L-functions and the Conformal Bootstrap

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Conformal field theory in general dimension ⟷ **Spectral theory of automorphic forms**

[Kravchuk, DM, Pal '21]

Why is there a connection?

- Both sides rooted in unitary representation theory of noncompact Lie groups.
- Conformal field theory: $G = \mathrm{SO}(1,d+1)$ and $G = \widetilde{\mathrm{SO}}(2,d)$.
- Automorphic forms: $G = \mathrm{SL}_2(\mathbb{R})$ (Poincaré) $\to$ G general (Langlands).

Applications:

- Conformal bootstrap $\Rightarrow$ new rigorous results in number theory and spectral geometry.
  1. Bounds on spectral gaps of hyperbolic manifolds
  2. Bounds on *L*-functions
- Insights into conformal field theory

c.f. tomorrow's talk by Nathan Benjamin

A brief history of unitary representations of non-compact Lie groups

- Wigner (1939), Dirac (1945)
- Bargmann (1947), Gelfand-Naimark (1947)
- Harish-Chandra
- Langlands
- ...

Langlands (referring to Dirac's 1945 paper): "This is as close as one comes to the source of the theory of infinite-dimensional representations of semisimple groups, which as it turned out were to be of limited physical significance but of great mathematical import."

Map of the spectral data

[Bonifacio, Hinterbichler ’20], [Bonifacio ’21+’21]

[Kravchuk, DM, Pal ’21]

Conformal field theory		Automorphic forms
Scaling dimensions of primary operators Δ_i	$\longleftrightarrow$	Laplace eigenvalues $\lambda_i = \Delta_i(d - \Delta_i)$
Structure constants of the operator algebra c_{ijk}	$\longleftrightarrow$	Integrals of products of eigenfunctions $c_{ijk} = \int f_i f_j f_k$
		Relation to L-functions: $\lvert c_{ijk} \rvert^2 \sim L(\frac{1}{2}, \pi_i \otimes \pi_j \otimes \pi_k)$

[Watson ’02], [Ichino ’08]

- Conformal bootstrap: well-developed methodology for bounding Δ_i, c_{ijk}.
- Thanks to the above mapping, it is directly applicable to problems in pure mathematics.
- The corresponding techniques sometimes already known to mathematicians, but often new!

Plan

1. Review of the connection

2. Spectral gaps of hyperbolic manifolds
   [Kravchuk, DM, Pal: 2111.12716], [Bonifacio, DM, Pal: 2308.11174]

3. Weyl bound on the triple product L-functions
   [Adve, Bonifacio, Kravchuk, DM, Pal, Radcliffe, Rogelberg: 2508.20576]

4. Completeness of the bootstrap for compact hyperbolic surfaces
   [Adve: 2509.17935]

What is conformal field theory?

- QFT = a "measure" μ on "functions" $\Phi(x)$ (scalars, fermions, gauge fields, discrete d.o.f.,...)
- μ can be far from the Gaussian measure.
- Define correlation functions $\langle \ldots \rangle := \int \ldots d\mu(\Phi)$.
- Locality: "$d\mu(\Phi) = \prod_x d\mu_x(\Phi)$".
- Local operators $\mathcal{O}_i(x)$ built out of $\Phi(x + \epsilon)$.
- $\mathcal{O}_i(x) \Rightarrow$ deformations of the measure $d\mu' = \exp\left[\int \sum_i g_i \mathcal{O}_i(x) dx\right] d\mu$.
- Space of quantum field theories Ω = space of couplings g_i.
- Rescaling $x \mapsto \lambda x$ leads to a group action $(\mathbb{R}, \cdot\,) \times \Omega \to \Omega$, the renormalization group flow.
- Fixed points = conformal field theories.

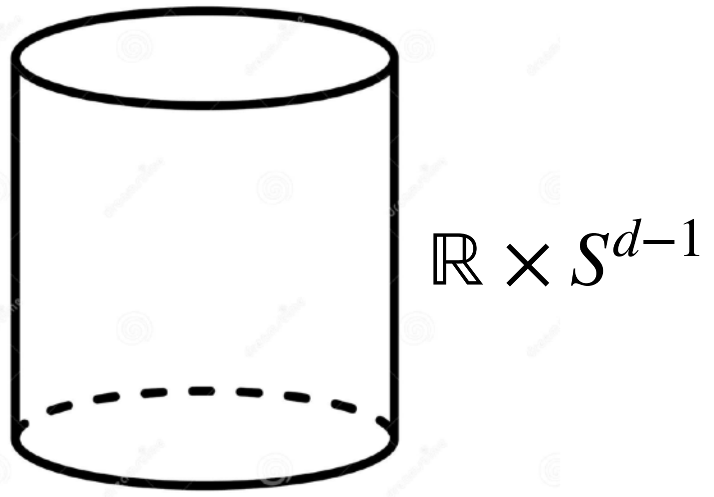
Important goal: Classify conformal field theories.

Problems:
1. No mathematically precise global notion of the space of couplings Ω.
2. RG flow not calculable when $g_i = O(1)$.

An alternative: The conformal bootstrap = an algebraic definition of conformal field theory in general d.

[Ferrara, Gatto, Grillo '73] [Polyakov '74] [Rattazzi, Rychkov, Tonni, Vichi '08]

Definition sketch *(unitary conformal field theory in d dimensions):*



1. State space: A Hilbert space V = a positive-energy, unitary representation of $\widetilde{\mathrm{SO}}(2,d)$.

2. Spectrum: V decomposes into irreducibles as $V = \bigoplus_{i=0}^{\infty} V_i$, where $V_i \simeq D_{\Delta_i, \rho_i}$. $V_0 \simeq \mathbb{C}$ (the vacuum).

3. State-operator map: There exist operator-valued distributions $\phi_i(x)$, such that $\phi_i(x)V_0$ generates V_i.

4. Microcausality: $[\phi_i(x), \phi_j(y)] = 0$ if x, y spacelike.

(5. Locality: The stress tensor representation $D_{d,2}$ is in the spectrum.)

Empirical fact: The bootstrap definition of a CFT $\Rightarrow$ strong constraints on the spectral data (Δ_i, ρ_i), c_{ijk}.

Resulting estimates on the critical $d = 3$ Ising model: $\Delta_1 = 0.518148806(24)$, $\Delta_2 = 1.41262528(29)$.

[Chang, Dommes, Erramilli, Homrich, Kravchuk, Liu, Mitchell, Poland, Simmons-Duffin '24]

$$\textbf{RG CFT} \overset{?}{\Leftrightarrow} \textbf{Bootstrap CFT}$$

- $\Rightarrow$ direction is expected to hold generically.
- $\Leftarrow$ direction more mysterious. Says that the conformal bootstrap is **complete**.
- If $\Leftarrow$ holds, one can expect further results, e.g.

1. Generic $d > 2$ unitary local bootstrap CFT are isolated.
2. Any $d = 4$, $\mathcal{N} = 4$ unitary local bootstrap CFT is the super Yang-Mills.
3. All $d > 6$ unitary local bootstrap CFTs are free.

Today: Completeness of the bootstrap holds rigorously for compact hyperbolic surfaces. [Adve '25]

What are automorphic forms?

- Let G be a Lie group and $\Gamma \subset G$ a discrete subgroup.
- $\Gamma \backslash G$ carries a G-invariant probability measure μ.
- $L^2(\Gamma \backslash G, \mu)$ is a unitary representation of G.
- Suppose that $\Gamma \backslash G$ is compact (not essential).
- $L^2(\Gamma \backslash G, \mu) = \bigoplus_i V_i$, where V_i are *unitary irreducible representations* of G.
- Let $K \subset G$ be a maximal compact subgroup.

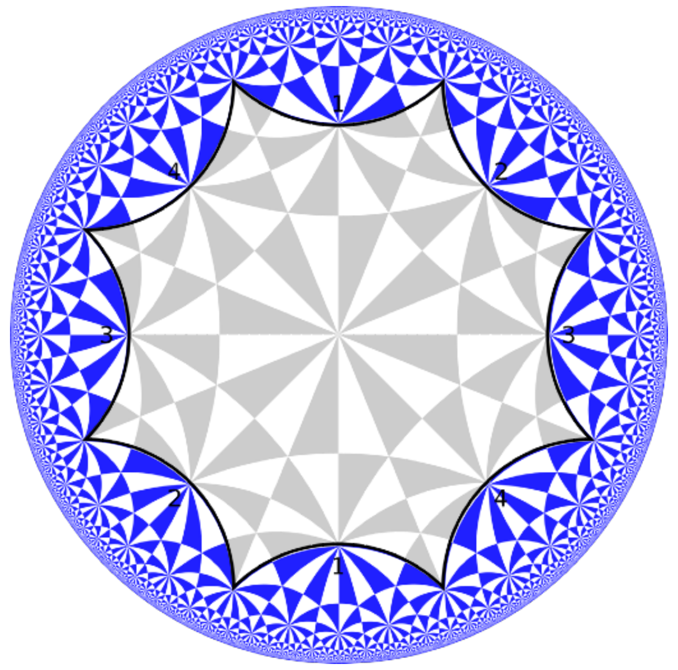
Definition: An *automorphic form* f is a smooth function on $\Gamma \backslash G$ transforming irreducibly under K and G.

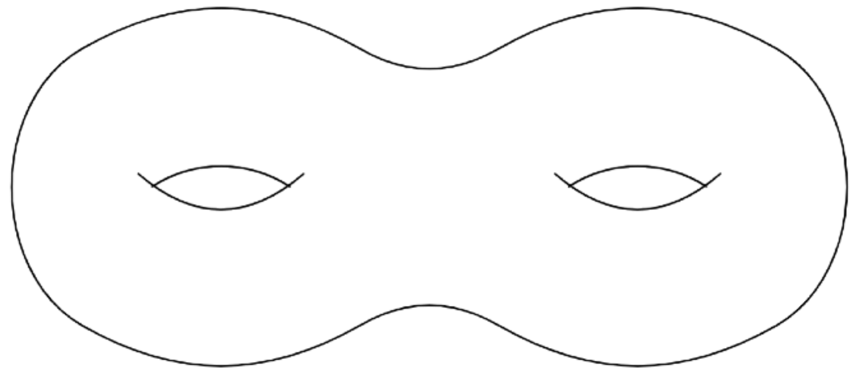
- Irreducibility under $K \Leftrightarrow f$ is a "function" on the locally symmetric space $\Gamma \backslash G / K$.
- Irreducibility under $G \Leftrightarrow f$ is an eigenfunction of Casimir differential operators.

Fundamental example: $G = \mathrm{PSL}_2(\mathbb{R}) = \mathrm{SO}^0(1,2)$

- $K = \mathrm{SO}(2)$
- $G/K = \mathbb{H}$ = upper half-plane with the hyperbolic metric.

$$ds^2 = \frac{dx^2 + dy^2}{y^2} \qquad z = x + iy \mapsto \frac{az+b}{cz+d}$$





- Γ = Fuchsian group such that $\Gamma \backslash G / K = \Gamma \backslash \mathbb{H}$ is a compact hyperbolic surface.

$$L^2(\Gamma \backslash G) = \mathbb{C} \oplus \bigoplus_{j=1}^{\infty} (D_{n_j} \oplus \overline{D}_{n_j}) \oplus \bigoplus_{k=1}^{\infty} P_{\lambda_k}$$

1. $\mathbb{C}$: trivial representation $\leftrightarrow$ f = constant functions.
2. D_n: discrete series $\leftrightarrow$ f = holomorphic modular form for Γ of weight $2n$: $f(z) = (cz+d)^{-2n} f(\frac{az+b}{cz+d})$.
3. P_λ: principal series $\leftrightarrow$ f = Laplace eigenfunction on $\Gamma \backslash \mathbb{H}$ with eigenvalue λ.

$$L^2(\Gamma\backslash G) = \mathbb{C} \oplus \bigoplus_{j=1}^{\infty} (D_{n_j} \oplus \overline{D}_{n_j}) \oplus \bigoplus_{k=1}^{\infty} P_{\lambda_k}$$

- Riemann-Roch theorem $\Rightarrow$ multiplicity$(D_n) = (2n-1)($genus$-1) + \delta_{n,1}$.
- Laplace spectrum $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$: moduli dependent, no closed form, quantum chaos as $j \to \infty$.
- Pointwise multiplication of functions provides a G-invariant product:

$$C^\infty(\Gamma\backslash G) \times C^\infty(\Gamma\backslash G) \to C^\infty(\Gamma\backslash G) \qquad (F_1, F_2) \mapsto F_1(g)F_2(g)$$

- Representation theory strongly constrains the structure constants of this product.
- For $G = \mathrm{PSL}_2(\mathbb{R})$, the space of G-invariant maps $V_i \otimes V_j \to V_k$ is at most 2-dimensional.

$$F_iF_j = \sum_k \Pi_k(F_iF_j) \qquad \Pi_k(F_iF_j) = c_{ijk}\, M_{ijk}(F_i, F_j) \qquad M_{ijk} : V_i \otimes V_j \to V_k$$

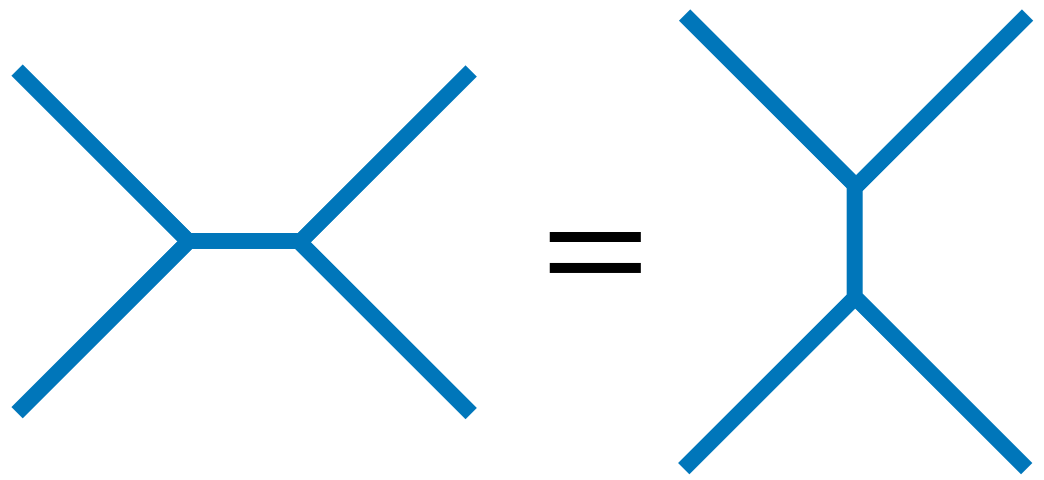
- The proportionality constant $c_{ijk} = \int_{\Gamma\backslash\mathbb{H}} f_if_jf_k$: an integral of a triple product of automorphic forms.

Associativity = Crossing Equation = Spectral Reciprocity Formula

[Bernstein, Reznikov '04, '10] [Reznikov '08]
[Kravchuk, DM, Pal '21]

- Consider $I = \int_{\Gamma \backslash G} F_i(g) F_j(g) F_k(g) F_\ell(g)\, dg$.
- Apply the spectral decomposition to $F_i F_j$ and $F_k F_\ell$: $I = \sum_m c_{ijm} c_{k\ell m} I^m_{ijk\ell}(F_i, F_j, F_k, F_\ell)$.
- $I^m_{ijk\ell} = (M_{ijm}(\,\cdot\,), M_{k\ell m}(\,\cdot\,))_m =$ *conformal block.* Determined by representation theory.
- Associativity = I has permutation symmetry in (i, j, k, ℓ).

$$\sum_m c_{ijm} c_{k\ell m} I^m_{ijk\ell}(F_i, F_j, F_k, F_\ell) = \sum_m c_{jkm} c_{\ell im} I^m_{jk\ell i}(F_j, F_k, F_\ell, F_i)$$



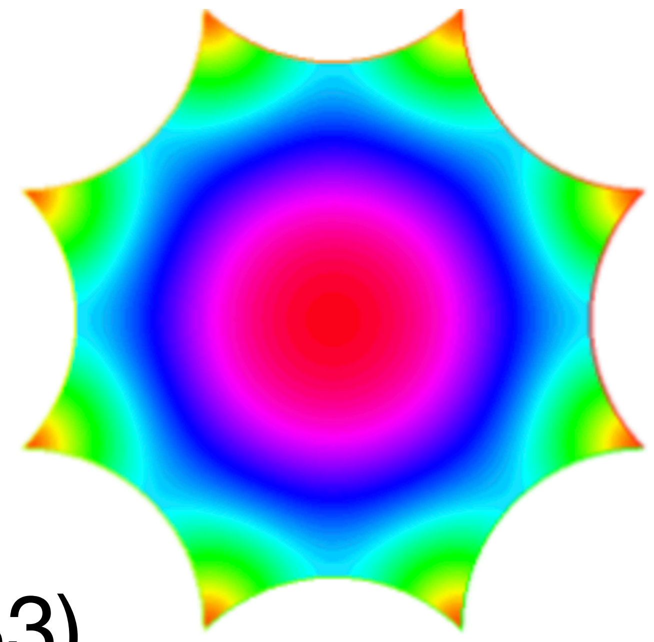
- i, j, k, ℓ range over the spectrum of irreps in $L^2(\Gamma \backslash G)$ and each F_i ranges over V_i.
- Infinite set of identities satisfied by the spectral data λ_i, c_{ijk}.

c.f. talk Francesco Bertucci

Spectral gaps of hyperbolic manifolds

Estimates on the spectral gaps of hyperbolic surfaces

Theorem: [Bonifacio '21] [Kravchuk, DM, Pal '21]



1. Every hyperbolic surface of genus two satisfies: $\lambda_1 \leq 3.8388977$.

   Bolza surface (regular octagon): $\lambda_1 \approx 3.838887258$

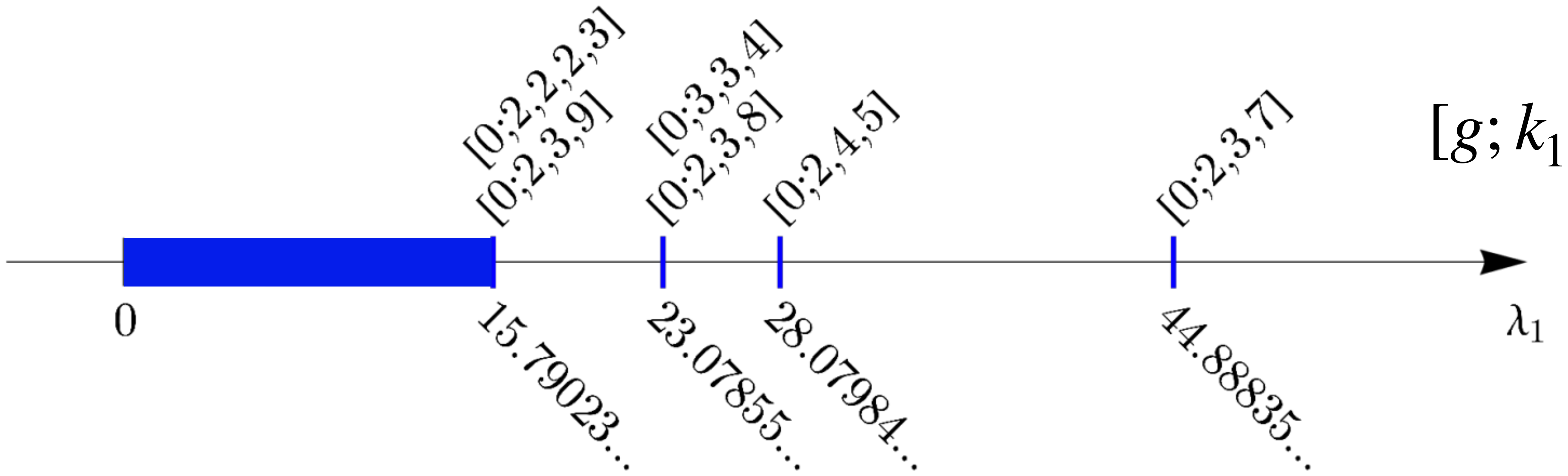
   previous record: $\lambda_1 \leq 4$ Yang-Yau (1980), Soufi-Ilias (1983)

2. Every hyperbolic surface of genus three satisfies: $\lambda_1 \leq 2.6784824$.

   Klein quartic: $\lambda_1 \approx 2.6779$

Question: What values does $\lambda_1(\Gamma)$ assume as Γ ranges over all cocompact subgroups of $\mathrm{PSL}_2(\mathbb{R})$?

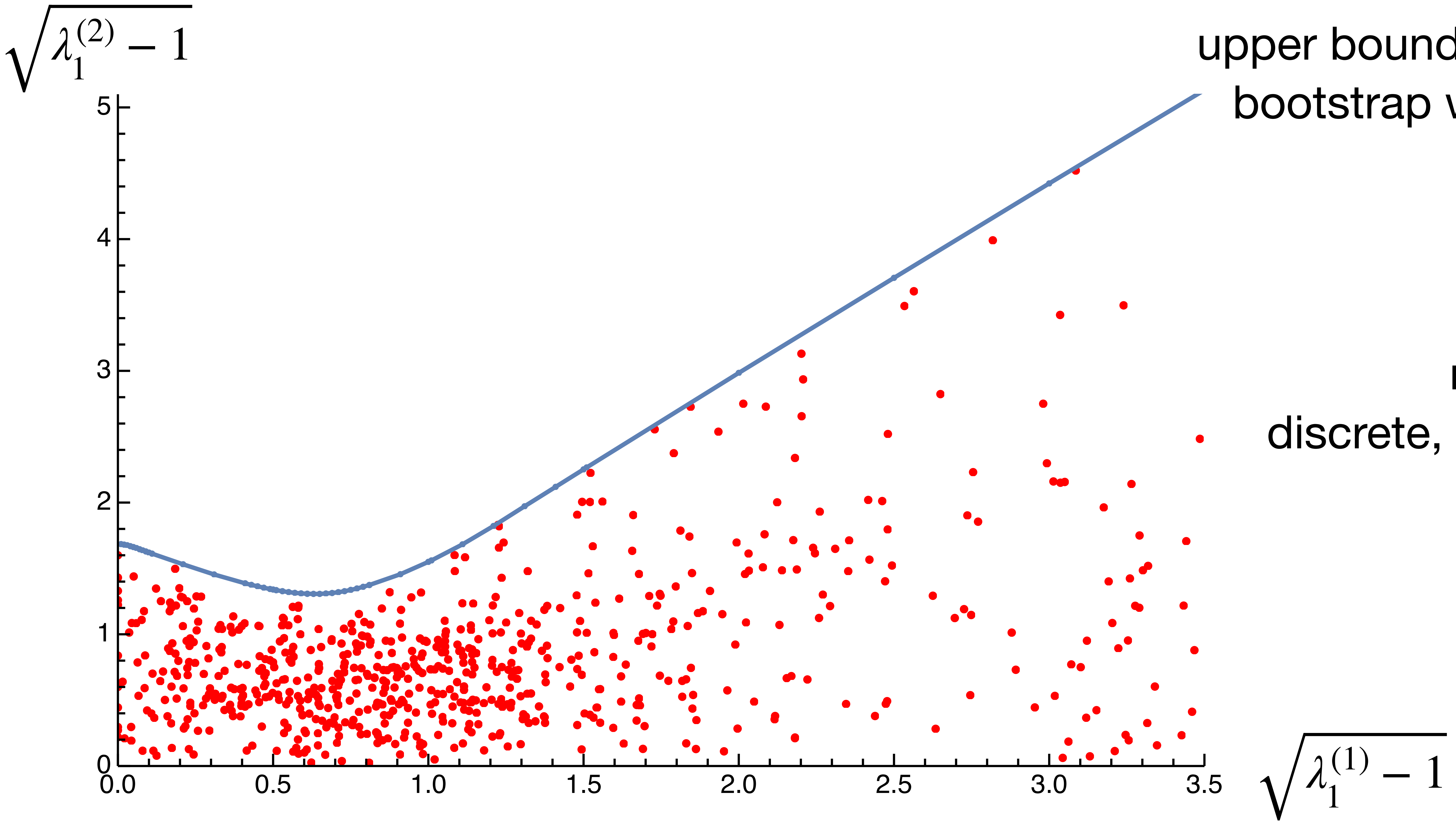
Answer:



$[g; k_1, \ldots, k_r]$: genus g, elliptic points of orders $k_1, \ldots, k_r$

Estimates on the spectral gaps of hyperbolic 3-manifolds

[Bonifacio, DM, Pal, Comm. Math. Phys. 406 (2025)]



upper bound from the conformal bootstrap with $G = \mathrm{SO}(3,1)$

red dots = discrete, cocompact $\Gamma \subset G$

$\lambda_1^{(J)}$ = (spectral gap of the Laplacian acting on symmetric traceless tensors of rank J).

Additional developments

- Spectral gaps of hyperbolic surfaces from LP applied to the Selberg trace formula

  [Fortier Bourque, Petri, arXiv: 2302.02540]

- Spectral gaps of Laplacian on spinors on hyperbolic surfaces

  [Gesteau, Pal, Simmons-Duffin, Xu, J.Assoc.Math.Res. 3 (2025)]

Weyl bound on the triple product *L*-functions

[Adve, Bonifacio, Kravchuk, DM, Pal, Radcliffe, Rogelberg ’25]

The Riemann zeta function

- The simplest *L*-function is the Riemann zeta function: $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. It satisfies:
    1. Euler product: $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$.
    2. Functional equation: $\Lambda(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ satisfes $\Lambda(s) = \Lambda(1-s)$.
- The complex-analytic properties of $\zeta(s)$ encode the distribution of prime numbers.
    1. Pole at $s = 1 \;\Rightarrow \infty$-many primes must exist.
    2. $\zeta(1+it) \neq 0 \;\Rightarrow$ (# of primes $\leq x$) $\sim \dfrac{x}{\log x}$
- Riemann hypothesis: $\Lambda(s) = 0$ only if $\mathrm{Re}(s) = 1/2$. Implies the optimal error term in 2.
- Implies the ***Lindelöf hypothesis***: $|L(\frac{1}{2}+it)| \lesssim t^{\epsilon}$ as $t \to \infty$, for any $\epsilon > 0$.

General L-functions

- General L-functions take the form of a Dirichlet series $L(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, $a_n \in \mathbb{C}$, and satisfy:

  1. $L(s)$ is meromorphic.

  2. Euler product: $L(s) = \prod_p L_p(s)$, were $L_p(s) = \prod_{j=1}^{d} (1 - \alpha_j(p)p^{-s})^{-1}$.

  3. Functional equation: $\Lambda(s) = \prod_{j=1}^{d} \hat{\Gamma}(s - \mu_j) \times L(s)$ satisfies $\Lambda(s) = \#\overline{\Lambda(1 - \overline{s})}$.

- The functional equation is a strong constraint on the *spectral data* $\alpha_j(p)$, μ_j.

- All L-functions are expected to arise from *automorphic representations*.

↑

a generalization of an irreducible subrepresentation of $L^2(\Gamma \setminus G)$

Conjectures about *L*-functions

- Many results in number theory (would) follow from analytic properties of *L*-functions.
- The Grand Riemann Hypothesis: All the zeros of $\Lambda(s)$ have $\mathrm{Re}(s) = 1/2$.
- Implies the Lindelöf hypothesis: $|L(\frac{1}{2}+it)| \lesssim t^{\epsilon}$ as $t \to \infty$, for any $\epsilon > 0$.
- This generalizes to families of *L*-functions $|L(\frac{1}{2}+it)| \lesssim C^{\epsilon}$, where $C = \prod_{j=1}^{d} |\mu_j + it|$ is the *conductor*.
- The *convexity bound* $|L(\frac{1}{2}+it)| \lesssim C^{1/4}$ is relatively easy to prove.
- Any further improvement of the exponent is referred to as a *subconvex* bound.

[Hardy, Littlewood, Landau, Bourgain, Sarnak, Bernstein, Reznikov, Michel, Venkatesh, Blomer, Nelson, ...]

Our work: A new subconvex bound from the conformal bootstrap.

L-functions and three-point functions

- For arithmetic hyperbolic surfaces, three-point functions proportional to *triple product L-functions*

$$|c_{ijk}|^2 \sim L(\tfrac{1}{2}, \pi_i \otimes \pi_j \otimes \pi_k)$$ [Watson '02, Ichino '08]

- In our case $c_{ijk} = \int_{\Gamma\backslash\mathbb{H}} f_i\, \overline{f_j}\, \varphi_k$. Here $f_{i,j}$ are holomorphic of weight $2n$, φ_k Laplace e.f. with e.v. $\lambda_k = t_k^2 + 1/4$.
- Want to estimate $|c_{ijk}|$ as $k \to \infty$.

- Subconvex bound of quality $\sigma \in (0,1)$: $L(\tfrac{1}{2}, \pi_i \otimes \pi_j \otimes \pi_k) \lesssim C^{(1-\sigma)/4}$ equivalent to $|c_{ijk}|^2 \lesssim t_k^{4n-2\sigma+\epsilon} e^{-\pi t_k}$.

State of the art:

Theorem (Bernstein, Reznikov, 2010): $\sigma \geq 1/6$ for general Γ, even non-arithmetic.

Theorem (Blomer, Jana, Nelson, 2021): $\sigma \geq 1/3$ for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. (Weyl quality)

Our result: $\sigma \geq 1/3$ for general Γ. [Adve, Bonifacio, Kravchuk, DM, Pal, Radcliffe, Rogelberg '25]

Weyl bound from the conformal bootstrap

- WLOG set $f_i = f_j = f$ and define $c_k = \int_{\Gamma\backslash\mathbb{H}} |f|^2 \varphi_k$.
- The crossing equation: $\sum_{k=0}^{\infty} |c_k|^2 H_{t_k}(z^2) = \sum_{k=0}^{\infty} |c_k|^2 H_{t_k}(z^{-2})$ valid for all $\mathrm{Re}(z) > 0$.
- $H_t(z) = z^n {}_2F_1(\frac{1}{2}+it, \frac{1}{2}-it; 1; 1-z)$ are the *conformal partial waves.*
- Convergence of the sum over $k \Rightarrow |c_k|^2 \lesssim e^{-(\pi-\epsilon)t_k}$ for any $\epsilon > 0$.

Idea: c.f. [Komargodski, Zhiboedov '12], [Fitzpatrick, Kaplan, Poland, Simmons-Duffin '12] [Qiao, Rychkov '17], [Mukhametzhanov, Zhiboedov '18, '19]

- $k \to \infty$ in the s-channel is reproduced by $k = 0$ (the identity) in the t-channel.
- Apply functionals so that the s-channel sum localizes to $[T-H, T+H]$ with $T \gg 1$.
- If we could isolate individual s-channel terms, it would imply Lindelöf.
- However, the identity only dominates in the t-channel if $T^{1/3} \lesssim H$.
- Lindelöf on average for $H \sim T^{1/3} \Rightarrow$ Weyl bound on individual terms.

Completeness of the Hyperbolic Bootstrap

[Adve, ’25]

Natural questions:

- Why is the conformal bootstrap method for hyperbolic manifolds so powerful?
- Is it in some sense complete?

↖ implying, in principle, any true statement about the spectral data of $\Gamma \backslash G$,

Theorem (Adve 2025)**:** Yes, at least for cocompact subgroups of $G = \mathrm{PSL}_2(\mathbb{R})$.

More precise statement: Every solution of the hyperbolic bootstrap equations (for all correlators) arises from $\Gamma \backslash \mathrm{PSL}_2(\mathbb{R})$ for some Γ.

A cautionary tale: Single-correlator bounds saturated to ≥ 7 digits fail to be exactly saturated.

[Radcliffe '24]

Sketch of the proof

1. The hyperbolic bootstrap equations axiomatize the following situation:
   - $V =$ a unitary representation of $G = \mathrm{SO}^0(1,2)$, with a discrete spectrum of irreducibles.
   - The smooth vectors in V carry a G-invariant, commutative, associative product.
2. Gelfand duality
   - { spaces } $\leftrightarrow$ { commutative algebras }
   - space $X \mapsto V =$ functions on X
   - commutative algebra $V \mapsto X = \mathrm{Hom}(V, \mathbb{C})$
3. In the setting of 1, Gelfand duality produces $X = \Gamma \backslash G$ for a discrete cocompact Γ.

	algebraic definition		geometric definition
a hyperbolic manifold	a solution of the hyperbolic bootstrap	$\Leftrightarrow$	$\Gamma \backslash \mathbb{H}^d$
a conformal field theory	a solution of the conformal bootstrap		?