Accurate boundary bootstrap for the 3d O(N) normal universality class

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Progress of Theoretical Bootstrap
YITP, Kyoto University
2025/11/14

R. Hu & WL, 2508.20854

Outline

- Boundary universality class
- O(N) boundary bootstrap
- Ising boundary bootstrap
- Summary

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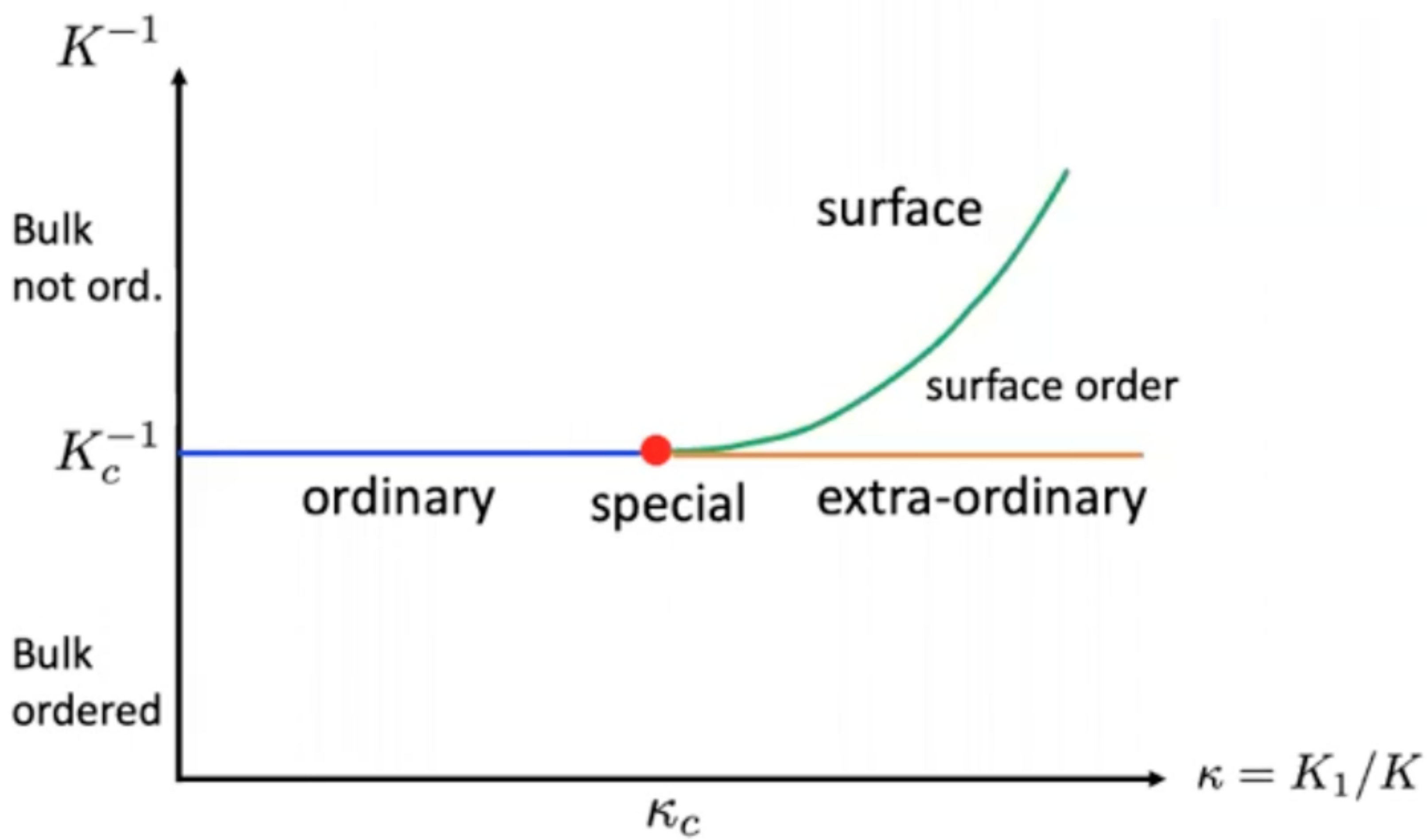
d>3 boundary phase diagram

Classical O(N) model with a boundary $$\frac{H}{k_B T} = -\sum_{\langle ij\rangle} K_{ij}\vec{S}_i \cdot \vec{S}_j$$



Max A. Metlitski, 2009.05119
Bootstrap Zoom 21

d=3 boundary phase diagram

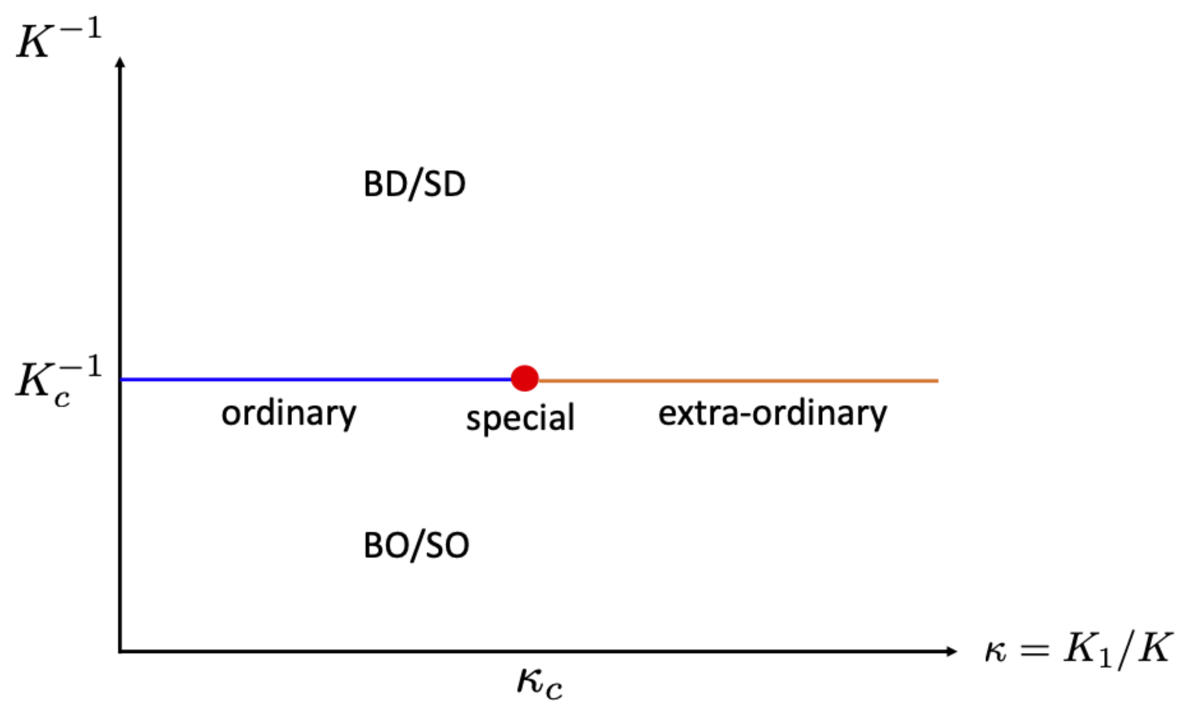
- Ising model (N=1)
  the same phase diagram as d>3

- XY model (N=2)
  the same topology as d>3
  but only **quasi**-long-range boundary order

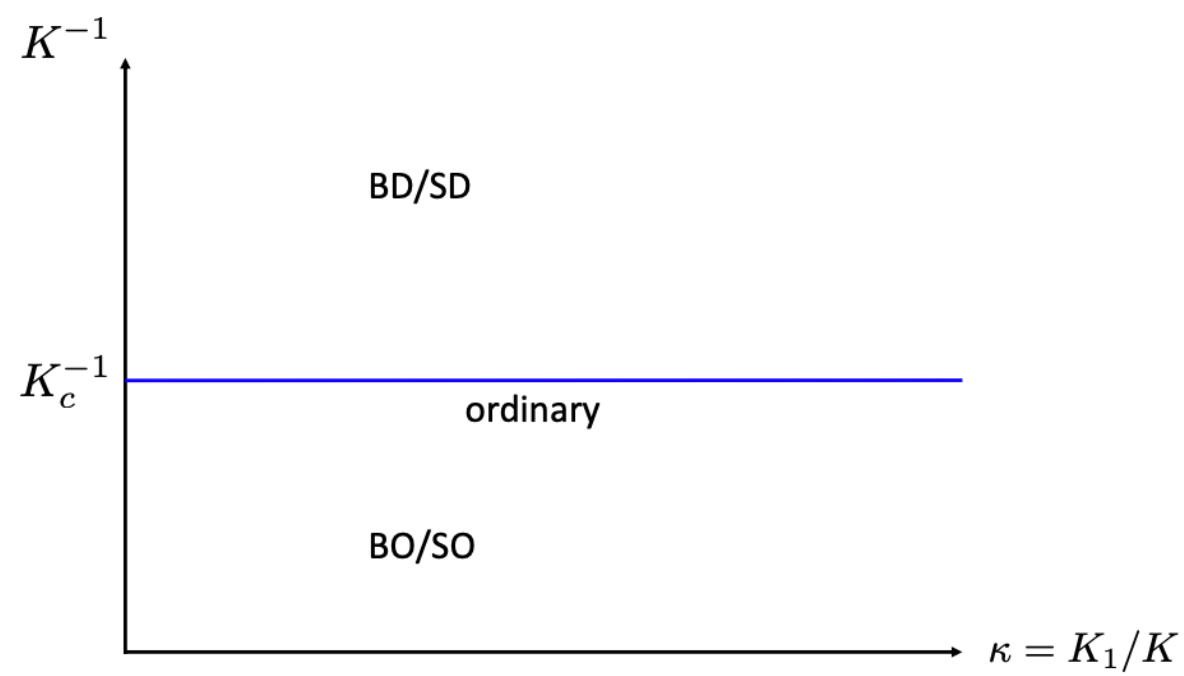
- Large N
  only ordinary transition, no special/extraordinary transition

- Mermin-Wagner theorem
  no spontaneous breaking of continuous symmetries for short-range interactions in two dimensions

d=3 boundary phase diagram



(a)



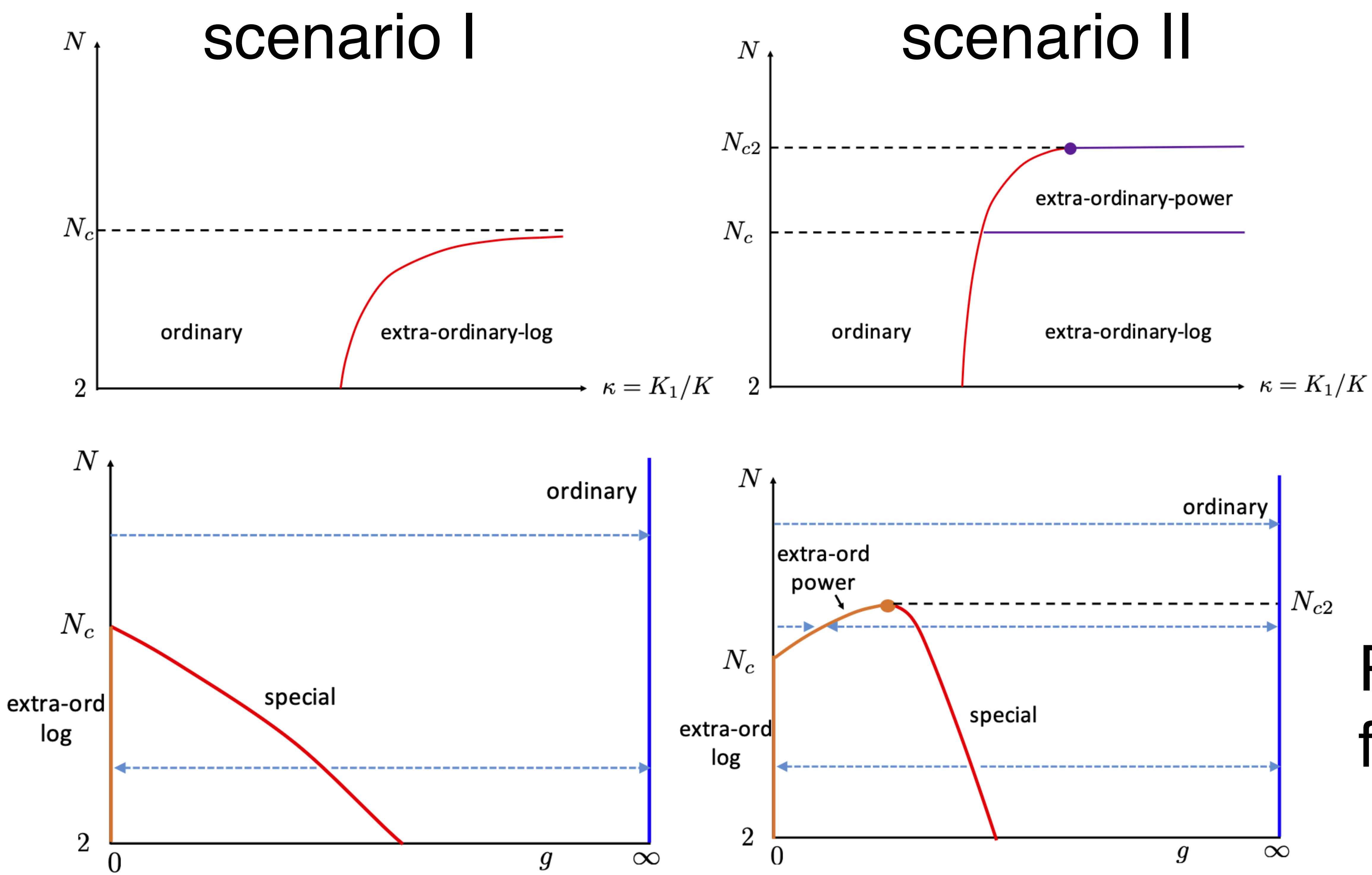
(b)

2 < N < Nc

Large N

extraordinary-log

d=3 boundary phase diagram



RG flow

From “normal” to “extraordinary”

- Normal boundary transition
  explicit symmetry breaking field on the boundary
  O(N) -> O(N-1)

- d>3: normal = extraordinary Bray, Moore, 1977; Diehl, 1994
  Goldstone modes decouple from the bulk with a normal boundary

- d=3 extraordinary-log

$$S_{IR} = S_{normal} + S_n - s \int d^{d-1}\mathrm{x}\, \pi_i(\mathrm{x}) \hat{\phi}_i(\mathrm{x}) + \delta S, \qquad \vec{n} = (\vec{\pi}, \sqrt{1-\vec{\pi}^2})$$

logarithmic RG flow around the “normal” fixed point
the coupling s is fixed by restoring O(N) symmetry.

From “normal” to “extraordinary”

- Logarithmic RG flow around the “normal” fixed point (g=0)
- β function $$\beta(g) = \alpha g^2 + O(g^3)$$

  nonlinear sigma model $$S_n = \int d^{d-1}\mathrm{x} \left(\frac{1}{2g} (\partial_\mu \vec{n})^2 - \vec{h} \cdot \vec{n} \right), \quad \vec{n}^2 = 1\,.$$

- What is Nc?

  fixed point stability depends on the sign of $$\alpha = \frac{1}{32\pi} \frac{a_\phi^2}{b_{\phi t}^2} - \frac{N-2}{2\pi}$$

  Bootstrap target

- Boundary correlation function

$$\langle \vec{S}_{\vec{x}} \cdot \vec{S}_{\vec{y}} \rangle \sim \frac{1}{(\log |\vec{x} - \vec{y}|)^q} \qquad q = \frac{N-1}{2\pi\alpha}$$

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Conformal field theory

- In the RG parameter

$$\alpha = \frac{1}{32\pi}\frac{a_\phi^2}{b_{\phi t}^2} - \frac{N-2}{2\pi}$$

$(a_\phi, b_{\phi t})$ are **universal amplitudes** in the normal transition

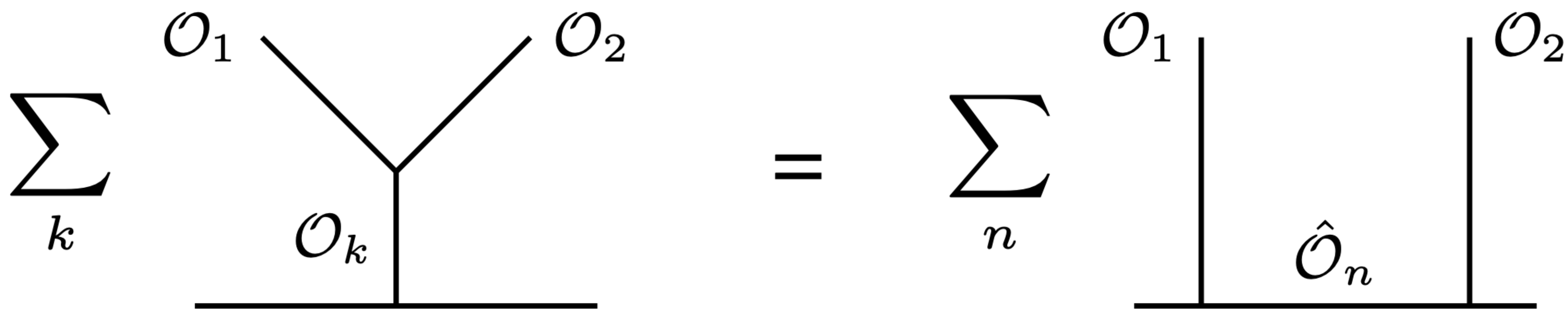
- In BCFT, $(a_\phi, b_{\phi t})$ are **B**oundary **O**perator **E**xpansion coefficients.

$$\mathcal{O}_k(x) = \sum_n b_{kn}\, D_{kn}(x_\perp, \partial_{x_\parallel})\, \hat{\mathcal{O}}_n(x_\parallel) \qquad a_k = b_{kI}$$

- Normal universality class is a natural bootstrap target
  a. leading bulk dimensions from the bulk bootstrap or Monte Carlo
  b. leading boundary dimensions are protected: displacement, tilt

Boundary bootstrap

Liendo, Rastelli, van Rees, 2012



$$\mathcal{O}_1(x)\mathcal{O}_2(y) = \sum_k \lambda_{12k}\, C_{12k}(x-y,\partial_y)\,\mathcal{O}_k(y) \qquad \mathcal{O}_k(x) = \sum_n b_{kn}\, D_{kn}(x_\perp,\partial_{x_\parallel})\,\hat{\mathcal{O}}_n(x_\parallel)$$

$$\langle \mathcal{O}_k(x)\rangle = \frac{a_k}{(2x_\perp)^{\Delta_k}}$$

- Bulk channel $$\phi_a \times \phi_b \sim \sum_S \delta_{ab}\,\mathcal{O} + \sum_T \mathcal{O}_{(ab)} + \sum_A \mathcal{O}_{[ab]}$$
- Boundary channel $$\phi_N \sim 1 + D + \sum_{\hat{\Delta}>3} \hat{\mathcal{O}}^{(\hat{S})}\,, \quad \phi_i \sim t_i + \sum_{\hat{\Delta}>2} \hat{\mathcal{O}}_i^{(\hat{V})}\,,$$

Boundary bootstrap

- Crossing equation for $\langle \mathcal{O}_1(x)\, \mathcal{O}_2(y) \rangle = \dfrac{G(\xi)}{(2x_\perp)^{\Delta_1}(2y_\perp)^{\Delta_2}}\, \xi^{-\frac{\Delta_1+\Delta_2}{2}}$

$$\sum_k \lambda_{12k} a_k\, f^{\Delta_{12}}_{\Delta_k}(\xi) - \xi^{\frac{\Delta_1+\Delta_2}{2}} \sum_n b_{1n} b_{2n}\, \hat{f}_{\hat{\Delta}_n}(\xi) = 0\,, \qquad \xi = \frac{(x-y)^2}{4x_\perp y_\perp}$$

- Conformal blocks McAvity, Osborn, 1995 $\Delta_{12} = \Delta_1 - \Delta_2$

bulk channel $f^{\Delta_{12}}_{\Delta}(\xi) = \xi^{\Delta/2} {}_2F_1\left[\dfrac{\Delta+\Delta_{12}}{2}, \dfrac{\Delta-\Delta_{12}}{2}; \Delta - \dfrac{d-2}{2}; -\xi\right]$

boundary channel $\hat{f}_{\hat{\Delta}}(\xi) = \xi^{-\hat{\Delta}}\, {}_2F_1\left[\hat{\Delta}, \hat{\Delta} - \dfrac{d}{2} + 1; 2\hat{\Delta} - d + 2; -1/\xi\right]$

1 cross ratio, no spin, simpler than 4-pt bulk crossing

- Positivity?

Truncated bootstrap

- Truncate the bootstrap equation to finitely many operators
  determinant: Gliozzi, 2013; Gliozzi, Rago, 2014
  singular value: Esterlis, Fitzpatrick, Ramirez, 2016

- η minimization Li, 2017

  cost function

$$\eta = \sum_j \sum_{m=0}^{M_j} \left| \partial_\xi^m \left(\text{bootstrap equation}_j \right) \right|_{\xi=1}^2$$

  solve the truncated bootstrap equations by local minimization

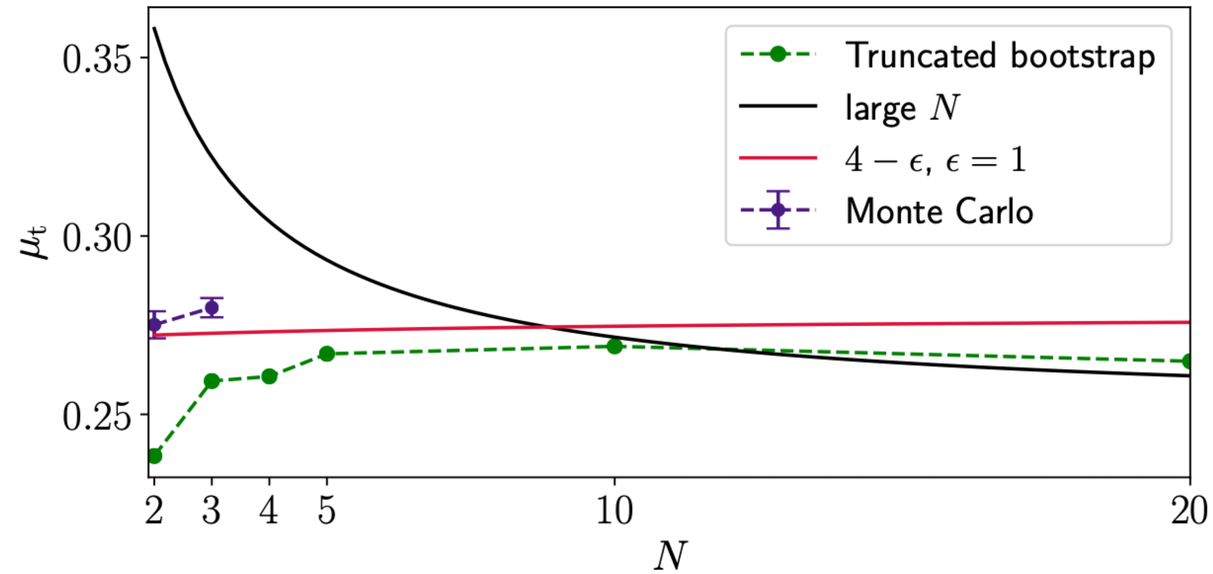
- Variants: artificial intelligence, analytic input, random weight
  Kántor, Niarchos, Papageorgakis, Richmond, Stapleton, Woolley, 2021-2025
  Li, 2023; Poland, Prilepina, Tadic, 2023-2025
  Barrat, Marchetto, Miscioscia, Pomoni, 2024

Truncated boundary bootstrap

- Previous truncated boundary bootstrap results are promising
  Gliozzi, Liendo, Meineri, Rago, 2015
  Padayasi, Krishnan, Metlitski, Gruzberg, Meineri, 2021

- **Tension** with Monte Carlo

  uncontrolled systematic error ?
  low truncation order ?



- The η minimization allows for significantly higher truncation orders

  We search for the **zeros** of η.
  They are **global** minima.

	$N=1$	$N=2$	$N=3$	$N=4$	$N=5$
This work	88, 68	76	68	60	88
[6, 38]	9, 8	10	10	9	9

Λ: # bootstrap constraints

How to find a zero?

- Low truncation order: **numerical algebraic geometry**
  rational approximation -> polynomial equations
  Mathematica's NSolve or HomotopyContinuation.jl
  huge number of solutions, few are physical

- High truncation order: **local minimization**
  Mathematica's FindMinimum with LevenbergMarquardt (trust region)
  interpolates between Gauss–Newton and gradient descent

- How to construct a **starting point**?
  low-lying operator dimensions change gently (effective description)
  most operator dimensions are from lower order solution
  scan high-lying operator dimensions (discrete set of values)
  coefficients of conformal blocks =1

Bulk O(N) input

N	Δ_ϕ	Δ_S	$\Delta_{S'}$	Δ_T
2	0.51908(1)[57]	1.51128(5)[57]	3.789(4)[54]	1.23629(**11**)[30]
3	0.518936(**67**)[31]	1.5948(2)[55]	3.759(2)[55]	1.20954(**32**)[31]
4	0.51812(4) [56]	1.66340(35)[56]	3.755(5)[56]	$1.1864^{+0.0024}_{-0.0034}$[23]
5	0.516985(45)[56]	1.7182(10)[56]	3.754(7)[56]	$1.1568^{+0.009}_{-0.010}$[23]

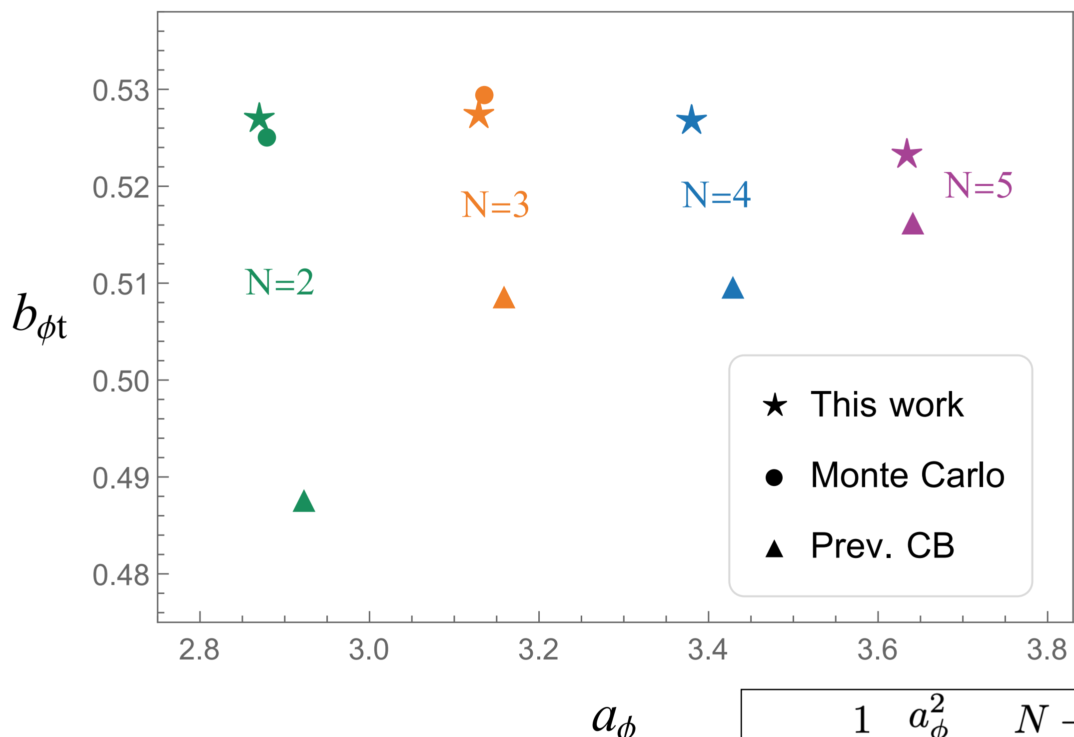
- Bulk bootstrap

[23] Kos, Poland, Simmons-Duffin, 2013
[30, 31] Chester, Landry, Liu, Poland, Simmons-Duffin, Su, Vichi, 2019 + 2020

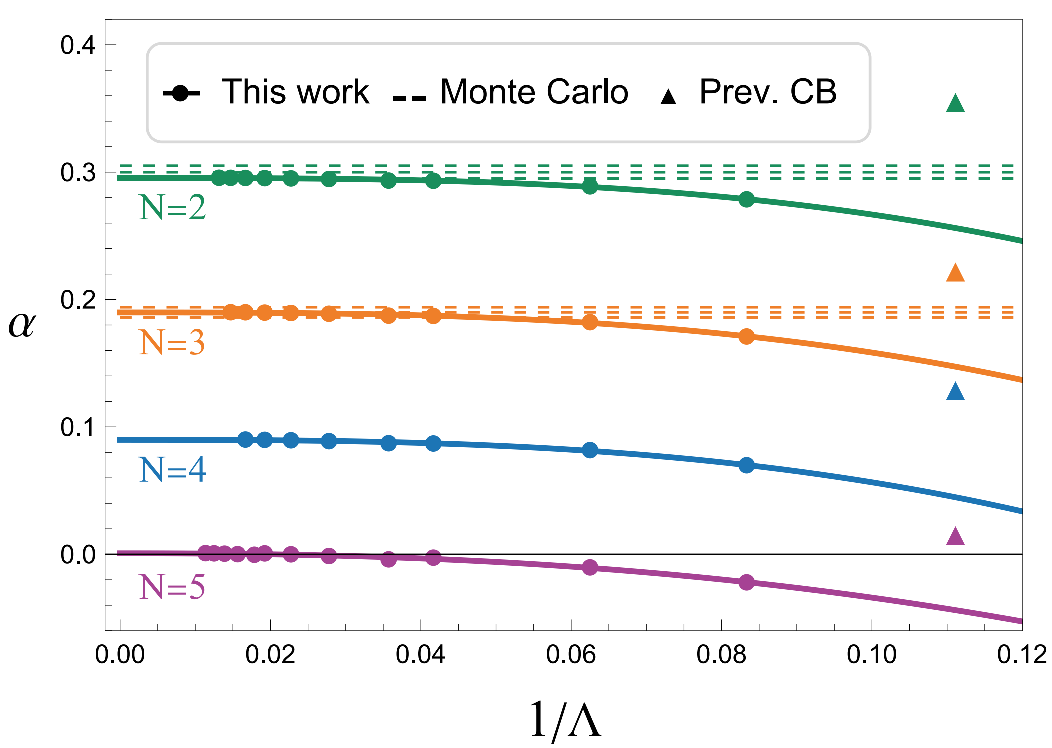
- Monte Carlo

[54-57] Hasenbusch, 2019 + 2020 + 2021 + 2025

Bootstrap results



$$\alpha = \frac{1}{32\pi}\frac{a_\phi^2}{b_{\phi t}^2} - \frac{N-2}{2\pi}$$



$N = 2$

Method	a_ϕ	$b_{\phi t}$	$b_{\phi D}$	α
This work	2.875(2)	0.5272(2)	0.2440(4)	0.2957(6)
MC [7]	2.880(2)	0.525(4)		0.300(5)
CB [6]	2.923	0.4882	0.2701	0.3567

$N = 3$

Method	a_ϕ	$b_{\phi t}$	$b_{\phi D}$	α
This work	3.129(2)	0.5278(2)	0.2406(6)	0.1903(7)
MC [7]	3.136(2)	0.529(3)		0.190(4)
CB [6]	3.159	0.5092	0.2690	0.2236

$N = 4$

Method	a_ϕ	$b_{\phi t}$	$b_{\phi D}$	α
This work	3.380(6)	0.5272(12)	0.2369(23)	0.0906(34)
CB [6]	3.429	0.5105	0.2758	0.1304

$N = 5$

Method	a_ϕ	$b_{\phi t}$	$b_{\phi D}$	α
This work	3.634(5)	0.5235(5)	0.2390(9)	0.002(2)
CB [6]	3.641	0.5166	0.2653	0.0166

Monte Carlo: Toldin, Metlitski, 2021

N=4: α = 0.097(3) Toldin et al., unpublished

3.386(2) 0.524(2)

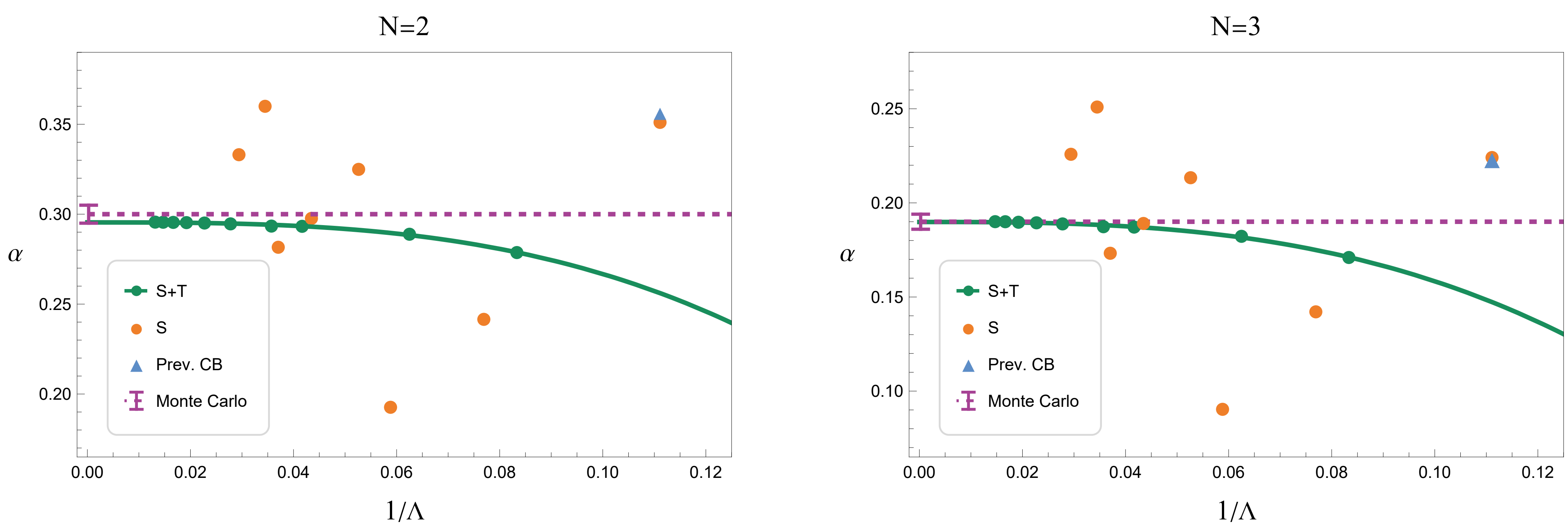
Importance of bulk O(N) traceless symmetric tensors

- <ϕa(x) ϕb(y)> involves two O(N-1) singlets

  a,b = (i,N) and i = 1,2,...,N−1

- Two crossing equations $\langle \phi_N(x)\,\phi_N(y)\rangle\,, \quad \sum_i \langle \phi_i(x)\,\phi_i(y)\rangle$

- Previously, the T contributions were projected out.
  Padayasi, Krishnan, Metlitski, Gruzberg, Meineri, 2021



Other bootstrap results

- Subleading O(N) traceless-symmetric dimensions

$\Delta_{T'} = 3.6484(22)\,,\ 3.559(4)\,,\ 3.49(3)\,,\ 3.354(16)\,,$ for N=2,3,4,5

consistent with the O(2) bulk bootstrap estimate 3.650(2)

- New bulk 1-point coefficients

$$\begin{aligned} N=2: &\quad a_S = 5.57(1)\,, \quad a_T = 3.897(5)\,, \\ N=3: &\quad a_S = 5.37(1)\,, \quad a_T = 8.406(12)\,. \end{aligned}$$

We used $\lambda_{\phi\phi S}$, $\lambda_{\phi\phi T}$ from the O(2), O(3) bulk bootstrap.

Chester, Landry, Liu, Poland, Simmons-Duffin, Su, Vichi, 2019 + 2020

Higher accuracy?

- We can systematically increase the truncation order.
- The accuracy is mainly limited by bulk input uncertainties.
- More precise bulk input, more accurate bootstrap results.
- Let’s demonstrate this by the Ising boundary bootstrap!
- Ising normal = Ising extraordinary
  The Ising normal transition is of interest on its own due to symmetry breaking.

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Bulk Ising input

- We consider <σ(x) σ(y)> and <σ(x) ϵ(y)>.
- The mixed correlator is nonzero due to symmetry breaking.
- High precision input from the bulk bootstrap

$$\Delta_\sigma^{\text{input}} = 0.518148806(\mathbf{24})\,,\ \Delta_\epsilon^{\text{input}} = 1.41262528(\mathbf{29})\,,$$
$$\lambda_{\sigma\sigma\epsilon}^{\text{input}} = 1.05185373(11)\,.$$

Chang, Dommes, Erramilli, Homeric, Kravchuk, Liu, Mitchell, Poland, Simmons-Duffin, 2024

- Main source of error $\Delta_{\epsilon'}^{\text{input}} = 3.82951(\mathbf{61})$ Reehorst, 2021

Ising boundary bootstrap

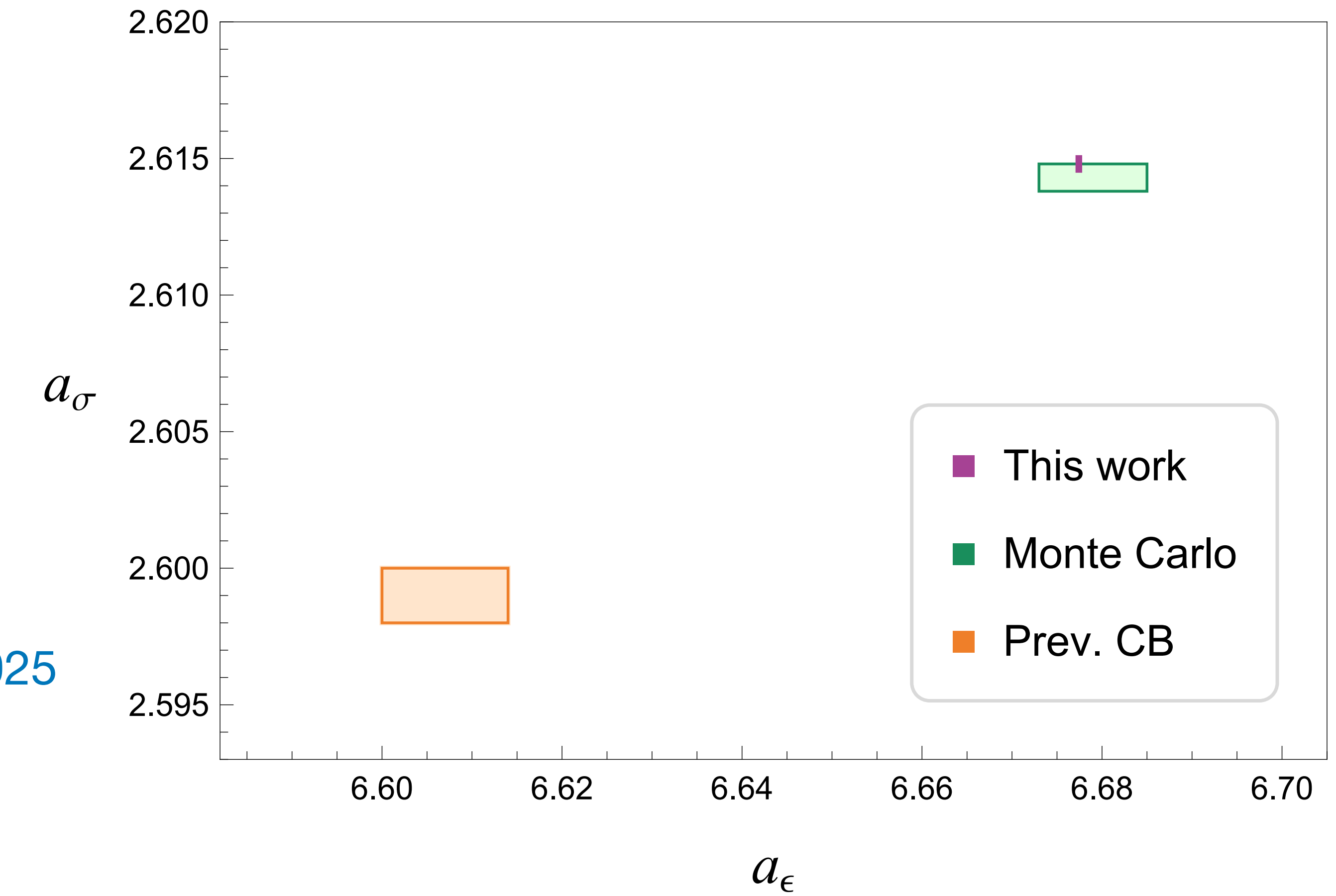
- Fusion rules

$$\sigma \times \epsilon \sim \sigma + \sigma' + \sigma'' + \dots$$

$$\sigma \times \sigma \sim 1 + \epsilon + \epsilon' + \epsilon'' + \dots$$

$$\sigma, \epsilon \sim 1 + D + \hat{N} + \hat{N}' + \dots$$

Monte Carlo: Przetakiewicz, Wessel, Toldin, 2025
Previous: Gliozzi, Liendo, Meineri, Rago, 2015



- In the bulk channel, the Z2 odd spectrum has a **large gap**.
  $\Delta\sigma \approx 0.52$, $\Delta\sigma' \approx 5.3$ (multiplet recombination)

  Highly accurate results from the <σ(x) ϵ(y)> crossing.

Bootstrap results

- Selected results

MC: Przetakiewicz, Wessel, Toldin, 2025
Fuzzy Sphere: Zhou, Zou, 2024
Previous: Gliozzi, Liendo, Meineri, Rago, 2015
Bulk Bootstrap: Simmons-Duffin, 2016

Method	a_ϵ	a_σ	$b_{\epsilon D}$	$b_{\sigma D}$
This work	6.677424(16)	2.6148(2)	1.7234(4)	0.24757(4)
MC [40]	6.679(6)	2.6143(5)	1.69(1)	0.242(2)
FS [61]	6.4(9)	2.58(16)	1.74(22)	0.254(17)
CB [38]	6.607(7)	2.599(1)	1.742(6)	0.25064(6)

- Boundary scaling dimensions
BOE coefficients

$$\hat{\Delta}_{\hat{N}} = 5.879(1)\,, \quad \hat{\Delta}_{\hat{N}'} = 8.086(23)\,,$$
$$b_{\epsilon\hat{N}} = 0.2147(23)\,, \quad b_{\epsilon\hat{N}'} = 0.046(4)\,,$$
$$b_{\sigma\hat{N}} = 0.00946(10)\,, \quad b_{\sigma\hat{N}'} = 0.0013(1)\,.$$

Fuzzy Sphere: 5.858 Dedushenko, 2024

Method	$\Delta_{\sigma'}$	$\Delta_{\epsilon''}$	$\Delta_{\sigma''}$
This work	5.28901(3)	6.873(7)	8.42915(36)
CB [29]	5.2906(11)	6.8956(43)	
CB [38]	5.49(1)	7.27(5)	10.6(3)

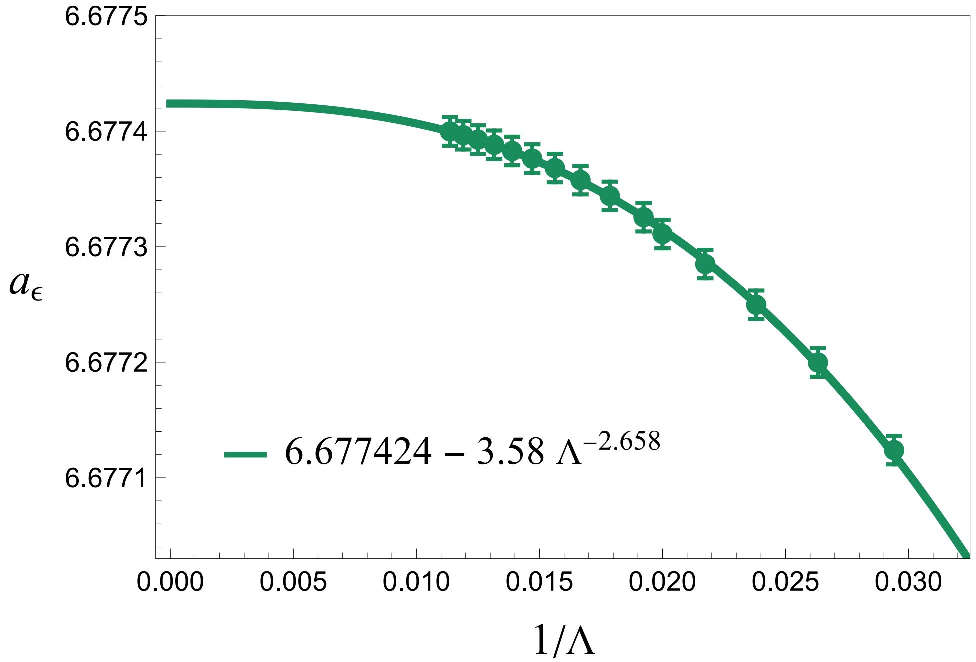
- New bulk 1-point coefficients

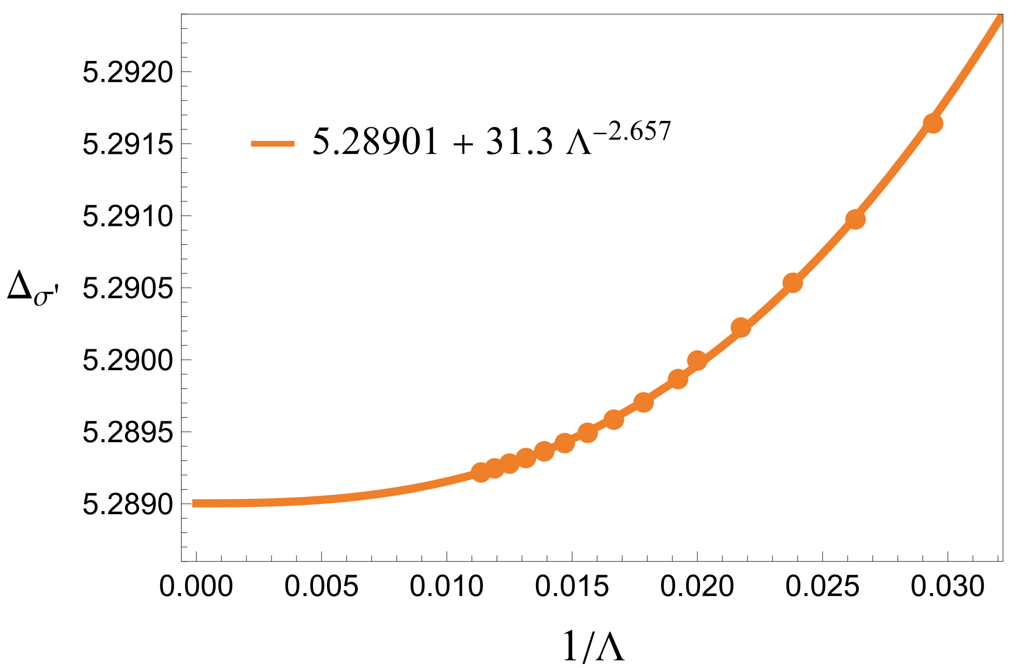
$$a_{\sigma'} = 110(3)\,,\ a_{\epsilon'} = 42.46(14)\,,\ a_{\epsilon''} = 267.8(14)$$

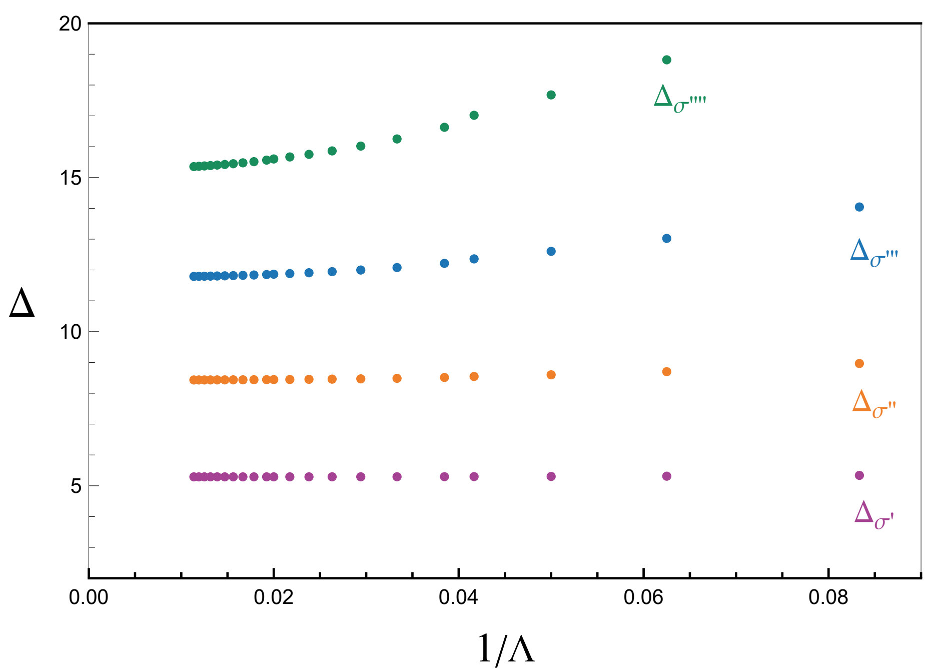
Use bulk OPE coefficients from
Simmons-Duffin, 2016
Reehorst, 2021

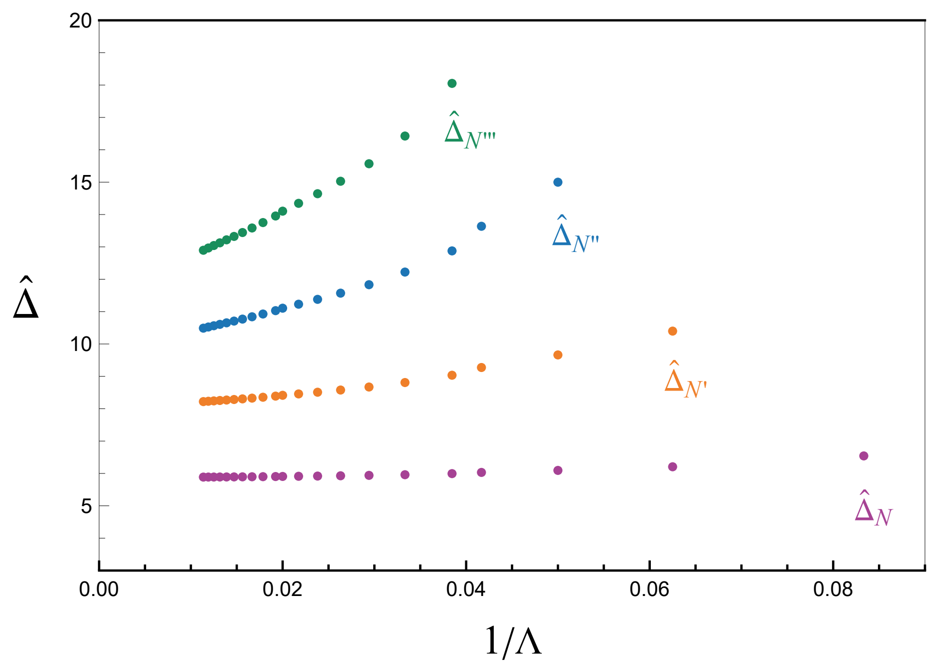
Convergence and extrapolation

Selected results from the <σ(x) ϵ(y)> crossing

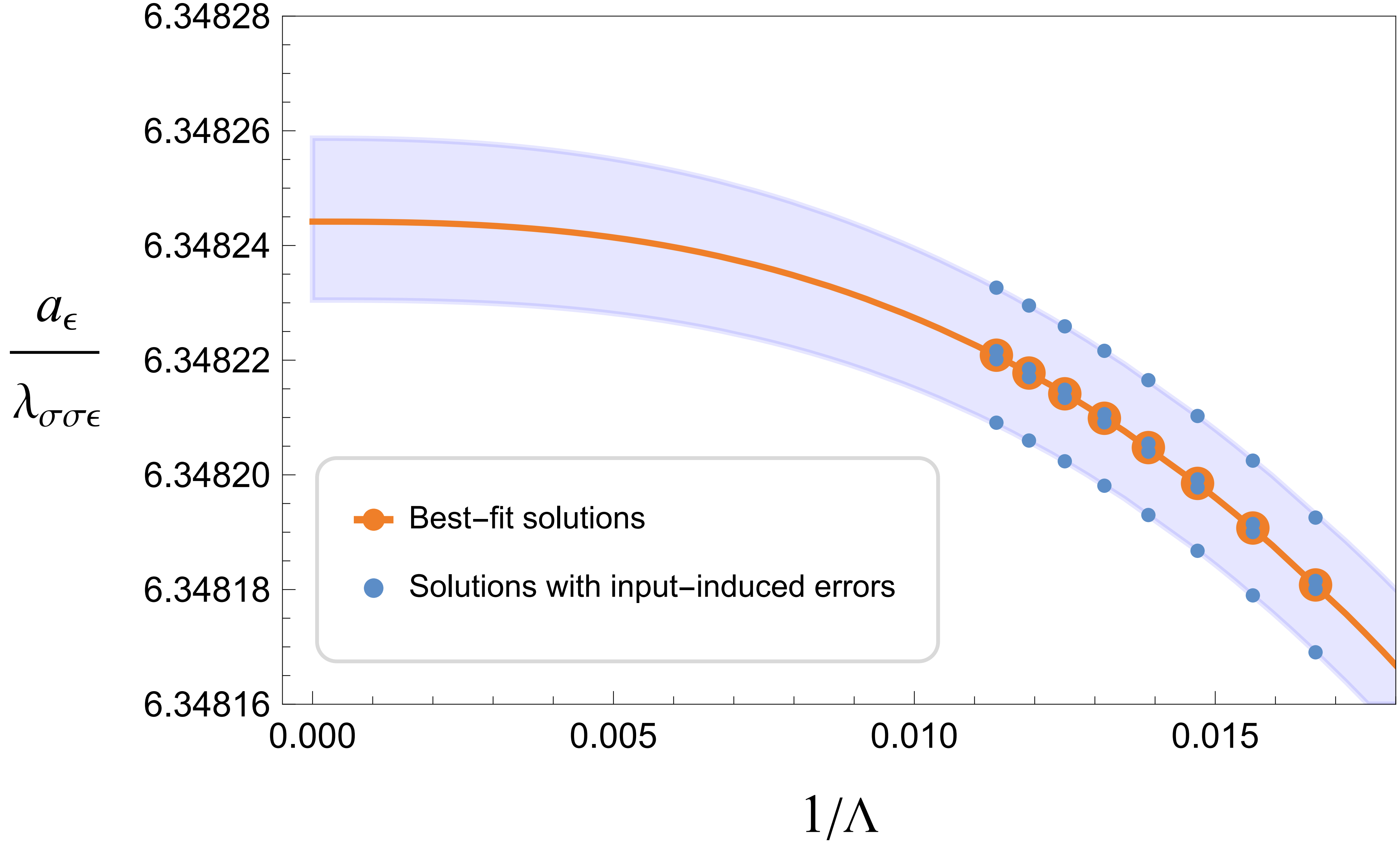








Error analysis



input induced error + extrapolation uncertainty

Are errors reliable?

- Bulk OPE coefficient from our boundary bootstrap results

$$\lambda_{\sigma\sigma\epsilon}^{\text{this work}} = \sqrt{\left(\frac{a_\epsilon}{\lambda_{\sigma\sigma\epsilon}}\right)^{-1} (\lambda_{\sigma\sigma\epsilon}\, a_\epsilon)} = 1.05184(13)$$

Bulk bootstrap determination $\lambda_{\sigma\sigma\epsilon}^{\text{input}} = 1.05185373(11)$

- Zamolodchikov norm of the displacement operator

$$C_D = \left(\frac{\Delta_{\mathcal{O}}\, a_{\mathcal{O}}}{4\pi\, b_{\mathcal{O}D}}\right)^2 = \begin{cases} 0.18966(9) & \mathcal{O} = \sigma \\ 0.18970(9) & \mathcal{O} = \epsilon\,. \end{cases}$$

Ward identity

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Summary

- Truncated boundary bootstrap becomes more systematic
  higher truncation orders, better error estimates, new bulk+boundary results

- Accurate bootstrap results from precise bulk input

  N=5 wanted!

  resolve Bootstrap-MC discrepancies, 2 orders more accurate than the latest MC

- Larger bootstrap system?
  correlators of higher points, boundary operators, other bulk operators (spinning?)

- Other defects? interface, line, ...

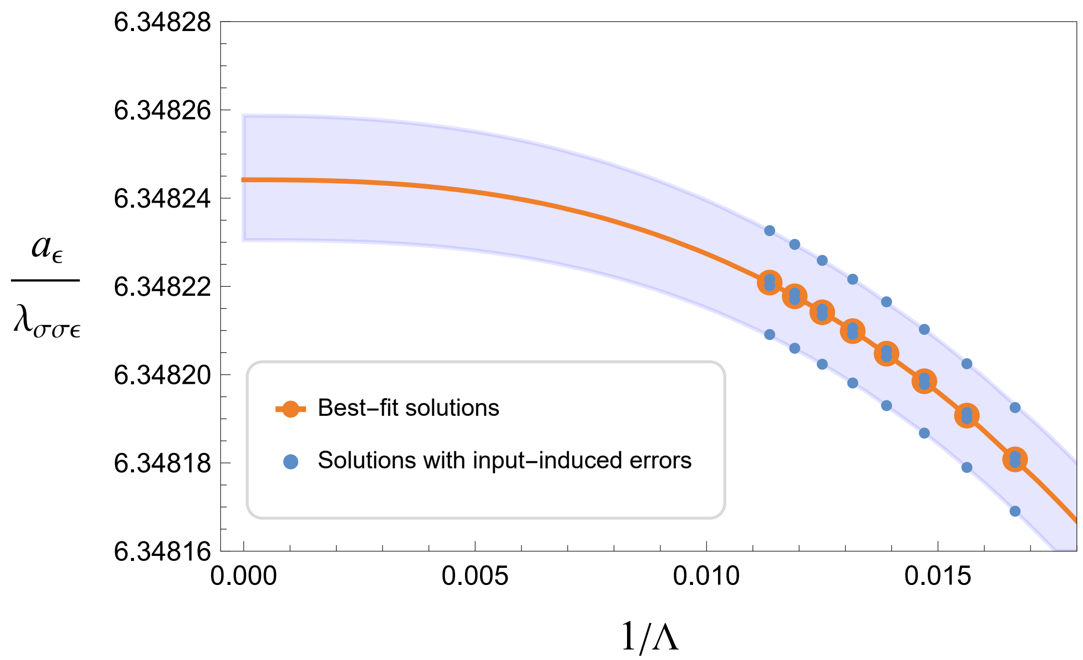
- Nontrivial manifolds? real projective space, ...

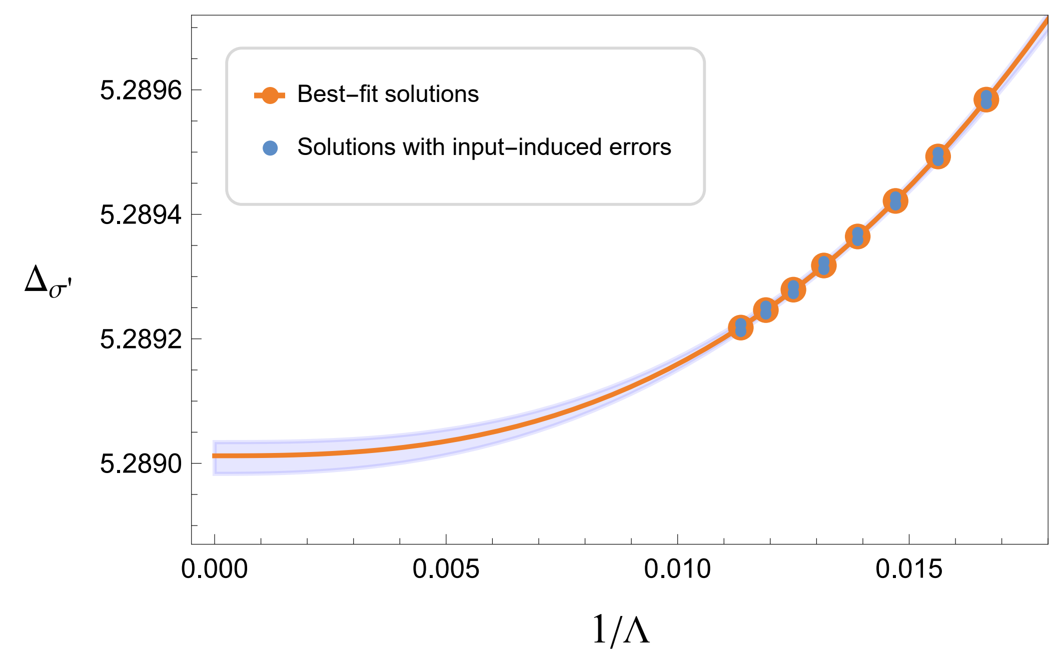
Thank you!

- Systematic non-unitary bulk bootstrap?
  Yang-Lee edge singularity, percolation, polymer, disorder, turbulence, ...

More error analysis

Ising





O(2)