Surface Defects & Integrability

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Based on 2503.22598 with C. Kristjansen and Chenliang Su

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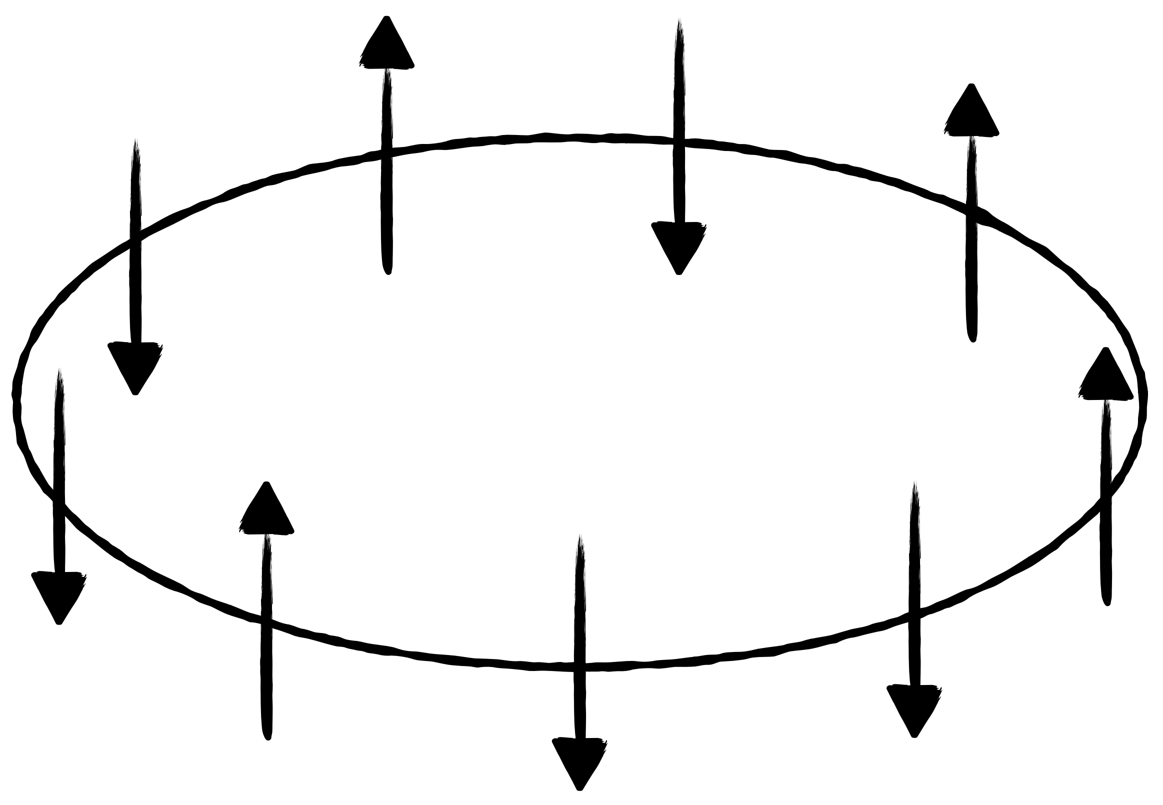
FONDO
ITALIANO
PER LA
SCIENZA

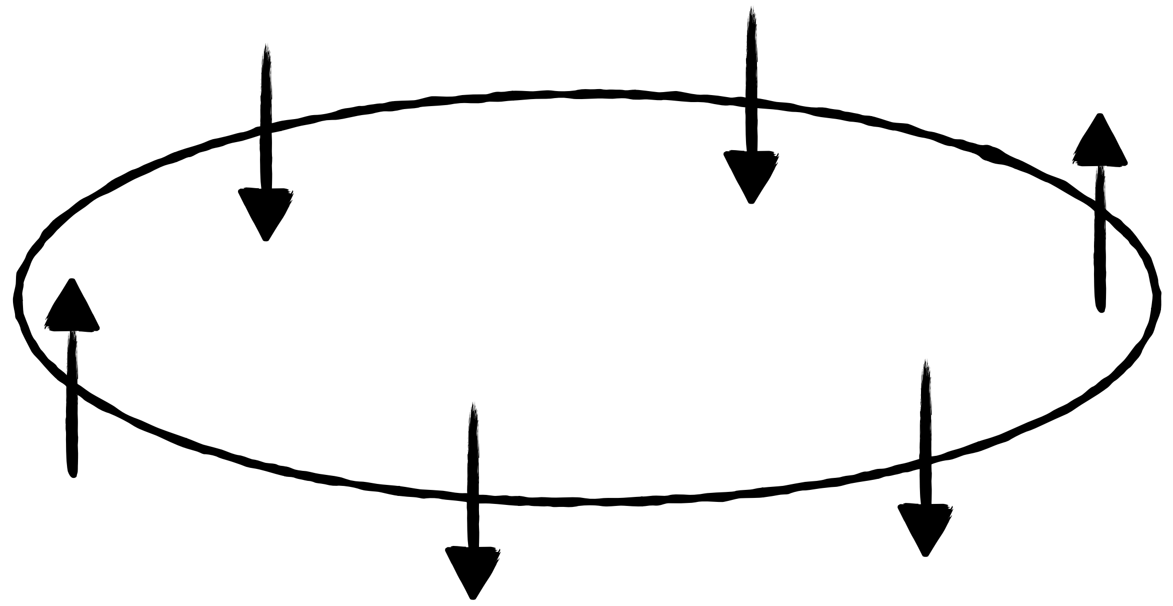
Integrability in *N*=4 SYM Theory at Large-N

- Local operators $\mathrm{Tr}\ \phi^{I_1}\cdots\phi^{I_L}$ of the same engineering dimension L can mix
- Diagonalise dilatation operator D to find scaling ops $\langle \mathcal{O}(x)\mathcal{O}(0)\rangle \sim |x|^{-2\Delta}$
- Minahan-Zarembo '02: $D_{1-loop} \leftrightarrow H$ Hamiltonian on a 1d spin chain
- H is integrable $\implies$ $\exists$ tower of conserved charges Q
- H diagonalisable via Bethe ansatz



Closed Sectors and Integrability

- SU(2) sector: simplest closed sub-sector at 1-loop consists of $\{Z, Y\}$ only
- Heisenberg spin chain with identification $Z = |\downarrow\rangle$ and $Y = |\uparrow\rangle$
- E.g. $\mathrm{Tr}\, ZZYZZY \quad \longleftrightarrow$



- Hamiltonian $H \propto \sum_{i=1}^{L} (1_{i,i+1} - P_{i,i+1})$
- Eigenstates $|\mathbf{u}\rangle$ found by solving Bethe equations $\left(\frac{u_j + i/2}{u_j - i/2}\right)^L \prod_{k\neq j}^{\mathcal{N}} \frac{u_j - u_k - i}{u_j - u_k + i} = 1$

Closed Sectors and Integrability

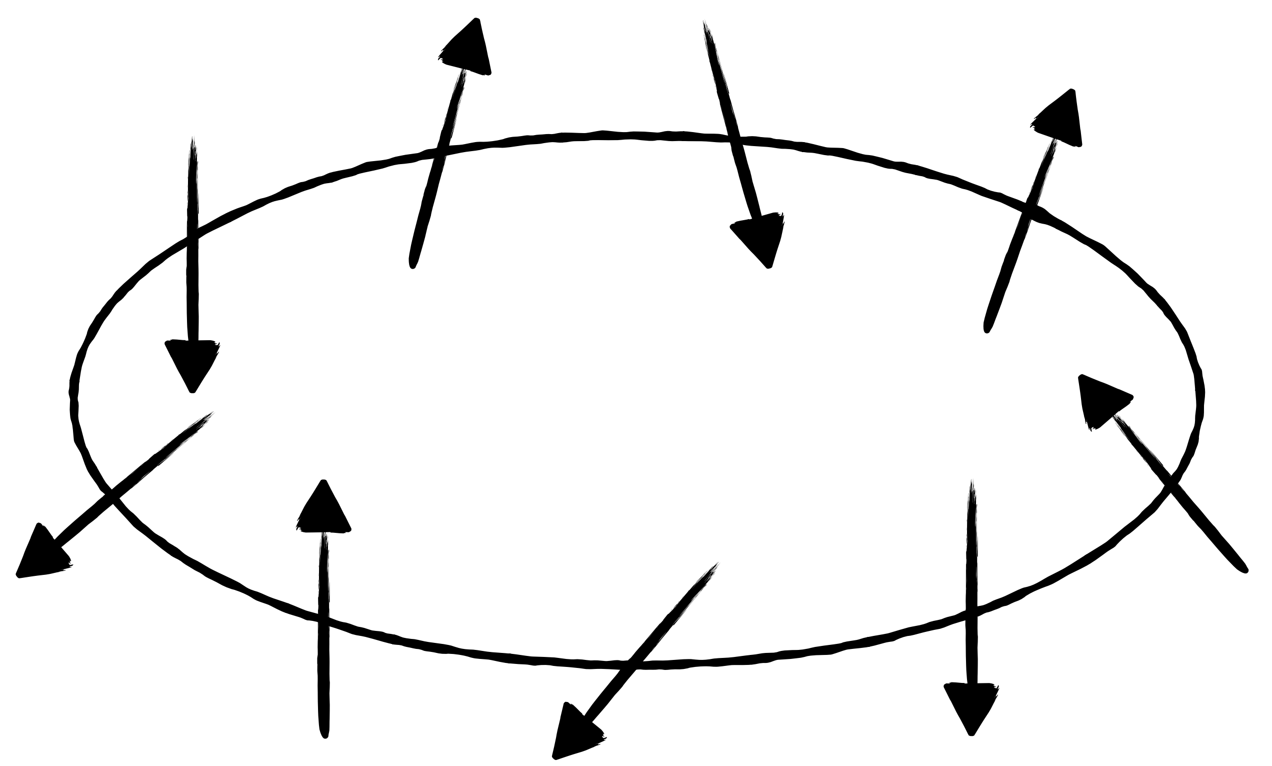
- SO(6) sector: $\{Z, Y, X, \bar{X}, \bar{Y}, \bar{Z}\}$

  $\longrightarrow$ each site in vector representation of so(6)

- SL(2) sector: $\{Z, DZ, D^2Z, \dots\}$ where $D = D_t + D_x$ and $D_\mu = \partial_\mu + i[A_\mu, \bullet\,]$

  $\longrightarrow$ infinite-dimensional Hilbert space at each site

- Integrability extends to full planar *N*=4 SYM beyond 1-loop



Defects in CFT

- In CFTs with defects, local operators acquire 1-pt functions $\langle \mathcal{O}_\Delta \rangle = \dfrac{a_{\mathcal{O}}}{r^\Delta}$
- Some defects described by singularity conditions

  e.g. $\phi^I = \dfrac{\omega^I}{r} +$ fluctuations
- At leading order $\langle \mathrm{Tr}\, \phi^{I_1} \cdots \phi^{I_L} \rangle$ obtained by substituting classical part
- But operators mix! $\longrightarrow$ Use integrability to find scaling ops

Integrable Defects

- In spin chain picture, encode defect as boundary state

$$\langle \mathscr{B} | = \mathrm{tr} \left(\sum_{I=1}^{d} \omega^I \langle I | \right)^{\otimes L}$$

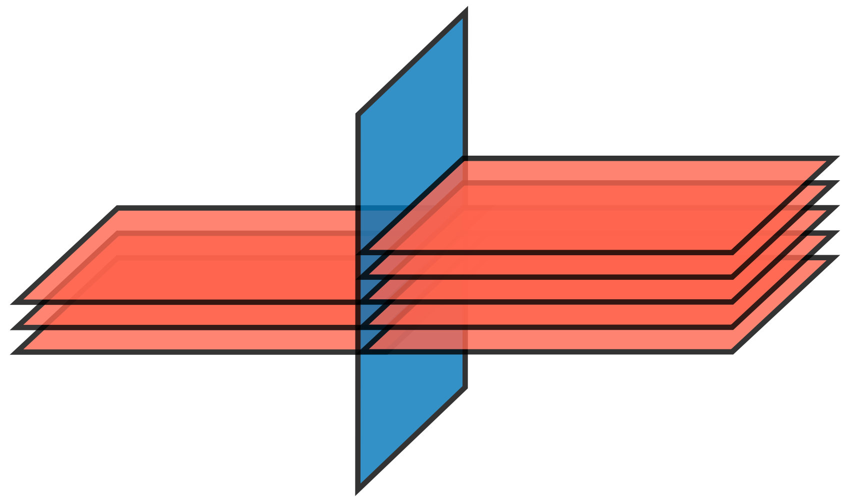
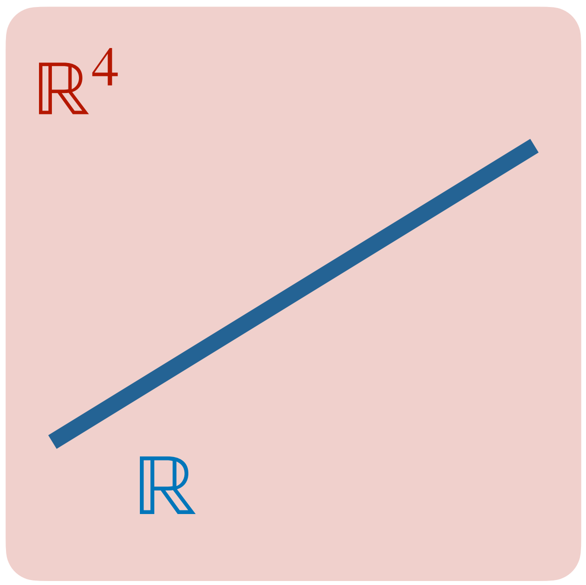
- LO 1-pt function of D-eigenstates $\langle \mathscr{O} \rangle \propto \langle \mathscr{B} \,|\, \mathbf{u} \rangle$

Specifies defect

Specifies $\mathscr{O}$ via Bethe roots $\mathbf{u}$

- Defect is integrable $\Longleftrightarrow Q^{\mathrm{odd}} | \mathscr{B} \rangle = 0$ and $\langle \mathscr{B} \,|\, \mathbf{u} \rangle$ has closed-form
- Sometimes integrability persists to all orders in λ

Integrable 1/2-BPS Defects in *N*=4 SYM

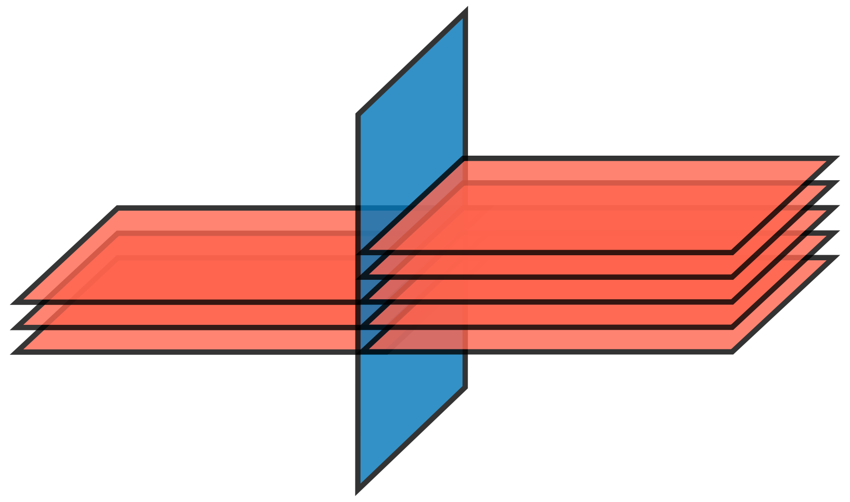
Co-dim 1	Co-dim 2	Co-dim 3
D3-D5 system		1/2-BPS ’t Hooft line
		
$\phi_i \to t_i^{(k)}/r$		$\phi^I \to n^I/r$

[Bajnok, Buhl-Mortensen, de Leeuw, Gombor, Ipsen, Komatsu, Kristjansen, Linardopoulos, Wang, Wilhelm, Zarembo, … ’15 — today]

Integrable 1/2-BPS Defects in *N*=4 SYM

Co-dim 1

D3-D5 system



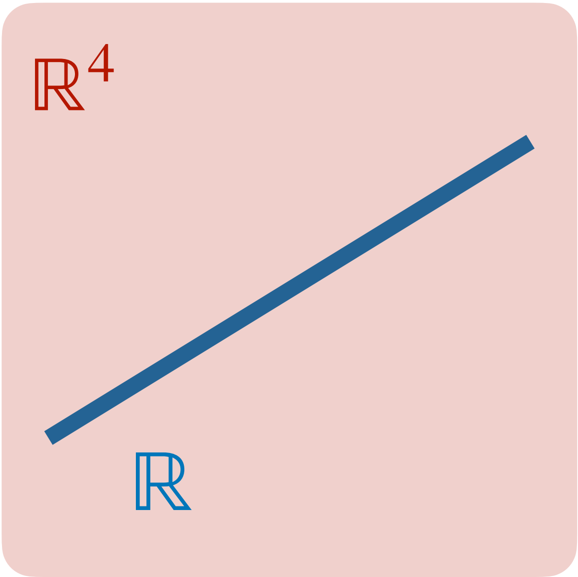
$$\phi_i \to t_i^{(k)}/r$$

Co-dim 2

?

Co-dim 3

1/2-BPS ’t Hooft line



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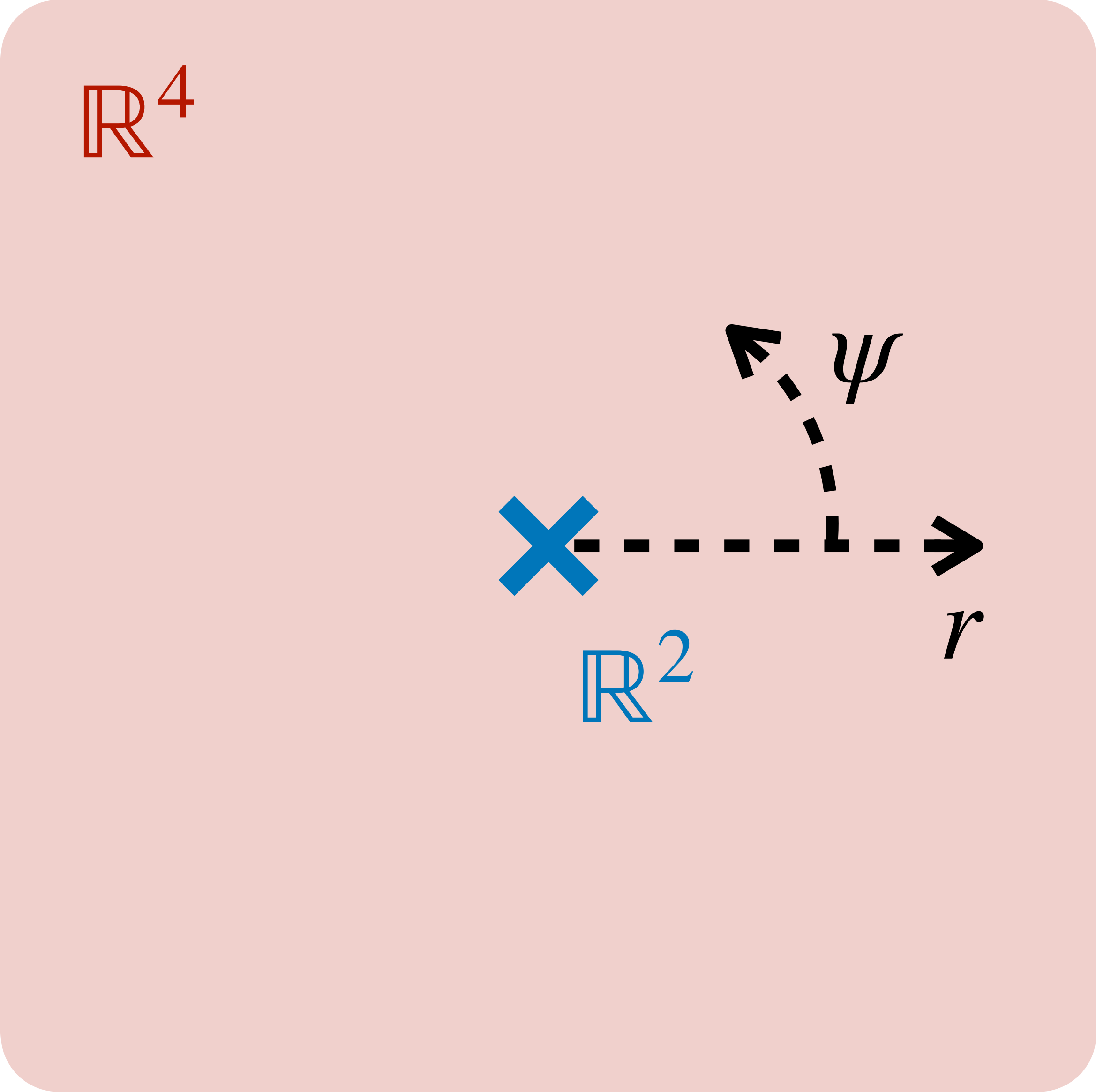
1/2-BPS Surface Defects

- 2d N=(4,4) SUSY ensured by [Gukov-Witten '06]

$$A = X_3(r)\,d\psi \quad \text{and} \quad \Phi = \frac{e^{-i\psi}}{r}(X_1(r) + iX_2(r))$$

satisfying Nahm equations

$$[X_i, X_j] + \epsilon_{ijk}\frac{\partial X_k}{\partial \log r} = 0$$



Gukov-Witten Defects

- 2d N=(4,4) surface defects come in two kinds

(1) Ordinary

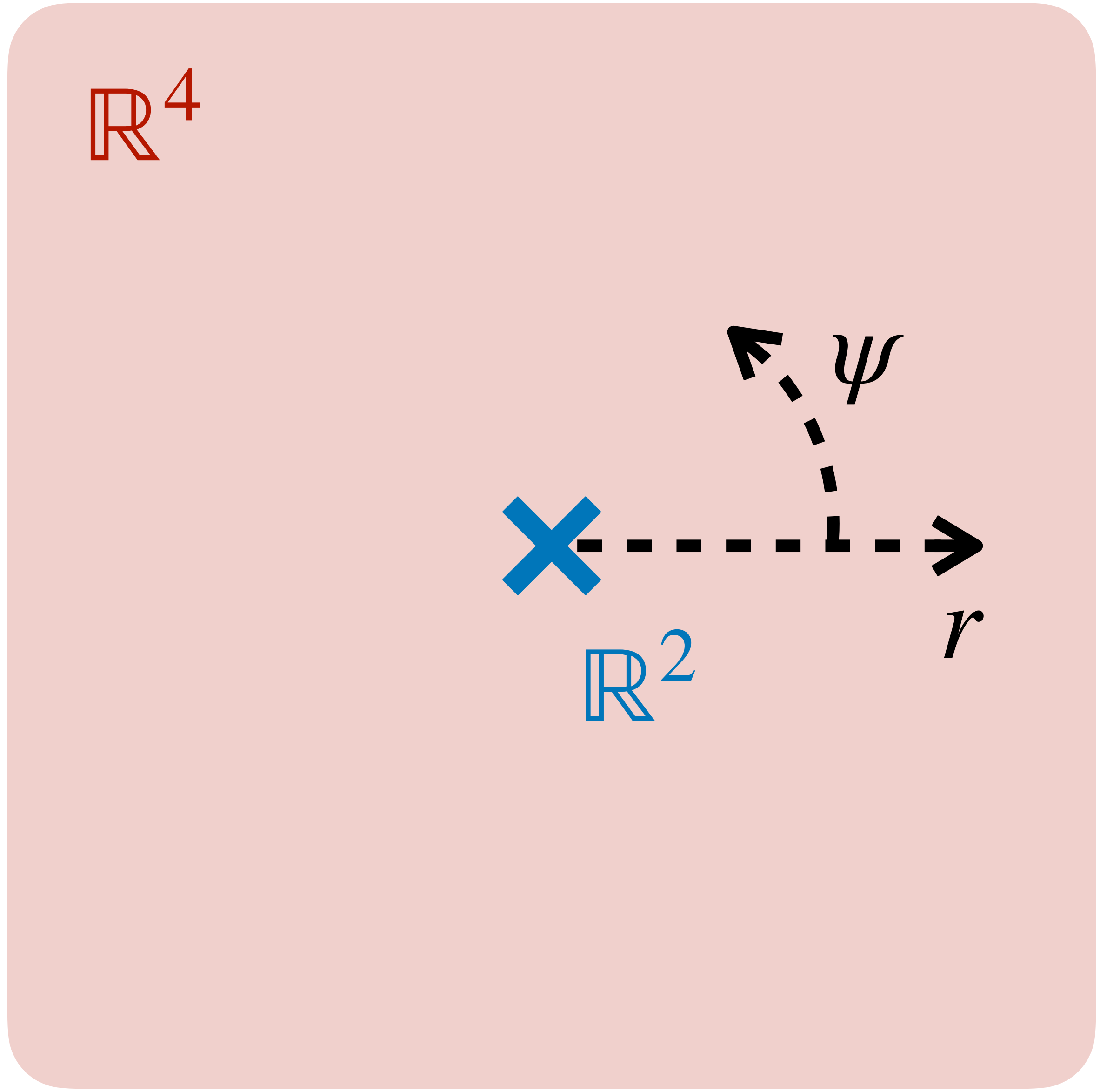
$$A = \alpha\, d\psi \ \text{ and } \ \Phi = \frac{e^{-i\psi}}{r}(\beta + i\gamma)$$

diagonal matrices

(2) Rigid $(\alpha, \beta, \gamma \to 0)$

$$A = \frac{t_3}{\log r} d\psi \ \text{ and } \ \Phi = \frac{e^{-i\psi}}{r \log r}(t_1 + it_2)$$

su(2) representation matrices



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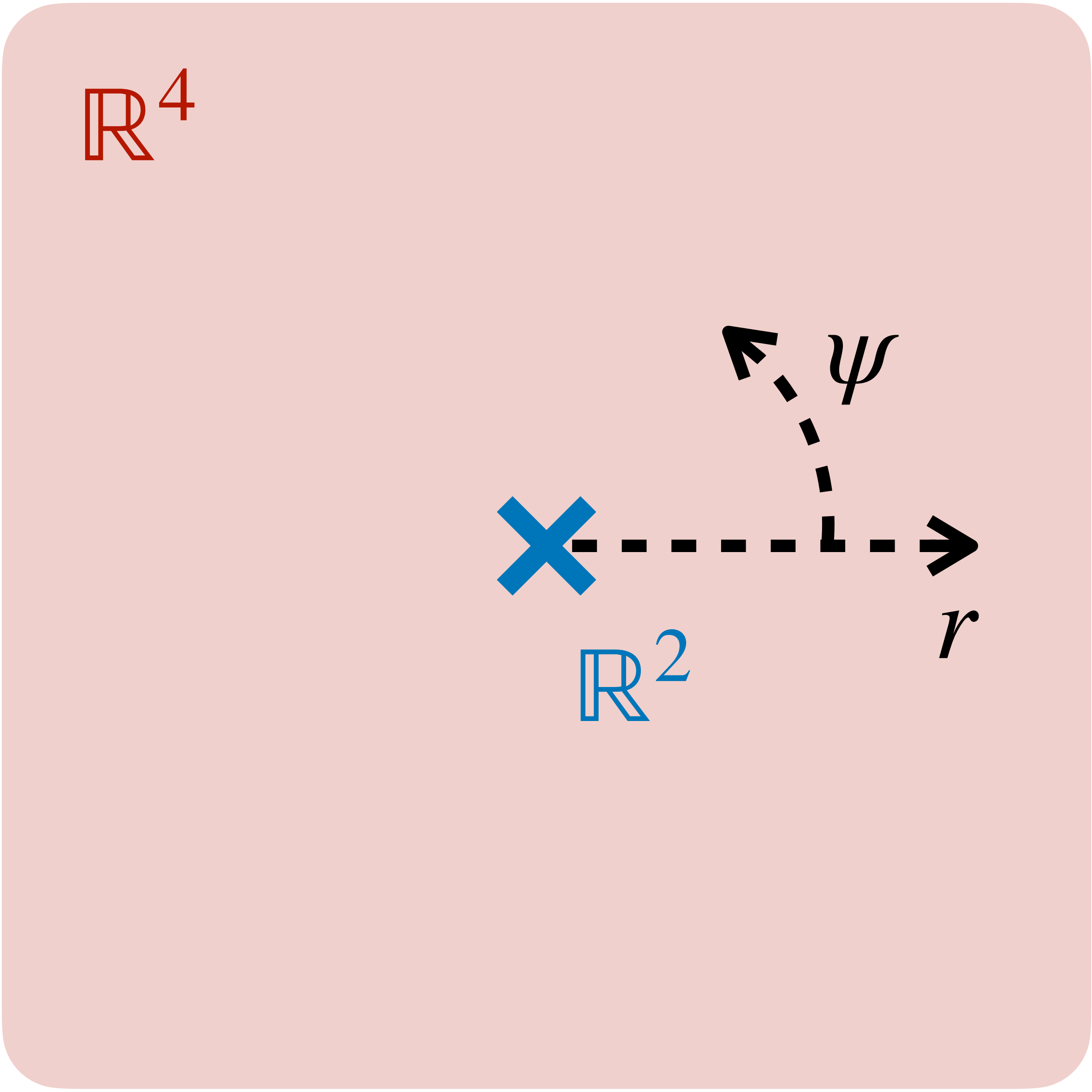
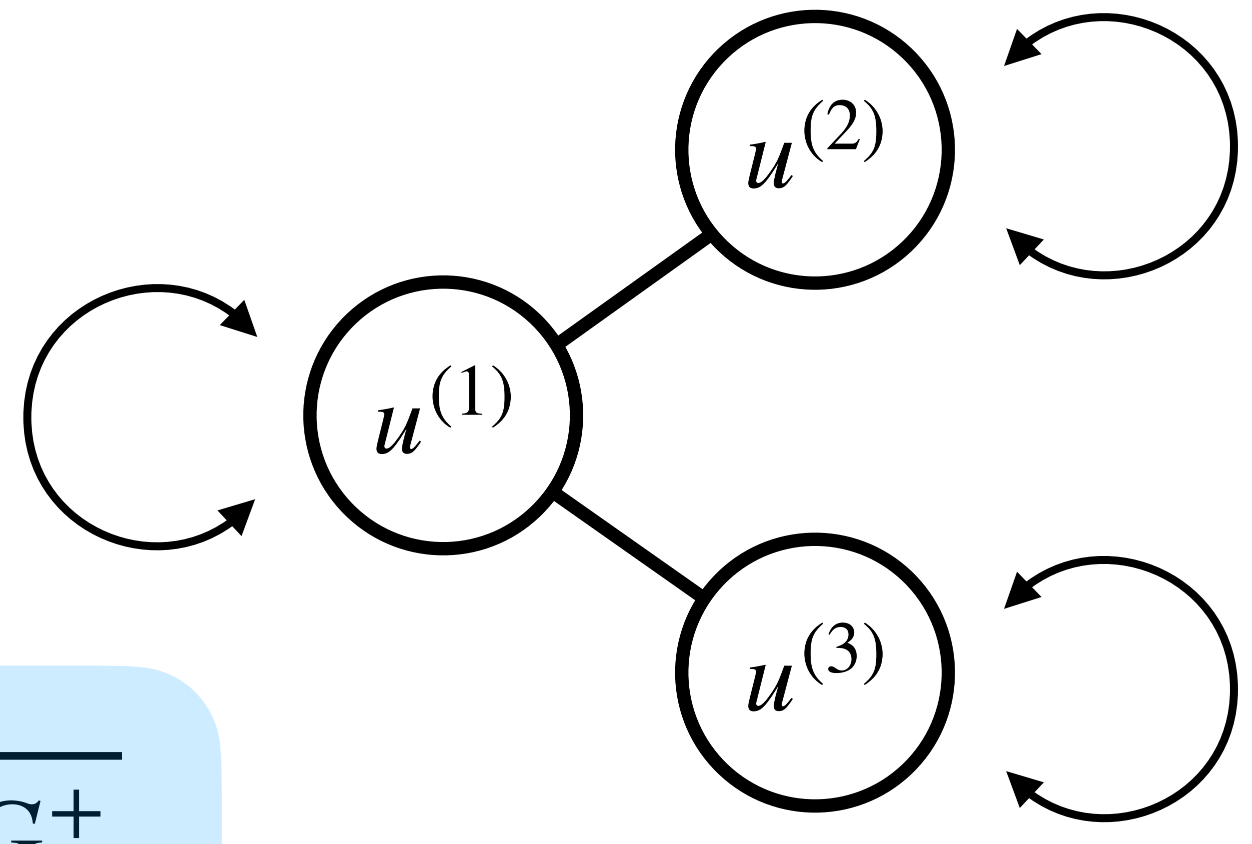


Overlaps for Ordinary Gukov-Witten Defect

- SO(6) sector: $\langle \mathscr{B} | \mathbf{u} \rangle = 0$ unless number of Bethe roots $\mathcal{N}_1 = 2\mathcal{N}_2 = 2\mathcal{N}_3$

  and roots are **chirally** paired $(u_1^{(i)}, -u_1^{(i)}, \ldots, u_{\mathcal{N}_i/2}^{(i)}, -u_{\mathcal{N}_i/2}^{(i)})$



$$\langle \mathscr{B} | \mathbf{u} \rangle = \sum_{m=1}^{M} N_m e^{-i(L-\mathcal{N}_1)\psi_m} (\beta_m^2 + \gamma_m^2)^{L/2} \sqrt{\frac{Q_1(0)Q_1(i/2)}{Q_2(0)Q_2(i/2)Q_3(0)Q_3(i/2)} \frac{\det G^+}{\det G^-}}$$

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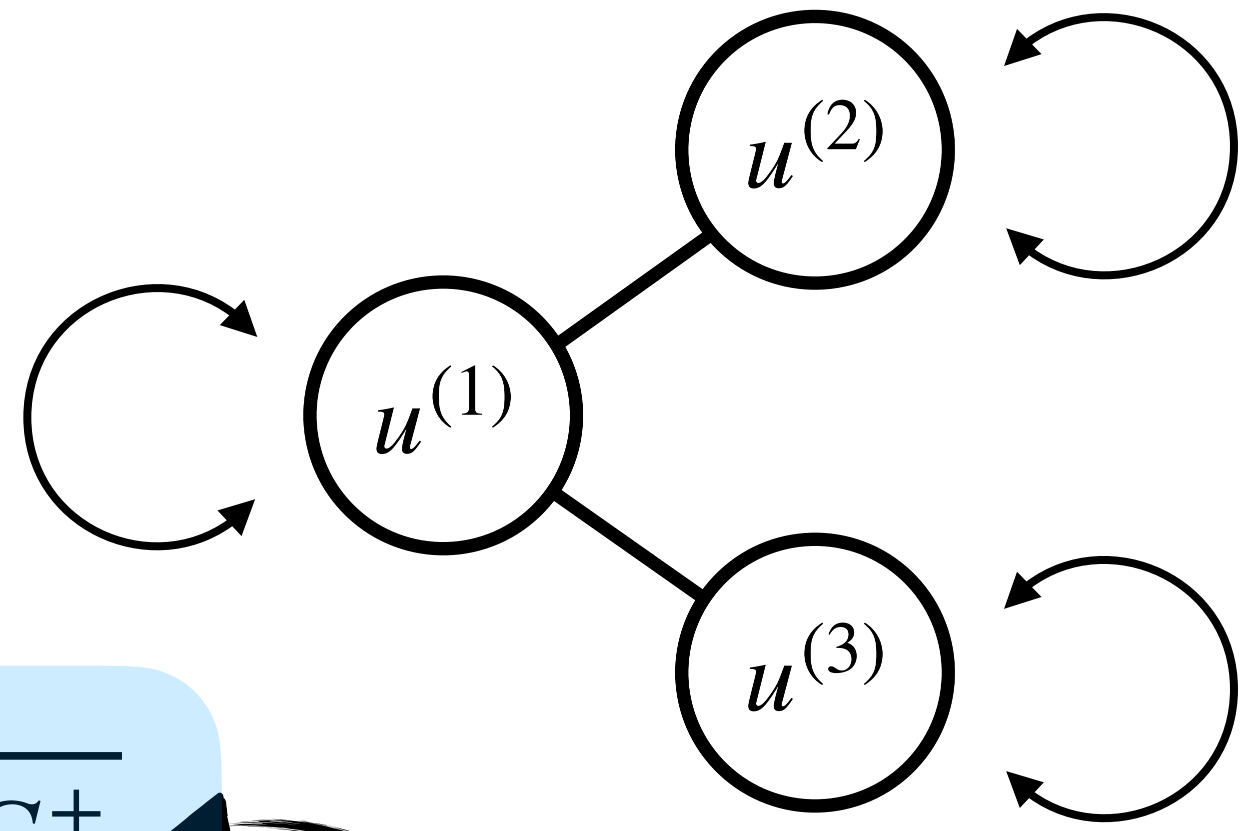

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$$\psi_m = \psi - \arg(\beta_m + i\gamma_m)$$

$$Q_i(x) = \prod\nolimits_a (x - u_a^{(i)})$$

$$G^\pm = \text{“}\frac{\partial}{\partial u} \log \text{(Bethe equations)”}$$

$$\Phi = \frac{e^{-i\psi}}{\sqrt{2}r} \mathrm{diag}((\beta_1 + i\gamma_1)1_{N_1}, \ldots, (\beta_M + i\gamma_M)1_{N_M})$$

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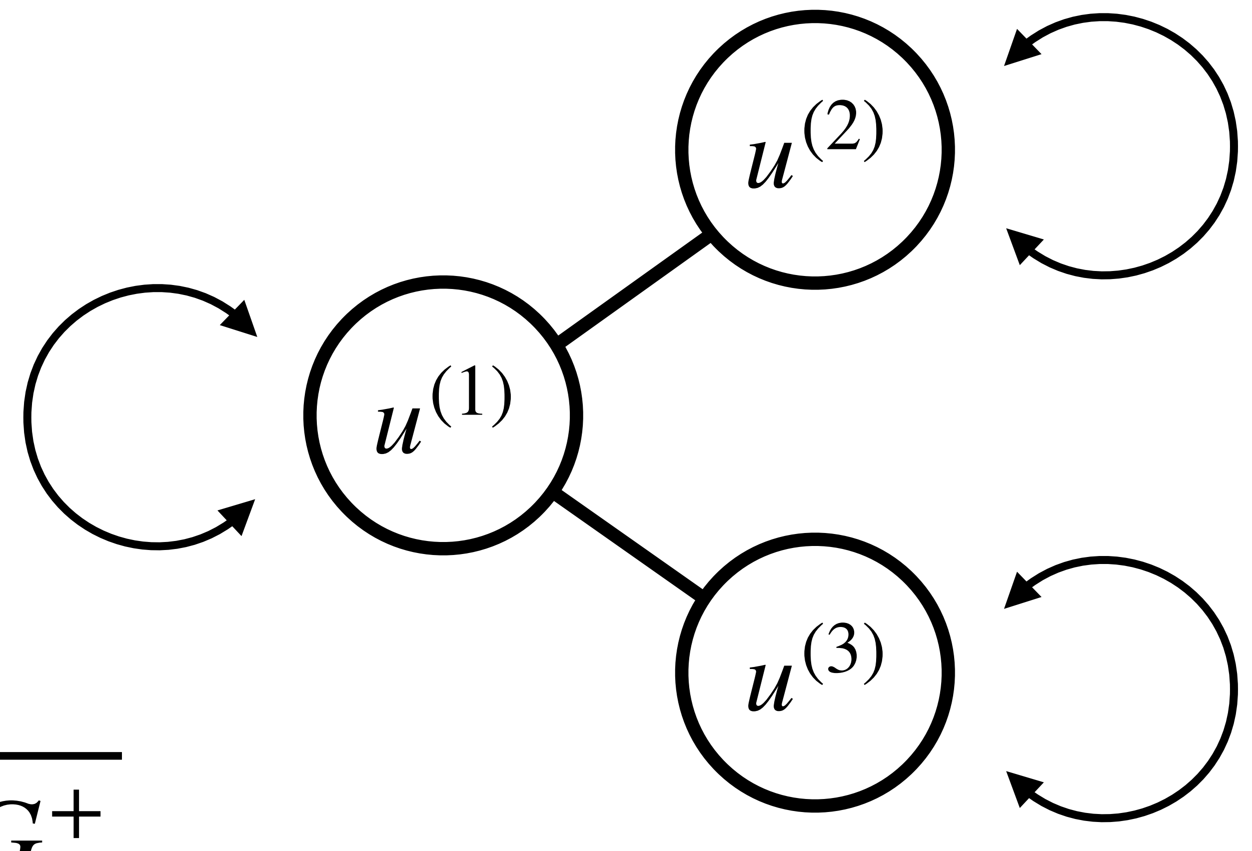

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- Trivial SU(2) sector since it is reached by taking $\mathcal{N}_2 = \mathcal{N}_3 = 0 \implies \mathcal{N}_1 = 0$

Overlaps for Ordinary Gukov-Witten Defect

- SL(2) sector $\{Z, DZ, D^2Z, \ldots\}$: depends on how we construct $|\mathscr{B}\rangle$
- $\exists$ choice for which $|\mathscr{B}\rangle$ not integrable! [see also Holguin-Kawai '25]

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[see also Holguin-Kawai ‘25]

SU(2)	SO(6)	SL(2)
0	✅	❌

just a coincidence at leading order

Gukov-Witten Defects

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$$A = \alpha\, d\psi \;\text{ and }\; \Phi = \frac{e^{-i\psi}}{r}(\beta + i\gamma)$$

diagonal matrices

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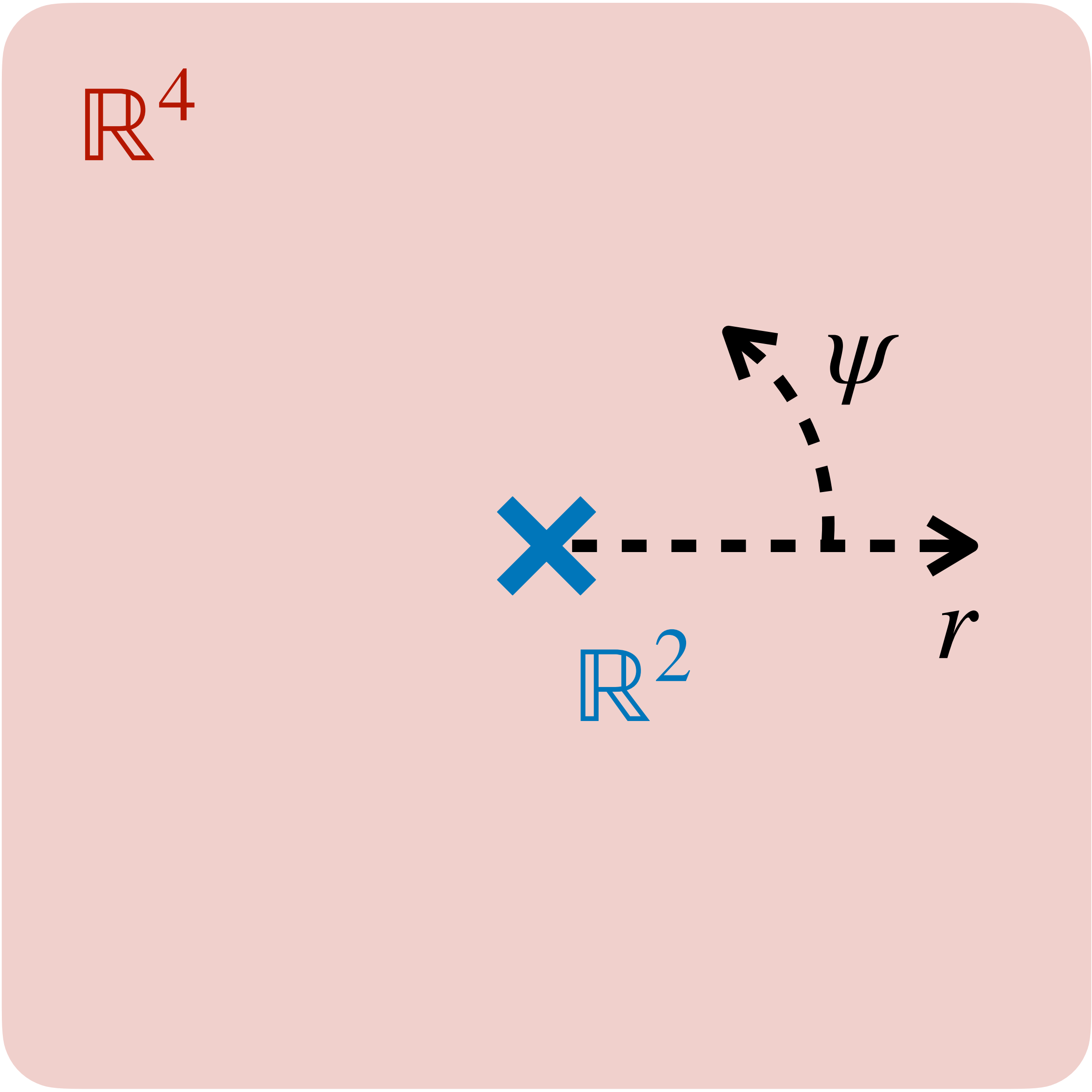
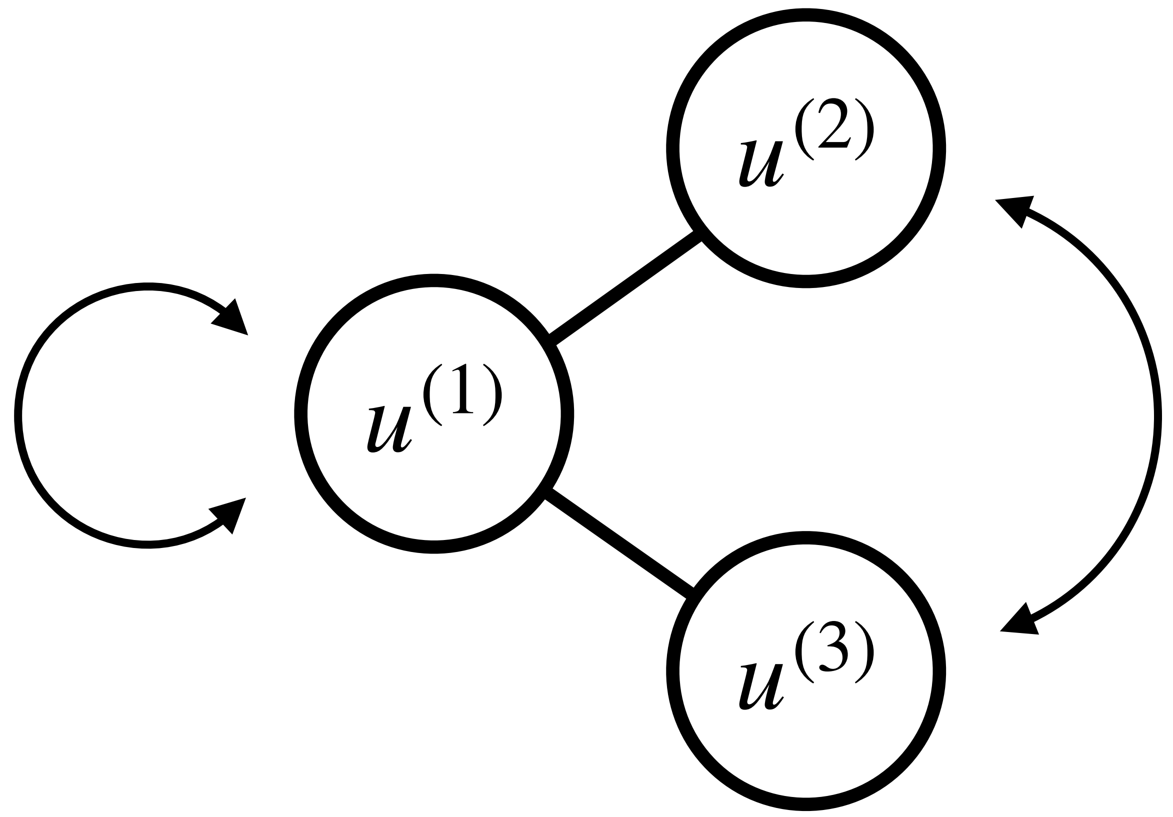


Overlaps for Rigid Gukov-Witten Defect

- SO(6) sector: $\langle \mathscr{B} \,|\, \mathbf{u} \rangle = 0$ unless roots are **achirally** paired.

  When $k = 2$ only, we find $\langle \mathscr{B} \,|\, \mathbf{u} \rangle = \dfrac{1}{2^{L-1}} \sqrt{\dfrac{Q_1(i/2)}{Q_1(0)} \, \dfrac{\det G^+}{\det G^-}}$



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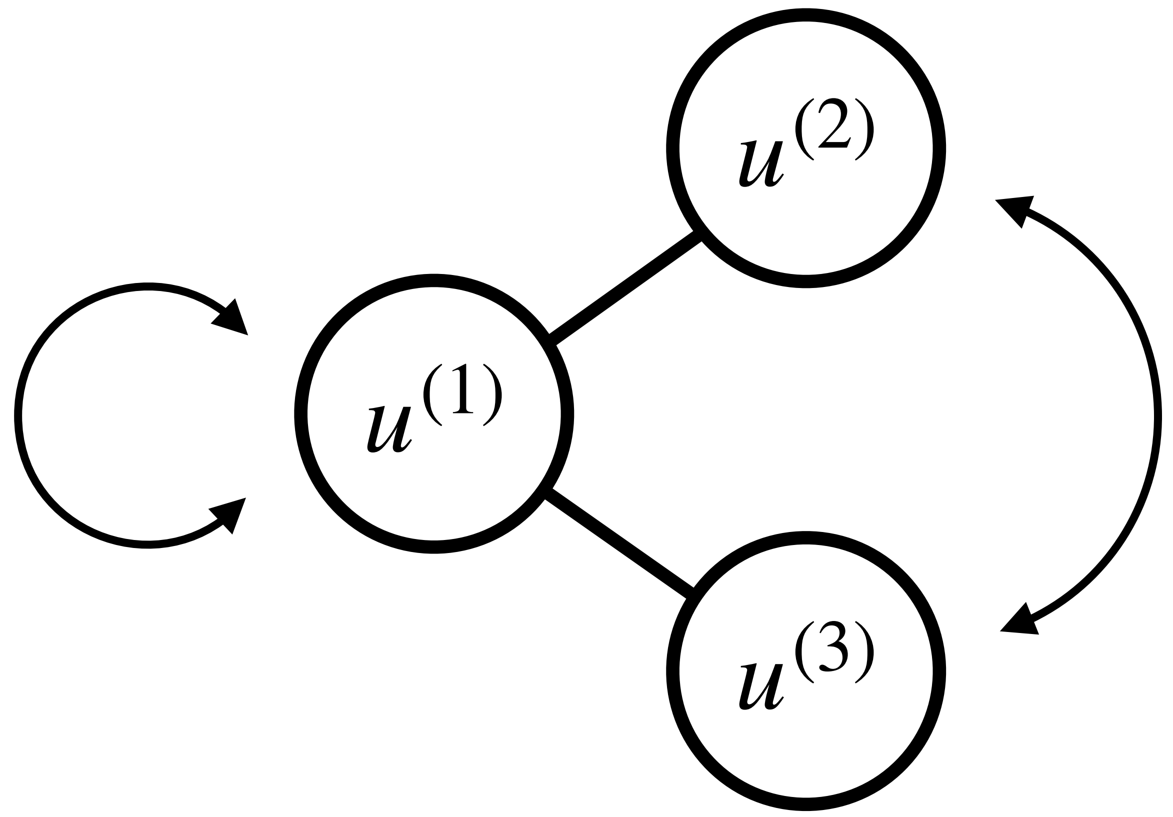

- SU(2) sector actually integrable for all k

$$|\langle \mathscr{B} \,|\, \mathbf{u} \rangle| = \mathbb{S}_k \, Q\left(\frac{ik}{2}\right) \sqrt{Q(i/2) Q(0) \frac{\det G^+}{\det G^-}} \,,$$

where $\mathbb{S}_k = \displaystyle\sum_{q=-\frac{k-1}{2}}^{\frac{k-1}{2}} \frac{q^L}{Q\left(\frac{2q+1}{2} i\right) Q\left(\frac{2q-1}{2} i\right)}$

Overlaps for Rigid Gukov-Witten Defect

- For SL(2) sector $\{Z, DZ, D^2Z, \ldots\}$ restrict to leading singularity $\sim 1/\log^L r$
- $\langle \mathscr{B} \,|\, \mathbf{u} \rangle = 0$ unless all $\mathscr{N}$ roots are paired,

$$\langle \mathscr{B} \,|\, \mathbf{u} \rangle = \frac{\sin^{\mathscr{N}} \psi}{r^{\mathscr{N}}} \frac{1}{2^{L-1}} \sqrt{\frac{Q(i/2)}{Q(0)} \frac{\det G^+}{\det G^-}}$$

for $k = 2$ only

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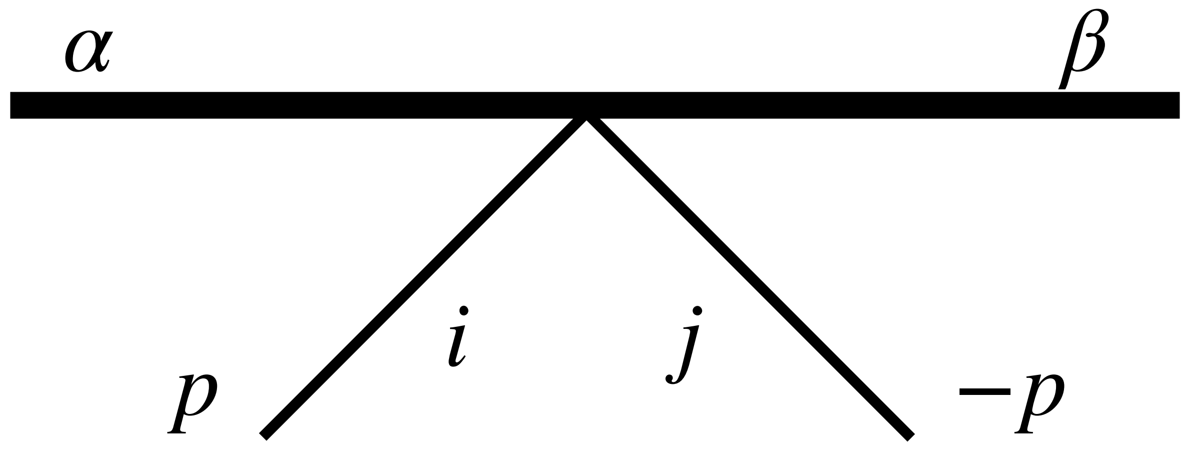
for $k = 2$ only

SU(2)	SO(6)	SL(2)
✅ all k	✅ k=2 only	✅ k=2 only

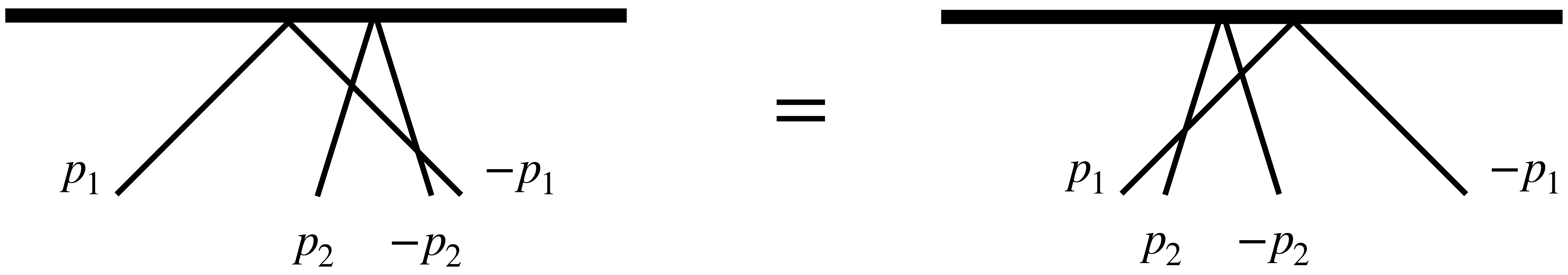
$k > 2$ just a coincidence at leading order

Sketch of Derivation

- Key ingredient is the K-matrix $(K_{i,j}(u))^{\alpha,\beta}$
- Amplitude of two excitations with opposite momenta annihilated by $|\mathscr{B}\rangle$



- Integrable scattering off a boundary needs to satisfy the K-Yang-Baxter eq



Sketch of Derivation

- Boundary state, K-matrix and monodromy matrix T satisfy KT-relation
- $K_{i,\ell}(u)\omega_I\mathscr{L}_{\ell,j;I,J}(u) = \omega_I K_{\ell,j}(u)\mathscr{L}_{i,\ell;I,J}(-u)$

$\langle\mathscr{B}| = \mathrm{tr}\left(\sum_{I=1}^{d}\omega^I\langle I|\right)^{\otimes L}$

$T_0 = \mathscr{L}_{0,L}\cdots\mathscr{L}_{0,1}$

- Combines K-Yang-Baxter equation and $Q^{\mathrm{odd}}|\mathscr{B}\rangle = 0$ condition

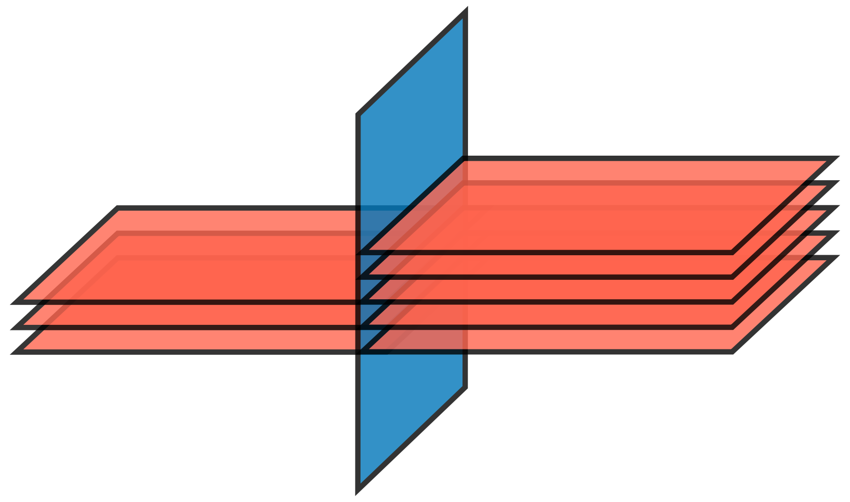
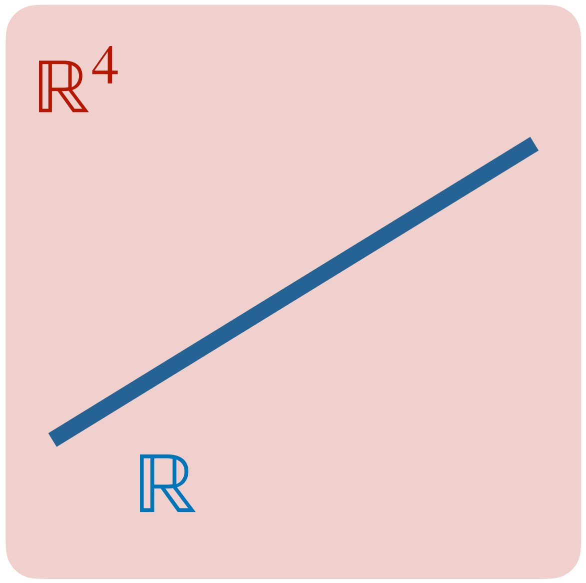
Sketch of Derivation

- Gombor '24: general method to find $\langle \mathscr{B} \,|\, \mathbf{u} \rangle$ from solution of the KT-relation

$$\langle \mathscr{B} \,|\, \mathbf{u} \rangle = \underbrace{\sum_{\gamma=1}^{k}}_{\text{size of matrix } \omega^I} \underbrace{\beta_\gamma}_{\text{eigenvalues of } \langle \mathscr{B} | 0 \rangle} \left(\underbrace{\prod_{a=1}^{n_+}}_{\text{\# distinct types of roots}} \underbrace{\prod_{j=1}^{n_a}}_{\text{\# distinct roots of a given type}} \underbrace{\tilde{\mathscr{F}}_\gamma^{(a)}}_{\text{constructed out of } K}(u_{a,j}) \right) \underbrace{\sqrt{\frac{\det G^+}{\det G^-}}}_{\text{universal part}}$$

- Pair structure determined by reflection algebra $Y(\mathfrak{g}, \mathfrak{h})$

Integrable 1/2-BPS Defects in $\mathcal{N}$=4 SYM

Co-dim 1	**Co-dim 2**	**Co-dim 3**
D3-D5 system		1/2-BPS ’t Hooft line
	?	
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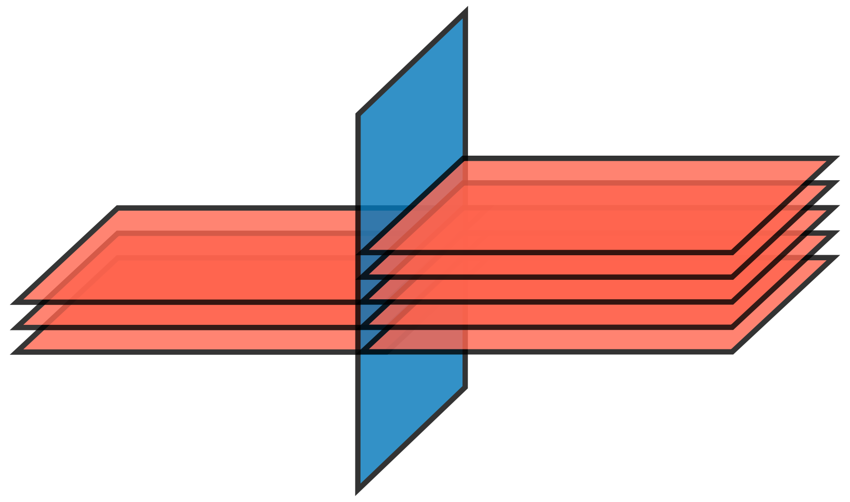
Integrable 1/2-BPS Defects in *N*=4 SYM

Co-dim 1

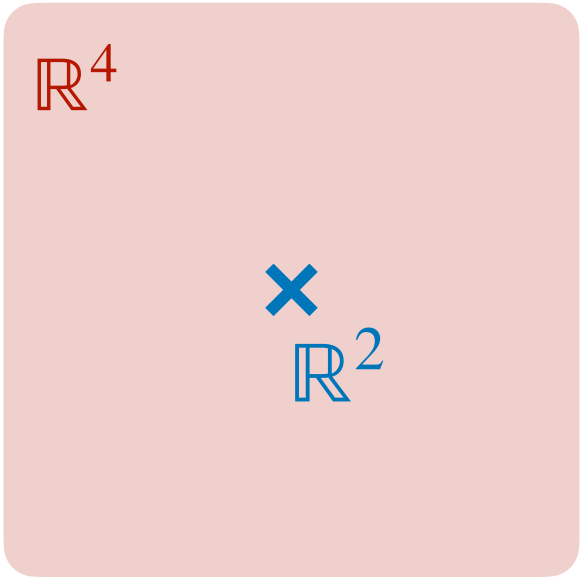
D3-D5 system



$$\phi_i \to t_i^{(k)}/r$$

Co-dim 2

Rigid Gukov-Witten defect?

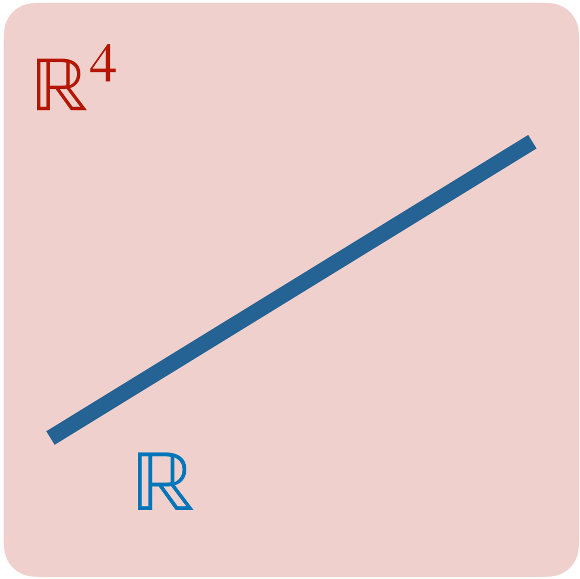


$$A_\psi \to \frac{t_3^{(2)}}{\log r},$$

$$\Phi \to \frac{e^{-i\psi}}{r \log r}(t_1^{(2)} + i t_2^{(2)})$$

Co-dim 3

1/2-BPS 't Hooft line



$$\phi^I \to n^I/r$$

Outlook

- Loop corrections?
- Rigid defect has same symmetries as $\langle \mathscr{D}\mathscr{D}\mathscr{O} \rangle$ [Jiang-Komatsu-Vescovi '19]

  $\longrightarrow$ bootstrap all-loop asymptotic formula?
- Rigid Gukov-Witten defects in holography, SUSY localisation, VOA, AGT?

  [WIP with R. Izquierdo Garcia, C. Kristjansen, A. O'Bannon, James Ratcliffe, R. Rodgers]
- Space of integrable defects? 2d N=(8,0) SUSY?