Yang-Lee Quantum Criticality In Various Dimensions

Erick Arguello, Igor Klebanov, Grigory Tarnopolsky, Yuan Xin

2505.06369

See also

2505.06342 Ruihua Fan, Junkai Dong, Ashvin Vishwanath

2505.07655 Joan Elias Miro, Olivier Delouche

Yuan Xin, November 5, 2025

Yang-Lee Criticality

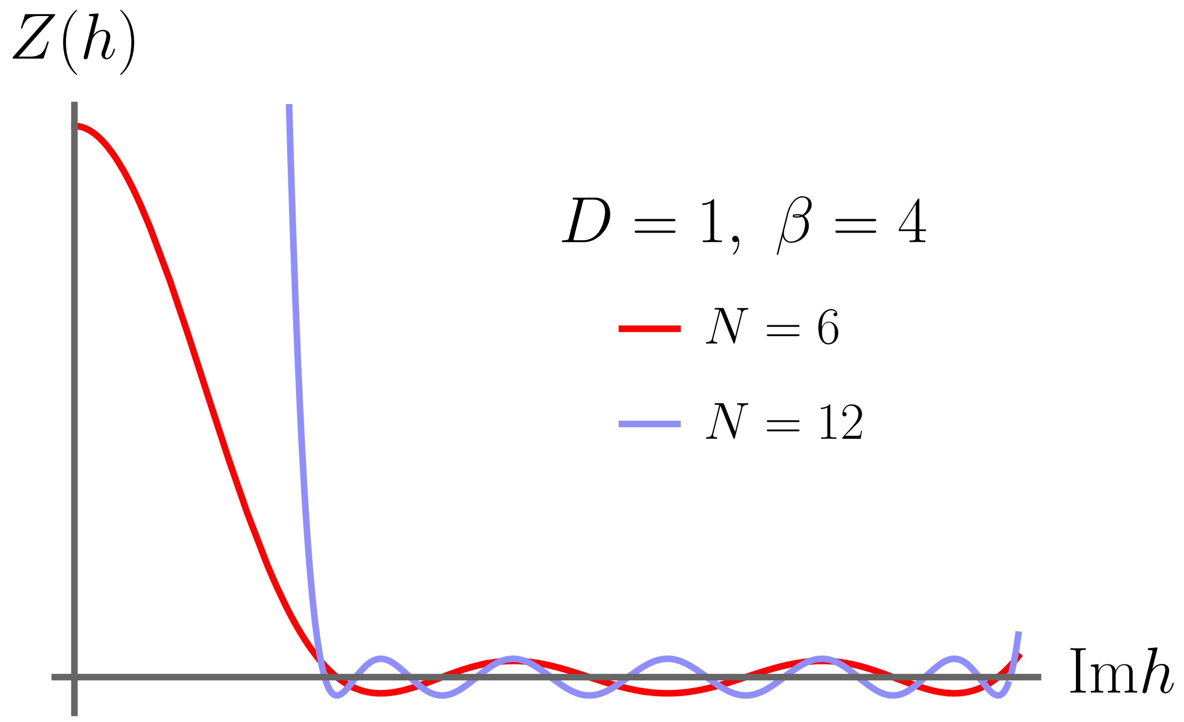
Singularity of Ising Model Partition Function

$$Z(h) = \sum_{\{s_i\}} e^{\beta \sum_{ij} s_i s_j + h \sum_i s_i}$$

- Lee and Yang (1952): Phase transition occurs when zeros of the partition function in the complex plane approach the real axis.

Statistical Theory of Equations of State and Phase Transitions. I. Theory of Condensation

C. N. Yang and T. D. Lee
Institute for Advanced Study, Princeton, New Jersey
(Received March 31, 1952)



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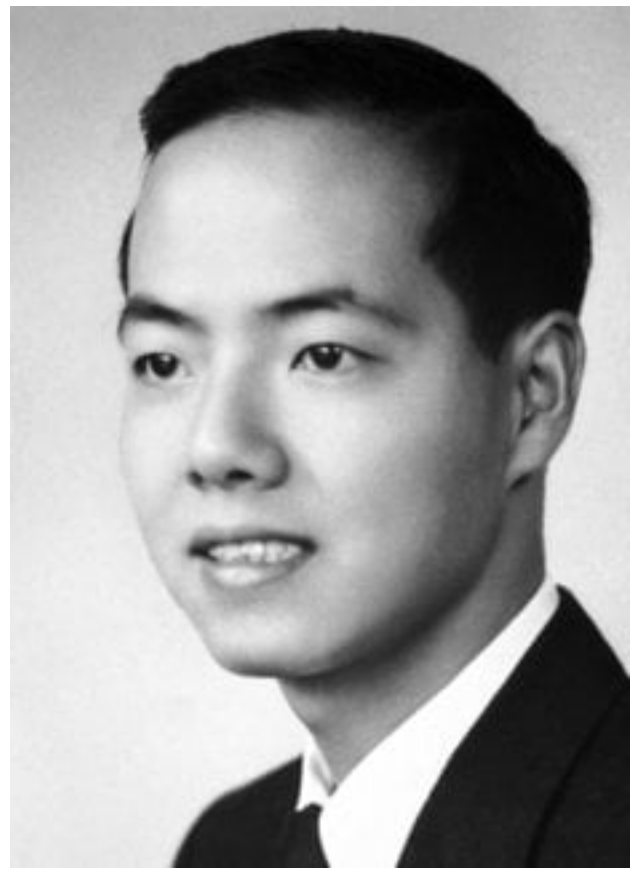
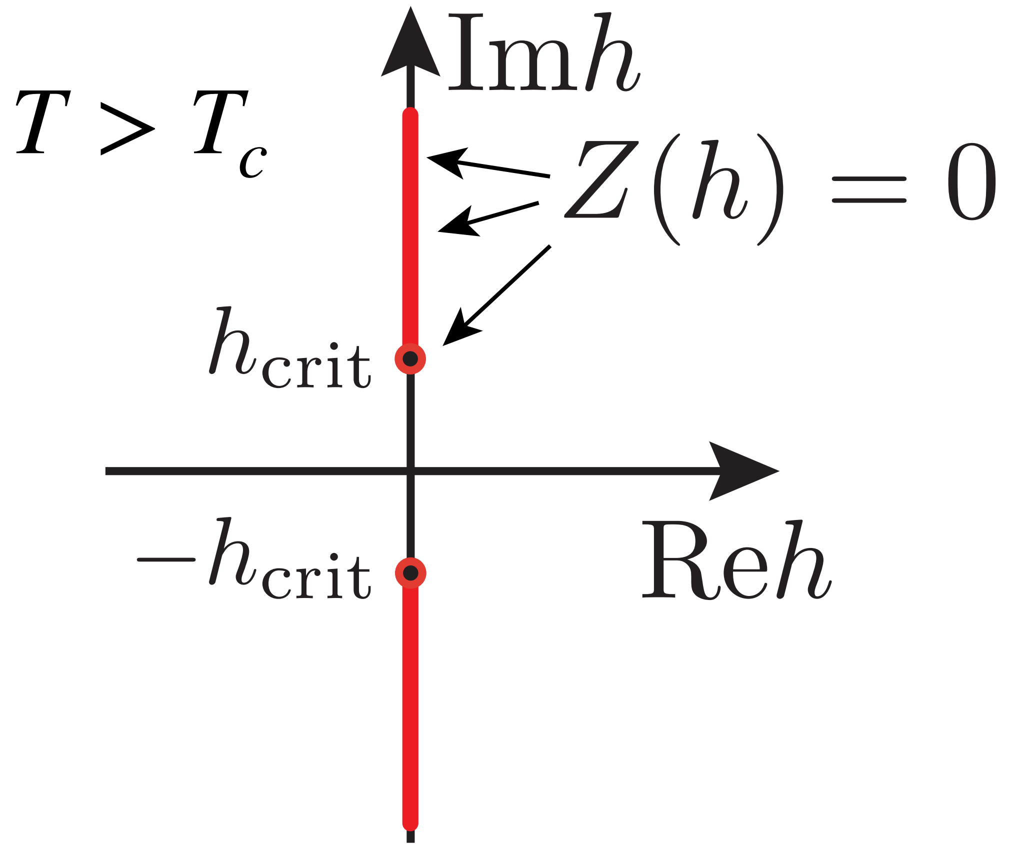
C. N. Yang
(1922-2025)



T. D. Lee
(1926-2024)



Yang-Lee Criticality

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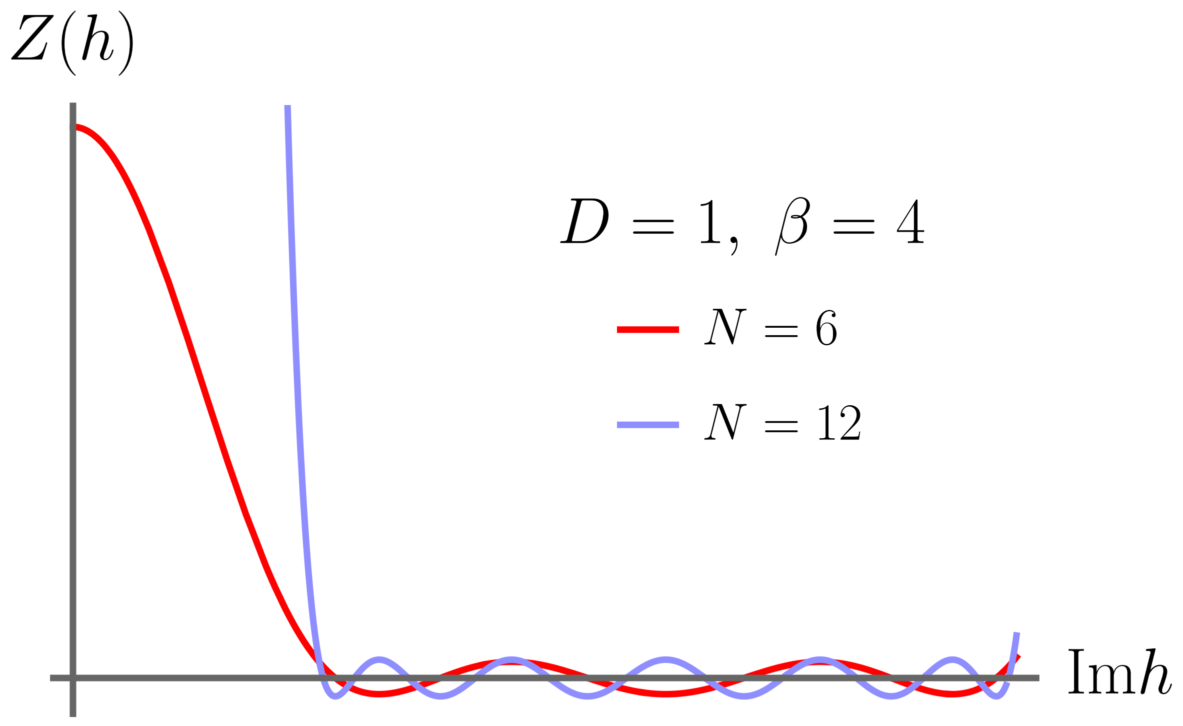
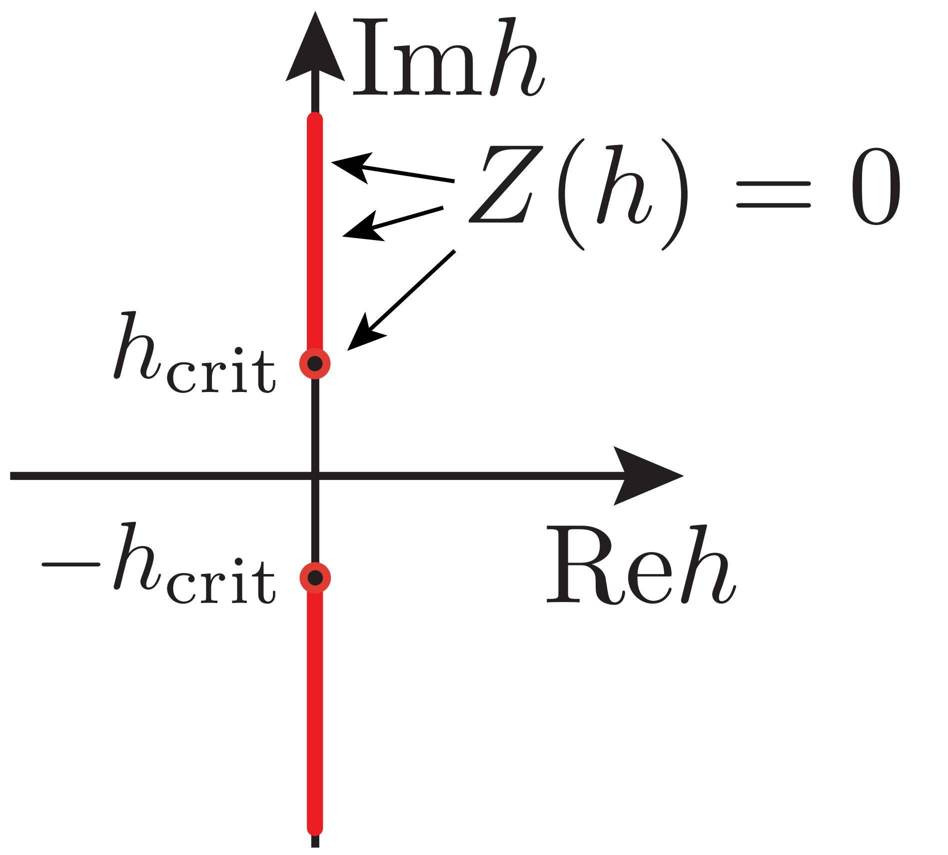


- Yang-Lee edge singularity: at $T > T_c$, the edge of the zeros describes a critical point.

- Kortman and Griffiths (1971): Numerical high temperature expansion reveals a power-law singularity near the edge-point h_{crit} .

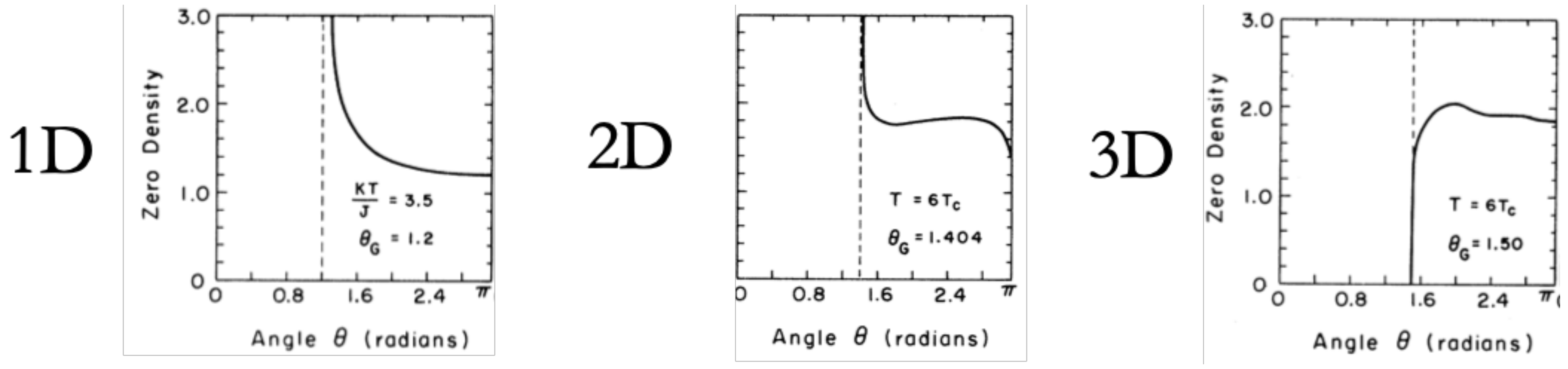


Density of Zeros on the Lee-Yang Circle for Two Ising Ferromagnets*

Peter J. Kortman† and Robert B. Griffiths
Department of Physics, Carnegie-Mellon University, Pittsburgh, Pennsylvania 15213

(Received 20 September 1971)

$$\rho(h'') \propto |h'' - h_{\text{crit}}(T)|^{\sigma}$$



$$h = h' + ih'' \qquad \theta = 2h''/T$$

Yang-Lee Criticality
Ginzburg-Landau Description for d<6

Yang-Lee Edge Singularity and φ^3 Field Theory

Michael E. Fisher
Baker Laboratory, Cornell University, Ithaca, New York 14853
(Received 20 April 1978)

- Fisher (1978): YL has a continuous description of $i\phi^3$ field theory

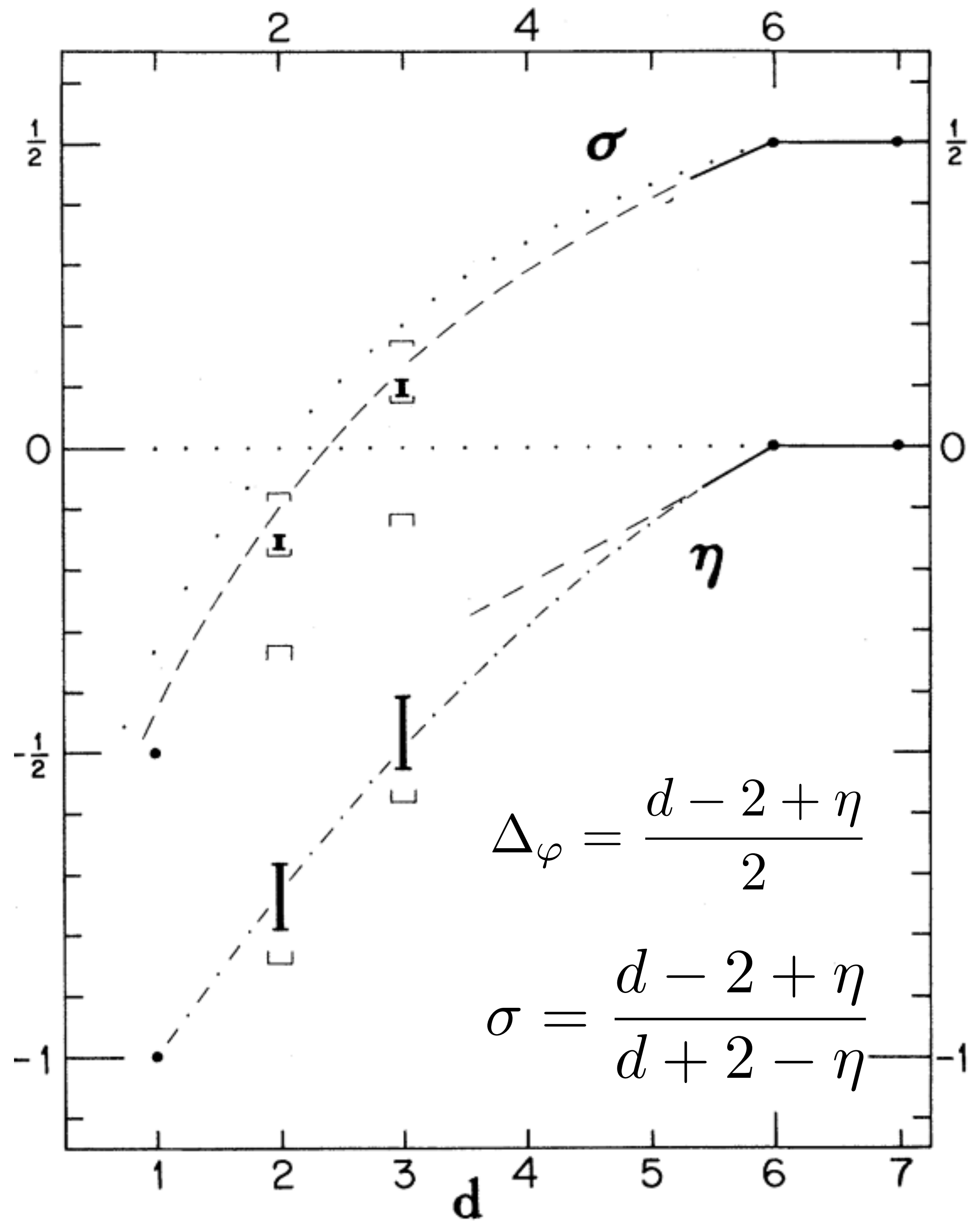
$$S = \int d^dx \left(\frac{1}{2}(\partial_\mu \varphi)^2 + i(h - h_c)\varphi + \frac{1}{3} ig\varphi^3 \right)$$

- Density of zeros analogous to magnetization
- Critical exponent estimated using $(6 - \epsilon)$ expansion.

$$\eta = -\frac{\epsilon}{9} - \frac{43\epsilon^2}{729}$$

$$\eta(d = 1) = -1$$

$$\Delta(d = 2) \approx -0.34$$



$$\rho(h'') \propto |h'' - h_{\text{crit}}(T)|^\sigma$$

$$M(h) \sim |h'' - h_{\text{crit}}(T)|^\sigma$$

Yang-Lee Criticality

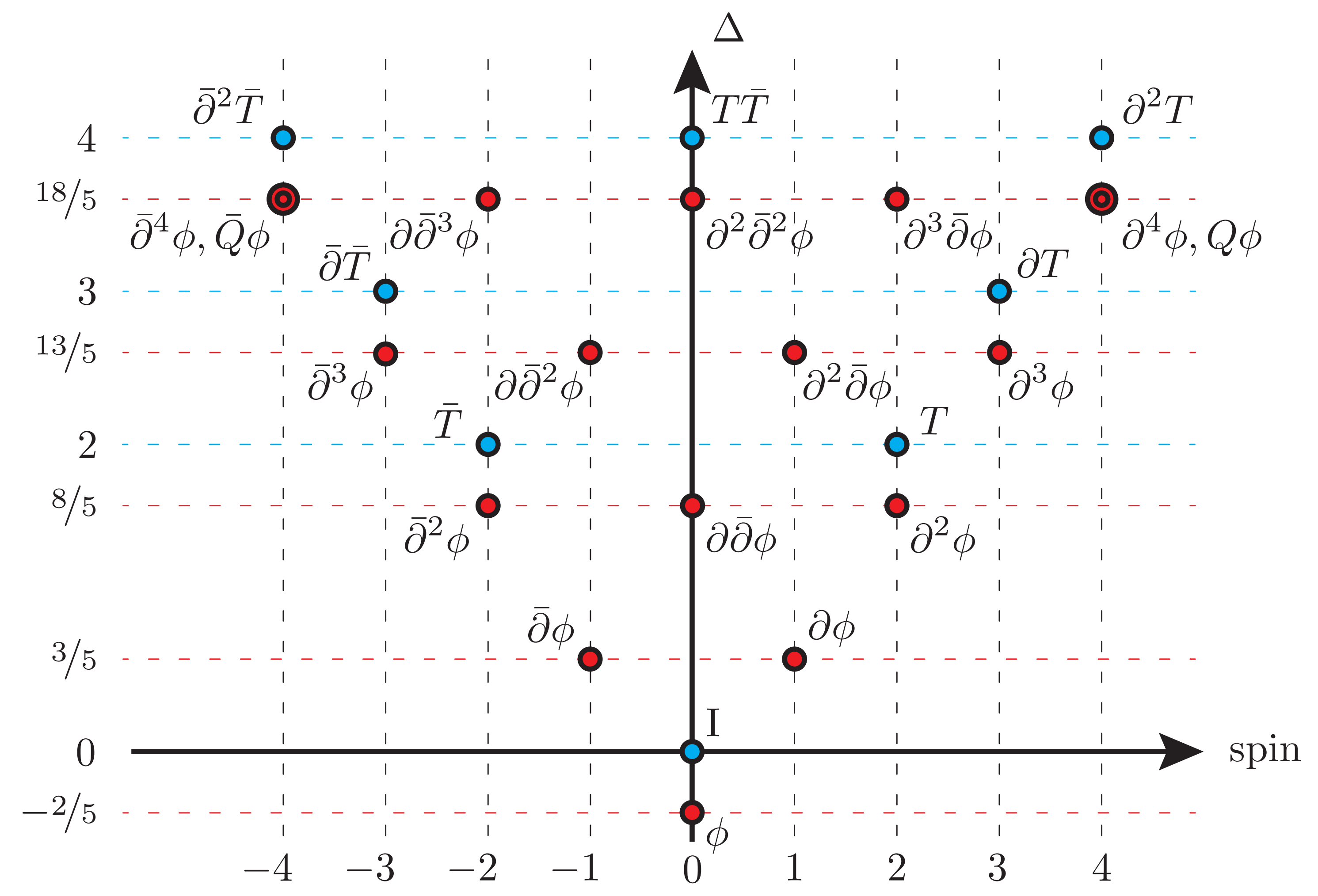
YL in 2D: exact minimal model solution

- In 2D, there are series of exactly solvable CFTs.

  [A.Belavin, A. Polyakov, A. Zamolodchikov ’84]

  $M(p, p+1)$ is unitary, everything else is non-unitary

- Cardy (1985): Yang-Lee is M(2,5)



$M(2,5)$

I, 0	φ, −1/5

$c = -22/5$

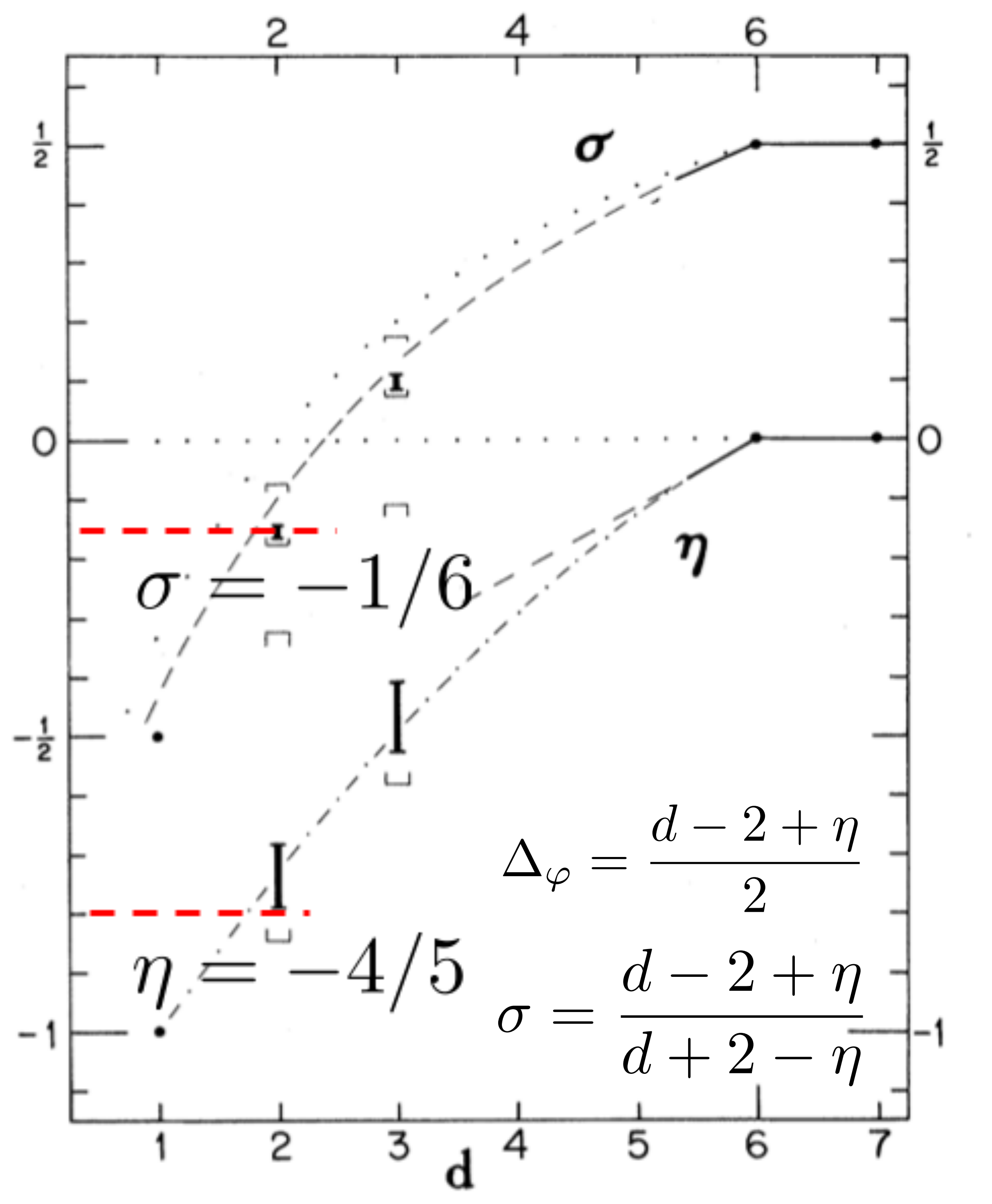
$$\langle \varphi(z_1)\varphi(z_2)\varphi(z_3)\rangle = C_{\varphi\varphi\varphi}|z_{12}z_{23}z_{31}|^{2/5}$$

$$C_{\varphi\varphi\varphi} = i\left(\frac{\Gamma(\frac{6}{5})^2\Gamma(\frac{1}{5})\Gamma(\frac{2}{5})}{\Gamma(\frac{3}{5})\Gamma(\frac{4}{5})^3}\right)^{1/2}$$

$M(3,4)$

$I, 0$	$\sigma, 1/16$	$\epsilon, 1/2$

$c = 1/2$



Conformal Invariance and the Yang-Lee Edge Singularity in Two Dimensions

John L. Cardy

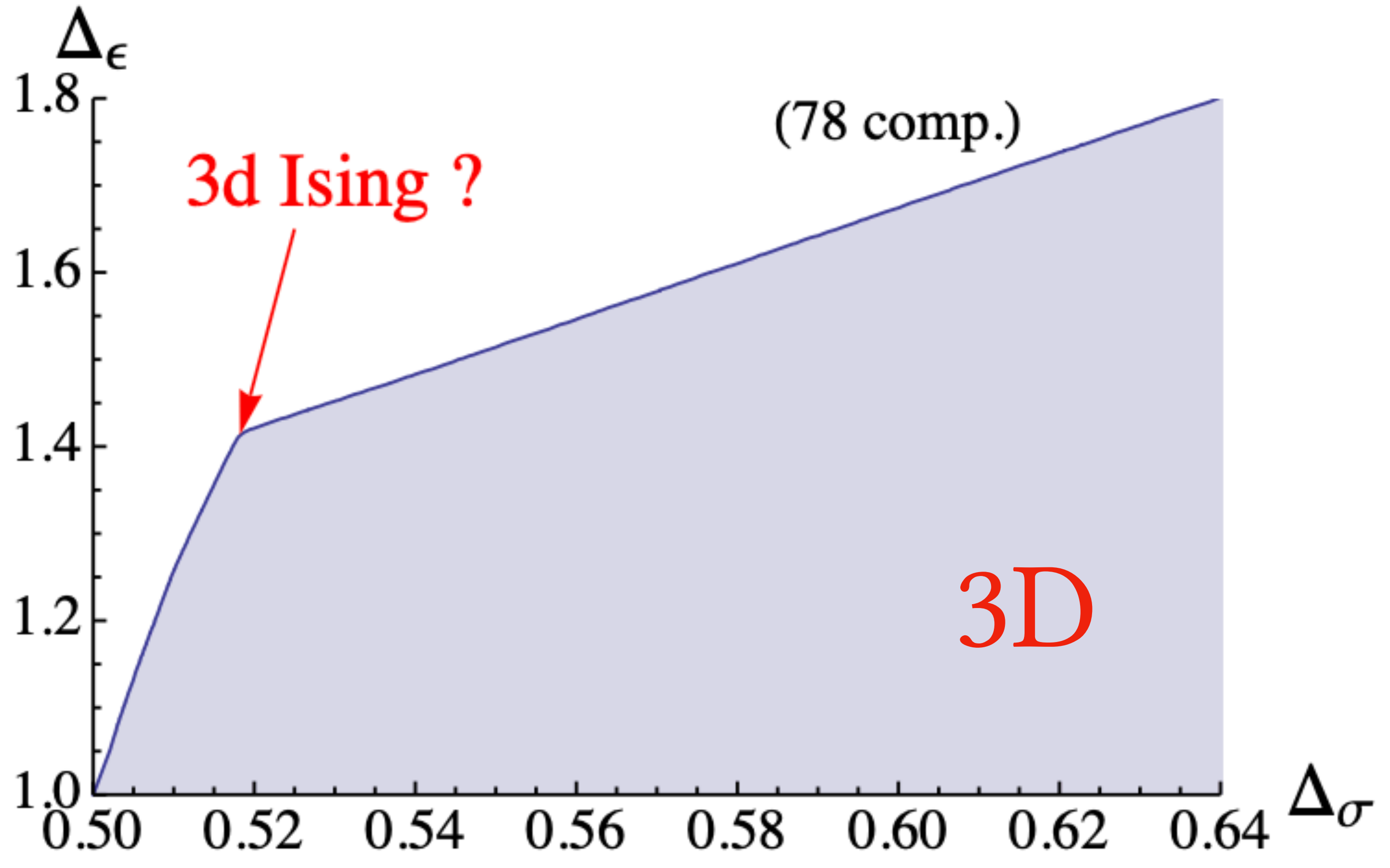
Department of Physics, University of California, Santa Barbara, California 93106

(Received 8 January 1985)

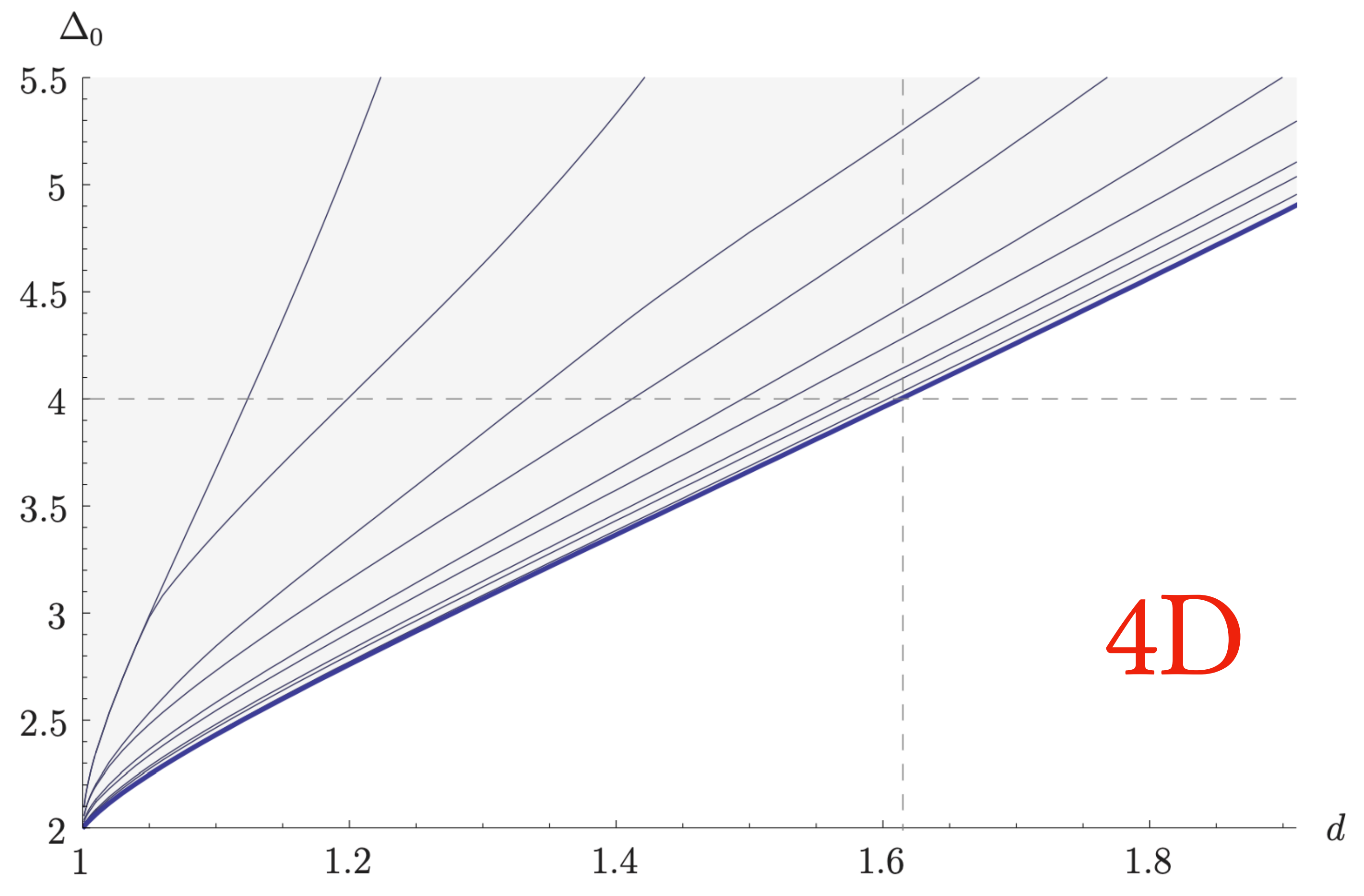
Motivation for Studying YL Criticality

Simplest interacting CFT that Lives in Many Dimensions

- The success of conformal bootstrap began from the analysis of the criticality of ϕ^4 theory - the Ising Model.
- ϕ^4 is no-longer relevant in 4D, but $i\phi^3$ remains interesting. However, a non-unitary CFT challenges the conformal bootstrap. We love challenges.



[S. El-Showk, M. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, A. Vichi, 1203.6064]



[D. Poland, D. Simmons-Duffin, A. Vichi, 1109.5176]

Same task, with modern tools

Monte-Carlo

Fuzzy Sphere

Conformal bootstrap

Exact Diag

Light-Cone

Hamiltonian

Bootstrap

Strongly coupled QFT/many body

TCSA

S-matrix bootstrap

DMRG

Perturbation

Many more...

Same task, with modern tools

Monte-Carlo ?

Conformal bootstrap

Bootstrap ?

S-matrix bootstrap

Yang-Lee criticality

Fuzzy Sphere

Exact Diag

Light-Cone

Hamiltonian

TCSA

DMRG

Perturbation

Many more...

Connecting Energy Levels in Various Dimensions

- $(6-\epsilon)$ expansion is known up to high loops orders.

$$\Delta_\phi = 2 - \frac{\epsilon}{2} + \gamma_\phi = 2 - \frac{5}{9}\epsilon - \frac{43}{1458}\epsilon^2 + \left(-\frac{8375}{472392} + \frac{8\zeta(3)}{243}\right)\epsilon^3 + \cdots$$

$T\bar{T}$

$$\Delta_{\phi^3} = d + \beta'(g_*) = 6 - \frac{125}{162}\epsilon^2 + \left(\frac{36755}{52488} + \frac{20\zeta(3)}{27}\right)\epsilon^3 + \cdots$$

$C_{\mu\nu\kappa\lambda}$

$$\Delta_Q = 8 - \frac{16}{15}\epsilon - \frac{871}{30375}\epsilon^2 + O(\epsilon^3)\,. \qquad Q = \phi\partial^4\phi$$

[Borinsky, Gracey, Kompaniets, Schnetz '21]
[Bonfim, Kirkham, McKane '80]
[Fei, Giombi, Klebanov, and Tarnopolsky '14]
(6 loops just published recently by Oliver SCHNETZ)

- One can study it better by putting in d=2 information through a two-sided Padé.

Operators $d=2$	Exact Δ $d=2$	Two-sided Padé for Δ			GL description
		$d=3$	$d=4$	$d=5$	
ϕ	$-2/5$	$0.218_{[3,3]},\ 0.218_{[4,2]}$	$0.827_{[3,3]},\ 0.827_{[4,2]}$	$1.425_{[3,3]},\ 1.425_{[4,2]}$	ϕ
$T\bar{T}$	4	$4.631_{[3,3]},\ 4.639_{[4,2]}$	$5.206_{[3,3]},\ 5.212_{[4,2]}$	$5.701_{[3,3]},\ 5.702_{[4,2]}$	$i\phi^3$
$Q\ ,\ \bar{Q}$	$18/5$	$4.681_{[1,2]},\ 4.709_{[2,1]}$	$5.793_{[1,2]},\ 5.815_{[2,1]}$	$6.910_{[1,2]},\ 6.916_{[2,1]}$	$Q_{\mu\nu\kappa\lambda}$

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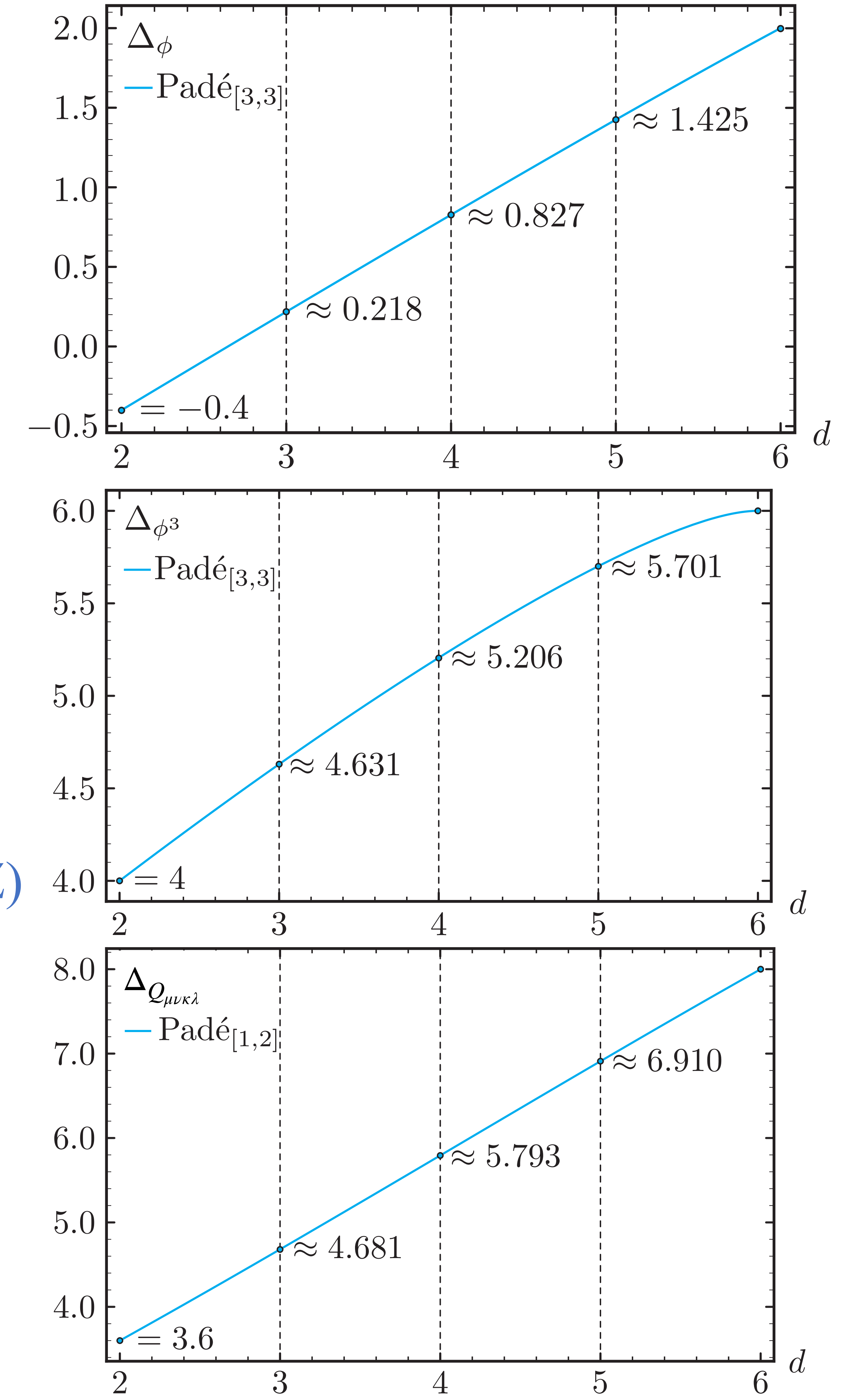

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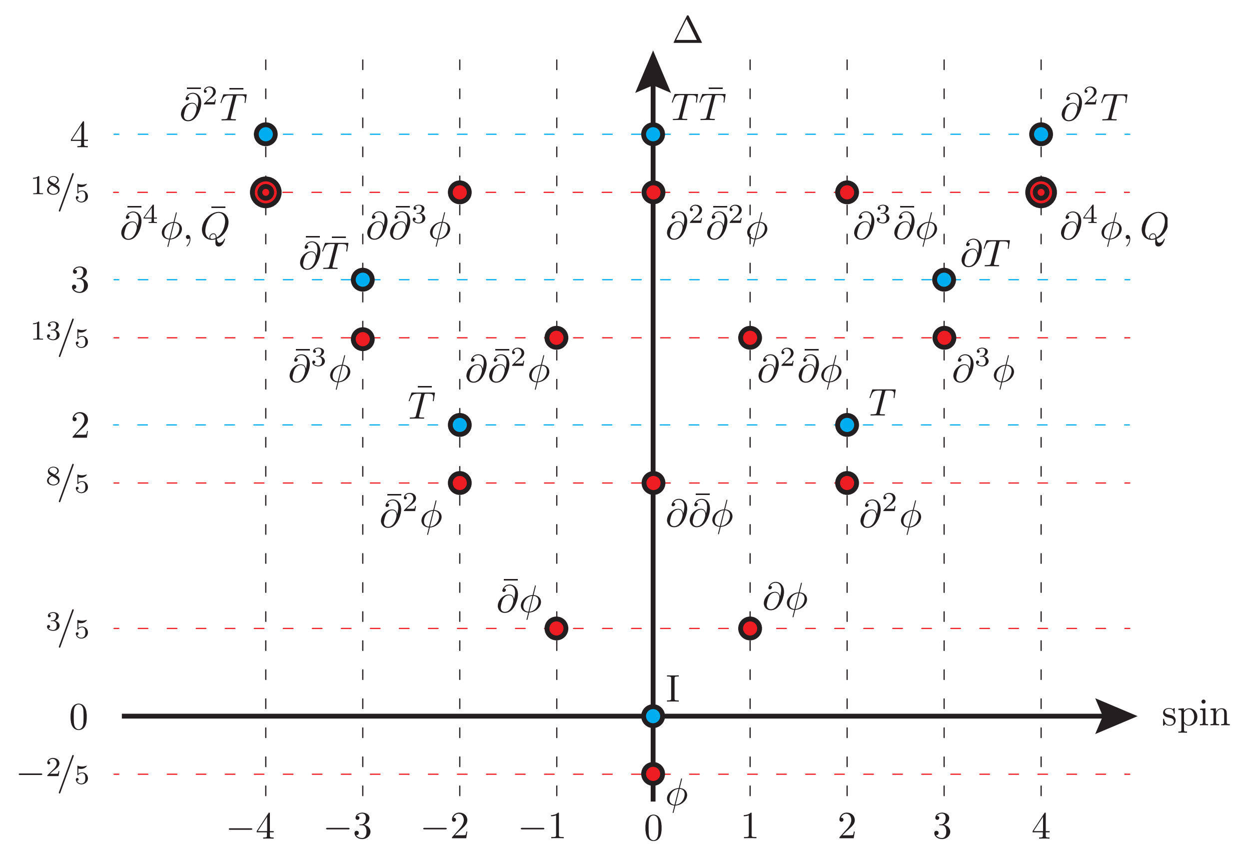

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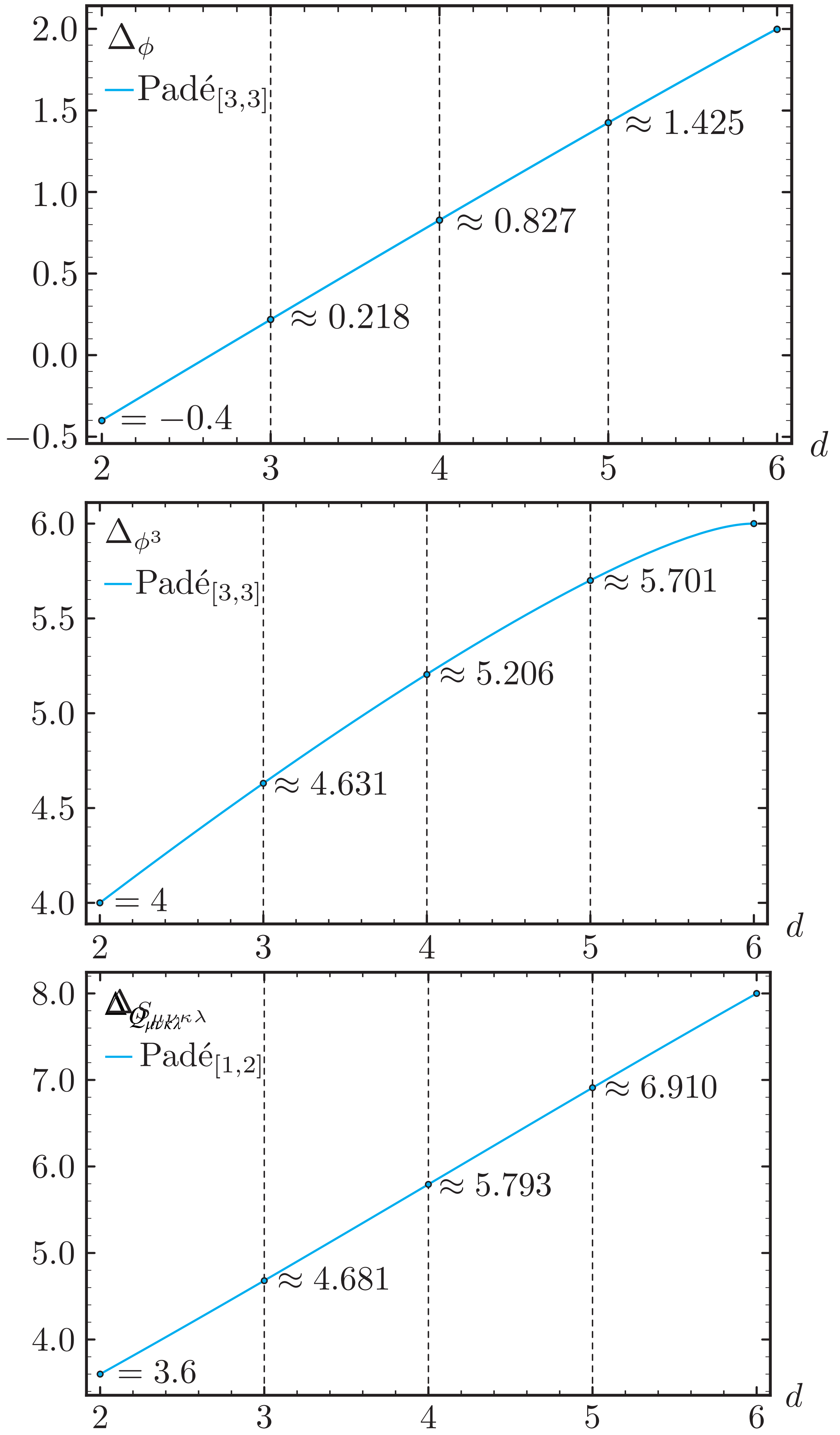

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$C_{\mu\nu\kappa\lambda}$

2D Criticality on Spin Chain

Past works by von Gehlen et al.

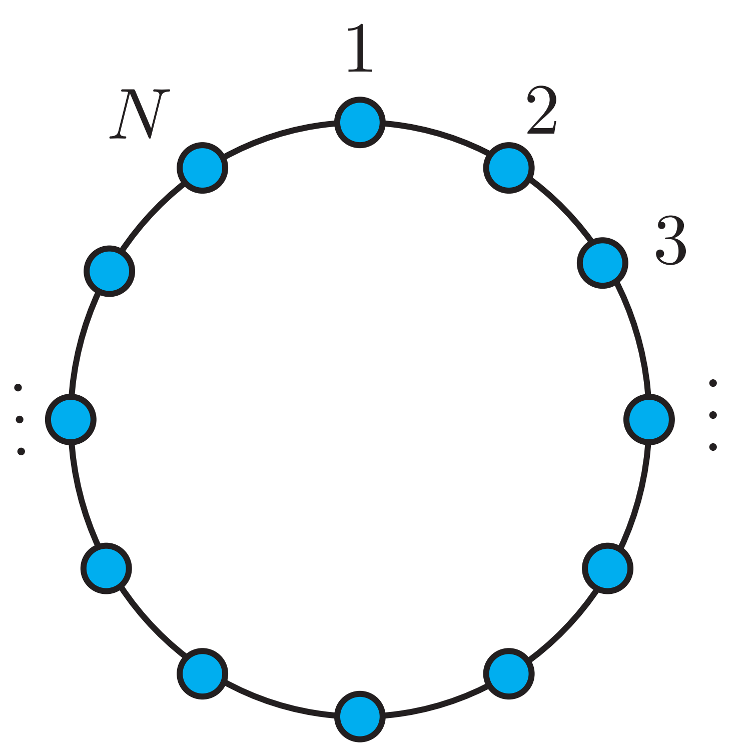
$$H_{\rm YL} = -J \sum_{\langle ij\rangle \in e} Z_i Z_j - h_x \sum_{i\in v} X_i - i h_z \sum_{i \in v} Z_i \,,$$

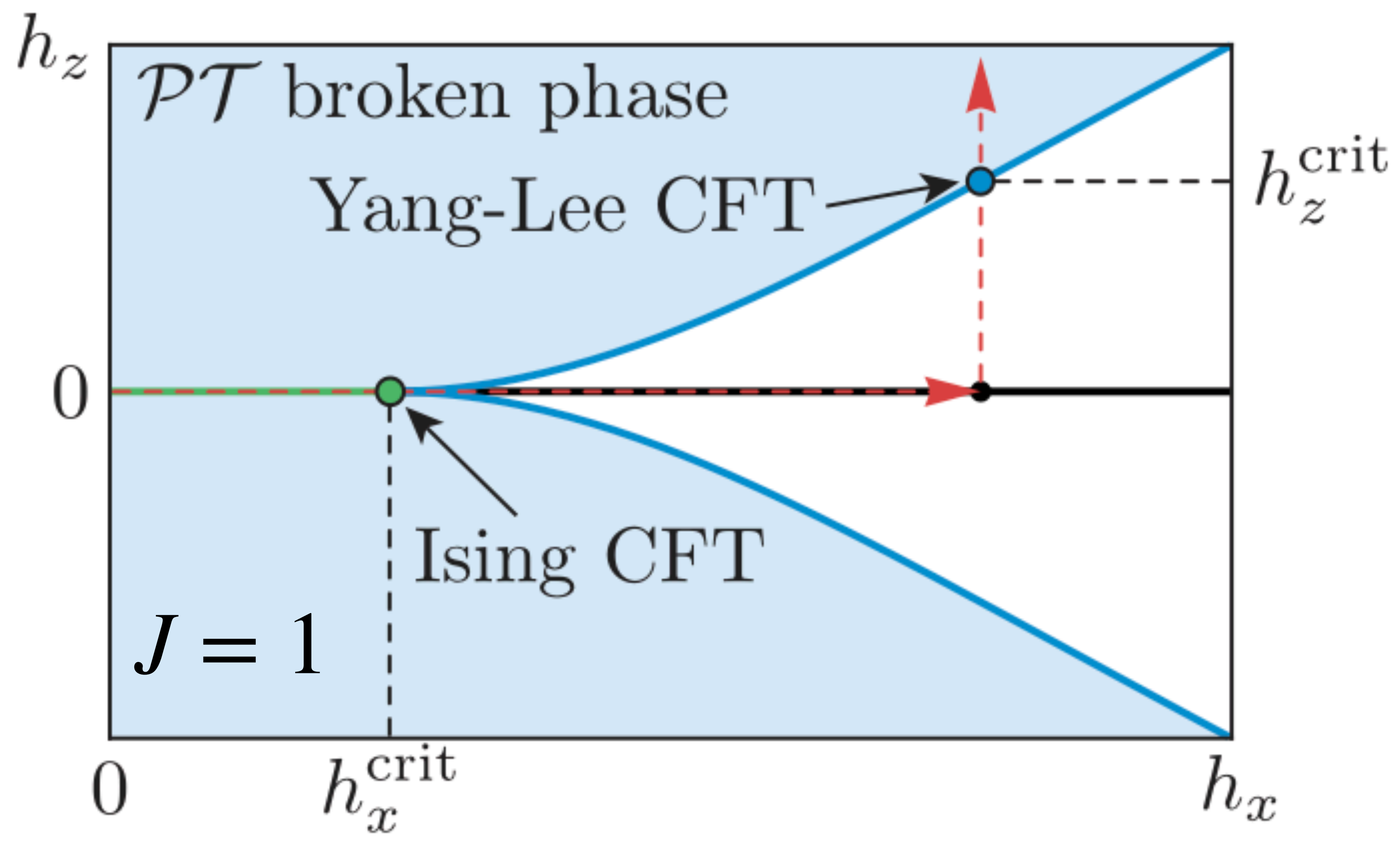
[Uzelac ’79-81, Gehlen ’91 and ’94]

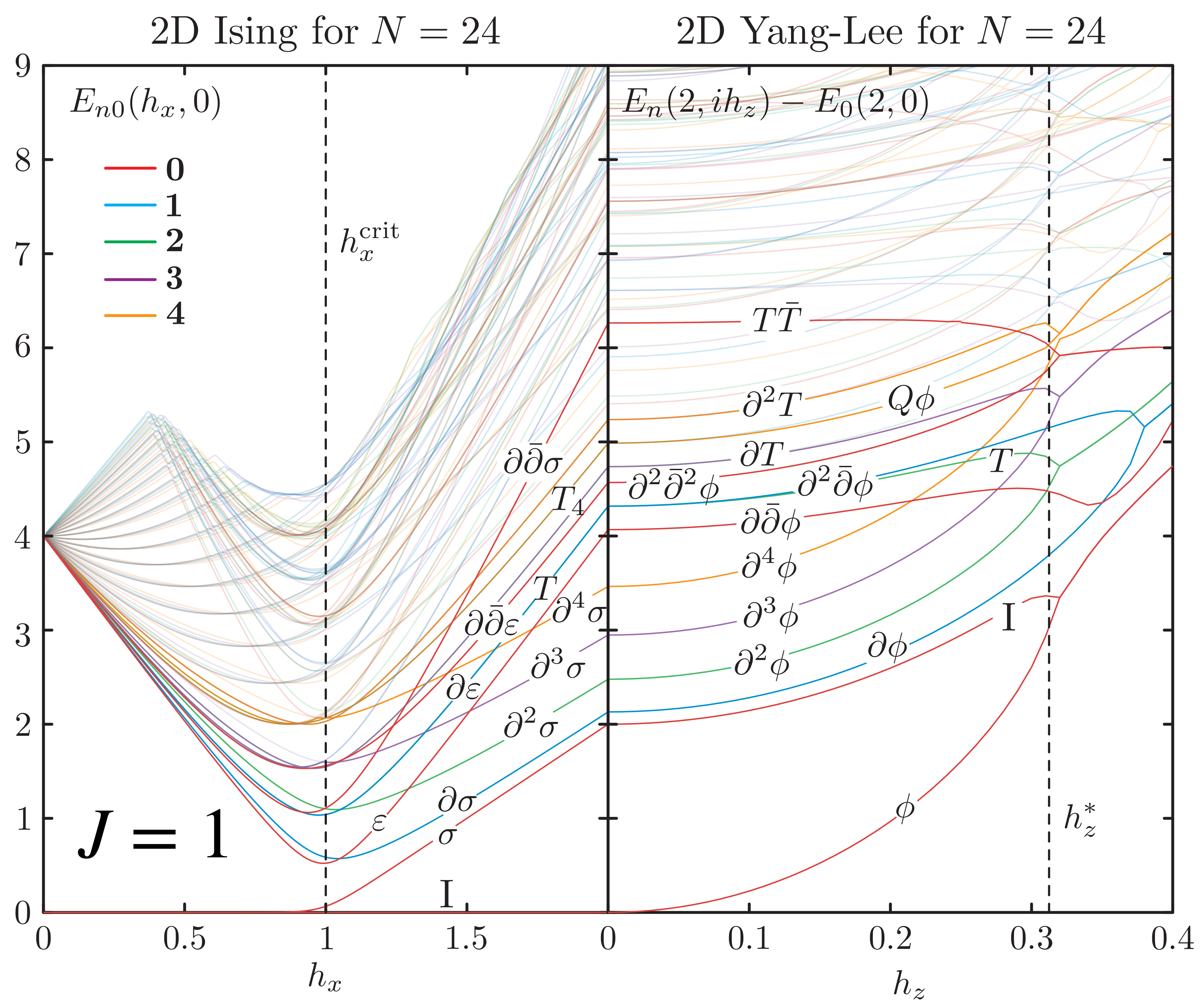
$$[\mathrm{PT}, H_{\rm YL}] = 0 \quad \mathrm{P} = \prod_{n=1}^{N} X_n \quad \mathrm{T}: i \to -i$$

[Castro-Alvaredo, Fing’09]

- Transverse Field Ising Model with an imaginary field. Theory is non-Hermitian but still PT-symmetric.
- Radial quantization: eigen val. $E \leftrightarrow$ scaling dim. Δ







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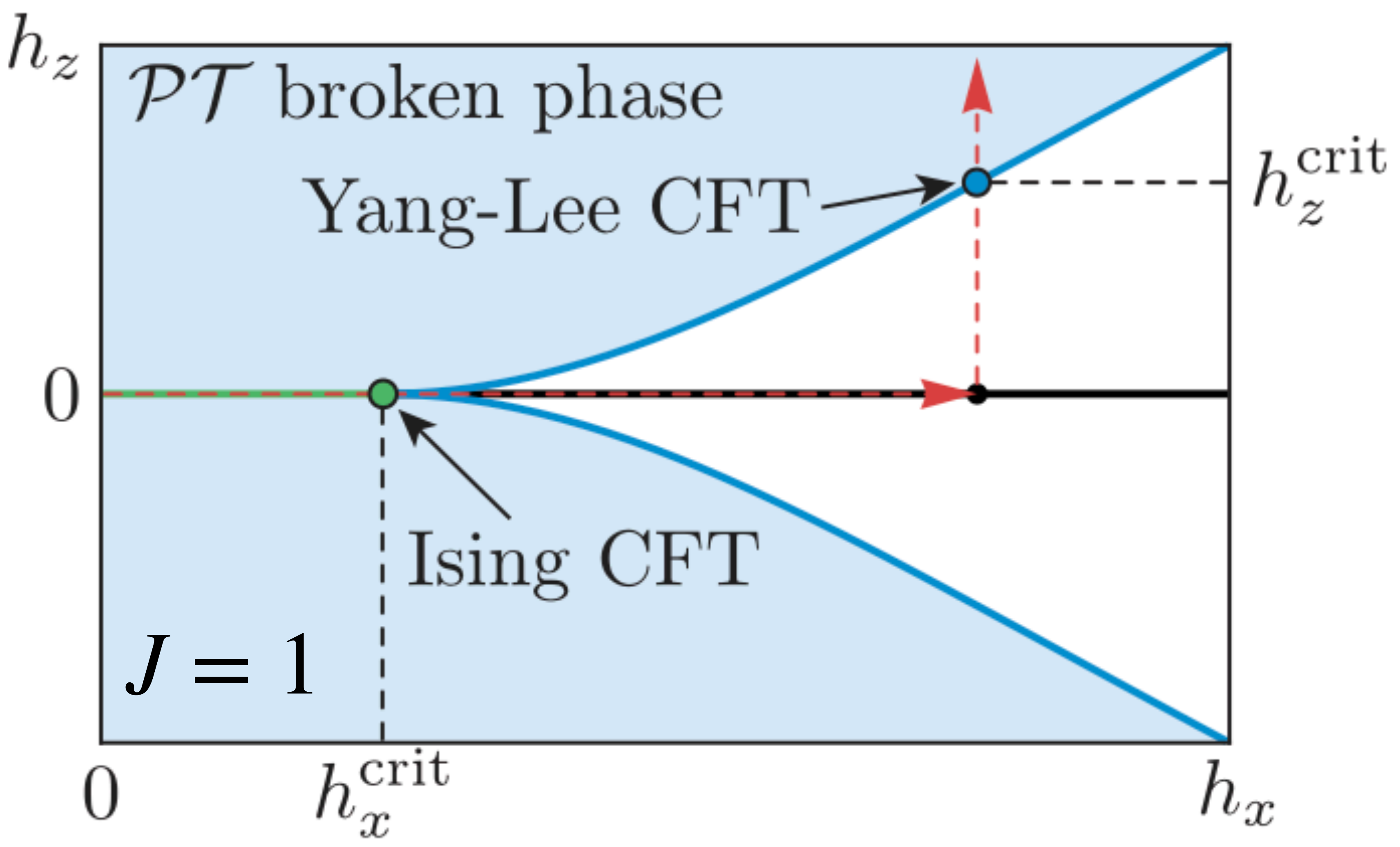
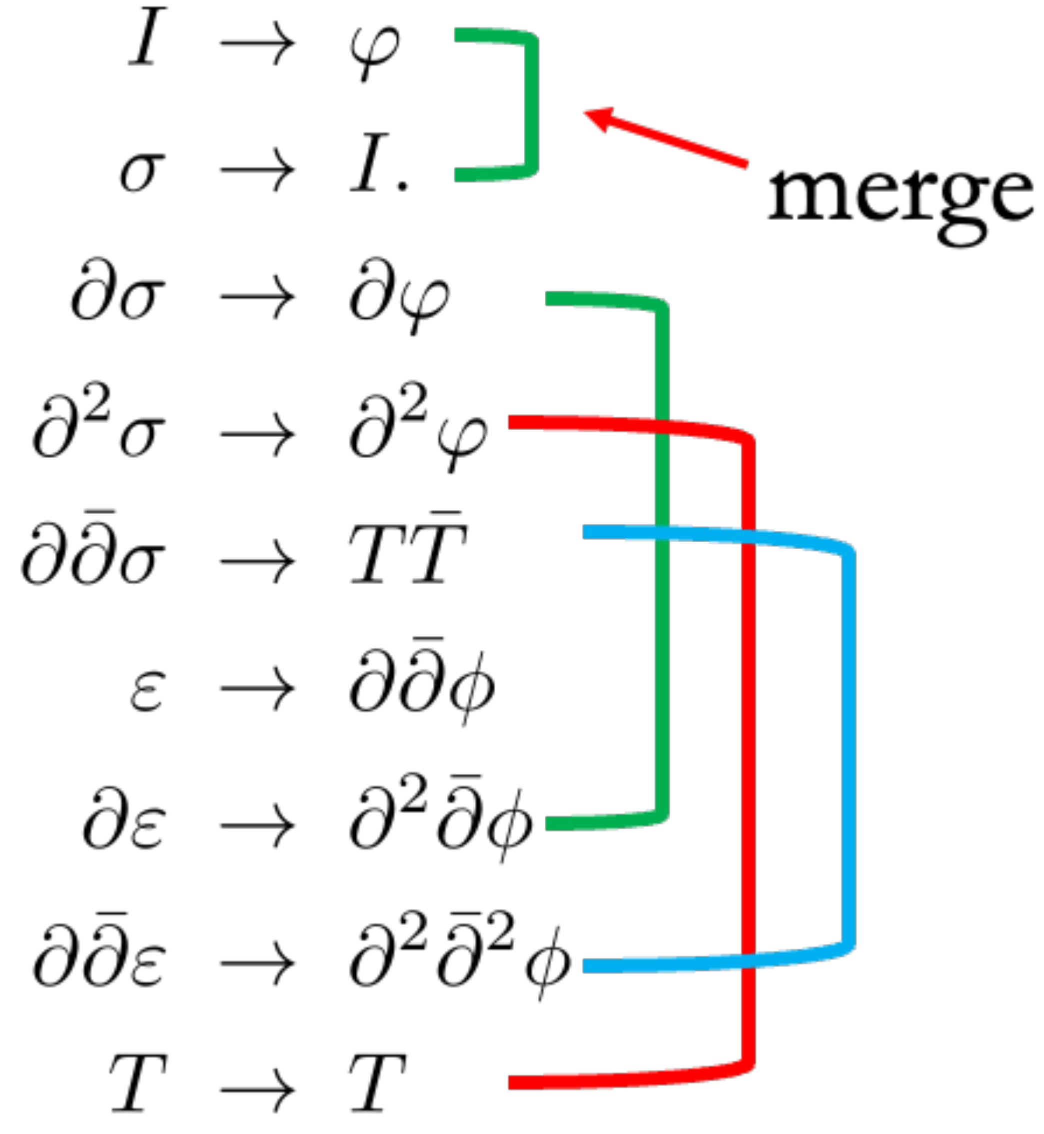
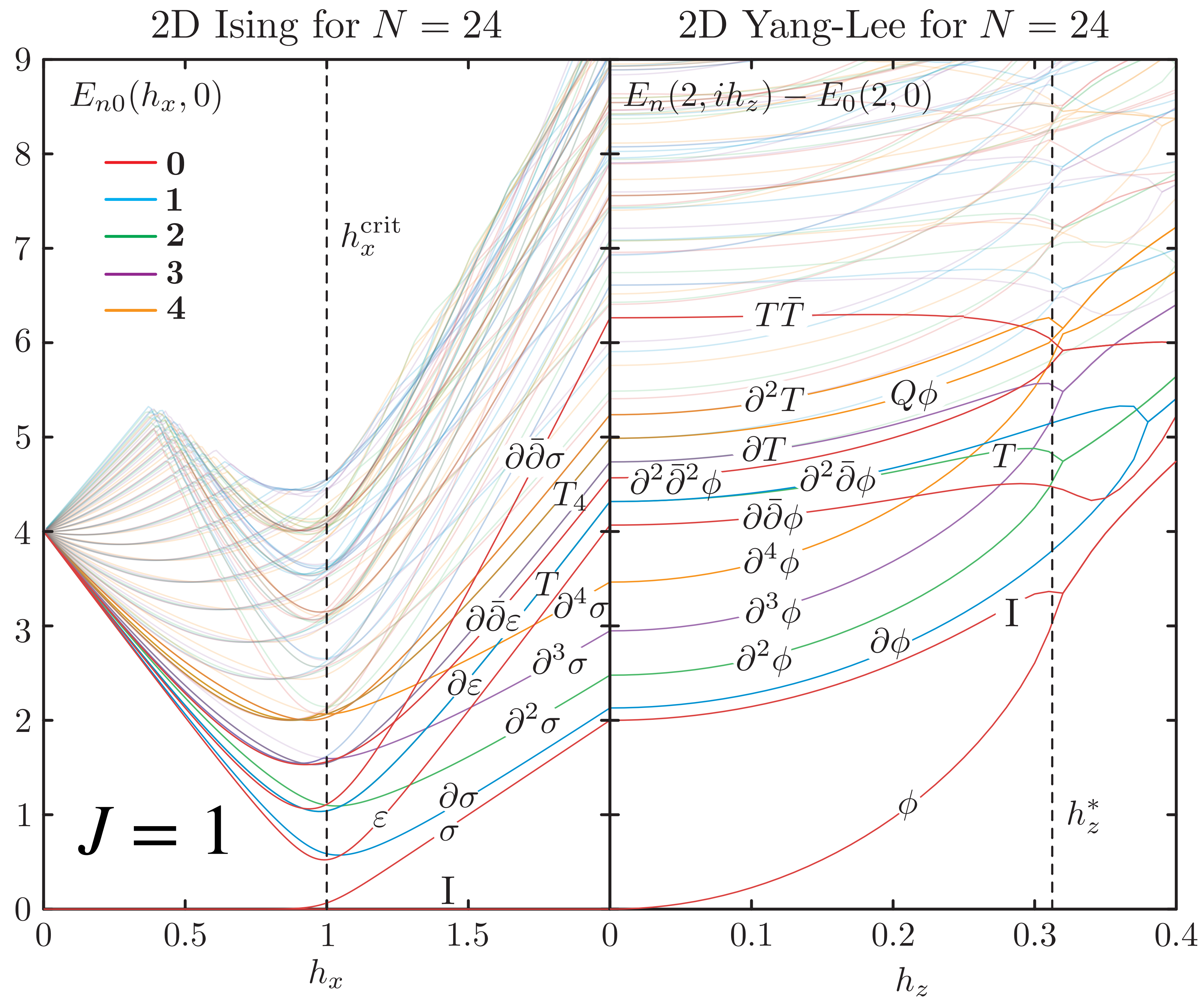




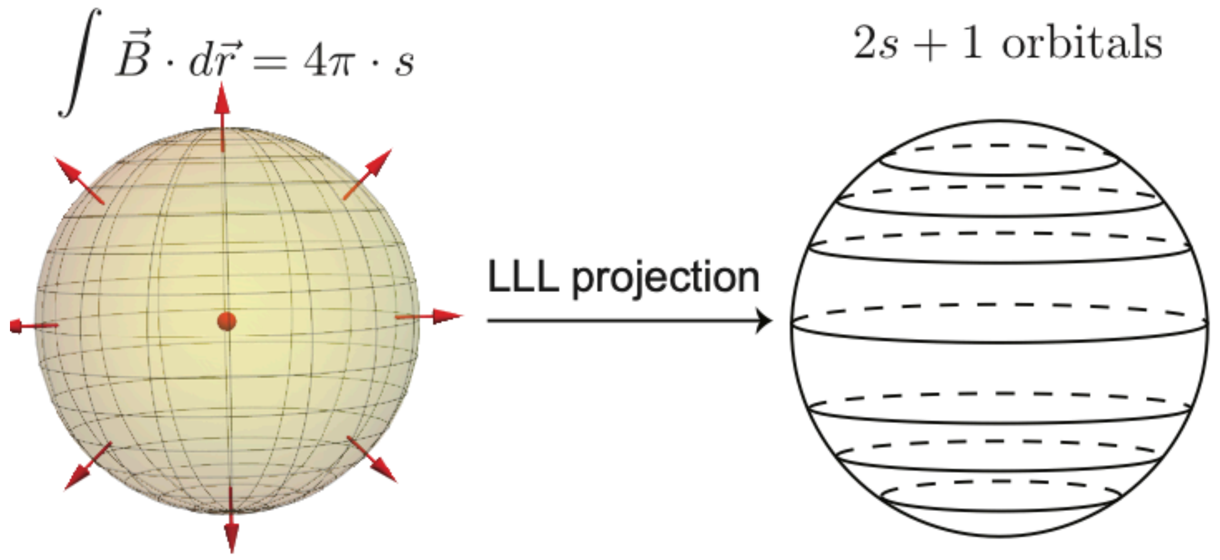


3D Criticality on Fuzzy Sphere

Fuzzy Sphere Ising Model with Imaginary Field

Fuzzy Sphere Ising Model

N fermions



[Zhu, Han, Huffman, Hofmann, He '23
Hu, He, Zhu '23, '24]

$$H = H_4 + H_2$$

$$H_4 = R^2 \int d^2\Omega \left[\lambda_n (\psi^\dagger \psi) U (\psi^\dagger \psi) - \lambda_{n,z} (\psi^\dagger \sigma_z \psi) U (\psi^\dagger \sigma_z \psi)\right]$$

$$H_2 = -R^2 \int d^2\Omega \left[h_x (\psi^\dagger \sigma_x \psi) + i h_z (\psi^\dagger \sigma_z \psi)\right]$$

$$U : V_0 + V_1 \nabla^2 + \cdots$$

2D lattice	3D Fuzzy Sphere
Hopping $Z_i Z_{i+1}$	Density-Density Interaction $(\psi^\dagger \sigma_z \psi) U (\psi^\dagger \sigma_z \psi)$
Density $X,\ Z$	Density $\psi^\dagger \sigma_x \psi,\ \psi^\dagger \sigma_z \psi$

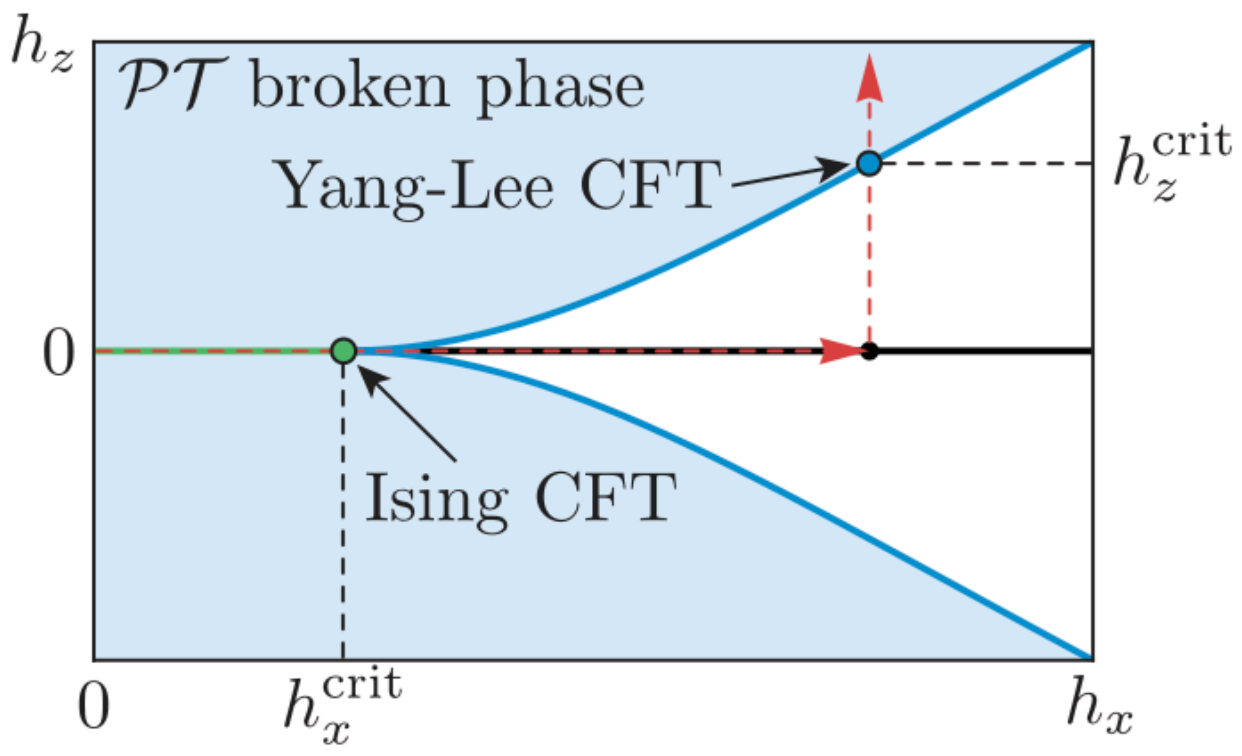
$$H(V_0, h_x, i h_z) = H_{\text{Ising}}(V_0, h_x) - i h_z H_Z$$

FuzzifiED.jl

[https://docs.fuzzified.world/
Zheng Zhou, 2503.00100]

3D Criticality on Fuzzy Sphere

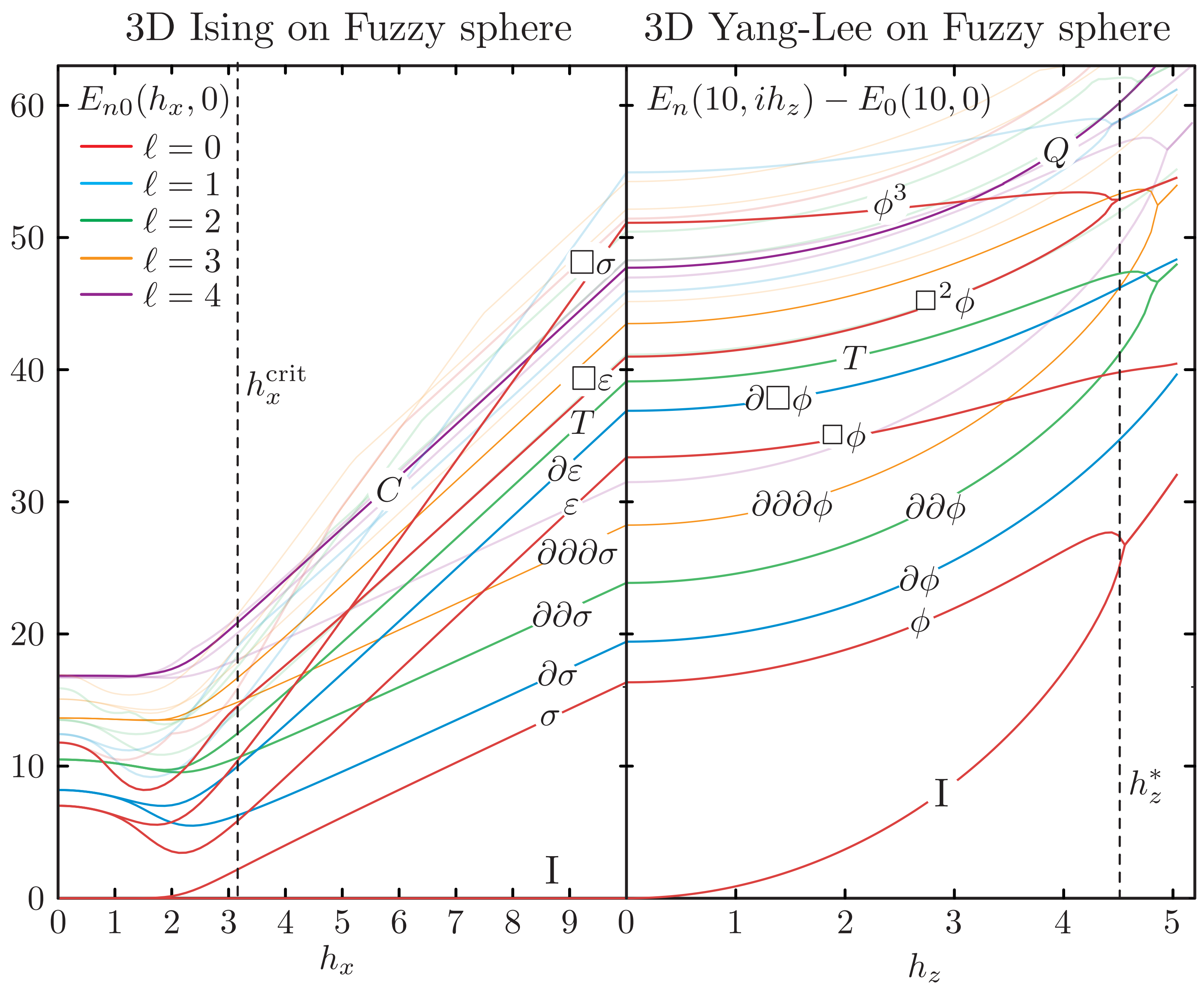
Finding the critical point



- Start in the paramagnetic phase, bring ih_z close to merging, fix $ih_z^*(N)$ using $r_T = 3$, and extrapolate.

$$H(V_0, h_x, ih_z) = H_{\rm Ising}(V_0, h_x) - ih_z H_Z$$

$V_0 = 4.75$ for the rest of the talk



Operator flow from 3D Ising CFT to 3D YL CFT

Ising ($\ell = 0$)	YL ($\ell = 0$)
I	I
σ	ϕ
ε	$\Box\phi$
$\Box\varepsilon$	$\Box^2\phi$
$\Box\sigma$	ϕ^3
ε'	ϕ^4

Ising ($\ell = 1$)	YL ($\ell = 1$)
$\partial_\mu\sigma$	$\partial_\mu\phi$
$\partial_\mu\varepsilon$	$\partial_\mu\Box\phi$

Ising ($\ell = 2$)	YL ($\ell = 2$)
$\partial_{\mu_1}\partial_{\mu_2}\sigma$	$\partial_{\mu_1}\partial_{\mu_2}\phi$
$T_{\mu\nu}$	$T_{\mu\nu}$
$T'_{\mu\nu}$	$T'_{\mu\nu}$ (2.15)

Ising ($\ell = 3$)	YL ($\ell = 3$)
$\partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\sigma$	$\partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\phi$
$\partial_\alpha T_{\mu\nu}$	$\partial_\alpha T_{\mu\nu}$

Ising ($\ell = 4$)	YL ($\ell = 4$)
$\partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\partial_{\mu_4}\sigma$	$\partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\partial_{\mu_4}\phi$
$\partial_{\alpha_1}\partial_{\alpha_2}T_{\mu\nu}$	$\partial_{\alpha_1}\partial_{\alpha_2}T_{\mu\nu}$
$C_{\mu_1\mu_2\mu_3\mu_4}$	$Q_{\mu_1\mu_2\mu_3\mu_4}$

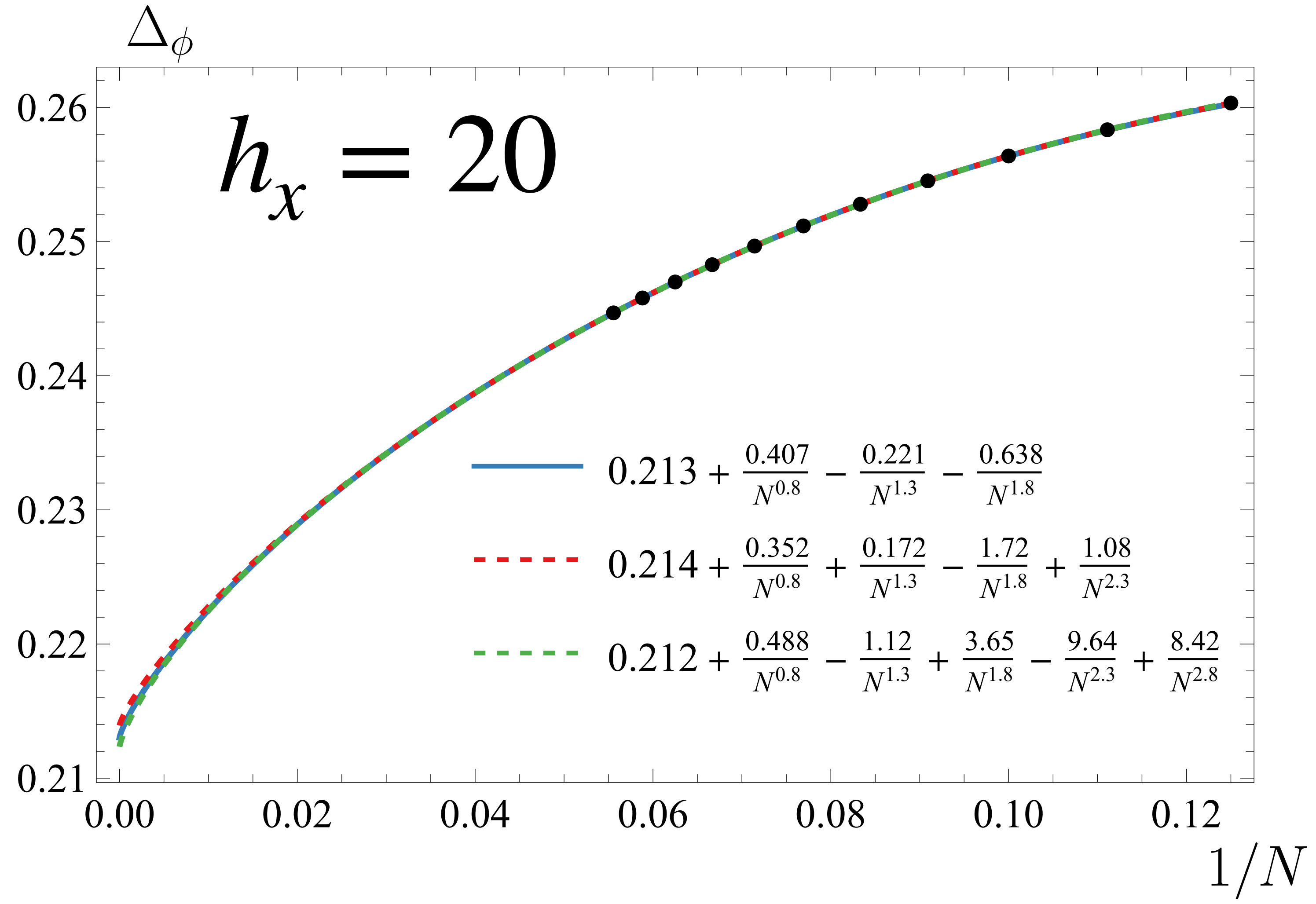
Merging states in 3D YL

Merger 1	Merger 2
I	ϕ
$\partial_\mu\phi$	$\partial_\mu\Box\phi$
$\partial_\mu\partial_\nu\phi$	$T_{\mu\nu}$
$\partial_\alpha\partial_\mu\partial_\nu\phi$	$\partial_\alpha T_{\mu\nu}$
$\Box^2\phi$	ϕ^3
$\partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\partial_{\mu_4}\phi$	$\partial_{\mu_1}\partial_{\mu_2}T_{\mu_3\mu_4}$

Same pattern as 2D

3D Criticality on Fuzzy Sphere

Extracting the spectrum from finite size scaling



$$\Delta^{(N)} = P_{w,K}(N) \equiv \Delta + \sum_{k=0}^{K} a_k \frac{1}{N^{w+k/2}}$$

- Finite size result is still far. One needs to fit.
- Error analysis provided by Conformal perturbation theory (CPT). [B.-X. Lao and S. Rychkov, 2307.02540, A. M. L¨auchli, L. Herviou, P. H. Wilhelm, and S. Rychkov, 2504.00842]

$$E_n = E_0 + \frac{\nu}{\sqrt{N}} \left(H_{\text{CFT}} + \frac{g_\phi}{N^{(\Delta_\phi - d)/2}} \int d^{d-1}\Omega\, \phi(\Omega) + \sum_i \frac{g_i}{N^{(\Delta_i - d)/2}} \int d^{d-1}\Omega\, \mathcal{O}_i(\Omega) \right) \qquad (N \sim R^2)$$

- Going to the pseudo critical point removes the leading power.

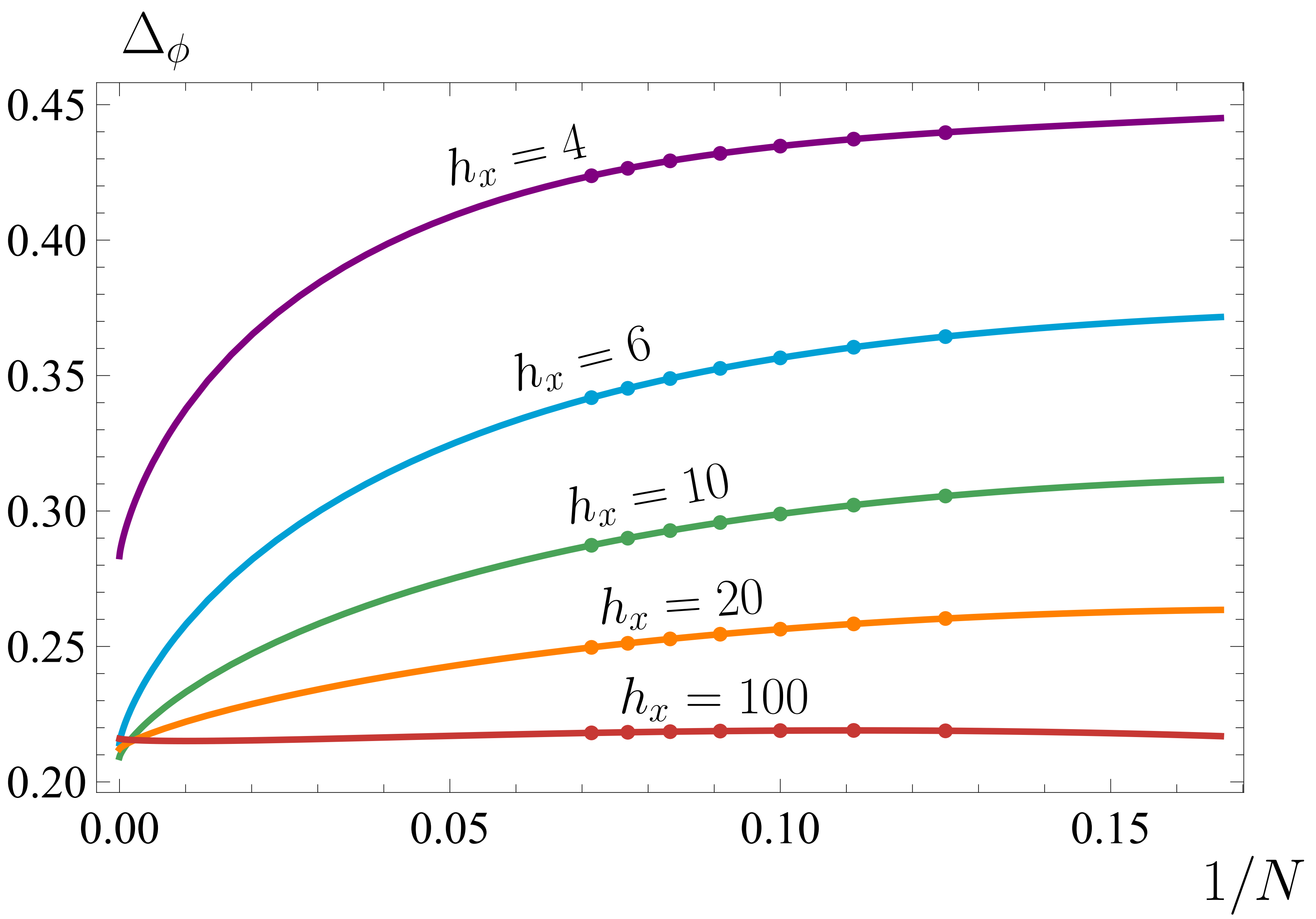
$$r_T = \frac{E_T(N) - E_{\text{I}}(N)}{E_{\partial\phi}(N) - E_\phi(N)} = 3 \qquad g_\phi \sim N^{\Delta_\phi - \Delta_{\mathcal{O}_I}} \qquad \Delta^{(N)} = \Delta + cN^{(3-\Delta_{\mathcal{O}_I})/2} + \cdots$$

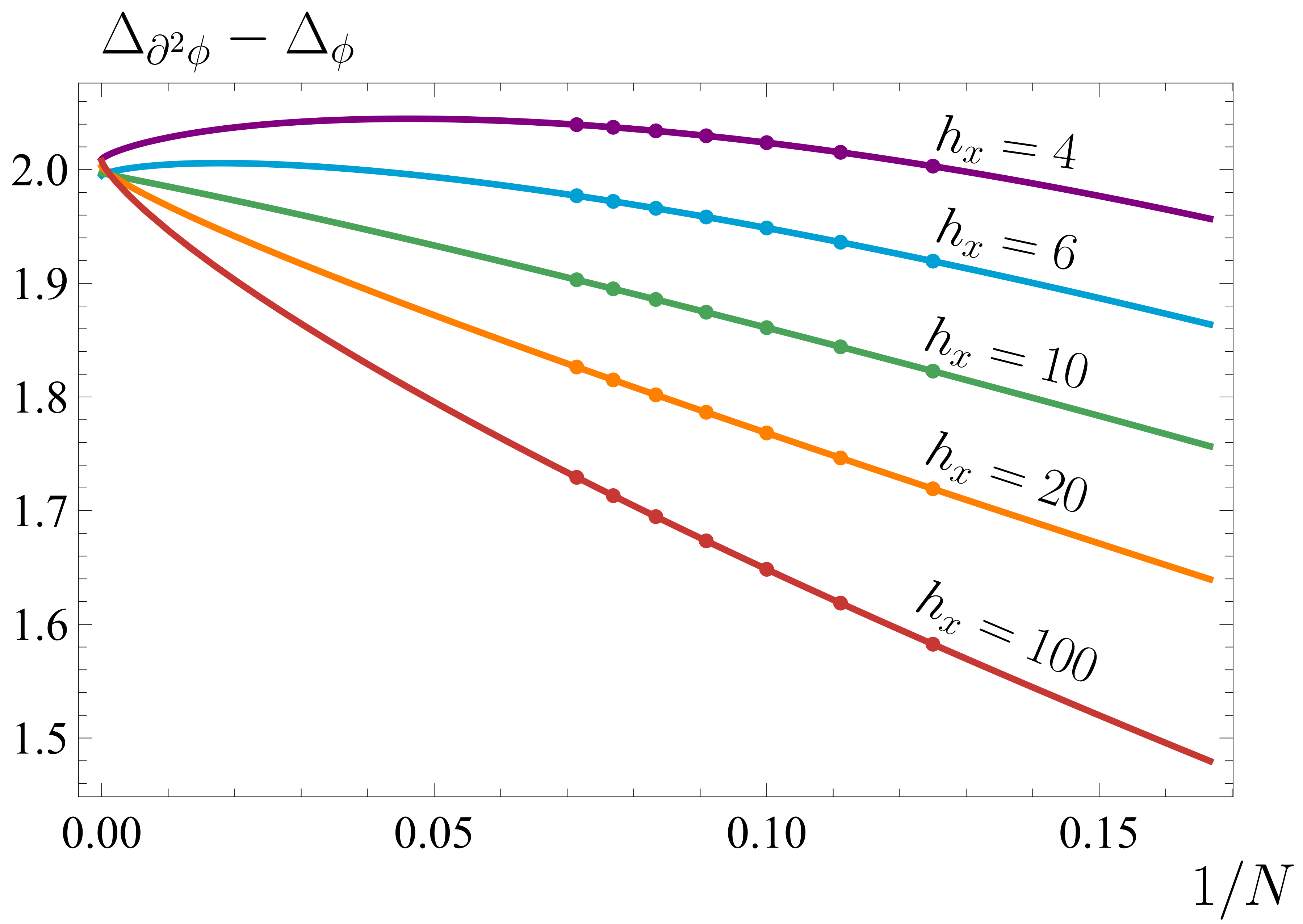
- Estimate the leading error $(\Delta_{\phi^3} - 3)/2 \approx 0.8$ $\quad (\Delta_{\phi^3} \approx 4.6)$

3D Criticality on Fuzzy Sphere

h_x universality

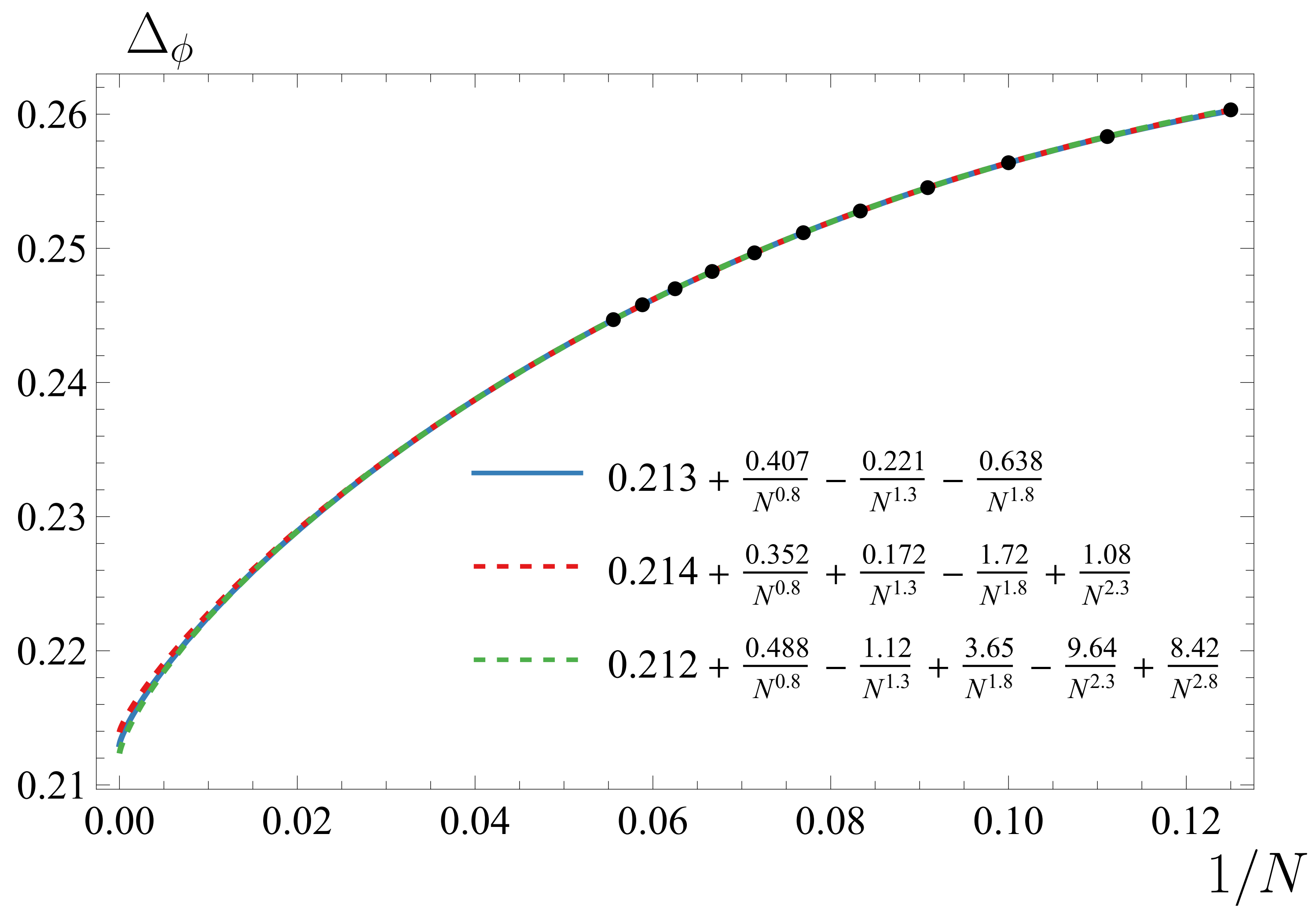
$$H(V_0, h_x, ih_z) = H_{\text{Ising}}(V_0, h_x) - ih_z H_Z$$

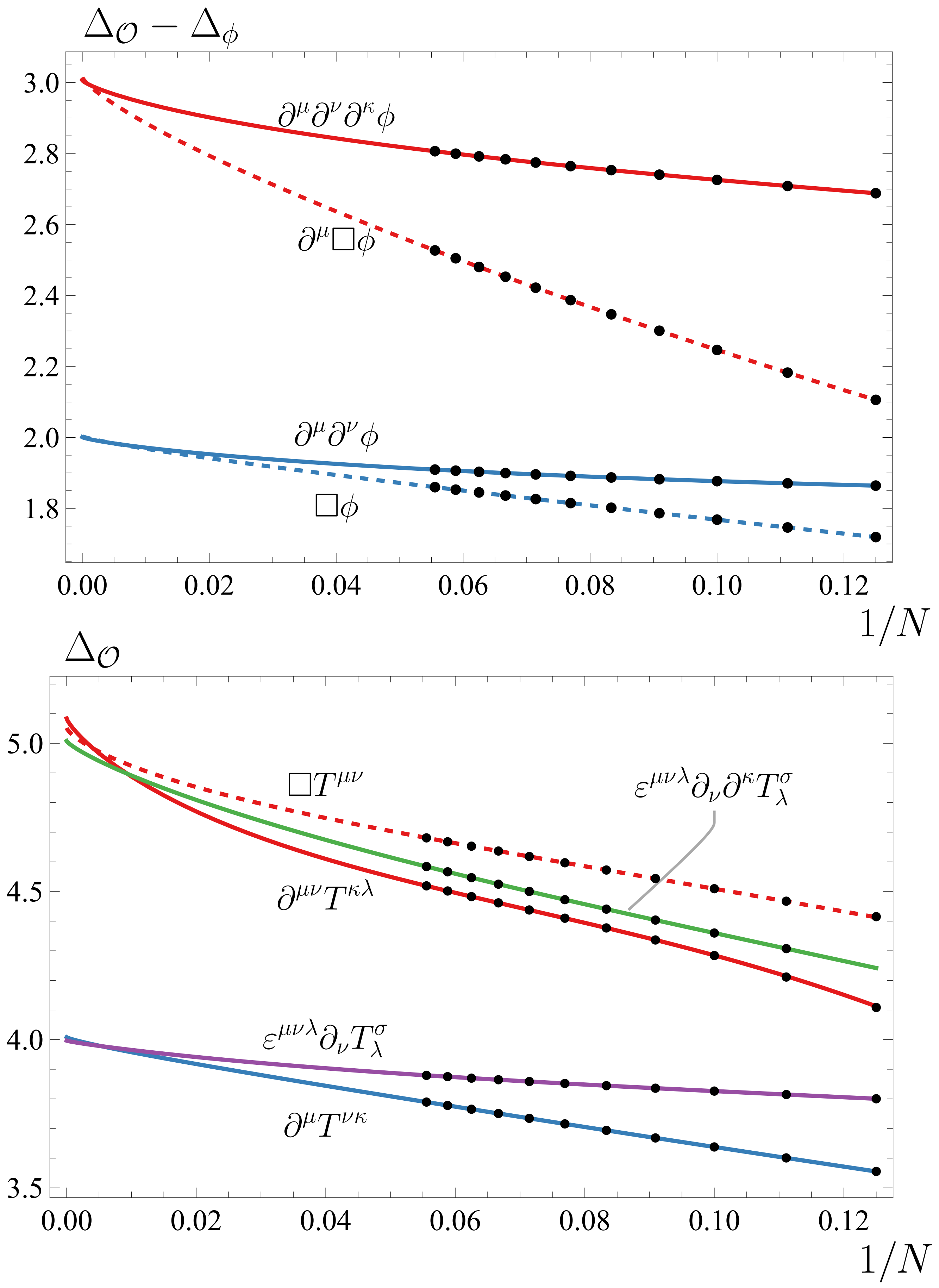


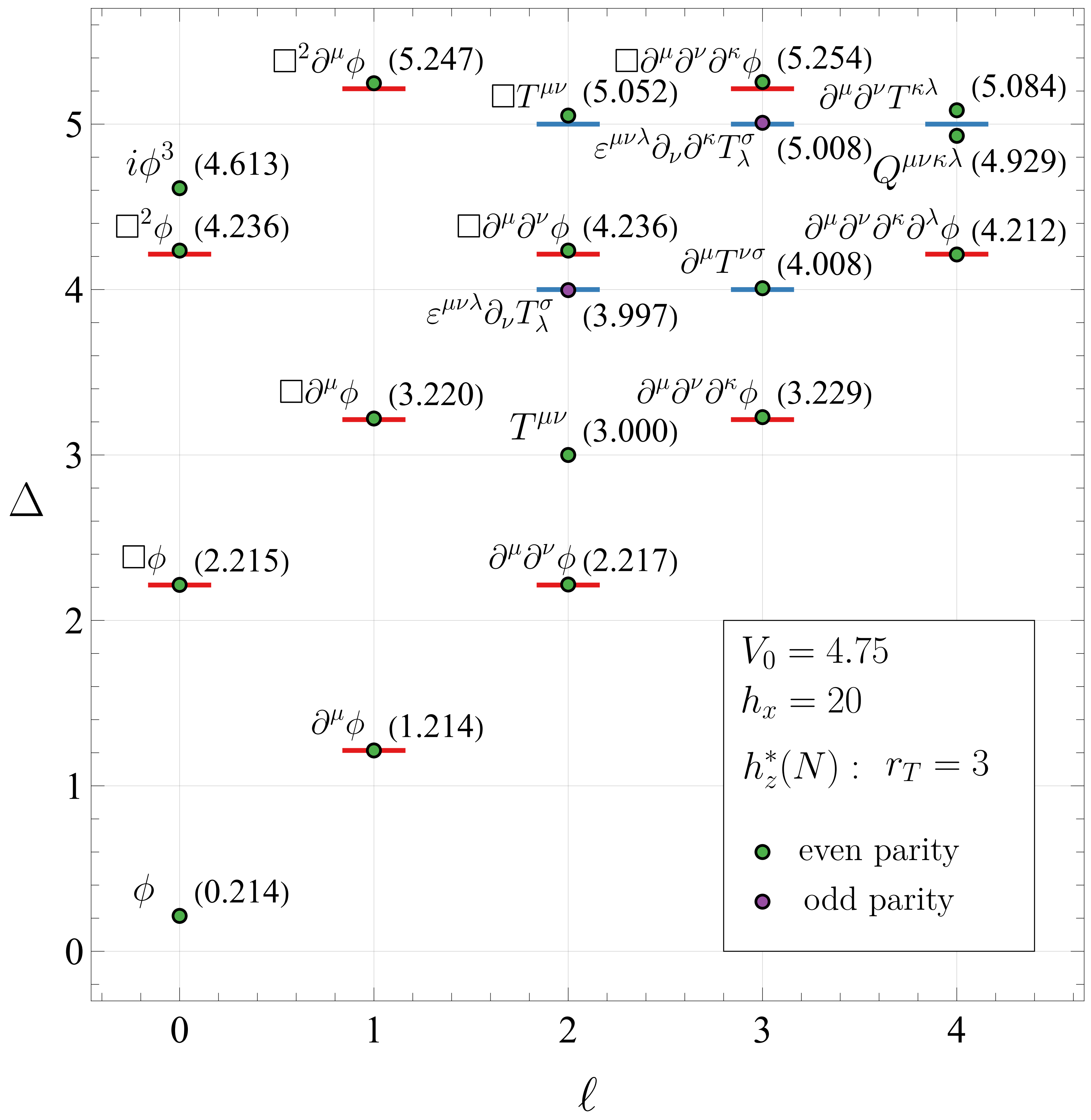


3D YL Criticality on Fuzzy Sphere

Extracting the scaling dimensions







3D Criticality on Fuzzy Sphere

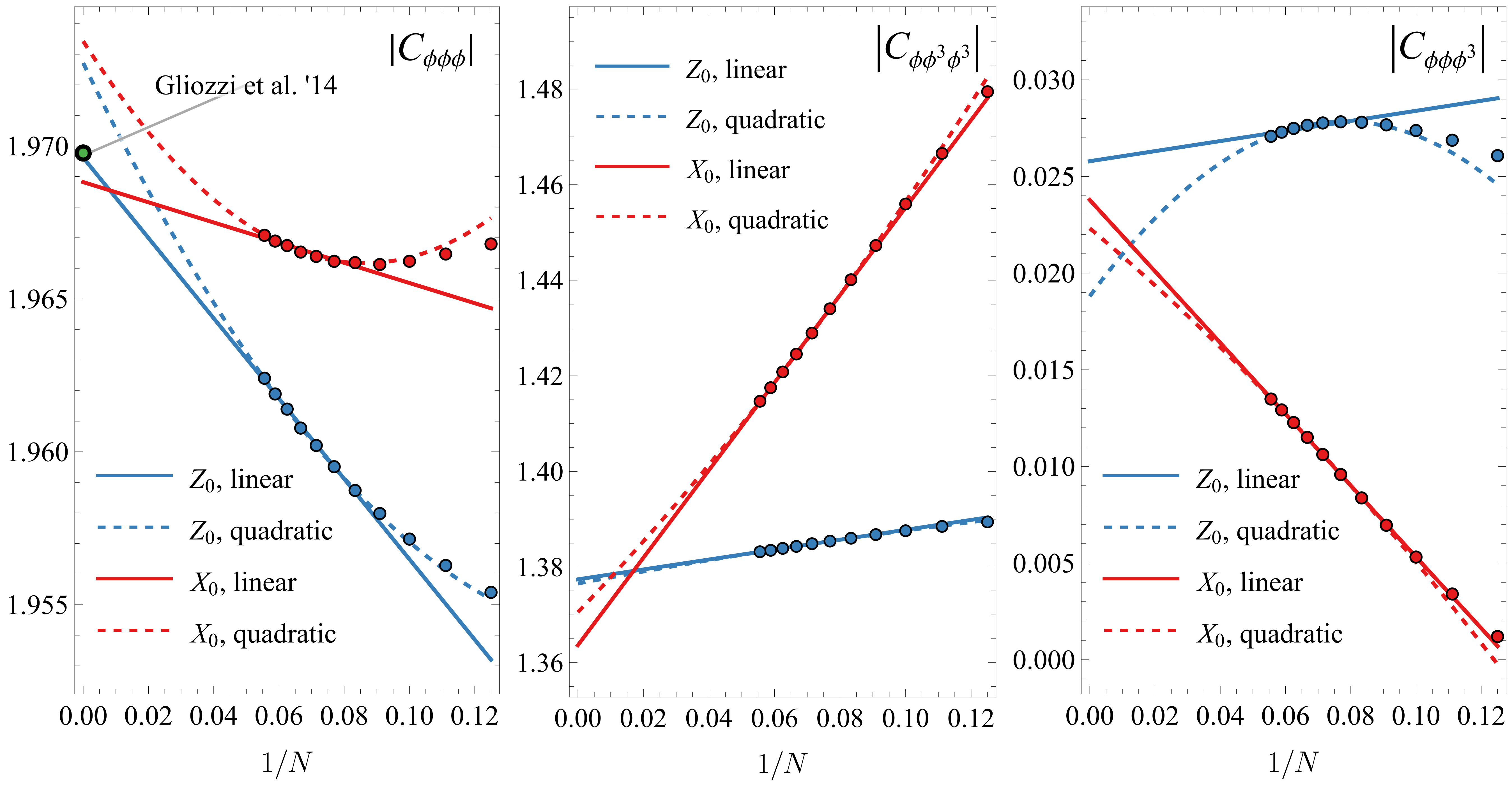
Compute the OPE coefficients

- Fuzzy sphere operators flow to CFT operators Hu, He, Zhu '23

$$X = \bar{\psi}\sigma^x\psi \sim \phi + \cdots \ , \quad X_\ell = \int d\Omega X(\Omega) Y_{\ell,0}$$

- Eigenstates ~ CFT local operators

- Leading errors are expected to come from $\partial^2\phi$.



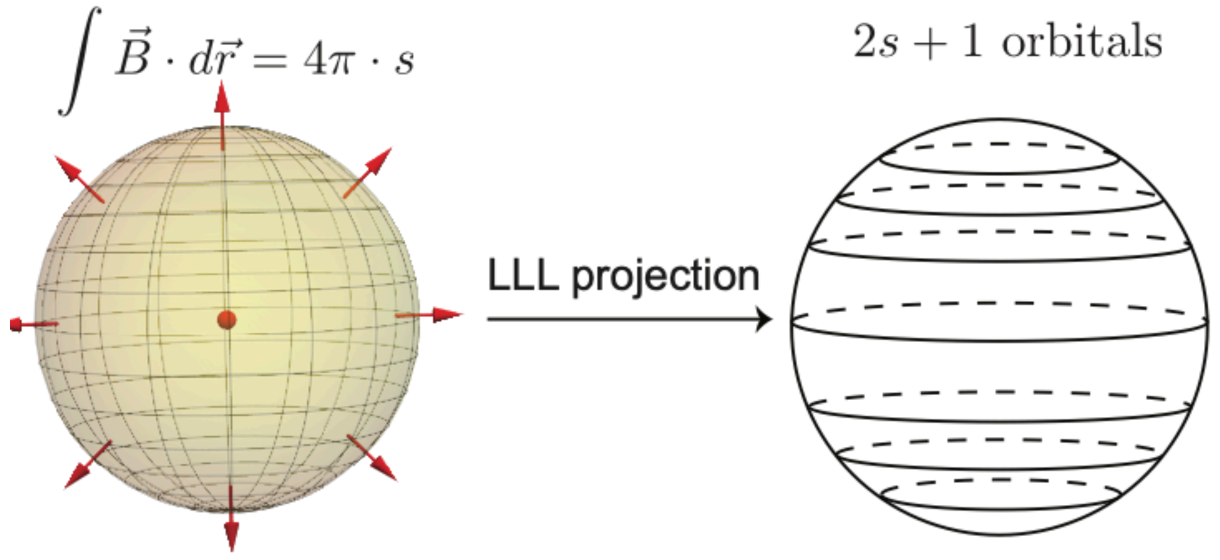
3D YL Criticality on Fuzzy Sphere

Compare Data

Observable	Fuzzy sphere	Padé	Two-sided Padé	5-loop All [41]	Truncated Bootstrap [43]	High-temperature expansion [52]
Δ_ϕ	$0.214(2)_{[E]}$ $0.2155(16)_{[Z]}$ $0.2151(8)_{[X]}$	$0.222_{[3,2]}$	$0.218_{[3,3]}$ $0.218_{[4,2]}$	0.215(10)	0.235(3) 0.174 [44]	0.214(6)
Δ_{ϕ^3}	$4.613(6)_{[E]}$	$4.766_{[3,2]}$	$4.631_{[3,3]}$ $4.639_{[4,2]}$	4.5(2)	5.0(1)	
$\Delta_{Q_{\mu\nu\kappa\lambda}}$	$4.9(1)_{[E]}$	$4.519_{[1,1]}$	$4.681_{[1,2]}$ $4.709_{[2,1]}$		4.75(1)	
$\lvert C_{\phi\phi\phi}\rvert$	$1.9696(31)_{[Z]}$ $1.969(5)_{[X]}$				1.9697(25)	
$\lvert C_{\phi\phi\phi^3}\rvert$	$0.026(7)_{[Z]}$ $0.0238(15)_{[X]}$					
$\lvert C_{\phi^3\phi\phi^3}\rvert$	$1.3774(9)_{[Z]}$ $1.364(7)_{[X]}$					
$\lvert C^{(0)}_{T\phi T}\rvert$	$1.2841(8)_{[Z]}$ $1.277(3)_{[X]}$					

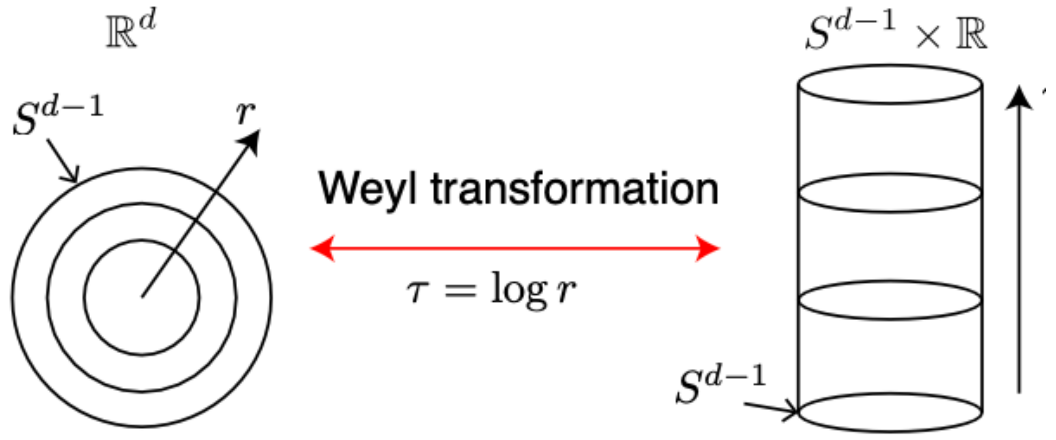
Radial Quantization: Fuzzy Sphere vs. Polyhedron

N fermions

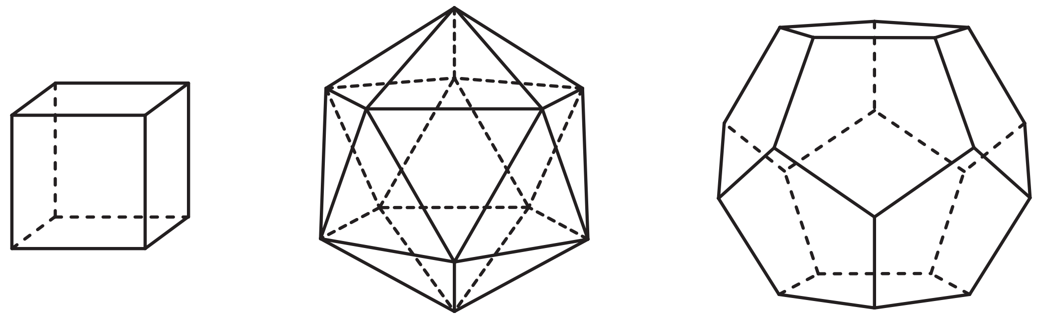


[Zhu, Han, Huffman, Hofmann, He ’23
Hu, He, Zhu ‘23, ‘24]

$$H = H_4 + H_2$$
$$H_4 = R^2 \int d^2\Omega \left[\lambda_n(\psi^\dagger\psi)U(\psi^\dagger\psi) - \lambda_{n,z}(\psi^\dagger\sigma_z\psi)U(\psi^\dagger\sigma_z\psi)\right]$$
$$H_2 = -R^2 \int d^2\Omega \left[h_x(\psi^\dagger\sigma_x\psi) + ih_z(\psi^\dagger\sigma_z\psi)\right]$$



$$E_n \sim \frac{\Delta_n}{R}$$
$$|E_n\rangle \sim O_n(0)\,|0\rangle$$



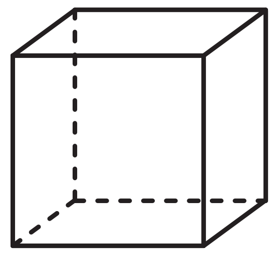
[Brower, Fleming, Neuberger ’12, ‘13
Gluck, Fleming, Brower, et all ‘23
Lao, Rychkov ‘23]

$$H_{\mathrm{YL}} = -J \sum_{\langle ij\rangle \in e} Z_i Z_j - h_x \sum_{i\in v} X_i - ih_z \sum_{i \in v} Z_i\,,$$

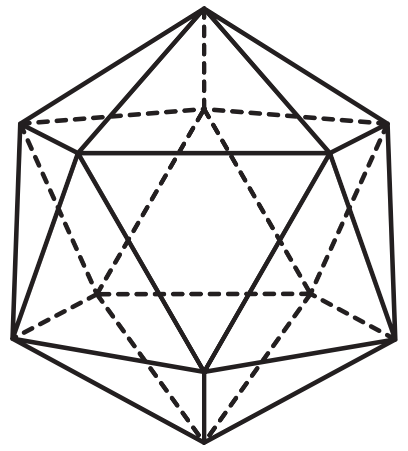
- Exact SO(3) symmetry.
- Locality is approximate.
- Free to change number of sites.

- Manifestly local interaction.
- SO(3) broken to finite groups.
- Number of sites is rigid.

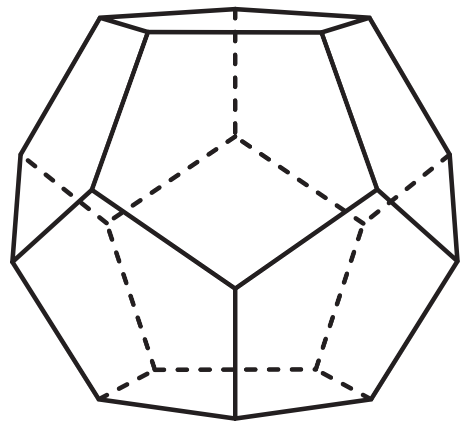
3D YL Criticality on Platonic solids



Cube



Icosahedron



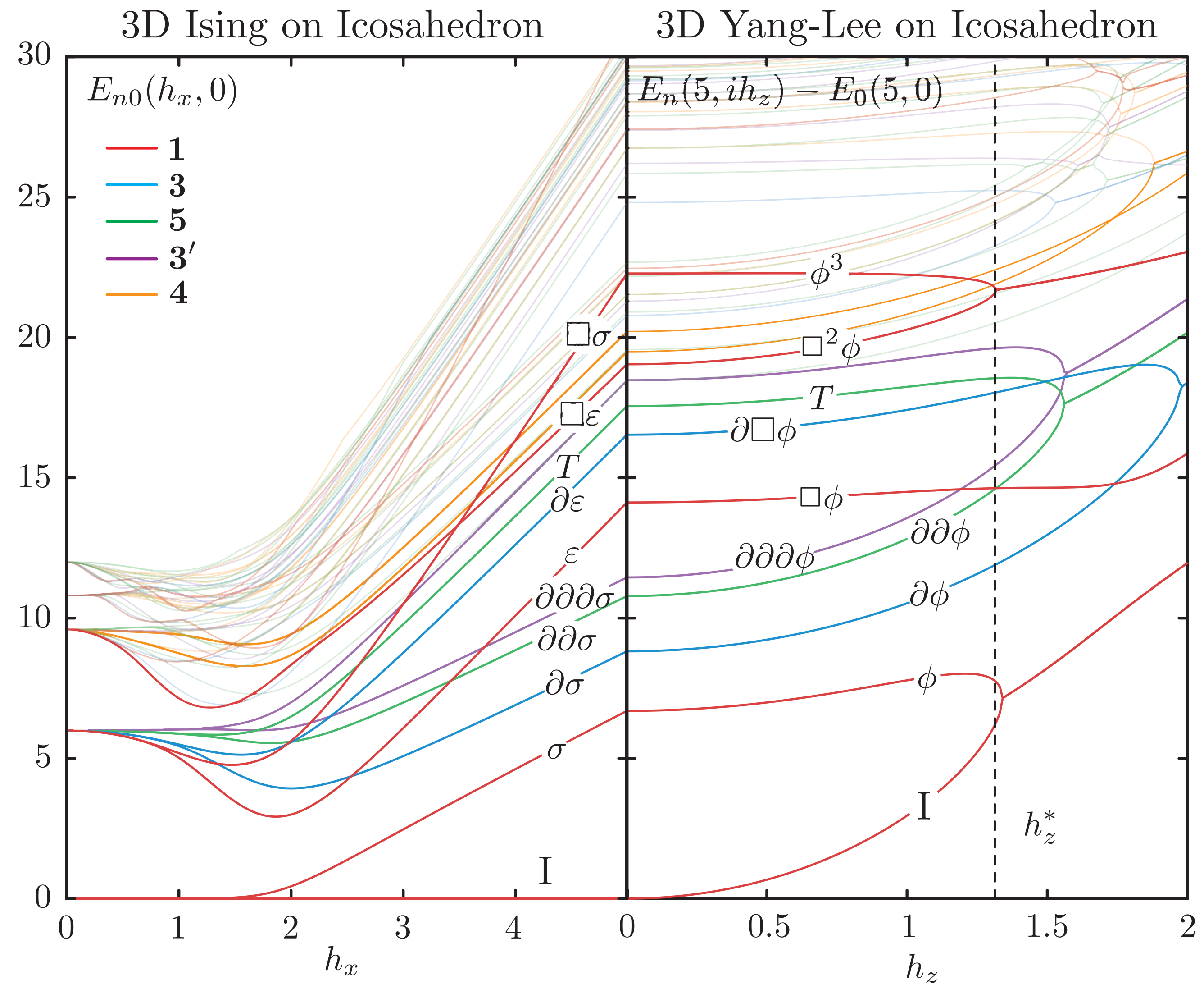
Dodecahedron

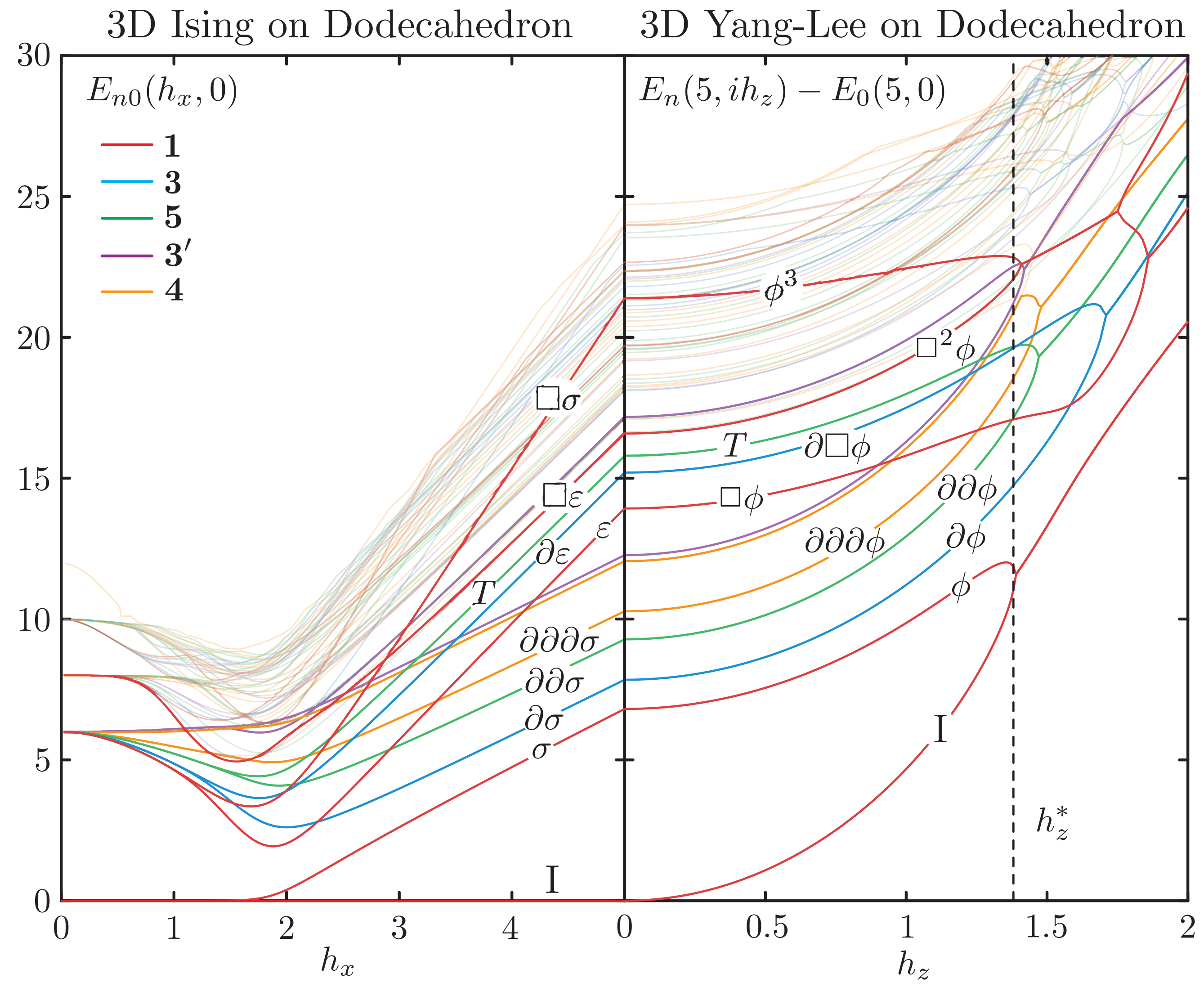
$$H_{\mathrm{YL}} = -J \sum_{\langle ij\rangle \in e} Z_i Z_j - h_x \sum_{i\in v} X_i - i h_z \sum_{i \in v} Z_i \,,$$

- Qualitatively same as fuzzy sphere.
- No easy extrapolation.
- Use stress tensor = 3 criterion.

$$\text{Cube}: \quad r_T = \frac{E_1^{(\mathbf{3}')} - E_0^{(\mathbf{1})}}{E_0^{(\mathbf{3})} - E_1^{(\mathbf{1})}}, \qquad \text{Icosahedron/Dodecahedron}: \quad r_T = \frac{E_1^{(\mathbf{5})} - E_0^{(\mathbf{1})}}{E_0^{(\mathbf{3})} - E_1^{(\mathbf{1})}}$$

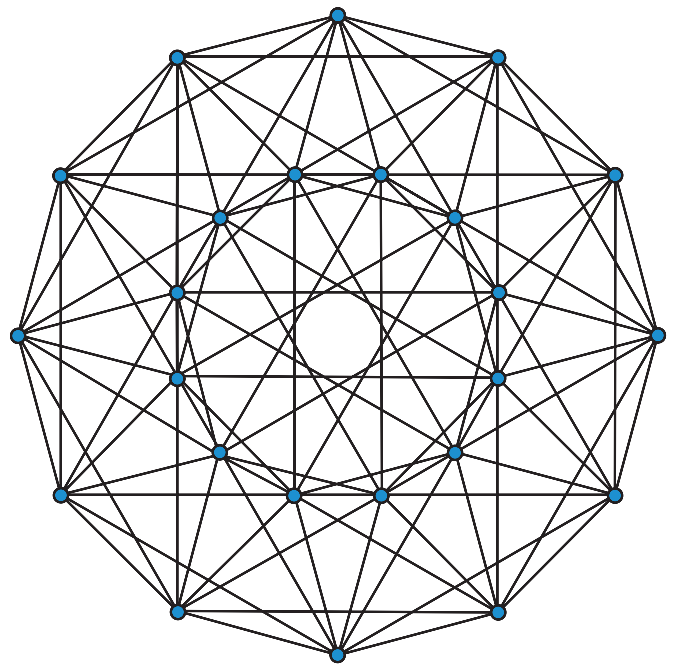
h_x	5	10	15	20	25	30	35	40	45	50
C: Δ_ϕ	0.424	0.346	0.323	0.312	0.305	0.300	0.297	0.294	0.292	0.290
I: Δ_ϕ	0.385	0.306	0.283	0.272	0.265	0.260	0.257	0.254	0.252	0.250
D: Δ_ϕ	0.289	0.244	0.233	0.228	0.224	0.222	0.221	0.220	0.219	0.218





4D YL Criticality on the 24-cell

Numerically accessible CFT beyond 3D

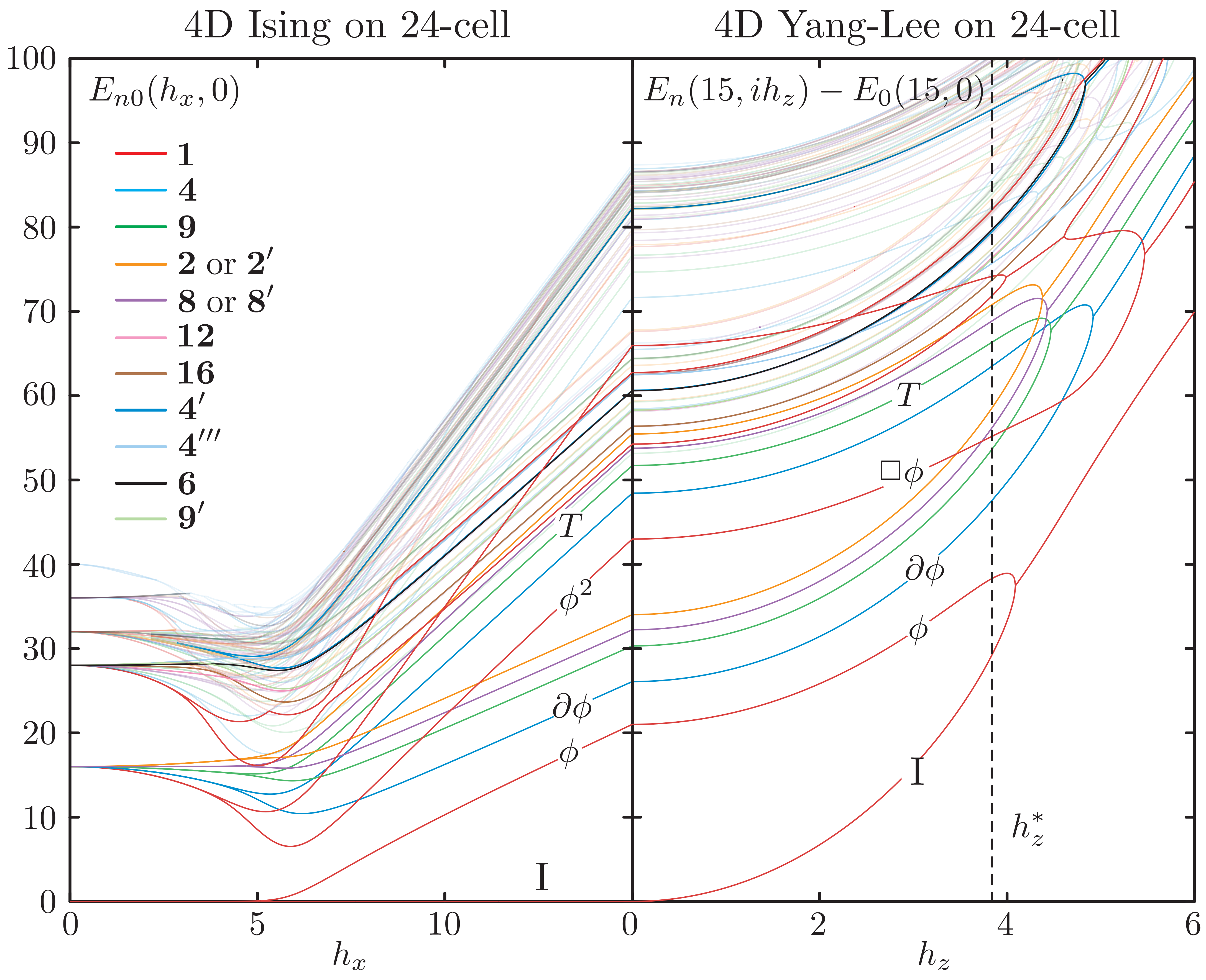


- Regular polytope in 4D with 24 vertices.
- Similar behavior as lower dimensions.

h_x	15	20	30	40	50	60	70	80	90	100
h_z^*	3.839	7.233	14.783	22.890	31.317	39.958	48.753	57.664	66.668	75.746
Δ_ϕ	0.976	0.944	0.914	0.900	0.891	0.886	0.881	0.878	0.876	0.874
$\Delta_{\Box\phi}$	2.796	2.876	2.951	2.987	3.009	3.023	3.034	3.042	3.049	3.054
$\Delta_{\Box^2\phi}$	4.593	4.607	4.638	4.658	4.672	4.682	4.691	4.697	4.702	4.707
Δ_{ϕ^3}	4.842	5.076	5.293	5.401	5.467	5.513	5.546	5.575	5.597	5.605

- Prediction from 2-sided Padé:

$$\Delta_\phi = 0.827 \qquad \Delta_{\phi^3} = 5.216 \sim 5.212$$



Outlook

- We obtained numerical solution to quantum YL criticality in various dimensions and they agree well with the $6-\epsilon$ expansion and are comfortably consistent with conformal symmetry.
- The ϕ^3 family has many members. E.g. $M(3,8)$ has a Ginzburg-Landau description of two scalars with imaginary cubic couplings.
- It would be interesting to combine fuzzy sphere and bootstrap study in 3D. Non-unitary bootstrap requires a initial guess with some precision, which fuzzy sphere can provide. Or could we figure out how to use positivity in open-systems?
- The fact that 24-cell gives us reasonable accuracy for 4D YL is encouraging. Could we generalize fuzzy sphere to 4D?

Thank you!