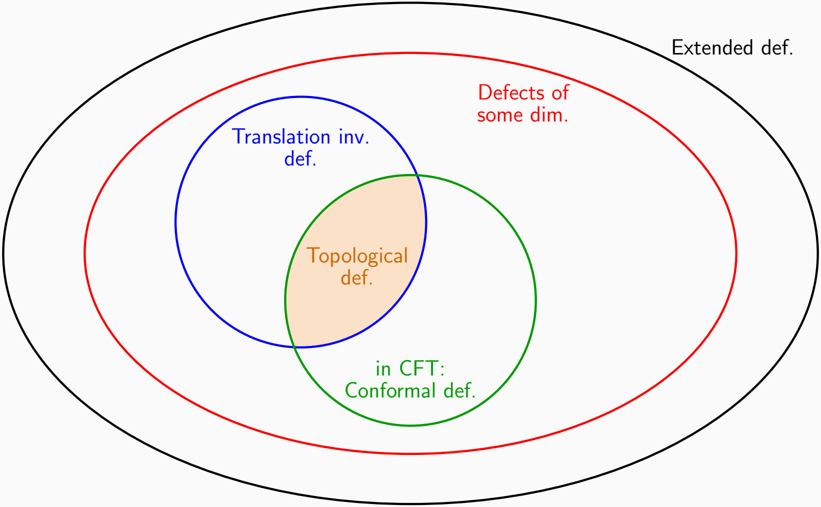
Translational invariant defects:

$$\left[\mathcal{D}(\lambda\psi + \frac{\widetilde{\lambda}}{\mu}\overline{\psi}), H(\mu)\right] = 0$$



- Admit non-singular fusion!
- $\mathcal{D}_{\text{top}} \subseteq \mathcal{D}^{\text{per}}_{\text{trans.inv.}} \subseteq \mathcal{D}_{\text{trans.inv.}}$
- Generalize topological defects: $\sim \mathcal{F}$ Fusion category
- Trans. Invariant: Drinfeld-Yetter Module $(\text{Rep}[\mathcal{F}])$
- Non-local conserved charges in QFT

General picture



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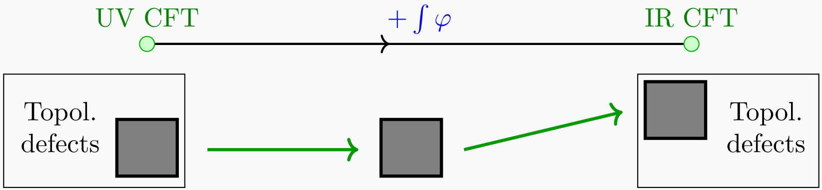

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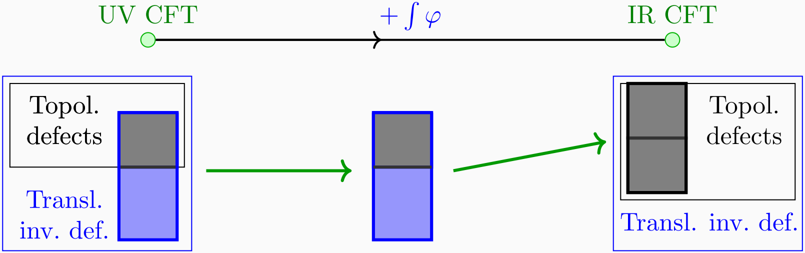

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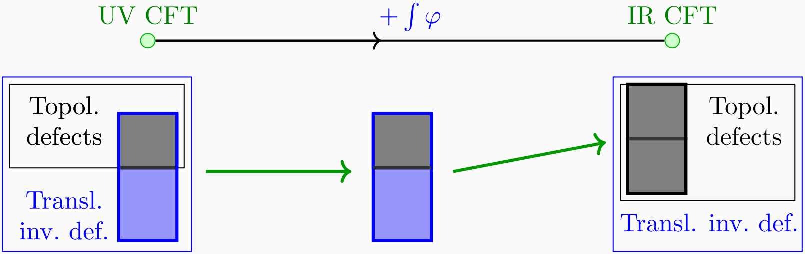

Emergent symmetries in the IR?

Where do those new defect flow?

New defects = New constraints

[FA, Runkel, Watts, Konenchy][Runkel '07]

Warm up Chiral deformation of $\mathcal{M}(p,q)$:

E.g. chiral $\psi = \psi_{(1,3)}(z)$, $\qquad \mathcal{D}_s(\lambda) := \mathcal{L}_{(1,s)}(\lambda\psi)$

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Chiral def breaks all lines $\mathcal{L}_{(1,s)}$!

$\mathcal{D}(\lambda\psi)$ satisfy Y-systems! [Runkel][FA, Runkel, Watts, Konechny][2]

$$\mathcal{D}_2(\lambda)\mathcal{D}_s(\zeta^{\pm s}\lambda) = \mathcal{D}_{s-1}(\zeta^{\pm(s+1)}\lambda) + \mathcal{D}_{s+1}(\zeta^{\pm(s-1)}\lambda)\,, \qquad \zeta = e^{i\pi p/q}$$

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- ⋆ In the IR: $\mathcal{D}_s(r^\alpha e^{i\theta}) \to \mathcal{D}_\sigma^\infty \exp(a_\sigma(\theta) r^\alpha)$ $\qquad \alpha = \frac{1}{2(1-p/q)}$
- ⋆ We can solve asymptotically! $\quad p-1$ Stokes $\mathcal{M}(p,-)$

Solutions of Hirota in $\mathcal{M}(p,q)$

$$\boxed{\mathcal{D}_{\mathsf{s}}(\zeta\lambda)\mathcal{D}_{\mathsf{s}}(\zeta^{-1}\lambda) = 1 + \mathcal{D}_{\mathsf{s}-1}(\lambda)\mathcal{D}_{\mathsf{s}+1}(\lambda)}$$

Form asymptotic $\mathbb{R}^{\max}$ semi-ring: $\mathcal{D}_{\mathsf{s}}(\lambda) \to \mathcal{D}_{\sigma}^{\infty} \exp(a_\sigma(\theta))$

- $\mathcal{M}(2,-)$
- $\mathcal{M}(3,-)$:
- etc... We can do any chiral on any defects ...
- We (**Gerard & Anatoly**) check with TCSA!

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- $\mathcal{M}(3,-)$: non trivial IR topological defects:

  $$\mathcal{D}_s(\lambda) \to \boldsymbol{Id} \exp\big(a_s \cos(\overline{\alpha\theta})\big) + \mathcal{D}_{q-1} \exp\Big(a_{q-s} \cos\Big(\overline{\alpha(\theta-\pi)}\Big)\Big)$$

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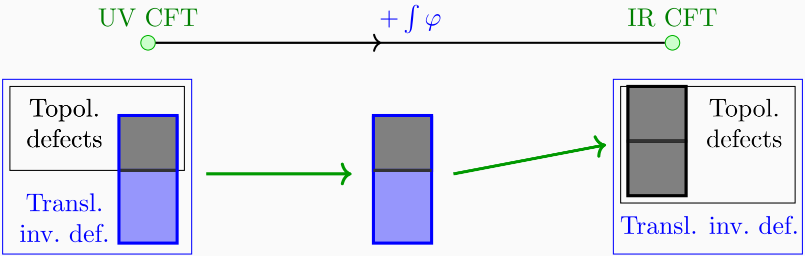
Explicit realization



Explicit realization in $\mathcal{M}(3,-)$ example

1. $\mathcal{D}_{(1,s)}$ broken by chiral $\phi_{(1,3)} = \psi_{(h,0)}$
2. Survives along flow as $\mathcal{D}(\lambda\psi)$
3. Re-emergence of topological symmetry in deep IR:

$$\mathcal{D}(\lambda\psi) \quad \rightsquigarrow \quad \mathrm{Id} \quad \text{or} \quad \mathcal{D}_{(1,q-1)}$$

Concluding remarks

News from 2d CFTs!![Belavin, Polyakov Zamolodchikov, 1984]

- **New flows via non-invertible symmetries**
  - → From Virasoro to $\mathcal{W}_N$ and beyond! 
  - ? Truncations $Y_{N,M,L} \cap Y_{0,P,Q}$, Grassmanian VOA[Eberhardt, Prochazka], compact irrational CFT [Antunes, Behan, Rong], WZW?[Levine. . .] etc...
- **Beyond the topological framework!**
- **New Transl invariant defects!** Higher dim?
- **Great responsibilities!** Dynamical constraints?
  - → Defects in IR from solving Y-systems
  - ? New exact solvability?
- Virasoro ADE classification[Cappelli,Itzykson,Zuber][Nakayama,Tanaka]
  - ? Modular invariant classification for $\mathcal{W}_N$? [Gannon] [WIP: FA, Behan]

Thanks for your attention!