Scalars in 2d CFT

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Based on 2505.18314 with Cyuan-Han Chang, Liam Fitzpatrick, Tobi Ramella

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Perturbative corrections. Non-perturbative corrections from (a) other light operators and (b) other SL(2,Z) saddles

- Another Question: What is the minimal theory of quantum gravity in AdS_3? Is there a “pure” theory?
- Related question: What is the largest gap allowed for a 2d CFT at large central charge?

- The main strategy we are going to be using is modular invariance of the torus partition function:

$$Z(\tau) = Z(\gamma\tau), \quad \gamma \in SL(2,\mathbb{Z})$$

- So τ is valued not in the upper half plane $\mathbb{H}$ (y>0), but its quotient $\mathbb{H}/SL(2,\mathbb{Z})$

- **(Notation:** $\tau = x + iy$**,** $Z(\tau)$ **is not assumed meromorphic)**

$$Z(\tau) := \mathrm{Tr}(e^{-2\pi y\Delta}e^{2\pi iJx})$$

$$Z(y) = \mathrm{Tr}(e^{-2\pi y \Delta}) = Z(1/y)$$

- From modular invariance of the partition function, it was shown at large c, $\Delta_{\text{gap}} \leq \#c$ — consistent scaling with black hole solutions

(Hellerman)

- This result is agnostic of spin. There must be some state with dimension less than ~c, but no statement on its spin

- The reason is the result is using the spinless bootstrap — the bootstrap equations used happen to not grade by spin

$$Z(y) = \mathrm{Tr}(e^{-2\pi y \Delta}) = Z(1/y)$$

- Can we say something more refined? E.g. there must be a state of spin 0 with certain properties? Simply grading by spin and trying similar bootstrap techniques do not produce **any** bound at large c

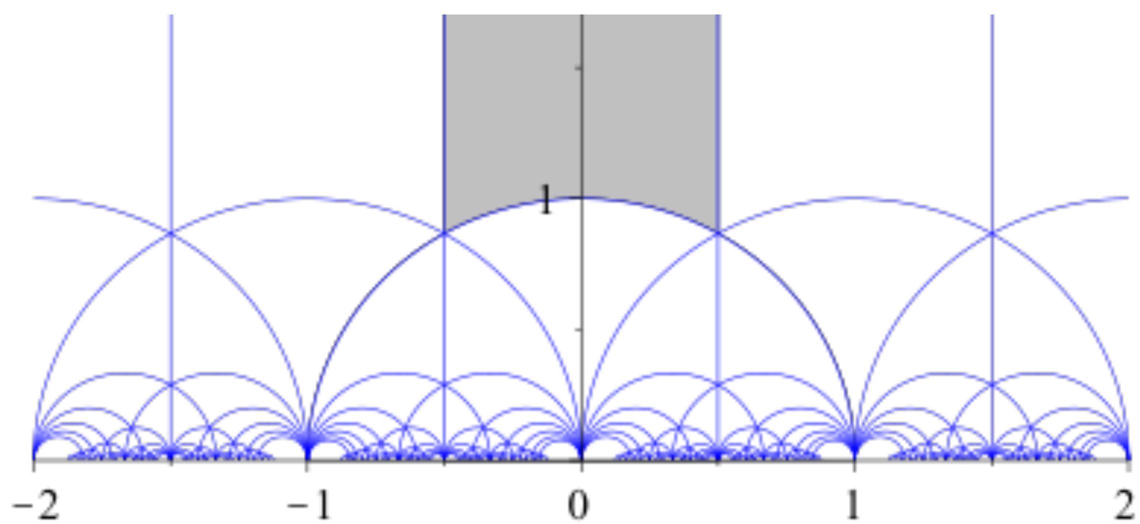
(Collier, Lin, Yin)

$$Z(\tau) := \mathrm{Tr}(e^{-2\pi y \Delta} e^{2\pi i J x})$$

- In this talk we will derive a new crossing equation acting only on **scalars** of CFTs
  - First use $U(1)^c$ theories as a benchmark then move on to general 2d CFT
  - Rephrasing of the Riemann hypothesis purely in terms of CFT
  - Numerical, spin-dependent (scalar) bounds on CFTs
- The technology used is harmonic analysis which we review

Harmonic analysis

- Let us review the fundamental domain $\mathbb{H}/SL(2,\mathbb{Z})$



$\tau = x + iy$

$\tau \in \mathbb{H}/SL(2,\mathbb{Z})$

- There is a natural metric on moduli space $ds^2 = \dfrac{dx^2 + dy^2}{y^2}$

- The Laplacian is $\Delta = -y^2(\partial_x^2 + \partial_y^2)$

- Also a natural inner product on this space called the Petersson inner product

$$(f, g) = \int_{\mathscr{F}} \frac{dxdy}{y^2} f(\tau)\overline{g(\tau)}$$

- This inner product will allow us to read off the basis coefficients when we decompose into eigenfunctions

- The idea is to decompose square-integrable modular invariant functions into eigenfunctions of the Laplacian
- There are three different types of eigenfunctions:
    - The **constant** function
    - An infinite **continuous family** called the real analytic Eisenstein series
    - An infinite **discrete family** called Maass cusp forms

- The real analytic Eisenstein series are defined as:

$$E_s(\tau) = \sum_{\gamma \in \Gamma_\infty \backslash SL(2,\mathbb{Z})} y^s |_\gamma$$

- This converges for Re$(s) > 1$ but can be analytically continued to all s in the complex plane

- It admits the following Fourier decomposition

$$E_s(\tau) = y^s + \frac{\Lambda(1-s)}{\Lambda(s)} y^{1-s} + \sum_{j=1}^{\infty} 4\cos(2\pi j x) \frac{\sigma_{2s-1}(j)}{j^{s-\frac{1}{2}}\Lambda(s)} \sqrt{y} K_{s-\frac{1}{2}}(2\pi j y)$$

where $\Lambda(s) = \pi^{-s}\Gamma(s)\zeta(2s)$

- It is an eigenfunction of Δ with eigenvalue $s(1-s)$

- The Maass cusp forms $\nu_n(\tau)$are a set of discrete eigenfunctions, $n = 1,2,\cdots$. They are modular invariant and take the following functional form

$$\nu_n(\tau) = \sum_{j=1}^{\infty} a_j^{(n)} \cos(2\pi j x)\sqrt{y} K_{iR_n}(2\pi j y)$$

- It has eigenvalue $\frac{1}{4} + R_n^2$

- $a_j^{(n)}$ and R_n are sporadic real numbers with extremely interesting statistics

- As an example here are the first five (even) Maass cusp forms:

n	R_n	$\lambda_n = R_n^2 + \frac{1}{4}$	$a_1^{(n)}$	$a_2^{(n)}$	$a_3^{(n)}$	$a_4^{(n)}$
1	13.780	190.132	1	1.549	0.247	1.400
2	17.739	314.907	1	−0.765	−0.978	−0.414
3	19.423	377.522	1	−0.693	1.562	−0.520
4	21.316	454.613	1	1.288	1.252	0.658
5	22.786	519.448	1	0.268	−0.585	−0.928

- These functions obey have chaotic properties. Mathematicians have computed these numbers to a lot of precision.

- Let us consider the Eisenstein series at $s = \frac{1}{2} + it$ together with the constant and the cusp forms:

$$E_s(\tau) = y^s + \frac{\Lambda(1-s)}{\Lambda(s)} y^{1-s} + \sum_{j=1}^{\infty} 4\cos(2\pi jx) \frac{\sigma_{2s-1}(j)}{j^{s-\frac{1}{2}}\Lambda(s)} \sqrt{y} K_{s-\frac{1}{2}}(2\pi jy)$$

$$\nu_n(\tau) = \sum_{j=1}^{\infty} a_j^{(n)} \cos(2\pi jx) \sqrt{y} K_{iR_n}(2\pi jy)$$

- These functions form a basis for square-integrable modular-invariant functions. By square-integrable:

$$\int_{\mathcal{F}} \frac{dxdy}{y^2} |f(\tau)|^2 < \infty$$

Roelcke-Selberg decomposition

$$f(\tau) = c + \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} ds E_s(\tau)(f, E_s) + \sum_{n=1}^{\infty} \frac{(f,\nu_i)}{(\nu_i,\nu_i)}\nu_i(\tau)$$

where (f, g) is the Petersson inner product

$$(f,g) = \int_{\mathcal{F}} \frac{dxdy}{y^2} f(\tau)\overline{g(\tau)}$$

and the constant c is $\quad \frac{3}{\pi}\int_{\mathcal{F}} \frac{dxdy}{y^2} f(\tau)$

$U(1)^c$ CFTs

- First consider theories with an extended chiral algebra of $U(1)^c$. These theories have a character given by $\dfrac{1}{\eta(q)^c}$

- Examples include Narain's family of c free bosons compactified on a lattice

- The primary counting function $\hat{Z} := y^{c/2}\,|\,\eta(q)\,|^{2c}\,Z$ admits a spectral decomposition

- For Narain theories it is counting the lattice theta function

$$\eta(q) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

- Ex: Free boson theories at c=2. These are labeled by two complex moduli $\rho, \sigma \in \mathbb{H}/SL(2,\mathbb{Z})$

- Schematically the spectral decomposition looks like:

$$\hat{Z}(\rho,\sigma;\tau) \sim \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} ds \frac{\text{stuff with moduli } (\rho,\sigma) \text{ dependence}}{\zeta(2s)\zeta(2-2s)} E_s(\tau)$$

↗

pole structure depends on zeros of zeta function!

- There’s an exact spectral decomposition at c=2

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$$
\begin{aligned}
Z^{(c=2)}(\tau;\rho,\sigma) = \alpha &+ \widehat{E}_1(\rho) + \widehat{E}_1(\sigma) + \widehat{E}_1(\tau) \\
&+ \frac{1}{4\pi i}\int_{\mathrm{Re}\, s=\frac{1}{2}} ds\, 2\frac{\Lambda(s)^2}{\Lambda(1-s)} E_s(\rho)E_s(\sigma)E_s(\tau) \\
&+ 8\sum_{\epsilon=\pm}\delta_\epsilon \sum_{n=1}^{\infty}(\nu_n^\epsilon,\nu_n^\epsilon)^{-1}\nu_n^\epsilon(\rho)\nu_n^\epsilon(\sigma)\nu_n^\epsilon(\tau),
\end{aligned}
$$

- For general c, the decomposition looks roughly similar

$$\hat{Z}^c(\mu;\tau) \sim \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} ds\ \varepsilon^{c}_{\frac{c}{2}-s}(\mu) E_s(\tau)$$

where the overlap $\varepsilon^{c}_{\frac{c}{2}-s}(\mu)$ has the following properties:

1. It is fully determined by the scalars of the CFT

2. It has poles at (roughly) the zeros of the zeta function.

- Property 1 (determined from scalars) is because we can "unfold" the integral over the fundamental domain by switching the integral and the sum

$$\int_{\mathcal{F}} \sum_{\gamma \in \Gamma_\infty \backslash SL(2,\mathbb{Z})} \gamma(y^s e^{-2\pi \Delta y}) = \int_{-1/2}^{1/2} dx \int_0^\infty dy (y^s e^{-2\pi\Delta y})$$

which changes the integral over the fundamental domain to the integral over the strip. (This is called the Rankin-Selberg transform.)

- This shows the overlap is given by roughly $\mathcal{E}_s^c(\mu) := \sum_{\Delta \in \mathcal{S}} (2\Delta)^{-s}$

- Property 2 is more subtle

- We can prove a gap in the spectrum prohibits poles in $\Gamma(s-\frac{1}{2})\zeta(2s-1)\varepsilon_{\frac{c}{2}-s}(\mu)$. So $\varepsilon_{\frac{c}{2}-s}$ can have poles at nontrivial zeros of $\zeta(2s-1)$ (and generically it does)

- Homework: This implies Re(2s—1) = 1/2 (or Re(s) = 3/4)

- The primary-counting partition function has the spectral decomposition:

$$\hat{Z}^c(\tau,\mu) = E_{c/2}(\tau) + 3\pi^{-\frac{c}{2}}\Gamma\left(\frac{c}{2}-1\right)\mathcal{E}^c_{\frac{c}{2}-1}(\mu) + \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} ds \pi^{s-\frac{c}{2}}\Gamma\left(\frac{c}{2}-s\right)\mathcal{E}^c_{\frac{c}{2}-s}(\mu)E_s(\tau)$$
$$+\sum_{n=1}^{\infty}\sum_{\epsilon=\pm}\frac{(\hat{Z}^c,\nu_n^\epsilon)(\mu)}{(\nu_n^\epsilon,\nu_n^\epsilon)}\nu_n^\epsilon(\tau).$$

$$\mathcal{E}^c_s(\mu) := \sum_{\Delta\in\mathcal{S}}(2\Delta)^{-s}$$

- The pole structure in s is constrained and the overlap with the Eisenstein series is **just a function of scalars**

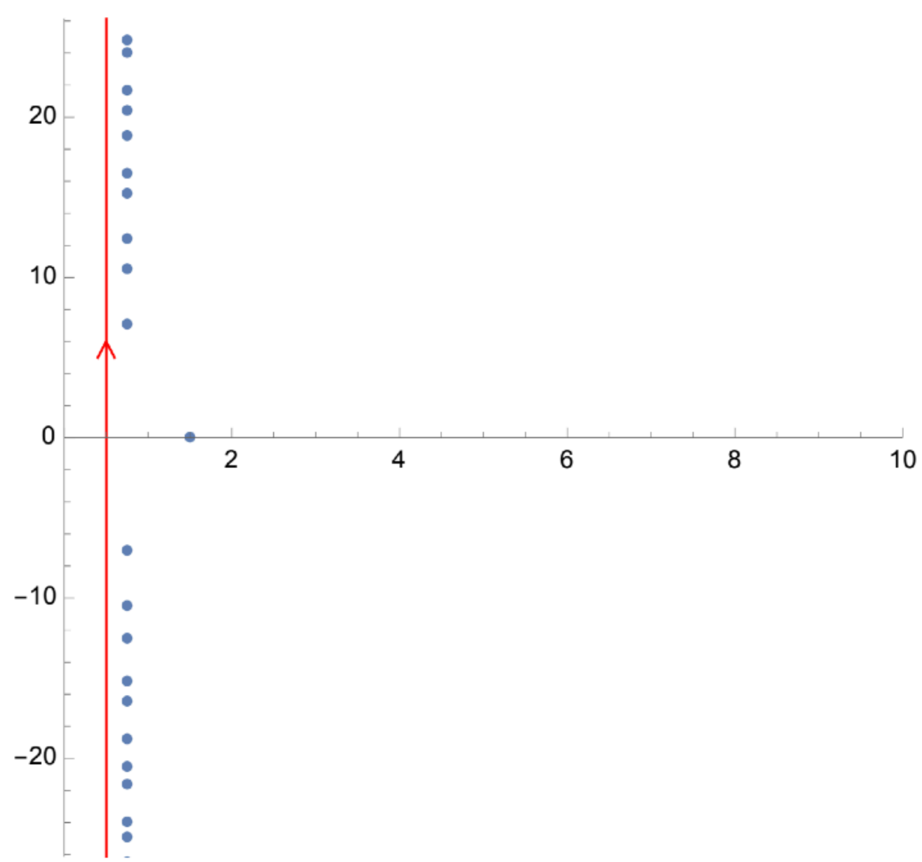
- If we take the scalar part of our spectral decomposition we then get

$$\sum_{\Delta\in\mathcal{S}} e^{-2\pi\Delta y} = \frac{\Lambda\left(\frac{c-1}{2}\right)}{\Lambda\left(\frac{c}{2}\right)} y^{1-c} + \varepsilon_c(\mu) y^{-\frac{c}{2}} + \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} ds \pi^{s-\frac{c}{2}} \Gamma\left(\frac{c}{2}-s\right) \mathcal{E}^{c}_{\frac{c}{2}-s}(\mu) y^{s-\frac{c}{2}},$$

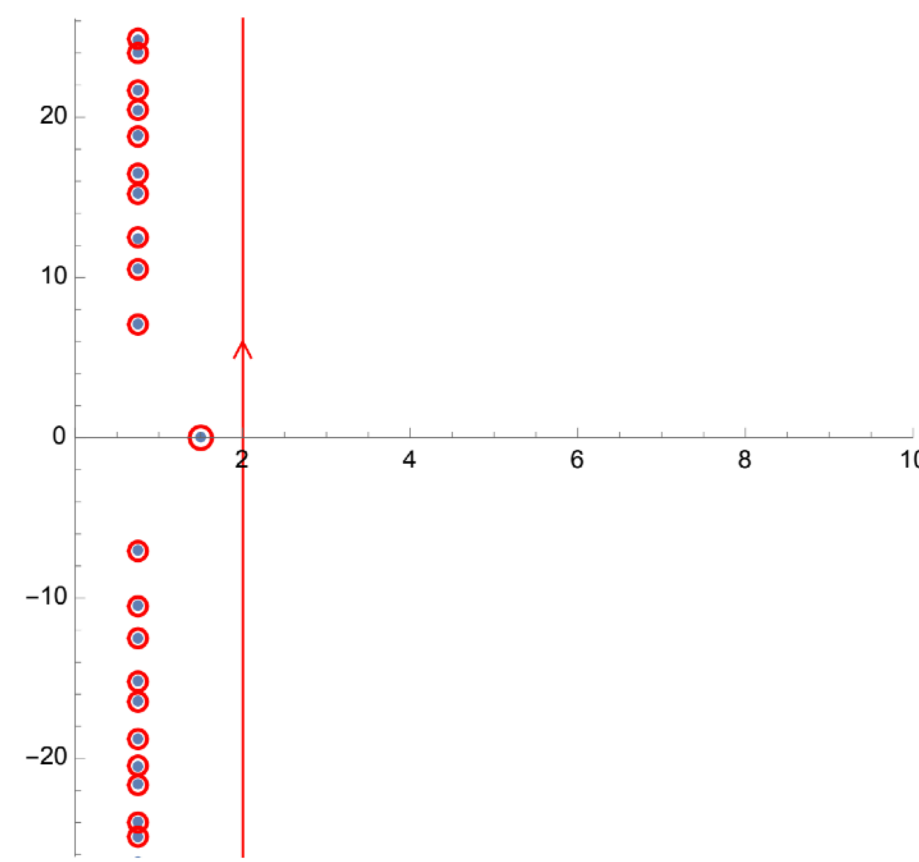
- We would like to rewrite the term $\mathcal{E}^{c}_{\frac{c}{2}-s}(\mu)$ in terms of the scalar operators S but this requires moving the contour

$$\mathcal{E}^c_s(\mu) \coloneqq \sum_{\Delta\in\mathcal{S}} (2\Delta)^{-s}$$

- The integral we want to do is: $\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} ds\pi^{s-\frac{c}{2}}\Gamma\left(\frac{c}{2}-s\right)\mathcal{E}^{c}_{\frac{c}{2}-s}(\mu)y^{s-\frac{c}{2}}$
- The pole structure of $\mathcal{E}^{c}_{\frac{c}{2}-s}(\mu)$ in the s-plane is completely determined
- There are poles at $s = c/2$ and $s = 1/2 + z_n/2$, where z_n are nontrivial zeros of the Riemann zeta function



(a)



(b)

- After the contour deformation we then get the following **scalar crossing equation:**

$$1+\sum_{\Delta\in\mathcal{S}} e^{-\frac{2\pi\Delta}{T}} = \frac{\Lambda\left(\frac{c}{2}-\frac{1}{2}\right)}{\Lambda\left(\frac{c}{2}\right)} T^{c-1} + aT^{\frac{c}{2}} + \sum_{n=1}^{\infty} \mathrm{Re}\left(b_n T^{\frac{c}{2}-1+\frac{z_n}{2}}\right)$$

$$+\frac{T^{c-1}}{\sqrt{\pi}} \sum_{\Delta\in\mathcal{S}} \sum_{n=1}^{\infty} d(n) n^{c-2}\, U\left(-\frac{1}{2}, \frac{c}{2}, 2\pi n^2 \Delta T\right) e^{-2\pi n^2 \Delta T}$$

$$d(n) = \sum_{s|n} s\mu(s)$$

$$\Lambda(s) = \pi^{-s}\Gamma(s)\zeta(2s)$$

- The subsubleading corrections at high temperature are controlled by z_n!

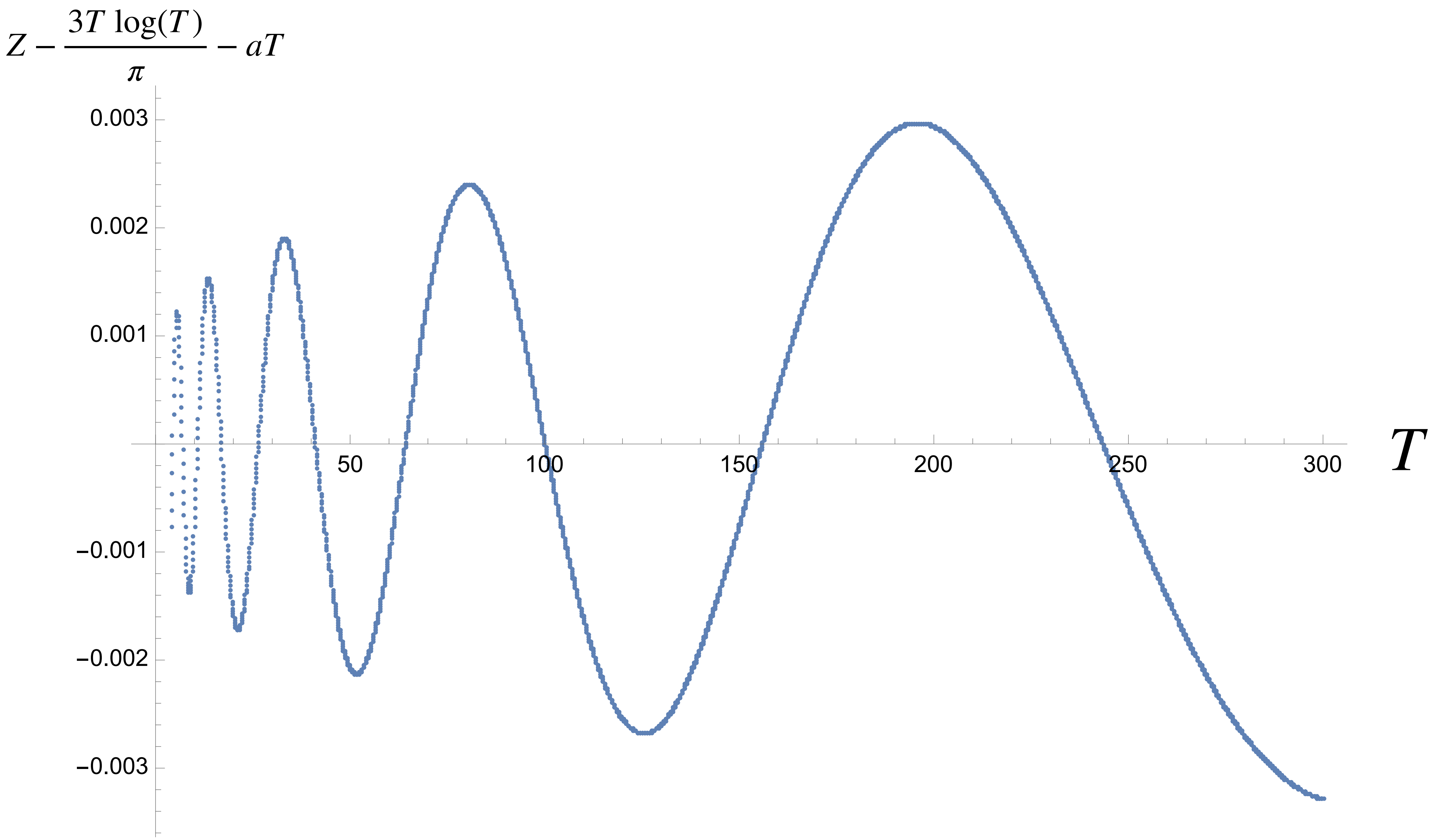
Just for fun….

$$Z^{c,\text{scalar}}(T) = \frac{\Lambda\left(\frac{c-1}{2}\right)}{\Lambda\left(\frac{c}{2}\right)} T^{c-1} + aT^{\frac{c}{2}} + \sum_{n=1}^{\infty} \text{Re}\left(b_n T^{\frac{c}{2}-1+\frac{z_n}{2}}\right) + \mathcal{O}\left(e^{-2\pi\Delta_{\text{gap}}T}\right)$$

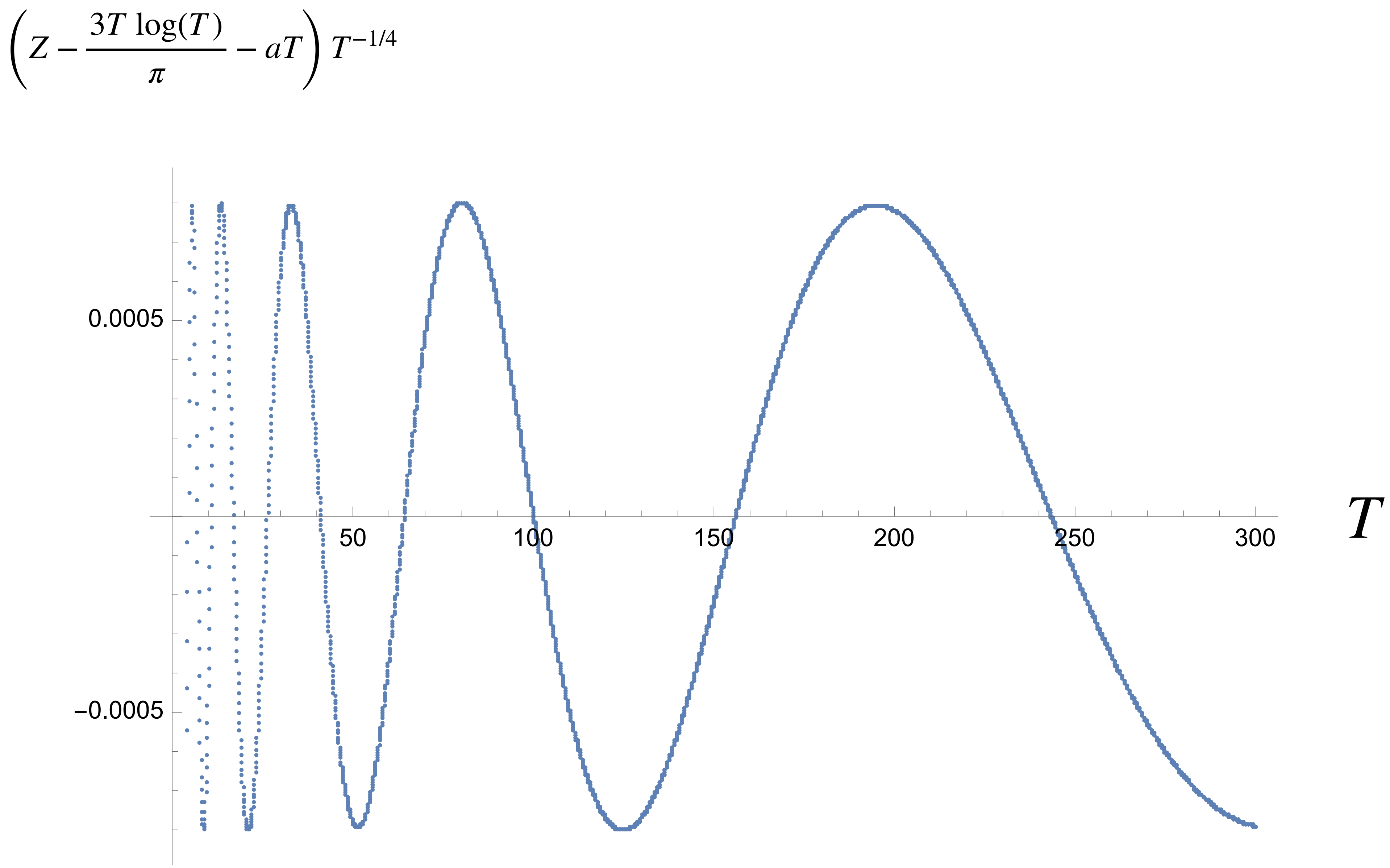
This term grows as $T^{\frac{c}{2}-\frac{3}{4}}$ iff the Riemann hypothesis is true!

It seems quite intriguing that the Riemann hypothesis can be rephrased as a statement about the behavior of scalars in $U(1)^c$ CFTs!

$SU(3)_1$ WZW model



$SU(3)_1$ WZW model rescaled



Bootstrapping scalar gaps

- Can we use our crossing equation to bound the **scalar gap**

- This is more refined information than the usual modular bootstrap

$$1+\sum_{\Delta\in\mathcal{S}} e^{-\frac{2\pi\Delta}{T}} = \frac{\Lambda\left(\frac{c}{2}-\frac{1}{2}\right)}{\Lambda\left(\frac{c}{2}\right)} T^{c-1} + aT^{\frac{c}{2}} + \sum_{n=1}^{\infty} \mathrm{Re}\left(b_n T^{\frac{c}{2}-1+\frac{z_n}{2}}\right)$$
$$+\frac{T^{c-1}}{\sqrt{\pi}} \sum_{\Delta\in\mathcal{S}} \sum_{n=1}^{\infty} d(n) n^{c-2}\; U\left(-\frac{1}{2},\frac{c}{2},2\pi n^2\Delta T\right) e^{-2\pi n^2 \Delta T}$$

- **PROBLEM:** Constants a, b_n unknown and *sign-indefinite.* Is there a linear functional we can apply to kill them?

- Here we borrow a technique from the sphere-packing community!

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Special thanks to Danylo Radchenko!

- We write down a family of kernels $\Phi^k(t)$ that we integrate our crossing equation against:

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$$\mathcal{F}^k\left[h(t)\right] = \int_0^\infty \frac{dt}{t} h(t) \Phi^k(t)$$

- We choose $\Phi^k(t)$ so that

$$\mathcal{F}^k\left[t^s\right] \propto \zeta(s)$$

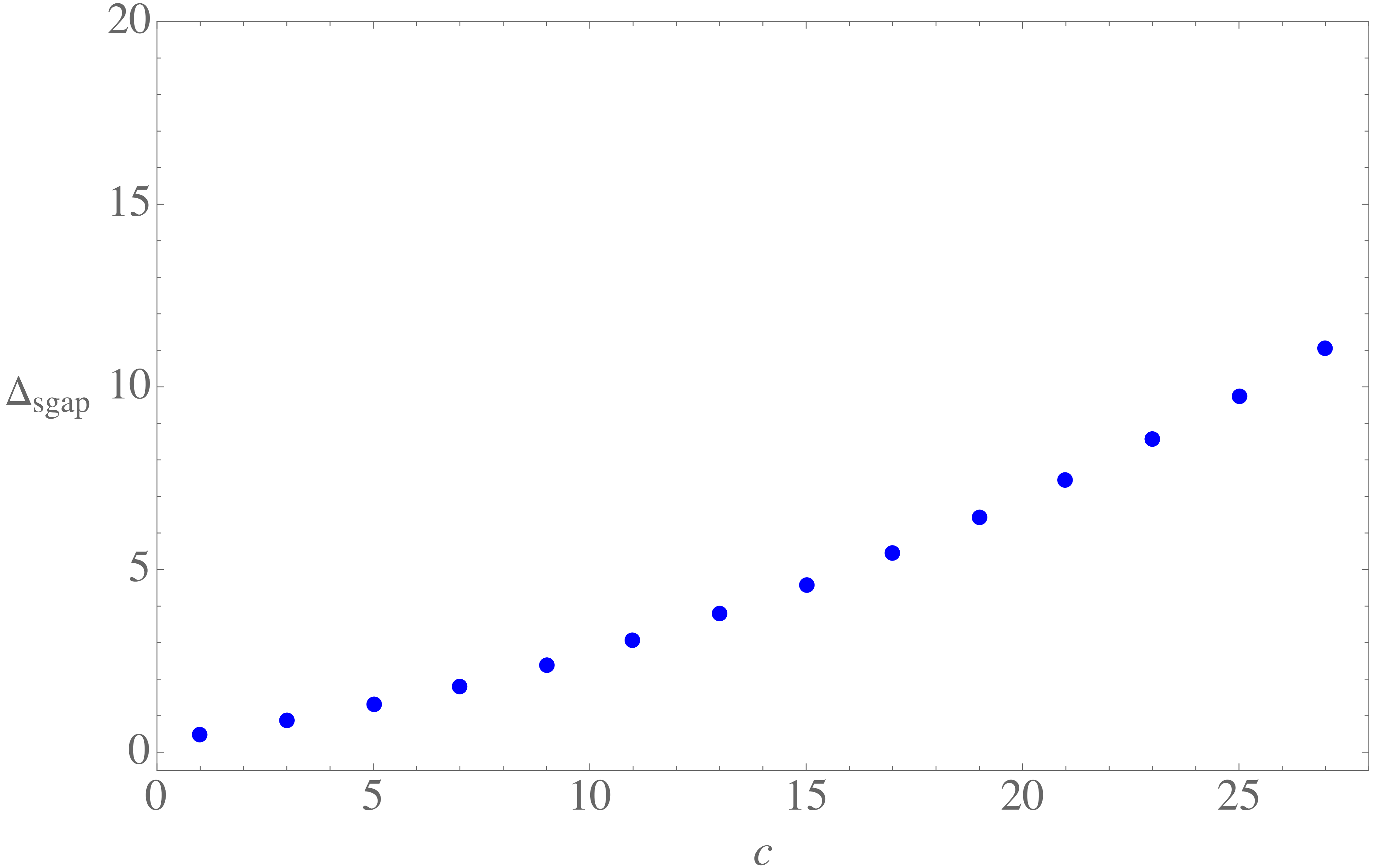
- Thus we are able to kill the zeros of the zeta function and get a positive sum rule!

- We get a positive sum rule acting **only on scalar operators**. For example for c=3, the sum rule looks like

$$\sum_{\Delta \in \mathcal{S}} \sum_{k} \alpha_k \left[\frac{\sqrt{2} - e^{2\sqrt{2}\pi\sqrt{k\Delta}}(\sqrt{2} - 2\pi\sqrt{k\Delta})}{2(-1 + e^{2\sqrt{2}\pi\sqrt{k\Delta}})^2\sqrt{\Delta}} + \frac{\pi e^{2\sqrt{2}\pi\sqrt{\frac{\Delta}{k}}}}{(-1 + e^{2\sqrt{2}\pi\sqrt{\frac{\Delta}{k}}})^2 k^{3/2}} \right]$$
$$= \frac{\pi}{6} \sum_{k} \alpha_k \left(\frac{3}{2} k^{\frac{1}{2}} + \frac{1}{2} k^{-\frac{3}{2}} \right).$$

- We can then choose the coefficients k, α_k cleverly and use positivity to bound the maximum scalar gap allowed

- Using the positive semidefinite solver SDPB we get the following bounds on the lightest scalar for $U(1)^c$ CFTs



- At large c, the scalar gap we get from this equation appears to scale quadratically with c

- This is in contrast with the usual spinless modular bootstrap which usually says there is an operator (of any spin) that is linear in c

- Can we improve our bound with other techniques? Or perhaps the scalar bound is qualitatively weaker than the spin-agnostic bound?

General Virasoro CFT

- So far our techniques have been applied to $U(1)^c$ CFTs. The reason these theories are simpler is because when we count the $U(1)^c$ primary operators, the resulting function is square-integrable

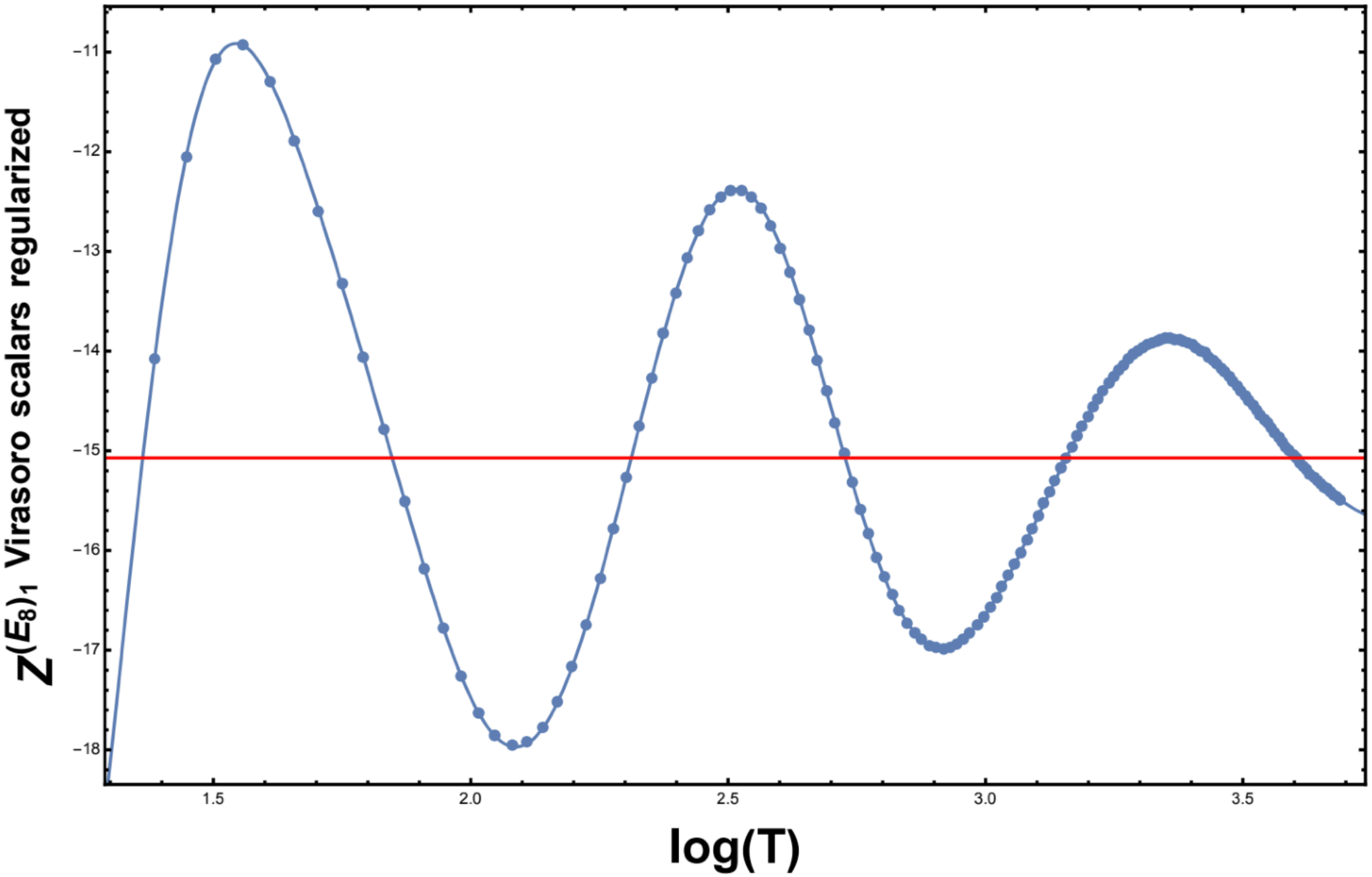
- However if we count Virasoro primary operators the function grows exponentially as $e^{\frac{c-1}{12}y}$ at large y

- Is there a scalar crossing equation for **Virasoro primary** operators? Yes!

- The spectral decomposition is more subtle. The integrals of the scalar partition function diverge due to the negative Casimir energy of the vacuum

- To fix this we subtract out the Poincare sums of (finitely many) light operators scalar part of SL(2,Z) sum of all operators with $\Delta < \frac{c-1}{12}$

$$Z_p^{\text{scalars}}(y) - Z_{j=0}^{\text{gravity}}(y) = \varepsilon + \sum_{k=1}^{\infty} \mathrm{Re}\left(\delta_k y^{\frac{1+z_k}{2}}\right)$$
$$+ (\text{perturbative terms fixed by light spectrum})(y) + O(e^{-\#/y})$$



- After applying the sphere packing functional we get:

$$f(\Delta, r) := \frac{\log(1 - e^{-2\sqrt{2}\pi r\sqrt{\Delta}})}{r - r^{-1}}$$

$$\int_{-\frac{c-1}{12}}^{\infty} d\Delta \rho_p^{\text{scalars}}(\Delta) \left[f(\Delta, r) + f(\Delta, r^{-1})\right] = \frac{\pi\varepsilon}{6}$$

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  If you have two functions of y that have the same SL(2,Z) sum, you can "unfold" the integral of any modular function in two ways

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- Another interpretation of this result:

If you have two functions of y that have the same SL(2,Z) sum, you can "unfold" the integral of any modular function in two ways

Classic example: y^s and y^{1-s} both give (approximately) $E_s(\tau)$. Unfold in two ways gives $s \leftrightarrow 1-s$ symmetry in Mellin transform of scalars

Another example is the c=1 free boson

$$\hat{Z}^{(c=1)}(x,y;r) = r \sum_{\gamma \in SL(2,\mathbb{Z})/\Gamma_\infty} \gamma \cdot \vartheta(r^2/y), \qquad \vartheta(t) \equiv \sum_{m=-\infty}^{\infty} e^{-\pi m^2 t} = \frac{1}{\sqrt{t}} \vartheta(\frac{1}{t}).$$

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The m=0 term is problematic for convergence; subtract it out

$$\int_0^\infty \frac{dy}{y^{3/2}} e^{-2\pi y \Delta} r(\vartheta(r^2/y) - 1) = 2\sum_{m=1}^{\infty} \frac{e^{-2\pi\sqrt{2}mr\sqrt{\Delta}}}{m} = -2\log(1 - e^{-2\sqrt{2}\pi r\sqrt{\Delta}})$$

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- At large Δ term in the brackets falls off as $e^{-\#\sqrt{\Delta}}$. For Narain CFTs this sum is clearly convergent, but for Virasoro CFTs the asymptotic density of scalars is given by $e^{2\pi\sqrt{\frac{\Delta(c-1)}{3}}}$

- Thus at large c (c>7) the sum rule does not converge

- Liouville theory

$$\rho(\Delta) \propto \frac{1}{\sqrt{\Delta}}$$

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$$\int_0^\infty d\Delta \frac{\log\left(\frac{1-e^{-2\sqrt{2}\pi r\sqrt{\Delta}}}{1-e^{-2\sqrt{2}\pi r^{-1}\sqrt{\Delta}}}\right)}{\sqrt{\Delta}(r-r^{-1})} = -\int_0^\infty \frac{du}{\sqrt{2}\pi}\log(1-e^{-u}) = \frac{\pi}{6\sqrt{2}},$$

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- Various theories (WZW models, etc) with $c < 7$

- Poincare sums of light operators

- The challenge is now to analytically continue the sum rule in *r* so that we can define a scalar sum rule for all Virasoro CFTs

- The integration of heavy operators against this kernel is convergent for sufficiently large Re(r) and the pole structure is exactly known (in terms of the light operators)

- There still is a crossing equation with zeros of the zeta function before applying a functional removing them. A fantasy would be the chaos of the zeros of the zeta function is related to black hole chaos???

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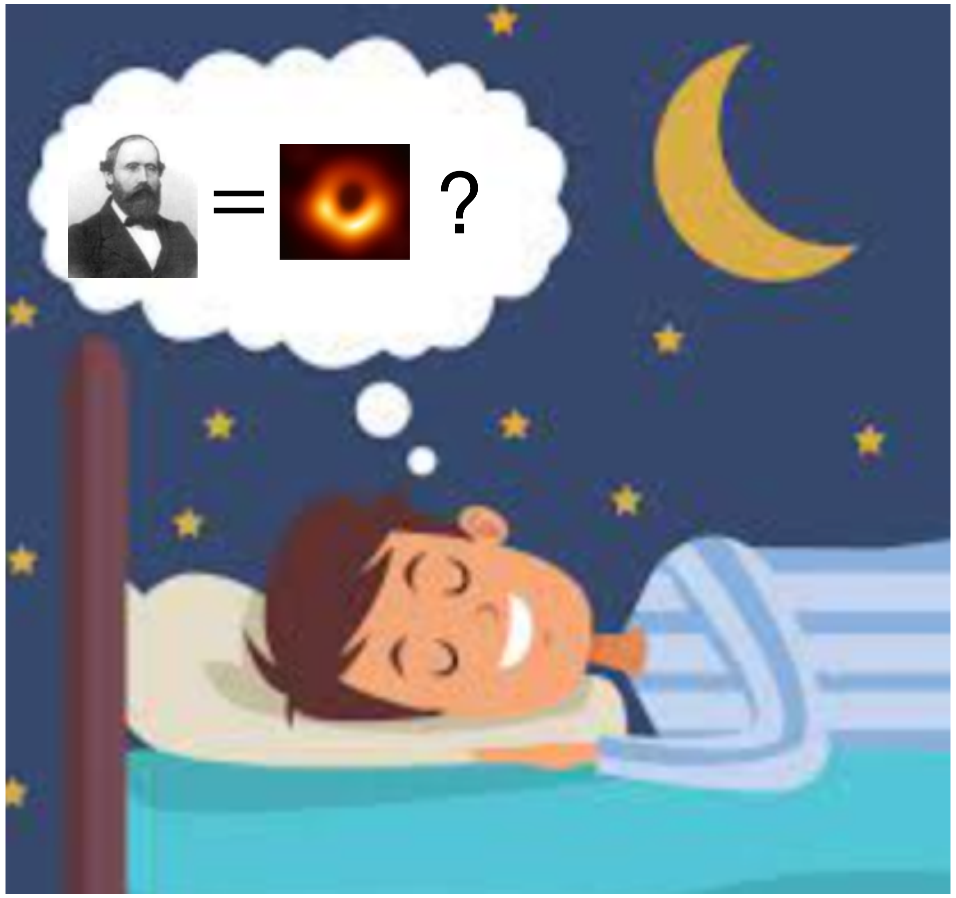

Higher spin

$$
\begin{aligned}
Z^{\text{spin j}}(y) &= \sqrt{y} \sum_{\Delta \geq 0} d(\Delta, j) e^{-2\pi \Delta y} \\
&= \frac{\sqrt{y}}{2} \sum_{\Delta \geq 0} d(\Delta, 0) \sum_{n=1}^{\infty} \mu(n) \sum_{k|j} e^{-\pi y j \left(\frac{2\Delta n^2 j}{k^2} + \frac{k^2}{2\Delta n^2 j} \right)} \\
&\quad + \sum_{k=1}^{\infty} \operatorname{Re}\left(\delta_k \frac{\sigma_{z_k}(j) K_{\frac{z_k}{2}}(2\pi j y) y^{\frac{1}{2}}}{\Lambda(\frac{z_k+1}{2}) j^{\frac{z_k}{2}}} \right) + \sum_{k=1}^{\infty} \frac{a_j^{(k)}(\hat{Z}, \nu_k)}{2(\nu_k, \nu_k)} \sqrt{y} K_{iR_n}(2\pi j y).
\end{aligned}
$$

Higher spin

$$
\begin{aligned}
Z^{\text{spin j}}(y) &= \sqrt{y}\sum_{\Delta\geq 0} d(\Delta,j)e^{-2\pi\Delta y} \\
&= \frac{\sqrt{y}}{2}\sum_{\Delta\geq 0} d(\Delta,0)\sum_{n=1}^{\infty}\mu(n)\sum_{k|j} e^{-\pi y j\left(\frac{2\Delta n^2 j}{k^2}+\frac{k^2}{2\Delta n^2 j}\right)} \\
&\quad + \sum_{k=1}^{\infty}\mathrm{Re}\left(\delta_k \frac{\sigma_{z_k}(j)K_{\frac{z_k}{2}}(2\pi j y)y^{\frac{1}{2}}}{\Lambda(\frac{z_k+1}{2})j^{\frac{z_k}{2}}}\right) + \sum_{k=1}^{\infty}\frac{a_j^{(k)}(\hat{Z},\nu_k)}{2(\nu_k,\nu_k)}\sqrt{y}K_{iR_n}(2\pi j y).
\end{aligned}
$$

- First line: naive spin j from scalars

Higher spin

$$Z^{\text{spin j}}(y) = \sqrt{y}\sum_{\Delta\geq 0} d(\Delta, j)e^{-2\pi\Delta y}$$

$$= \frac{\sqrt{y}}{2}\sum_{\Delta\geq 0} d(\Delta, 0)\sum_{n=1}^{\infty}\mu(n)\sum_{k|j} e^{-\pi y j\left(\frac{2\Delta n^2 j}{k^2}+\frac{k^2}{2\Delta n^2 j}\right)}$$

$$+\sum_{k=1}^{\infty}\text{Re}\left(\delta_k\frac{\sigma_{z_k}(j)K_{\frac{z_k}{2}}(2\pi j y)y^{\frac{1}{2}}}{\Lambda(\frac{z_k+1}{2})j^{\frac{z_k}{2}}}\right)+\sum_{k=1}^{\infty}\frac{a_j^{(k)}(\hat{Z},\nu_k)}{2(\nu_k,\nu_k)}\sqrt{y}K_{iR_n}(2\pi j y).$$

- First line: naive spin j from scalars
- Second line: correction of these “fake spin j” piece that must be discrete!

Summary

- From the spectral decomposition of CFT partition functions and contour manipulation, we wrote down a crossing equation the scalar operators must obey

- This equation is intimately related to the zeros of the Riemann zeta function, and the Riemann hypothesis can be restated as properties of the growth of scalar operators in such CFTs

- By applying a functional we can get a positive sum rule on these states and bound the scalar gap

- Entropy of scalars is related to zeta zeros

Questions

- Is there a physics argument why the subtracted corrections of the “scalar free energy” should scale as $T^{-1/4}$?
- Relate chaos of zeros of ζ function to black hole physics?
- Analytically continue our sum rule for Virasoro to large c?
- Can we generalize to arbitrary spin, not just scalars? (Warning: need functional for Maass cusp forms)
- Can we generalize to four-point functions?

Thank you!

Bonus slide: poles in $\varepsilon_s^c(\mu)$

The scalar partition function is given by:

$$y^{\frac{c}{2}}\left(1+\sum_{\Delta\in\mathcal{S}} e^{-2\pi\Delta y}\right) = y^{\frac{c}{2}} + \frac{\Lambda\left(\frac{c-1}{2}\right)}{\Lambda\left(\frac{c}{2}\right)} y^{1-\frac{c}{2}} + 3\pi^{-\frac{c}{2}}\Gamma\left(\frac{c}{2}-1\right)\mathcal{E}^{c}_{\frac{c}{2}-1}(\mu) + \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} ds\,\pi^{s-\frac{c}{2}}\Gamma\left(\frac{c}{2}-s\right)\mathcal{E}^{c}_{\frac{c}{2}-s}(\mu) y^{s}.$$

We can take the inverse Laplace transform to get the density of states. This must be a sum of delta functions

Bonus slide: poles in $\varepsilon_s^c(\mu)$

In particular it is given by the following. Note at small Δ it must vanish

$$= \frac{2\pi^c\zeta(c-1)\Delta^{c-1}}{(c-1)\Gamma(\frac{c}{2})^2\zeta(c)} + 12\frac{2^{\frac{c}{2}}\Delta^{\frac{c}{2}}\mathcal{E}^c_{\frac{c}{2}-1}(\mu)}{c(c-2)}$$
$$+ \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} ds \frac{2^{\frac{c}{2}-s}\Delta^{\frac{c}{2}-s}\Gamma(s)\Gamma(s+\frac{c}{2}-1)\zeta(2s)\mathcal{E}^c_{\frac{c}{2}+s-1}(\mu)}{\pi^{2s-\frac{1}{2}}\Gamma(\frac{c}{2}+1-s)\Gamma(s-\frac{1}{2})\zeta(2s-1)}$$

We need poles to cancel the first two terms. This gives a pole at s=1-c/2. There can be no other poles unless canceled by a zero of $\zeta(2s)$.

Bonus slide: poles in $\varepsilon_s^c(\mu)$

$$(\hat{Z}^c - E_{\frac{c}{2}}, E_s) = \pi^{s-\frac{c}{2}} \Gamma\left(\frac{c}{2} - s\right) \mathcal{E}^c_{\frac{c}{2}-s}(\mu).$$

$$\mathcal{E}^c_{\frac{c}{2}-s}(\mu) = \frac{\Gamma(s)\Gamma(s+\frac{c}{2}-1)\zeta(2s)}{\pi^{2s-\frac{1}{2}}\Gamma(\frac{c}{2}-s)\Gamma(s-\frac{1}{2})\zeta(2s-1)} \mathcal{E}^c_{\frac{c}{2}+s-1}(\mu).$$

Bonus slide: Functionals

$$\Phi(t) := \sum_{n=1}^{\infty} \varphi(nt).$$

$$\mathcal{F}^{\varphi}[h(t)] := \int_0^{\infty} \frac{dt}{t} h(t)\Phi(t).$$

$$\begin{aligned}
\mathcal{F}^{\varphi}[t^s] &= \int_0^{\infty} dt\ t^{s-1}\Phi(t) \\
&= \int_0^{\infty} dt\ t^{s-1} \sum_{n=1}^{\infty} \varphi(nt) \\
&= \sum_{n=1}^{\infty} n^{-s} \int_0^{\infty} dt\ t^{s-1}\varphi(t) \\
&= \zeta(s) \int_0^{\infty} dt\ t^{s-1}\varphi(t),
\end{aligned}$$

- Some fine print…
  - $\varphi(t)$ and $\hat{\varphi}(t)$ both decay rapidly (faster than any polynomial) at infinity
  - $\varphi(t)$ and $\hat{\varphi}(t)$ have no singularities at finite t
  - $\varphi(0) = \hat{\varphi}(0) = 0$
  - $\int_0^\infty \frac{dt}{t}\varphi(t)t^s$ admits an analytic continuation to all $s \in \mathbb{C}$ (which we will call $M_\varphi(s)$)

$$\Phi(t) = -\frac{1}{2}\varphi(0) + \frac{1}{2t}\hat{\varphi}(0) + \frac{1}{t}\sum_{n=1}^{\infty}\hat{\varphi}\left(\frac{n}{t}\right)$$

$$= \frac{1}{t}\sum_{n=1}^{\infty}\hat{\varphi}\left(\frac{n}{t}\right).$$