Bootstrapping Euclidean lattices

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Progress of Theoretical Bootstrap

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Plan of the presentation

1. Introduction
2. Flat tori
3. Some results

Introduction

$\mathcal{M}$ is a flat orientable manifold that is compact and without boundary.
Laplace eigenvalue equation:

$$-\nabla_a \nabla^a \phi_i^{(t)} = \lambda_i \phi_i^{(t)},$$

where $\phi_i^{(t)}$ is a divergence-free traceless symmetric tensors.

Spectrum is discrete and non-negative. Eigenfunctions form a basis for square-integrable tensors. The inner product is

$$\langle \phi^{(t)}, \phi^{(t)} \rangle = \int_{\mathcal{M}} \phi^{(t)}_{a_1 \dots a_t} \phi^{a_1 \dots a_t \, (t)} dV.$$

The bases are orthonormal.

Completeness:

$$\phi_i\phi_j = \frac{1}{V}\delta_{ij} + \sum_{k=1}^{\infty} c_{ijk}\phi_k, \qquad \text{where} \qquad c_{ijk} = \int_{\mathcal{M}} \phi_i\phi_j\phi_k \, dV.$$

Consider a 4pt integral

$$I = \int_{\mathcal{M}} \partial_a\phi_i\partial^a\phi_i\phi_i\phi_i \, dV$$

and use associativity to get the sum rule

$$\frac{1}{V}\lambda_i + \sum_{k=1}^{\infty}\left(\lambda_i - \frac{3}{4}\lambda_k\right)(c_{iik})^2 = 0$$

More derivatives $\Rightarrow$ more sum rules.

Introduction

By analyzing these sum rules numerically, we can extract information about the spectrum $\{\lambda_i\}$ and triple overlaps c_{ijk}.

Related topics:

- scattering amplitudes in KK theories [Bonifacio,Hinterbichler '19]
- other geometries (Einstein [Bonifacio,Hinterbichler '20], hyperbolic [Bonifacio,Gesteau,Kravchuk,Mazáč,Pal,Radcliffe,Simmons-Duffin,Xu et al.])
- rep. theory approach [Bonifacio,Gesteau,Kravchuk,Mazáč,Pal,Simmons-Duffin,Xu et al.]
- study of lattices (sphere packing [Viazovska '16, Mazáč '18, Hartman,Mazáč,Rastelli '19], kissing number)
- characterization of isospectral non-isometric manifolds [Adve '25]
- lattice CFTs [Dymarsky,Shapere '20]

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A flat n-dimensional torus is the quotient $T^n \equiv \mathbb{R}^n/\Lambda$, where Λ is a lattice:

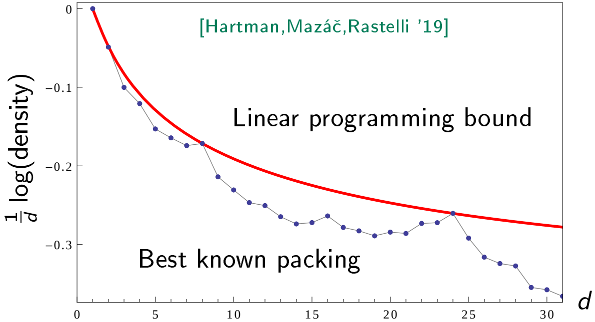
$$\Lambda = \left\{ \sum_{i=1}^{n} \xi_i v_i \mid \xi_i \in \mathbb{Z} \right\},$$

with $\{v_i\}_{i=1}^n$ a basis of $\mathbb{R}^n$.

It is sufficient to specify a generator matrix

$$A = \begin{pmatrix} v_{11} & \cdots & v_{1n} \\ v_{21} & \cdots & v_{2n} \\ \vdots & \ddots & \vdots \\ v_{n1} & \cdots & v_{nn} \end{pmatrix}.$$

- n-dimensional cubic lattice Z^n: $A = \mathrm{diag}(1, 1, \ldots, 1)$
- hexagonal lattice A_2: $A = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$
- Face-centered cubic lattice $A_3 \cong D_3$
- E_8 lattice: all coordinates are integer or all coordinates are half integers, and the sum of the coordinates is even
- Leech lattice Λ_{24}



The dual lattice of Λ is

$$\Lambda^* = \{k \in \mathbb{R}^n \mid k \cdot x \in \mathbb{Z}, \quad \forall x \in \Lambda\}\,.$$

The generator matrix of Λ^* is $(A^{-1})^T$.

Examples: $(Z^n)^* = Z^n, \quad (A_2)^* \cong A_2, \quad (E_8)^* = E_8, \quad (\Lambda_{24})^* = \Lambda_{24}$.

The theta series of a lattice Λ is

$$\Theta_\Lambda(z) = \sum_{x \in \Lambda} q^{x \cdot x} = \sum_{m=0}^{\infty} N_m q^m, \qquad \text{with} \qquad q = e^{i\pi z}.$$

It tells us how many lattice vectors have (squared) norm m.

Flat tori

Laplacian and eigenspectrum

The Laplacian on a flat torus T^n is

$$\Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

Eigenvalue equations:

$$\Delta \phi_i = \lambda_i \phi_i.$$

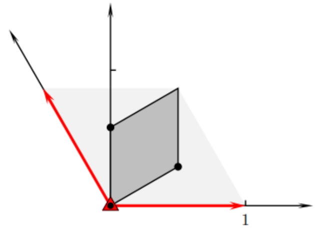
The (scalar) eigenspectrum is:

$$\phi_i(x) = e^{2\pi i \, k_i \cdot x}, \qquad \lambda_i = 4\pi^2 \left\| k_i \right\|^2, \qquad \forall\, x \in T^n, \forall\, k_i \in \Lambda^*.$$

Spectrum and eigenfunctions are known: we can compute triple overlaps c_{ijk} explicitly and compare with bounds!

We can further take quotients of flat tori to obtain flat orbifolds $\mathcal{O} \cong T^n/r(\Gamma)$.

Example [Nilse '06]: $\mathbb{R}^2/p3 = T^2_{A_2}/\mathbb{Z}_3$.



The eigenmodes are just sum over orbits:

$$\phi_i(x) = \sum_{\gamma \in \Gamma/\Lambda} e^{2\pi i\, k_i \cdot \gamma x}, \qquad \lambda_i = 4\pi^2 \|k_i\|^2, \qquad \forall\, x \in T^n, \forall\, k_i \in \Lambda^*.$$

Recall that (scalar) triple overlaps are

$$c_{ijk} = \int_{T^n} \phi_i \phi_j \phi_k dV.$$

For identical external eigenmodes we compute c_{iik}.

Due to eigenmode multiplicity we really compute

$$\tilde{c}_{iik} = \sqrt{\sum_p (c_{iip})^2}.$$

p runs over the multiplicity of the kth eigenmode: this is given by the theta series.

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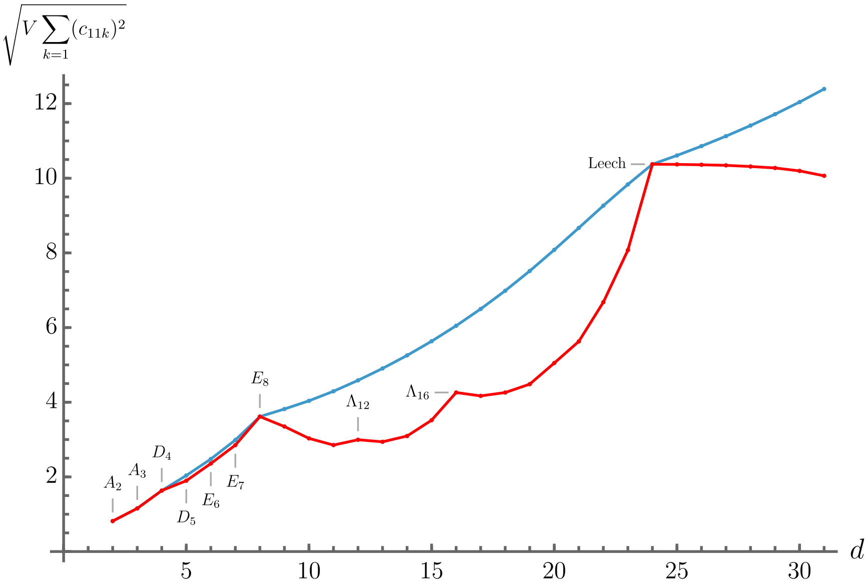
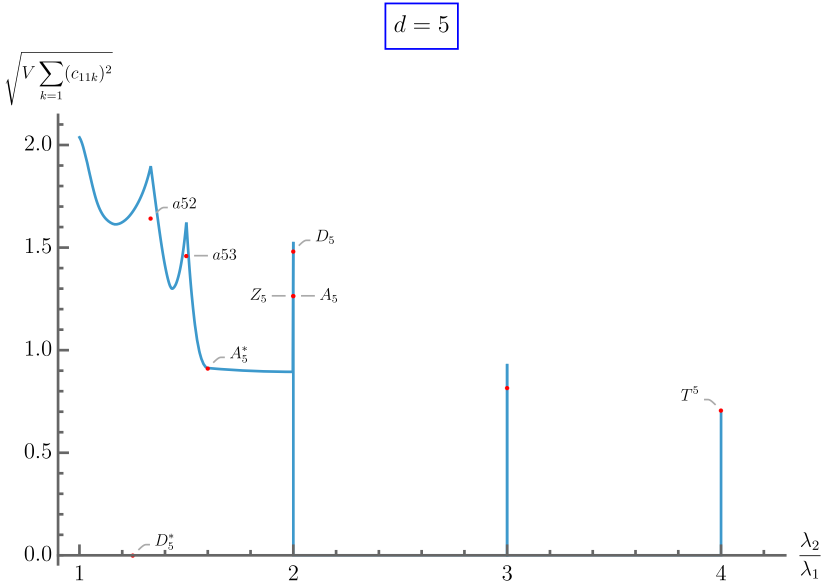
Some results: $\tilde{c}_{111}$



Some results: $\tilde{c}_{112}$ in 5d



So far $t = 0$.

The sum rules also involve spinning overlaps

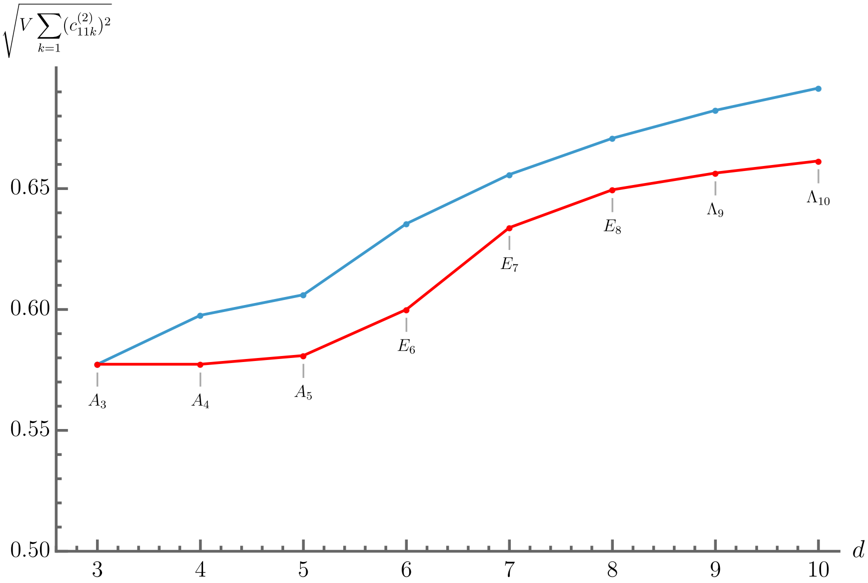
$$c_{ijk}^{(t)} = \int_{\mathcal{M}} \phi_i \nabla^{a_1} \cdots \nabla^{a_t} \phi_j \phi_{k,a_1 \dots a_t}^{(t)} dV.$$

For flat tori:

$$\text{eigenmode} = e^{2\pi i\, k \cdot x} \times \text{tensor structure}.$$

The eigenvalues do not change.

Some results: $\tilde{c}_{111}^{(2)}$



Conclusions

Summary

- we can bootstrap flat manifolds in any dimension
- sharp bounds on triple overlaps c_{ijk}
- some lattices are special

Future directions: non-identical externals, lattice CFTs, spinning manifolds?

Thanks for listening!