Insights from the crosscap state
toward a rigorous formulation of non-orientable TQFTs

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Motivation

(2+1)D bosonic **Topological quantum field theory (TQFT)** is a QFT which

- depends only on the topology of the spacetime manifolds
- is characterized by anyon data.

Fact: Anyon data $\xleftrightarrow{1:1}$ Modular tensor category $\mathcal{C}$ (MTC)

$\rightarrow$ We can construct TQFT mathematically using MTC.
But only on orientable manifolds!

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But only on orientable manifolds!

In general, it is understood that

QFT with time-reversal symmetry = QFT on non-orientable manifolds

$\rightarrow$ It is natural to expect that we can construct non-orientable TQFTs.

Question: Is there any implication from physics? $\rightarrow$ Yes!

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- TQFT and anyon
- Time reversal on anyon
- Introduction of crosscap
- Insights from crosscap

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TQFT and anyon

Example: $U(1)$ Chern–Simons theory

$$S_{\mathsf{CS}}[A] = \frac{k}{4\pi} \int_M A \wedge dA$$

where $k \in \mathbb{Z}$ is a level and A is a $U(1)$ gauge field.

$\rightarrow \frac{\delta S_{\mathsf{CS}}}{\delta g_{\mu\nu}} = 0$: independent of metric.

$\rightarrow Z[M] := \int \mathcal{D}A \exp(iS_{\mathsf{CS}})$ gives an invariant of M (i.e. TQFT!).

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Physical observable: Wilson loop labeled by anyon:

$$W(\gamma, a) = \exp\left(ia \oint_\gamma A\right)$$

where γ is a loop in M and $a \in \mathbb{Z}/k\mathbb{Z} \cong \{0, 1, \cdots, k-1\}$

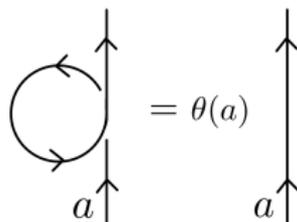$\rightarrow \langle W(\gamma, a)\rangle$ is comuputed as a knot invariant.

TQFT and anyon

Anyons: Generalized concepts of bosons and fermions.
Fusion rule:

$$a \times b = \sum_c N_{ab}^c \, c, \quad a, b, c \in \{\text{Anyons}\} =: \mathcal{C}, \quad N_{ab}^c \in \mathbb{Z}_{\geq 0}$$

Topological spin: $\theta : \mathcal{C} \to U(1)$ encoding spin statistics of anyons:

Point: Modular Tensor Category $\mathcal{C}$ naturally encodes these data.
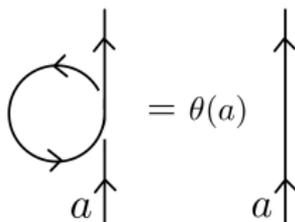
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Remark: TQFT is said to be abelian if $a \times b = c$ uniquely for all fusions.

TQFT and anyon

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$$S_{\mathsf{CS}}[A] = \frac{k}{4\pi} \int_M A \wedge dA$$

where $k \in \mathbb{Z}$ is a level and A is a $U(1)$ gauge field.

Input data: $k \in \mathbb{Z}$

Anyons: $a, b, \cdots \in \mathbb{Z}/k\mathbb{Z}$

Fusion rule: $a + b \in \mathbb{Z}/k\mathbb{Z}$

Topological spin: $\theta(a) := \exp\big(\pi i a^2 / k\big)$

TQFT and anyon

Example: $U(1)^N$ Chern–Simons theory

$$S_{\text{CS}}[A] = \frac{1}{4\pi} \int_M K_{IJ} A^I \wedge dA^J$$

where $K = (K_{IJ}) \in \text{Mat}_N(\mathbb{Z})$ is a K matrix and $\{A^I\}_{I=1,\cdots,N}$ are $U(1)$ gauge fields.

Input data: $k \in \mathbb{Z} \to K \in \text{Mat}_N(\mathbb{Z})$

Anyons: $a, b, \cdots \in \mathbb{Z}/k\mathbb{Z} \to a, b, \cdots \in \mathbb{Z}^N/K\mathbb{Z}^N$

Fusion rule: $a + b \in \mathbb{Z}/k\mathbb{Z} \to a + b \in \mathbb{Z}^N/K\mathbb{Z}^N$

Topological spin: $\theta(a) := \exp(\pi i a^2/k) \to \theta(a) := \exp(\pi i a^T K^{-1} a)$

TQFT and anyon

Example: Toric code $\sim K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} U(1)^2$ Chern–Simons theory

$\{1, e, m, f\}$ with $e \times e = m \times m = f \times f = 1, \ e \times m = f$

Input data: $K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$

Anyons: $\left\{ 1 := \begin{pmatrix} 0 \\ 0 \end{pmatrix}, e := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, m := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, f := \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \in \mathbb{Z}^2 / K\mathbb{Z}^2$

Fusion rule: $e \times m = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = f$

Topological spin: $\theta(f) = \exp\left(\pi i \, (1, 1) \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = -1$

Contents

Time reversal on anyon

With time-reversal, we have

$$\frac{1}{4\pi} \int_M K^{IJ} A_I \wedge dA_J \mapsto \frac{1}{4\pi} \int_{-M} K^{IJ} A_I \wedge dA_J = \frac{1}{4\pi} \int_M -K^{IJ} A_I \wedge dA_J$$

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Example: Consider Toric code and $\mathsf{T} \in \mathrm{Mat}_2(\mathbb{Z})$ as follows:

$$K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad \mathsf{T} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ satisfying } \mathsf{T}^T K \mathsf{T} = -K$$

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Then we obtain

$$\begin{aligned}
S &= \frac{1}{4\pi} \int_{-M} \begin{pmatrix} A_1, A_2 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} dA_1 \\ dA_2. \end{pmatrix} \\
&= \frac{1}{4\pi} \int_M \begin{pmatrix} A_1, A_2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} dA_1 \\ dA_2. \end{pmatrix} \\
&= \frac{1}{4\pi} \int_M \begin{pmatrix} A_2, -A_1 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} dA_2 \\ -dA_1 \end{pmatrix}.
\end{aligned}$$

Time reversal on anyon

Observation: Time-reversal induces $(A_1, A_2) \mapsto (A_2, -A_1)$

$$\mathsf{T} : \exp\left(i \oint_\gamma A_1 \right) \mapsto \exp\left(i \oint_\gamma A_2 \right).$$

Or more simply put, T permutes e and m

$$\mathsf{T}e = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} = m$$

How do we describe time reversal on anyons categorically and what additional data do we need?

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How do we describe time reversal on anyons categorically and what additional data do we need? $\rightarrow$ Symmetry fractionalization
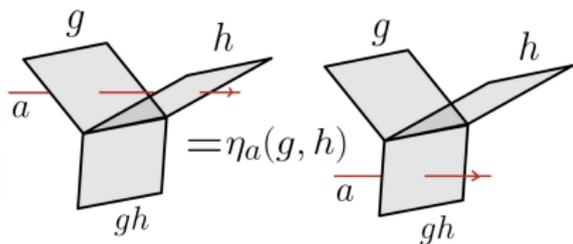
Time reversal on anyon

Symmetry fractionalization:
Finite group $G \curvearrowright \{\text{Anyons}\} = \mathcal{C}$
$\rightarrow$ Additional data like projective phase

$$\eta_a(g, h) \in U(1) \quad \text{for } a \in \mathcal{C}, g, h \in G$$

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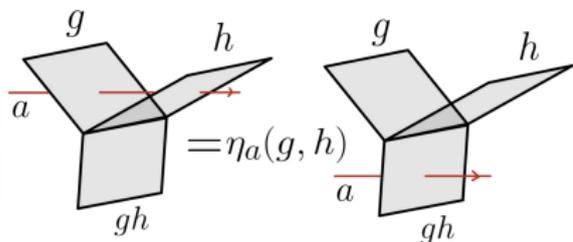$$\eta_a(g, h) \in U(1) \quad \text{for } a \in \mathcal{C}, g, h \in G$$

Remark: It is not always possible to define such notions.
$\rightarrow$ Symmetry fractionalization is not well-defined in the presence of
H^3 obstruction $e \in H^3(G, \mathcal{A})$, where $\mathcal{A} \subseteq \mathcal{C}$ is the set of abelian
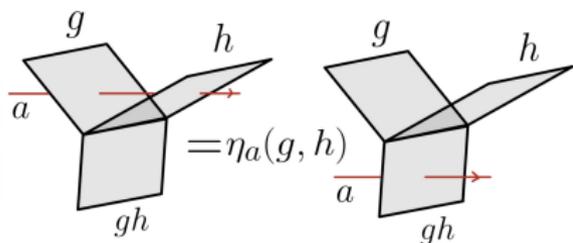anyons [Barkeshli et al. 1410.4540, 1612.07792].

Time reversal on anyon

Symmetry fractionalization:
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H^3 obstruction $e \in H^3(G, \mathcal{A})$, where $\mathcal{A} \subseteq \mathcal{C}$ is the set of abelian
anyons [Barkeshli et al. 1410.4540, 1612.07792].

For time reversal,
- H^3 obstruction vanishes for all abelian TQFTs [IO 2509.07368].
- Nontrivial examples are known for nonabelian TQFTs [Barkeshli et al. 1706.09464].

We will denote $\eta_a(\mathsf{T}, \mathsf{T}) =: \eta(a)$ from now on.

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- Introduction of crosscap
- Insights from crosscap

Introduction of crosscap

Recall: Algebraically, $N_{ab}^c = \sum_d S_{ad} S_{bd} \overline{S_{cd}} / S_{0d} \in \mathbb{Z}_{\geq 0}$ is nontrivial.
From TQFT perspectives, $N_{ab}^c = \dim V(S^2(a, b, \overline{c})) \in \mathbb{Z}_{\geq 0}$ is natural.

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Given $\mathcal{C}$ and time-reversal $\mathbb{Z}_2^{\mathsf{T}} \curvearrowright \mathcal{C}$ without H^3-obstruction:

$$M_a := \sum_{b = \mathsf{T}b} S_{ab} \eta(b)$$

RHS is purely determined with MTC. Surprisingly, we have the followings:

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Fact:

- $M_a \in \mathbb{Z}_{\geq 0}$ for every known example.
- $M_a \in \mathbb{Z}_{\geq 0}$ is shown for all abelian TQFTs [IO 2502.13532].

$\rightarrow$ These are highly nontrivial and give some implications.

Introduction of crosscap

Recall: By the definition of TQFT Z,

- $Z(M) \in V(\partial M)$: Path integral defines a state at a time-slice
- $Z(\Sigma \times S^1) = \text{Tr}_{V(\Sigma)} Z(\Sigma \times [0,1]) = \text{Tr}_{V(\Sigma)} \text{id}_{V(\Sigma)} = \dim V(\Sigma)$

where M is a three-manifold, Σ is a closed surface, and $V(\Sigma)$ is a Hilbert space on Σ assigned by TQFT.

$\rightarrow$ We can construct bases on $V(T^2)$ as follows:

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$\rightarrow$ We can construct bases on $V(T^2)$ as follows:

- **Constructing a basis:**
$$|a\rangle := Z(D^2(a) \times S^1) = Z(\,\raisebox{-2pt}{}\, \times S^1) = Z(\,\raisebox{-2pt}{}\,) \in V(T^2),$$
$$\rightarrow \quad V(T^2) = \mathrm{span}_{\mathbb{C}} \{|a\rangle \mid a \in \mathcal{C}\}.$$

- **Another basis via modular S:**
$$S|a\rangle := Z(S^1 \times D^2(a)) \in V(T^2),$$
$$\rightarrow \quad V(T^2) = \mathrm{span}_{\mathbb{C}} \{S|a\rangle \mid a \in \mathcal{C}\}.$$

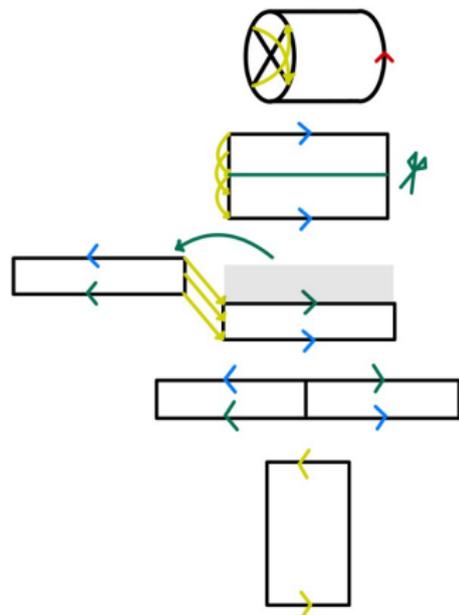## Introduction of crosscap

The **Crosscap state** is defined as:

$$|\mathbb{CC}\rangle = Z(\,\otimes\!\!\!\longrightarrow \times S^1\,)$$
$$= \sum_a M_a \,|a\rangle$$
$$= \sum_a \eta(a) S \,|a\rangle\,,$$

where M_a and $\eta(a)$ are coefficients.
Then, we have

$$M_a = \langle a|\sum_b \eta(b) S \,|b\rangle = \sum_b S_{ab}\eta(b)$$
$$\eta(a) = \langle a|\, S^{-1}\,|\mathbb{CC}\rangle$$

Remark: $\eta(a)$ is understood as symmetry fractionalization.
[Tachikawa-Yonekura 1611.01601] $\rightarrow$ Why $M_a \in \mathbb{Z}$?

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Insights from crosscap

We can compute M_a as follows:

$$\begin{aligned}
M_a &= \langle a | \mathbb{CC} \rangle \\
&= Z(\;\text{⬭} \times S^1) Z(\;\text{⬭} \times S^1) \\
&= Z(\;\text{⬭} \text{⬭} \times S^1) \\
&= Z(\;\text{⬭} \times S^1) \\
&= \dim V(\mathbb{RP}^2(a))
\end{aligned}$$

where we used $Z(\Sigma \times S^1) = \dim V(\Sigma)$ for closed surfaces.

If TQFT exists, $M_a \in \mathbb{Z}_{\geq 0}$ has a natural interpretation.
$\rightarrow$ Suggestion for the existence of TQFTs on non-orientable manifolds.

Insights from crosscap

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If TQFT exists, $M_a \in \mathbb{Z}_{\geq 0}$ has a natural interpretation.
$\rightarrow$ Suggestion for the existence of TQFTs on non-orientable manifolds.

$\rightarrow$ How about $\eta(a)$?

Insights from crosscap

Reshetikhin–Turaev (RT) invariant:
$\langle \ \rangle : M \to \mathbb{C}$ where M is an orientable closed 3-dim'l manifold.

$\to$ We can construct Reshetikhin–Turaev TQFT from this invariant.

Lickorish–Wallace Theorem:
We can get any orientable 3-dim'l manifold M
from S^3 with some link L:
$$S^3 + \exists L \to S^3_L \cong \forall M$$

$\to$ We can construct RT invariant by evaluating knot invariants with the use of MTC!
$$M \cong S^3_L \to \langle M \rangle \sim \sum_{a \in \mathcal{C}} \langle L(a) \rangle$$

Point: S^3 is the building block of orientable manifolds

Get back to our set up,
We can compute $\eta(a)$ as follows: [Tachikawa-Yonekura 1611.01601]

$$\begin{aligned}
\eta(a) &= \langle a | \, S^{-1} \, | \mathbb{C}\mathbb{C} \rangle \\
&= \cdots \\
&= Z \left(\frac{S^2(a, \overline{\mathrm{T}a}) \times [0,1]}{(0,x) \sim (1,-x)} \right) \\
&=: Z(X^3(a))
\end{aligned}$$

Recall: $\eta(a)$ is understood as symmetry fractionalization.
But it has another crucial implication.

X^3 is the building block of non-orientable manifolds ! [Lickorish 1963]

Future work:

- $\forall M$:orientable, $S^3 \xrightarrow{\exists L} S_L^3 \cong M \xrightarrow{\mathcal{C}} \langle M \rangle \rightarrow$ TQFT.

- $\forall M$:non-orientable, $X^3 \xrightarrow{\exists L} X_L^3 \cong M \xrightarrow{\mathbb{Z}_2^{\mathsf{T}} \curvearrowright \mathcal{C}} \langle M \rangle \rightarrow$ TQFT ?

Here, the red part remains to be understood.

Physics gives us the clue!

Question: Is there any implication from physics? $\rightarrow$ Yes!

Thank you for listening.